PLISKA
STUDIA MATHEMATICA

ПЛИСКА
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. **30** (2019), 127–146

PLISKA
STUDIA MATHEMATICA

ESTIMATES IN MORREY-CAMPANATO SPACES OF A SUITABLE WEAK SOLUTION OF THE NAVIER-STOKES EQUATIONS, SATISFYING AN EXTRA-CONDITION

Jmmy Alfonso Mauro

to my wife Elena on occasion of our marriage

We consider a study of regularity, by means of the theory of Morrey-Campanato spaces, of suitable weak solutions of the Cauchy problem for the non-stationary Navier-Stokes equations, which satisfy a suitable extra-condition.
According to what is known at the moment, the extra-conditions which we consider don't assure the regularity of the suitable weak solution.

1. Introduction

We consider the Cauchy problem for the non-stationary Navier-Stokes equations with unit viscosity and zero body force

$$
\begin{aligned}
&v_t - \Delta v + (v\cdot\nabla)v = -\nabla\pi && \forall\ (x,t) \in \mathbb{R}^n \times (0,T), \\
&\nabla\cdot v = 0 && \forall\ (x,t) \in \mathbb{R}^n \times (0,T), \qquad (1)\\
&v(x,0) = v_0(x) && \forall\ x \in \mathbb{R}^n,
\end{aligned}
$$

with $n \geq 3$; v and π represent the unknown velocity and pressure, respectively.
In our notation $(v\cdot\nabla)v = (\nabla v)v$.
If $n = 3$, the system (1) describes the motion of a Newtonian fluid that fills all the space $\mathbb{R}^3$.

2010 *Mathematics Subject Classification:* 76D05, 35Q30, 76D03.
Key words: Navier-Stokes equations, suitable weak solutions, Morrey-Campanato spaces.

The initial data v_0 should satisfy the compatibility condition $\nabla \cdot v_0 = 0$ in $\mathbb{R}^n$, at least in weak form. Moreover, we also assume the following condition at infinity

$$\lim_{|x|\to\infty} v(x,t) = 0 \qquad \forall\ t \in [0,T)\,.$$

The existence of weak solutions to the initial value problem (1) was proved by J. Leray in [12]; in particular, he introduced the first notion of weak solution for the Navier-Stokes system (cf. Definition 1).
In [11] E. Hopf proved the existence of weak solutions on any smooth enough domain $\Omega \subset R^n$, with $n \geq 2$; nevertheless, such solutions are slightly different to Leray's ones (cf. Definition 2).

Ever since, much effort has been made to establish results on the uniqueness and regularity of weak solutions; however, such questions remain mostly open so far. In particular, till now, it is not known whether or not a Leray weak solution or a Hopf weak one can develop singularities in a finite time, even if the initial data are smooth. The uniqueness problem is strictly related to the regularity one. Indeed, it is well-known that if the solution is smooth enough, then it is unique.

In a series of papers (e.g. see [27, 28]), V. Sheffer introduced the notions of *suitable* weak solution for the Navier-Stokes equations (see Definition 3) and of *generalized energy inequality* (8); he and other authors after (L. Caffarelli, R. Kohn, and L. Nirenberg in [2]) used them in developing the partial regularity theory of the Navier-Stokes system. Some recent improvements of this theory were obtained in [3, 4, 5].

In this paper, we consider a study of regularity of suitable weak solutions, satisfying a suitable extra-condition, by means of the theory of Morrey-Campanato spaces. Indeed, we prove that if v is a suitable weak solution of the Cauchy problem (1) which belongs to $L^p\big(0,T;L^q(\mathbb{R}^n)\big)$ for some pair (p,q) such that

$$p,q \in [3,\infty] \quad \text{and} \quad \frac{n}{q} + \frac{2}{p} = \lambda\,, \qquad \text{with} \quad 1 < \lambda < \frac{n+1}{3}\,,$$

then, there hold the estimates (15) and (16), from which we can deduce

$$\begin{aligned} &\nabla v \in L^{2,k}\big(\Omega \times (\varepsilon,T)\big), \qquad \text{with } k = \frac{n+1-3\lambda}{n+2}, \\ &v \in L^\infty\big(\varepsilon,T;L^{2,h}(\Omega)\big), \qquad \text{with } h = \frac{n+1-3\lambda}{n}, \end{aligned} \tag{2}$$

where $\varepsilon \in (0,T)$, $\Omega \subset \mathbb{R}^n$ is an arbitrary bounded domain which satisfies the cone condition, $L^{2,k}\big(\Omega \times (\varepsilon,T)\big)$ and $L^{2,h}(\Omega)$ are Morrey spaces.

According to what is known at the moment, the extra-conditions which we consider don't assure the regularity of the suitable weak solution.

The result presented in this paper is based on the Ph.D. Thesis [17] that the author defended at the University of Pisa, under the supervision of Prof. Vladimir Georgiev.
The author expresses his gratitude to Prof. Vladimir Georgiev and to Prof. Paolo Maremonti for their useful discussions during the preparation of this work.

1.1. Notations

Throughout this paper, we assume that Ω is a domain in $\mathbb{R}^n$, with $n \geq 3$, which satisfies one of the following conditions:

$(D1)$ $\Omega \equiv \mathbb{R}^n$;

$(D2)$ Ω is a bounded domain in $\mathbb{R}^n$;

$(D3)$ Ω is an exterior domain in $\mathbb{R}^n$.

Moreover, if Ω satisfies condition $(D2)$ or $(D3)$, its bounded boundary $\partial\Omega$ is required to be (at least) of class C^m, where m is an even positive integer such that $2m > n$.

For $1 \leq p \leq \infty$, let $L^p(\Omega)$ be the Lebesgue space of vector valued functions on Ω. The norm in $L^p(\Omega)$ is indicated by $\|\cdot\|_p$ and we use the notation $\langle u, v\rangle = \int_\Omega u \cdot v\, dx$ for any vector fields u, v for which the right hand side makes sense.
For $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, let $W^{m,p}(\Omega)$ be the Sobolev space of functions $u : \Omega \to \mathbb{R}^n$ in $L^p(\Omega)$ with distributional derivatives in $L^p(\Omega)$ up to order m included; the norm in $W^{m,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{m,p}(\Omega)}$.

By $C_0^\infty(\Omega)$ we denote the space of all infinitely differentiable vector valued functions with compact support in Ω. By $\mathscr{C}_0(\Omega)$ we denote the class of all solenoidal vector fields $\varphi(x) \in C_0^\infty(\Omega)$; for $1 < p < \infty$, $J^p(\Omega)$ and $J^{1,p}(\Omega)$ are the closure of $\mathscr{C}_0(\Omega)$ in $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively. We set $J(\Omega) \equiv J^2(\Omega)$.

For $T \in (0, \infty)$ and for a given Banach space $\mathbb{X}$, with associated norm $\|\cdot\|_{\mathbb{X}}$, $L^p\big(0, T; \mathbb{X}\big)$ is the linear space of functions $f : (0, T) \to \mathbb{X}$ such that $\int_0^T \|u(\tau)\|_{\mathbb{X}}^p\, d\tau < \infty$, if $1 \leq p < \infty$, or $\operatorname*{ess\,sup}_{\tau\in(0,T)} \|u(\tau)\|_{\mathbb{X}} < \infty$, if $p = \infty$.
If I is a real interval, we denote by $C\big(I; \mathbb{X}\big)$ the class of continuous functions from I to $\mathbb{X}$.

For every $T \in (0,\infty)$, we set $\Omega_T = \Omega\times[0,T)$ and we define

$$\mathscr{C}_0(\Omega_T) = \{\varphi \in C_0^\infty\big(\Omega_T;\mathbb{R}^n\big) \ : \ \nabla\cdot\varphi = 0 \ \text{ in } \ \Omega_T\}\,.$$

In this work, we use the same symbol to denote functional spaces of scalar or vector valued functions. Moreover, the symbol c denotes a generic positive constant whose numerical value is not essential to our aims. It may assume several different values in a single computation.

1.2. The parabolic metric

Let $\mathrm{d}(x,y) = |x-y|$ the Euclidean metric in $\mathbb{R}^n$; in $\mathbb{R}^n \times (0,\infty)$ we consider the following *parabolic metric*

$$\delta\big((x,t),\,(y,\tau)\big) = \max\Big\{\mathrm{d}(x,y)\,;\,\sqrt{|t-\tau|}\Big\} \qquad \forall\ (x,t),\ (y,\tau) \in \mathbb{R}^n \times (0,\infty)\,.$$

For $(\bar{x},\bar{t}) \in \mathbb{R}^n \times (0,\infty)$ and $r > 0$, we denote by

$$B_r(\bar{x}) = \{x \in \mathbb{R}^n \ \mid \ |x-\bar{x}| < r\}\,;$$

$$Q_r(\bar{x},\bar{t}) = B_r(\bar{x}) \times (\bar{t}-r^2, \bar{t}+r^2)\,;$$

$$Q_r^*(\bar{x},\bar{t}) = B_r(\bar{x}) \times (\bar{t}-r^2, \bar{t})\,;$$

So, $Q_r(\bar{x},\bar{t})$ is the ball of radius $r > 0$, centered at $(\bar{x},\bar{t})$, with respect to the metric δ, which we also call *parabolic cylinder.*

Of course,

$$\begin{matrix} Q_\rho(\bar{x},\bar{t}) \subset Q_r(\bar{x},\bar{t}) \\ Q_\rho^*(\bar{x},\bar{t}) \subset Q_r^*(\bar{x},\bar{t}) \end{matrix} \qquad \text{for} \quad 0<\rho<r\,. \tag{3}$$

By μ we denote the Lebesgue measure; then, we have

$$\mu\big(Q_r(\bar{x},\bar{t})\big) = r^{n+2}\,\mu\big(Q_1(\bar{x},\bar{t})\big) = 2\,\omega_n\,r^{n+2}\,;$$

$$\mu\big(Q_r^*(\bar{x},\bar{t})\big) = \omega_n\,r^{n+2}\,.$$

with $\omega_n = \mu\big(B_1\big)$.

For a bounded domain $\Omega \subset \mathbb{R}^n$ we denote

$$\mathrm{diam}(\Omega) = \sup\big\{|x-y|\ ;\ \ x,y \in \Omega\big\}\,.$$

For $x \in \Omega$ and $0 < \rho \le \operatorname{diam}(\Omega)$ we set

$$\Omega(x,\rho) = \Omega \cap B(x,\rho)\,.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which satisfies the cone condition; for $1 \le p < \infty$ and $0 \le \lambda \le 1$, by $L^{p,\lambda}(\Omega)$ we denote the linear space of all functions $f \in L^p(\Omega)$ such that the quantity

$$\|f\|_{L^{p,\lambda}(\Omega)} = \left\{ \sup_{\substack{x\in\Omega \\ 0<\rho<\operatorname{diam}(\Omega)}} \frac{1}{\big[\mu(\Omega(x,\rho))\big]^{\lambda}} \int_{\Omega(x,\rho)} |f(x)|^p\, dx \right\}^{\frac{1}{p}}$$

is finite. The linear space $L^{p,\lambda}(\Omega)$, equipped with the norm $\|\cdot\|_{L^{p,\lambda}(\Omega)}$, is called Morrey space.

2. Weak solutions: definitions and properties

We give three different definitions of weak solution of the Cauchy problem (1) and we collect some their properties. Let $\Omega \equiv \mathbb{R}^n$, with $n \ge 3$.

Definition 1. *Let $v_0 \in J(\Omega)$. A vector field $v : \Omega \times (0,\infty) \to \mathbb{R}^n$ is said a* Leray weak solution *of problem* (1) *with initial data v_0, if it satisfies the following conditions for all $T \in (0,\infty)$*

1. $v \in L^\infty\big(0,T; J(\Omega)\big) \cap L^2\big(0,T; J^{1,2}(\Omega)\big)$;
2. $\forall\, \varphi \in \mathscr{C}_0(\Omega_T)$

$$\int_0^T \big[\langle v,\varphi_t\rangle - \langle \nabla v,\nabla\varphi\rangle - \langle (v\cdot\nabla)v,\varphi\rangle\big]\, dt = -\langle v_0,\varphi_0\rangle\,; \tag{4}$$

3. *there holds the following energy inequality*

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2\, d\tau \le \|v(s)\|_2^2 \tag{5}$$

for $s = 0$, a.e. $s > 0$ and $\forall\, t \ge s$.

Definition 2. *Let $v_0 \in J(\Omega)$. A vector field $v : \Omega \times (0,\infty) \to \mathbb{R}^n$ is said a* Hopf weak solution *of problem* (1) *with initial data v_0, if it satisfies, for all $T \in (0,\infty)$, conditions* 1, 2 *of Definition 1 and if the energy inequality* (5) *holds only for $s = 0$ and for all $t \ge 0$.*

If Ω is a bounded or exterior domain in $\mathbb{R}^n$ or $\Omega \equiv \mathbb{R}^n$ (with $n = 2, 3, 4$), for any initial data $v_0 \in J(\Omega)$ there exists at least a Leray weak solution of problem (1). Whereas, if Ω is an arbitrary domain in $\mathbb{R}^n$ (with $n \geq 2$), for any initial data $v_0 \in J(\Omega)$ there exists at least a Hopf weak solution (cf. [12, 11, 9, 20], see also [8, Section 3]).
Obviously, every Leray weak solution is a Hopf weak one too.

Remark 1. If v is a Hopf weak solution, by the energy inequality (5) we have

$$\|v\|_{L^\infty(0,\,\infty;\,J(\Omega))} \leq \|v_0\|_2\,, \qquad \|\nabla v\|_{L^2(0,\,\infty;\,L^2(\Omega))} \leq \tfrac{1}{2}\,\|v_0\|_2\,;$$

moreover, by Gagliardo-Nirenberg interpolation inequality, $v \in L^p\big(0, \infty; L^q(\Omega)\big)$ for every pair of exponents (p, q) such that

$$\frac{n}{q} + \frac{2}{p} = \frac{n}{2} \quad \text{and} \quad q \in [2, q^*]\,, \quad \text{with } \frac{1}{q^*} = \frac{1}{2} - \frac{1}{n}\,, \quad \text{for } n \geq 3 \tag{6}$$

and there holds the following estimate

$$\|v\|_{L^p(0,\,\infty;\,L^q(\Omega))} \leq c\,\|v_0\|_2\,,$$

where the positive constant c does not depend on v.

Definition 3. *Let $v_0 \in J(\Omega)$ and $T \in (0, \infty]$. A pair (v, π), having as first component a vector field $v : \Omega \times (0, T) \to \mathbb{R}^n$ and as second component a scalar function $\pi : \Omega \times (0, T) \to \mathbb{R}$, is said a* suitable weak solution *of problem* (1)*, in $\Omega \times (0, T)$, with initial data v_0, if the following conditions are satisfied*

1. $v \in L^\infty\big(0, T; J(\Omega)\big) \cap L^2\big(0, T; J^{1,2}(\Omega)\big)$ *and* $\pi \in L^{\frac{5}{3}}\big(\Omega \times (0, T)\big)$*;*
2. *the energy inequality* (5) *holds, at least, for $s = 0$ and for all $t \in (0, T)$;*
3. $\forall\, \phi \in C_0^\infty\big(\Omega_T; \mathbb{R}^n\big)$

$$\int_0^T \big[\langle v, \phi_t\rangle - \langle \nabla v, \nabla \phi\rangle - \langle (v\cdot\nabla)\, v, \phi\rangle\big]\, dt = -\int_0^T \langle \pi, \nabla\cdot\phi\rangle\, dt - \langle v_0, \phi_0\rangle\,; \tag{7}$$

4. *for every non-negative, scalar valued function $\sigma \in C_0^\infty\big(\Omega_T; \mathbb{R}\big)$ there holds the following generalized energy inequality*

$$\begin{aligned} \int_\Omega |v(t)|^2 \sigma(t)\, dx + 2\int_s^t\!\!\int_\Omega |\nabla v|^2 \sigma\, dx\, d\tau \leq \int_\Omega |v(s)|^2 \sigma(s)\, dx \\ + \int_s^t\!\!\int_\Omega |v|^2 (\sigma_\tau + \Delta\sigma)\, dx\, d\tau + \int_s^t\!\!\int_\Omega (|v|^2 + 2\pi) v\cdot\nabla\sigma\, dx\, d\tau \end{aligned} \tag{8}$$

for $s = 0$, a.e. $s \in (0, T)$ and $\forall\, t \in (s, T)$.

The existence of suitable weak solutions to the Cauchy problem (1) was proved in Theorem A.1, in the Appendix of [2].

Definition 4. *A point $(x,t) \in \Omega \times (0,T)$ is called* singular *for a solution v of system* (1) *iff the vector field v is not essentially bounded [i.e. $v \notin L^\infty(I_{(x,t)})$] on any neighborhood $I_{(x,t)}$ of (x,t).*

For a suitable weak solution (v,π), there holds the following result (cf. [2, Proposition 1, Proposition 2], [14, Theorem 3.1, Theorem 3.3], [34, Theorem 2]).

Theorem 1. *If $n = 3$, there exist universal constants δ_1^*, δ_2^* such that the following property holds for any suitable weak solution (v,π) of problem* (1)*, in $\Omega \times (0,T)$, with $\pi \in L^{\frac{3}{2}}\big(\Omega \times (0,T)\big)$. Let $(\bar{x},\bar{t})$ be in $\Omega \times (0,T)$ and such that*

$$\limsup_{r\to 0^+} \frac{1}{r} \int_{\bar{t}-r^2}^{\bar{t}+r^2} \int_{B_r(\bar{x})} |\nabla v|^2 \, dx\, dt \le \delta_1^* \,,$$

or

$$\limsup_{r\to 0^+} \frac{1}{r^2} \int_{\bar{t}-r^2}^{\bar{t}+r^2} \int_{B_r(\bar{x})} \big[|v|^3 + |\pi|^{\frac{3}{2}}\big] \, dx\, dt \le \delta_2^* \,,$$

then, v is bounded in a neighborhood of $(\bar{x},\bar{t})$ (i.e. $(\bar{x},\bar{t})$ is a regular point).

Some local regularity results for suitable weak solutions are also obtained in [33], with slightly different hypothesis.

As a consequence of Theorem 1, for a suitable weak solution (v,π), there holds the following local partial regularity result (cf. [2, Theorem B] and [14]); here $\mathscr{P}^1$ denotes a measure on $\mathbb{R}^3 \times (0,\infty)$ analogous to one-dimensional Hausdorff measure $\mathscr{H}^1$, but defined using parabolic cylinders instead of Euclidean balls (cf. [2, Section 2D]).

Theorem 2. *Let Ω be an arbitrary domain in $\mathbb{R}^3$ and let $T \in (0,\infty]$; for any suitable weak solution (v,π) of problem* (1) *in $\Omega \times (0,T)$, with $\pi \in L^{\frac{3}{2}}\big(\Omega \times (0,T)\big)$, the associated set $\mathcal{S}$ of possible singular points satisfies $\mathscr{P}^1(\mathcal{S}) = 0$.*

In the previous theorem, the hypothesis $\pi \in L^{\frac{3}{2}}\big(\Omega \times (0,T)\big)$ can be weakened to $\pi \in L^{\frac{5}{4}}\big(0,T; L^{\frac{5}{4}}_{\mathrm{loc}}(\Omega)\big)$ (cf. [2, Section 2C] and [31]).

3. Estimates in Morrey-Campanato spaces

In this section, we consider a study of regularity of suitable weak solutions, satisfying a suitable extra-condition, by means of the theory of Morrey-Campanato spaces.

Let $n \geq 3$ and $T \in (0, \infty)$; let (p, q) be a pair such that

$$p, q \in [3, \infty] \quad \text{and} \quad \frac{n}{q} + \frac{2}{p} = \lambda \,, \qquad \text{with} \quad 1 < \lambda < \frac{n+1}{3} \,. \tag{9}$$

Let v be a suitable weak solution of the Cauchy problem (1), in $\mathbb{R}^n \times (0, T)$, with initial data $v_0 \in J(\mathbb{R}^n)$, such that $v \in L^p\big(0, T; L^q(\mathbb{R}^n)\big)$.
In a similar way as in Remark 1, by Gagliardo-Nirenberg interpolation inequality and the energy inequality (5), we have that any suitable weak solution w of the Cauchy problem (1), in $\mathbb{R}^n \times (0, T)$, is in $L^r\big(0, T; L^s(\mathbb{R}^n)\big)$ for every pair of exponents (r, s) satisfying (6). Since $\dfrac{n+1}{3} < \dfrac{n}{2}$ for $n \geq 3$, a priori, any suitable weak solution w of the Cauchy problem (1), in $\mathbb{R}^n \times (0, T)$, doesn't satisfy the summability property $w \in L^p\big(0, T; L^q(\mathbb{R}^n)\big)$. Thus, we have to consider it as an extra-condition.

According to what is known at the moment, such extra-condition doesn't assure the regularity of the suitable weak solution, for any pair of exponents (p, q) satisfying (9) (see Remark 4).

For $(\bar{x}, \bar{t}) \in \mathbb{R}^n \times (0, T]$ and $0 < r < \sqrt{\bar{t}}$, we set

$$F_{Q_r^*}(v) = \sup_{\bar{t}-r^2 \leq t \leq \bar{t}} \int\limits_{B_r(\bar{x})} |v(t)|^2 \, dx \,; \qquad G_{Q_r^*}(v) = \iint\limits_{Q_r^*(\bar{x}, \bar{t})} |\nabla v|^2 \, dx \, dt \,;$$

$$K_{Q_r^*}(v) = \left\{ \int_{\bar{t}-r^2}^{\bar{t}} \|v(t)\|^p_{L^q(B_r(\bar{x}))} \, dt \right\}^{\frac{1}{p}} ; \qquad H_{Q_r^*}(\pi) = \left\{ \int_{\bar{t}-r^2}^{\bar{t}} \|\pi(t)\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(B_r(\bar{x}))} \, dt \right\}^{\frac{2}{p}} ;$$

for $(\bar{x}, \bar{t}) \in \mathbb{R}^n \times (0, T)$ and $0 < r < \min\{\sqrt{\bar{t}}; \sqrt{T - \bar{t}}\,\}$, we set

$$F_{Q_r}(v) = \sup_{\bar{t}-r^2 \leq t \leq \bar{t}+r^2} \int\limits_{B_r(\bar{x})} |v(t)|^2 \, dx \,; \qquad G_{Q_r}(v) = \iint\limits_{Q_r(\bar{x}, \bar{t})} |\nabla v|^2 \, dx \, dt \,;$$

$$K_{Q_r}(v) = \left\{ \int_{\bar{t}-r^2}^{\bar{t}+r^2} \|v(t)\|^p_{L^q(B_r(\bar{x}))} \, dt \right\}^{\frac{1}{p}} ; \qquad H_{Q_r}(\pi) = \left\{ \int_{\bar{t}-r^2}^{\bar{t}+r^2} \|\pi(t)\|^{\frac{p}{2}}_{L^{\frac{q}{2}}(B_r(\bar{x}))} \, dt \right\}^{\frac{2}{p}} .$$

Lemma 1. *Let* $(\bar{x},\bar{t}) \in \mathbb{R}^n \times (0,T]$*; for* $0 < r < \sqrt{\bar{t}}$ *and* $0 < \rho \leq \frac{r}{2}$*, there holds*

$$\frac{F_{Q^*_\rho}(v)}{\rho^{n+1-3\lambda}} \leq c\left(K_{Q^*_r}(v)\right)^2 r^{\lambda-1} + c\left(K_{Q^*_r}(v)\right)^3 + c\,K_{Q^*_r}(v)\,H_{Q^*_r}(\pi)\,; \tag{10a}$$

$$\frac{G_{Q^*_\rho}(v)}{\rho^{n+1-3\lambda}} \leq c\left(K_{Q^*_r}(v)\right)^2 r^{\lambda-1} + c\left(K_{Q^*_r}(v)\right)^3 + c\,K_{Q^*_r}(v)\,H_{Q^*_r}(\pi)\,. \tag{10b}$$

Let $(\bar{x},\bar{t}) \in \mathbb{R}^n \times (0,T)$*; for* $0 < r < \min\{\sqrt{\bar{t}};\sqrt{T-\bar{t}}\}$ *and* $0 < \rho \leq \frac{r}{2}$*, there holds*

$$\frac{F_{Q_\rho}(v)}{\rho^{n+1-3\lambda}} \leq c\left(K_{Q_r}(v)\right)^2 r^{\lambda-1} + c\left(K_{Q_r}(v)\right)^3 + c\,K_{Q_r}(v)\,H_{Q_r}(\pi)\,; \tag{10c}$$

$$\frac{G_{Q_\rho}(v)}{\rho^{n+1-3\lambda}} \leq c\left(K_{Q_r}(v)\right)^2 r^{\lambda-1} + c\left(K_{Q_r}(v)\right)^3 + c\,K_{Q_r}(v)\,H_{Q_r}(\pi)\,. \tag{10d}$$

P r o o f. Let $h \in C_0^\infty\big((-1,1);\mathbb{R}\big)$ and $g \in C_0^\infty\big(B(0;1);\mathbb{R}\big)$ such that

$$\begin{aligned} g(x) &\in [0,1] \ \ \forall \ x \in B(0;1) & \qquad h(t) &\in [0,1] \ \ \forall \ t \in (-1,1) \\ g(x) &= 1 \text{ for } |x| \leq \frac{1}{2} & h(t) &= 1 \text{ for } |t| \leq \frac{1}{2} \\ g(x) &= 0 \text{ for } |x| \geq 1 & h(t) &= 0 \text{ for } |t| \geq 1 \end{aligned}$$

If we set $\sigma_r(x,t) = h\left(\frac{t-\bar{t}}{r^2}\right)\,g\left(\frac{x-\bar{x}}{r}\right)$ for $r>0$, we get $\sigma_r \in C_0^\infty\big(Q_r(\bar{x},\bar{t});[0,1]\big)$ and for every $(x,t) \in Q_r(\bar{x},\bar{t})$

$$\left|\frac{\partial \sigma_r}{\partial t}(x,t)\right| \leq \frac{C}{r^2}, \qquad \left|\nabla \sigma_r(x,t)\right| \leq \frac{C}{r}, \qquad \left|\mathrm{D}^2\sigma_r(x,t)\right| \leq \frac{C}{r^2}. \tag{11}$$

Let $(\bar{x},\bar{t}) \in \mathbb{R}^n \times (0,T]$ and $0 < r < \sqrt{\bar{t}}$, if we consider $Q^*_r(\bar{x},\bar{t})$; let $(\bar{x},\bar{t}) \in \mathbb{R}^n \times (0,T)$ and $0 < r < \min\{\sqrt{\bar{t}};\sqrt{T-\bar{t}}\}$, if we consider $Q_r(\bar{x},\bar{t})$). For $0 < \rho \leq \frac{r}{2}$, we can choose $\sigma_{2\rho}$ as test function in the generalized energy inequality (8); so, since $\sigma_{2\rho}(x,t) = 1$ for every $(x,t) \in \overline{Q_\rho}(\bar{x},\bar{t})$ and $\mathrm{supp}(\sigma_{2\rho}) \subset Q_{2\rho}(\bar{x},\bar{t})$, bearing in mind (3) we have

$$\iint\limits_{Q^*_\rho(\bar{x},\bar{t})} |\nabla v|^2\, dx\, dt = \iint\limits_{Q^*_\rho(\bar{x},\bar{t})} |\nabla v|^2 \sigma_{2\rho}\, dx\, dt$$

$$\leq \int\limits_{B_{2\rho}(\bar{x})} |v(\bar{t})|^2 \sigma_{2\rho}(\bar{t})\, dx + 2 \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} |\nabla v|^2 \sigma_{2\rho}\, dx\, dt$$

$$\leq \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho})\, dx\, dt + \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho}\, dx\, dt\,;$$

and

$$\iint\limits_{Q_\rho(\bar{x},\bar{t})} |\nabla v|^2\, dx\, dt = \iint\limits_{Q_\rho(\bar{x},\bar{t})} |\nabla v|^2 \sigma_{2\rho}\, dx\, dt \leq 2 \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} |\nabla v|^2 \sigma_{2\rho}\, dx\, dt$$

$$\leq \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho})\, dx\, dt + \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho}\, dx\, dt\,;$$

For every $t \in [\bar{t} - \rho^2, \bar{t}]$

$$\int\limits_{B_\rho(\bar{x})} |v(t)|^2\, dx = \int\limits_{B_\rho(\bar{x})} |v(t)|^2 \sigma_{2\rho}(t)\, dx$$

$$\leq \int\limits_{B_{2\rho}(\bar{x})} |v(t)|^2 \sigma_{2\rho}(t)\, dx + 2 \int\limits_{\bar{t}-4\rho^2}^{t} \int\limits_{B_{2\rho}(\bar{x})} |\nabla v|^2 \sigma_{2\rho}\, dx\, dt$$

$$\leq \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho})\, dx\, dt + \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho}\, dx\, dt\,;$$

So

$$\sup_{\bar{t}-\rho^2 \leq t \leq \bar{t}} \int\limits_{B_\rho(\bar{x})} |v(t)|^2\, dx \leq \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho})\, dx\, dt + \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho}\, dx\, dt\,;$$

Similarly, for every $t \in [\bar{t}-\rho^2, \bar{t}+\rho^2]$

$$\int\limits_{B_\rho(\bar{x})} |v(t)|^2 \, dx \le \int\limits_{B_{2\rho}(\bar{x})} |v(t)|^2 \sigma_{2\rho}(t) \, dx + 2 \int\limits_{\bar{t}-4\rho^2}^{t} \int\limits_{B_{2\rho}(\bar{x})} |\nabla v|^2 \sigma_{2\rho} \, dx \, dt$$
$$\le \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho}) \, dx \, dt + \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho} \, dx \, dt \,,$$

and

$$\sup_{\bar{t}-\rho^2 \le t \le \bar{t}+\rho^2} \int\limits_{B_\rho(\bar{x})} |v(t)|^2 dx \le \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho}) dx \, dt + \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho} dx \, dt \,.$$

Then we have

$$\text{(12a)} \quad F_{Q^*_\rho}(v) \le \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho}) \, dx \, dt + \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho} \, dx \, dt \,;$$

$$\text{(12b)} \quad G_{Q^*_\rho}(v) \le \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho}) \, dx \, dt + \iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho} \, dx \, dt \,.$$

$$\text{(12c)} \quad F_{Q_\rho}(v) \le \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho}) \, dx \, dt + \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho} \, dx \, dt \,;$$

$$\text{(12d)} \quad G_{Q_\rho}(v) \le \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} |v|^2 (\partial_t \sigma_{2\rho} + \Delta \sigma_{2\rho}) \, dx \, dt + \iint\limits_{Q_{2\rho}(\bar{x},\bar{t})} (|v|^2 + 2\pi) v \cdot \nabla \sigma_{2\rho} \, dx \, dt \,.$$

Since $v \in L^p\big(0,T; L^q(\mathbb{R}^n)\big)$, from [2, Section 2c] there follows

$\pi \in L^{\frac{p}{2}}\big(0,T;L^{\frac{q}{2}}(\mathbb{R}^n)\big)$; so, using estimates (11) and Hölder's inequality, we obtain

$$\iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})}|v|^2(\partial_t\sigma_{2\rho}+\Delta\sigma_{2\rho})\,dx\,dt+\iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})}(|v|^2+2\pi)v\cdot\nabla\sigma_{2\rho}\,dx\,dt$$

$$\le \frac{c}{\rho^2}\iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})}|v|^2\,dx\,dt+\frac{c}{\rho}\iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})}\big(|v|^3+|\pi|\,|v|\big)\,dx\,dt$$

$$\le \frac{c}{\rho^2}\big(K_{Q^*_{2\rho}}(v)\big)^2\rho^{n(1-\frac{2}{q})+2(1-\frac{2}{p})}$$

$$+\frac{c}{\rho}\Big[\big(K_{Q^*_{2\rho}}(v)\big)^3+K_{Q^*_{2\rho}}(v)\,H_{Q^*_{2\rho}}(\pi)\Big]\rho^{n(1-\frac{3}{q})+2(1-\frac{3}{p})}$$

$$= c\big(K_{Q^*_{2\rho}}(v)\big)^2\rho^{n-2(\frac{n}{q}+\frac{2}{p})}$$

$$+c\Big[\big(K_{Q^*_{2\rho}}(v)\big)^3+K_{Q^*_{2\rho}}(v)\,H_{Q^*_{2\rho}}(\pi)\Big]\rho^{n+1-3(\frac{n}{q}+\frac{2}{p})}$$

Since $0<2\rho\le r$ for $0<\rho\le\dfrac{r}{2}$, recalling $\dfrac{n}{q}+\dfrac{2}{p}=\lambda$, we get

$$\iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})}|v|^2(\partial_t\sigma_{2\rho}+\Delta\sigma_{2\rho})\,dx\,dt+\iint\limits_{Q^*_{2\rho}(\bar{x},\bar{t})}(|v|^2+2\pi)v\cdot\nabla\sigma_{2\rho}\,dx\,dt$$

$$\le c\big(K_{Q^*_r}(v)\big)^2\rho^{n-2\lambda}+c\Big[\big(K_{Q^*_r}(v)\big)^3+K_{Q^*_r}(v)\,H_{Q^*_r}(\pi)\Big]\rho^{n+1-3\lambda}\,.$$

In a similar way, we also get

$$\iint\limits_{Q_{2\rho}(\bar{x},\bar{t})}|v|^2(\partial_t\sigma_{2\rho}+\Delta\sigma_{2\rho})\,dx\,dt+\iint\limits_{Q_{2\rho}(\bar{x},\bar{t})}(|v|^2+2\pi)v\cdot\nabla\sigma_{2\rho}\,dx\,dt$$

$$\le c\big(K_{Q_r}(v)\big)^2\rho^{n-2\lambda}+c\Big[\big(K_{Q_r}(v)\big)^3+K_{Q_r}(v)\,H_{Q_r}(\pi)\Big]\rho^{n+1-3\lambda}\,.$$

Then, since

$$\lambda>1 \qquad\Longrightarrow\qquad n-2\lambda>n+1-3\lambda\,,$$

$$\lambda<\frac{n+1}{3} \qquad\Longrightarrow\qquad n+1-3\lambda>0\,,$$

from (12) we obtain (10) for every $0<\rho\le\dfrac{r}{2}$. □

Lemma 2. *Let $(\bar{x},\bar{t}) \in \mathbb{R}^n \times (0,T]$; for $0<\rho\le r<\sqrt{\bar{t}}$, there holds*

$$\frac{F_{Q^*_\rho}(v)}{\rho^{n+1-3\lambda}} \le c\,\frac{F_{Q^*_r}(v)}{r^{n+1-3\lambda}} + c\left(K_{Q^*_r}(v)\right)^2 r^{\lambda-1} + c\left(K_{Q^*_r}(v)\right)^3 + c\,K_{Q^*_r}(v)\,H_{Q^*_r}(\pi)\,; \tag{13a}$$

$$\frac{G_{Q^*_\rho}(v)}{\rho^{n+1-3\lambda}} \le c\,\frac{G_{Q^*_r}(v)}{r^{n+1-3\lambda}} + c\left(K_{Q^*_r}(v)\right)^2 r^{\lambda-1} + c\left(K_{Q^*_r}(v)\right)^3 + c\,K_{Q^*_r}(v)\,H_{Q^*_r}(\pi)\,. \tag{13b}$$

P r o o f. For $0<\rho\le\frac{r}{2}$, (13a) is a consequence of (10a), while (13b) is a consequence of (10b). For $\frac{r}{2}<\rho\le r$

$$\frac{G_{Q^*_\rho}(v)}{\rho^{n+1-3\lambda}} = \frac{1}{\rho^{n+1-3\lambda}} \iint\limits_{Q^*_\rho(\bar{x},\bar{t})} |\nabla v|^2\,dx\,dt < \left(\frac{2}{r}\right)^{n+1-3\lambda} \iint\limits_{Q^*_\rho(\bar{x},\bar{t})} |\nabla v|^2\,dx\,dt$$

$$\le \frac{c}{r^{n+1-3\lambda}} \iint\limits_{Q^*_r(\bar{x},\bar{t})} |\nabla v|^2\,dx\,dt = c\,\frac{G_{Q^*_r}(v)}{r^{n+1-3\lambda}}\,,$$

from which there follows (13b).

Similarly, for $\frac{r}{2}<\rho\le r$ and for every $t\in[\bar{t}-\rho^2,\bar{t}]$

$$\frac{1}{\rho^{n+1-3\lambda}} \int\limits_{B_\rho(\bar{x})} |v(t)|^2\,dx < \left(\frac{2}{r}\right)^{n+1-3\lambda} \int\limits_{B_\rho(\bar{x})} |v(t)|^2\,dx \le \frac{c}{r^{n+1-3\lambda}} \int\limits_{B_r(\bar{x})} |v(t)|^2\,dx$$

$$\le \frac{c}{r^{n+1-3\lambda}} \sup_{\bar{t}-r^2\le t\le\bar{t}} \int\limits_{B_r(\bar{x})} |v(t)|^2\,dx = c\,\frac{F_{Q^*_r}(v)}{r^{n+1-3\lambda}}\,.$$

So, for $\frac{r}{2}<\rho\le r$,

$$\frac{F_{Q^*_\rho}(v)}{\rho^{n+1-3\lambda}} \le c\,\frac{F_{Q^*_r}(v)}{r^{n+1-3\lambda}}\,,$$

from which there follows (13a). □

Theorem 3. *Let $n \geq 3$ and $T \in (0,\infty)$; let (p,q) be a pair such that*

$$p, q \in [3,\infty] \quad \text{and} \quad \frac{n}{q} + \frac{2}{p} = \lambda\,, \qquad \text{with} \quad 1 < \lambda < \frac{n+1}{3}\,. \tag{14}$$

If v is a suitable weak solution of the Cauchy problem (1), in $\mathbb{R}^n \times (0,T)$, with initial data $v_0 \in J(\mathbb{R}^n)$, such that $v \in L^p\big(0,T;L^q(\mathbb{R}^n)\big)$, then there hold the following estimates

$$\begin{aligned} \sup_{\substack{(\bar{x},\bar{t})\in\Omega\times(\varepsilon,T)\\ \rho>0}} \ \frac{1}{\rho^{n+1-3\lambda}} \iint\limits_{\Omega^\varepsilon((\bar{x},\bar{t});\rho)} |\nabla v|^2\, dx\, dt &\leq \frac{c}{\varepsilon^{\frac{n+1-3\lambda}{2}}} \, \|\nabla v\|^2_{L^2(0,\,T;\,L^2(\mathbb{R}^n))} \\ &\quad + c\,\varepsilon^{\frac{\lambda-1}{2}} \|v\|^2_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} + c\,\|v\|^3_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} \\ &\quad + c\,\|v\|_{L^p(0,\,T;\,L^q(\mathbb{R}^n))}\,\|\pi\|_{L^{p/2}(0,\,T;\,L^{q/2}(\mathbb{R}^n))}\,; \end{aligned} \tag{15}$$

$$\begin{aligned} &\sup_{\substack{(\bar{x},\bar{t})\in\Omega\times(\varepsilon,T)\\ \rho>0}} \ \frac{1}{\rho^{n+1-3\lambda}} \sup_{t\in(\bar{t}-\rho^2,\bar{t}+\rho^2)\cap(\varepsilon,T)} \int\limits_{B_\rho(\bar{x})\cap\Omega} |v(t)|^2\, dx \\ &\quad \leq \frac{c}{\varepsilon^{\frac{n+1-3\lambda}{2}}} \, \|v\|^2_{L^\infty(0,\,T;\,J(\mathbb{R}^n))} + c\,\varepsilon^{\frac{\lambda-1}{2}} \|v\|^2_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} \\ &\quad + c\,\|v\|^3_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} + c\,\|v\|_{L^p(0,\,T;\,L^q(\mathbb{R}^n))}\,\|\pi\|_{L^{p/2}(0,\,T;\,L^{q/2}(\mathbb{R}^n))}\,. \end{aligned} \tag{16}$$

for every bounded domain $\Omega \subset \mathbb{R}^n$ which satisfies the cone condition and for every $\varepsilon \in (0,T)$.

Remark 2. For an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ which satisfies the cone condition and for $\varepsilon \in (0,T)$, from estimate (15) there follows

$$\nabla v \in L^{2,k}\big(\Omega \times (\varepsilon,T)\big), \quad \text{with} \quad k = \frac{n+1-3\lambda}{n+2} \ \in \ \big(0\,,\ \tfrac{n-2}{n+2}\big)\,;$$

similarly to how we obtained estimate (16), we can also prove

$$v \in L^\infty\big(\varepsilon,T;L^{2,h}(\Omega)\big), \quad \text{with} \quad h = \frac{n+1-3\lambda}{n} \ \in \ \big(0\,,\ \tfrac{n-2}{n}\big)\,.$$

Remark 3. Let $n \geq 3$ and $T \in (0, \infty)$; let v be a suitable weak solution of the Cauchy problem (1), in $\mathbb{R}^n \times (0,T)$, with initial data $v_0 \in J(\mathbb{R}^n)$, such that $v \in L^4\big(0,T; L^4(\mathbb{R}^n)\big)$. Then, v satisfies the following energy equality

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2 \, d\tau = \|v(s)\|_2^2, \qquad \forall \; s,t \in [0,T] \,, \quad s \leq t. \tag{17}$$

Moreover v belongs to $C\big([0,T]; J(\Omega)\big)$. This result was originally proved by G. Prodi for a Hopf weak solution, in the two-dimensional and three-dimensional case (cf. [22, 23], [13, Théorème 1], [8, Theorem 4.1]).
In Theorem 3, for $p = q = 4$ we get

$$1 < \lambda = \frac{n+2}{4} < \frac{n+1}{3} \qquad \text{for} \quad n \geq 3\,;$$

then, by Remark 2, we have

$$\nabla v \in L^{2,k}\big(\Omega \times (\varepsilon, T)\big), \qquad \text{with } \; k = \frac{n-2}{4(n+2)}\,,$$

$$v \in L^{\infty}\big(\varepsilon, T; L^{2,h}(\Omega)\big), \quad \text{with } \; h = \frac{n-2}{4n}\,,$$

for every bounded domain $\Omega \subset \mathbb{R}^n$ which satisfies the cone condition and for every $\varepsilon \in (0,T)$.

Remark 4. If v is a suitable weak solution such that $v \in L^p\big(0,T; L^q(\mathbb{R}^3)\big)$, for some pair (p,q) such that $\dfrac{n}{q} + \dfrac{2}{p} = 1$ and $q > n$, then v is regular in $\mathbb{R}^3 \times (0,T)$ (cf. [25, 26], [30, Theorem 3.1], [10, Theorem 5−ii]); for a survey of regularity results see also [8, Section 5] and [6].
Recently, in [16, Theorem 2] P. Maremonti proved that, in the three-dimensional case, a Hopf weak solution v to the IBVP in Ω is regular if for all $\varepsilon > 0$ $v \in L^p\big(\varepsilon, T; L^q(\Omega)\big)$, with p, q such that $\dfrac{3}{q} + \dfrac{2}{p} = 1$ and $q > 3$.

According to what is known at the moment, the extra-condition considered in Theorem 3 doesn't assure the regularity of the suitable weak solution v, for any pair of exponents (p,q) satisfying (14).

Proof. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain which satisfies the cone condition and let $\varepsilon \in (0,T)$. For $(x,t) \in \Omega \times (\varepsilon, T)$ and $r > 0$ we set

$$\begin{aligned}\Omega^{\varepsilon}\big((x,t);r\big) &= Q_r(x,t) \cap \big(\Omega \times (\varepsilon,T)\big) \\ &= \big(B_r(x) \cap \Omega\big) \times \big((t - r^2, t + r^2) \cap (\varepsilon, T)\big).\end{aligned}$$

Let $(\bar{x},\bar{t}) \in \Omega \times (\varepsilon, T)$ be fixed; for $0 < \rho \le r = \sqrt{\dfrac{\varepsilon}{2}}$, we have

$$\begin{aligned}\Omega^{\varepsilon}\big((\bar{x},\bar{t});\rho\big) &= Q_{\rho}(\bar{x},\bar{t}) \cap \big(\Omega \times (\varepsilon, T)\big)\\ &\subseteq Q^*_{\rho}(\bar{x},\bar{t}) \cup Q^*_{\rho}(\bar{x},\bar{t}_{\rho}) \cup \big(B_{\rho}(\bar{x}) \times \{\bar{t}\}\big)\end{aligned}$$

with $\bar{t}_{\rho} = \min\{T; \bar{t} + \rho^2\}$. Since $B_{\rho}(\bar{x}) \times \{\bar{t}\}$ is a Lebesgue measurable subset of $\mathbb{R}^n \times (0,T)$ with zero measure, we have

$$\frac{1}{\rho^{n+1-3\lambda}} \iint\limits_{\Omega^{\varepsilon}((\bar{x},\bar{t});\rho)} |\nabla v|^2 \, dx\, dt \le \frac{G_{Q^*_{\rho}(\bar{x},\bar{t})}(v)}{\rho^{n+1-3\lambda}} + \frac{G_{Q^*_{\rho}(\bar{x},\bar{t}_{\rho})}(v)}{\rho^{n+1-3\lambda}}$$

and

$$\frac{1}{\rho^{n+1-3\lambda}} \sup_{t \in (\bar{t}-\rho^2, \bar{t}+\rho^2) \cap (\varepsilon, T)} \int\limits_{B_{\rho}(\bar{x}) \cap \Omega} |v(t)|^2 \, dx \le \frac{F_{Q^*_{\rho}(\bar{x},\bar{t})}(v)}{\rho^{n+1-3\lambda}} + \frac{F_{Q^*_{\rho}(\bar{x},\bar{t}_{\rho})}(v)}{\rho^{n+1-3\lambda}}\,,$$

for $0 < \rho \le r = \sqrt{\dfrac{\varepsilon}{2}}$.

Then, since $\varepsilon < \bar{t} < \bar{t}_{\rho} \le T$, by (13b) we have

$$\begin{aligned}\frac{1}{\rho^{n+1-3\lambda}} \iint\limits_{\Omega^{\varepsilon}((\bar{x},\bar{t});\rho)} |\nabla v|^2 \, dx\, dt &\le c\, \frac{G_{Q^*_{r}(\bar{x},\bar{t})}(v) + G_{Q^*_{r}(\bar{x},\bar{t}_{\rho})}(v)}{r^{n+1-3\lambda}}\\ &\quad + c\left[\big(K_{Q^*_{r}(\bar{x},\bar{t})}(v)\big)^2 + \big(K_{Q^*_{r}(\bar{x},\bar{t}_{\rho})}(v)\big)^2\right] r^{\lambda-1}\\ &\quad + c\,\big(K_{Q^*_{r}(\bar{x},\bar{t})}(v)\big)^3 + c\,\big(K_{Q^*_{r}(\bar{x},\bar{t}_{\rho})}(v)\big)^3\\ &\quad + c\, K_{Q^*_{r}(\bar{x},\bar{t})}(v)\, H_{Q^*_{r}(\bar{x},\bar{t})}(\pi) + c\, K_{Q^*_{r}(\bar{x},\bar{t}_{\rho})}(v)\, H_{Q^*_{r}(\bar{x},\bar{t}_{\rho})}(\pi)\\ &\le \frac{c}{\varepsilon^{\frac{n+1-3\lambda}{2}}}\, \|\nabla v\|^2_{L^2(0,\,T;\,L^2(\mathbb{R}^n))} + c\,\varepsilon^{\frac{\lambda-1}{2}}\, \|v\|^2_{L^p(0,\,T;\,L^q(\mathbb{R}^n))}\\ &\quad + c\,\|v\|^3_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} + c\,\|v\|_{L^p(0,\,T;\,L^q(\mathbb{R}^n))}\, \|\pi\|_{L^{p/2}(0,\,T;\,L^{q/2}(\mathbb{R}^n))}\end{aligned}$$

for $0 < \rho \le r = \sqrt{\dfrac{\varepsilon}{2}}$.

If $\rho > r = \sqrt{\frac{\varepsilon}{2}}$, we easily obtain

$$\frac{1}{\rho^{n+1-3\lambda}} \iint\limits_{\Omega^\varepsilon((\bar{x},\bar{t});\rho)} |\nabla v|^2 \, dx \, dt \leq \frac{c}{\varepsilon^{\frac{n+1-3\lambda}{2}}} \|\nabla v\|^2_{L^2(0,\,T;\,L^2(\mathbb{R}^n))} .$$

Since the previous estimates are uniform with respect to ρ and $(\bar{x}, \bar{t})$, then there follows (15).

Similarly, by (13a) we have

$$\begin{aligned} \frac{1}{\rho^{n+1-3\lambda}} \sup_{t\in(\bar{t}-\rho^2,\bar{t}+\rho^2)\cap(\varepsilon,T)} \int\limits_{B_\rho(\bar{x})\cap\Omega} |v(t)|^2 \, dx &\leq c \, \frac{F_{Q_r^*(\bar{x},\bar{t})}(v) + F_{Q_r^*(\bar{x},\bar{t}_\rho)}(v)}{r^{n+1-3\lambda}} \\ &+ c \left[\left(K_{Q_r^*(\bar{x},\bar{t})}(v)\right)^2 + \left(K_{Q_r^*(\bar{x},\bar{t}_\rho)}(v)\right)^2 \right] r^{\lambda-1} \\ &+ c \left(K_{Q_r^*(\bar{x},\bar{t})}(v)\right)^3 + c \left(K_{Q_r^*(\bar{x},\bar{t}_\rho)}(v)\right)^3 \\ &+ c \, K_{Q_r^*(\bar{x},\bar{t})}(v) \, H_{Q_r^*(\bar{x},\bar{t})}(\pi) + c \, K_{Q_r^*(\bar{x},\bar{t}_\rho)}(v) \, H_{Q_r^*(\bar{x},\bar{t}_\rho)}(\pi) \\ \leq \frac{c}{\varepsilon^{\frac{n+1-3\lambda}{2}}} \|v\|^2_{L^\infty(0,\,T;\,J(\mathbb{R}^n))} &+ c \, \varepsilon^{\frac{\lambda-1}{2}} \|v\|^2_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} \\ &+ c \, \|v\|^3_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} + c \, \|v\|_{L^p(0,\,T;\,L^q(\mathbb{R}^n))} \, \|\pi\|_{L^{p/2}(0,\,T;\,L^{q/2}(\mathbb{R}^n))} \end{aligned}$$

for $0 < \rho \leq r = \sqrt{\frac{\varepsilon}{2}}$.

If $\rho > r = \sqrt{\frac{\varepsilon}{2}}$, we easily obtain

$$\frac{1}{\rho^{n+1-3\lambda}} \sup_{t\in(\bar{t}-\rho^2,\bar{t}+\rho^2)\cap(\varepsilon,T)} \int\limits_{B_\rho(\bar{x})\cap\Omega} |v(t)|^2 \, dx \leq \frac{c}{\varepsilon^{\frac{n+1-3\lambda}{2}}} \|v\|^2_{L^\infty(0,\,T;\,J(\mathbb{R}^n))} .$$

Since the previous estimates are uniform with respect to ρ and $(\bar{x}, \bar{t})$, then there follows (16). $\square$

REFERENCES

[1] R. A. Adams, J. J. F. Fournier. Sobolev Spaces, 2nd edition. Amsterdam, Elsevier/Academic Press 2003.

[2] L. Caffarelli, R. Kohn, L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Commun. Pure Appl. Math.* **35**, *6* (1982), 771–831.

[3] F. Crispo, P. Maremonti. On the spatial asymptotic decay of a suitable weak solution to the Navier-Stokes Cauchy problem. *Nonlinearity* **29**, *4* (2016), 1355–1383.

[4] F. Crispo, P. Maremonti. A remark on the partial regularity of a suitable weak solution to the Navier-Stokes Cauchy problem. *Discrete Contin. Dyn. Syst.* **37**, *3* (2017), 1283–1294.

[5] F. Crispo, P. Maremonti. Some remarks on the partial regularity of a suitable weak solution to the Navier-Stokes Cauchy problem. arXiv:1809.08998, 2018.

[6] R. Farwig. On regularity of weak solutions to the instationary Navier-Stokes system: a review on recent results. *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **60**, *1* (2014), 91–122.

[7] G. P. Galdi. An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-state problems. 2nd edition. Springer Monographs in Mathematics. New York, Springer, 2011

[8] G. P. Galdi. An introduction to the Navier-Stokes initial-boundary value problem. In: Fundamental directions in mathematical fluid mechanics (Eds G. P. Galdi, J. G. Heywood, R. Rannacher) Adv. Math. Fluid Mech., Basel, Birkhauser, 2000

[9] G. P. Galdi, P. Maremonti. Monotonic decreasing and asymptotic behavior of the kinetic energy for weak solutions of the Navier-Stokes equations in exterior domains. *Arch. Rational Mech. Anal.* **94**, *3* (1986), 253–266.

[10] Y. Giga. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Differential Equations* **62**, *2* (1986), 186–212.

[11] E. Hopf. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.* **4** (1951), 213–231.

[12] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* **63**, *1* (1934), 193–248.

[13] J. L. Lions. Sur la régularité et l'unicité des solutions turbulentes des équations de Navier Stokes. *Rend. Sem. Mat. Univ. Padova* **30**, (1960), 16–23.

[14] F. H. Lin. A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Commun. Pure Appl. Math.* **51**, *3* (1998), 241–257.

[15] P. Maremonti. Partial regularity of a generalized solution to the Navier-Stokes equations in exterior domain. *Commun. Math. Phys.* **110**, *1* (1987), 75–87.

[16] P. Maremonti. A note on Prodi-Serrin conditions for the regularity of a weak solution to the Navier-Stokes equations. *J. Math. Fluid Mech.* **20**, *2* (2018), 379–392.

[17] J. A. Mauro. Some Analytic Questions in Mathematical Physics Problems. Ph. D. Thesis, University of Pisa, Italy, 2010.
`http://etd.adm.unipi.it/t/etd-12232009-161531/`

[18] J. A. Mauro. On the regularity properties of the pressure field associated to a Hopf weak solution to the Navier-Stokes equations. *Pliska Stud. Math. Bulgar.* **23**, (2014), 95–118.

[19] J. A. Mauro. Partial regularity of Hopf weak solutions of the Navier-Stokes equations, which satisfy a suitable extra-condition. *Pliska Stud. Math.* **29**, (2018), 93–108.

[20] T. Miyakawa, H. Sohr. On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains. *Math. Z.* **199**, *4* (1988), 455–478.

[21] M. O'Leary. Conditions for the local boundedness of solutions of the Navier-Stokes System in three dimensions. *Comm. Partial Differential Equations* **28**, *3–4* (2003), 617–636.

[22] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stockes. *Ann. Mat. Pura Appl.* **48**, *1* (1959), 173–182.

[23] G. Prodi. Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale. *Rend. Sem. Mat. Univ. Padova* **30**, (1960), 1–15.

[24] G. A. Serëgin. Estimates of suitable weak solutions to the Navier-Stokes equations in critical Morrey spaces. *J. Math. Sci. (N.Y.)* **143**, *2* (2007), 2961–2968.

[25] J. Serrin. On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.* **9**, (1962), 187–195.

[26] J. Serrin. The initial value problem for the Navier-Stokes equations, R. E. Langer ed. *Nonlinear Problems* (Proc. Sympos., Madison, Wis., 1962), University of Wisconsin Press, Madison **9**, (1963), 69–98.

[27] V. Sheffer., in *Turbulence and Navier-Stokes equations. Lecture Notes in Math.* **565**, (1976), 94–112.

[28] V. Sheffer. Hausdorff measure and the Navier-Stokes equations. *Commun. Math. Phys.* **55**, (1977), 97–112.

[29] H. Sohr. The Navier-Stokes equations, An elementary functional analytic approach. Modern Birkhäuser Classics. Basel, Birkhauser/Springer Basel AG, 2001.

[30] S. Takahashi. On interior regularity criteria for weak solutions of the Navier-Stokes equations. *Manuscripta Math.* **69**, *3* (1990), 237–254.

[31] Y. Taniuchi. On generalized energy equality of the Navier-Stokes equations. *Manuscripta Math.* **94**, *3* (1997), 365–384.

[32] R. Temam. Navier-Stokes equations. Theory and numerical analysis, 3rd (revised) edition. Studies in Mathematics and its Applications, vol. **2**. Amsterdam, North-Holland Publishing Co., 1984.

[33] G. Tian, Z. Xin. Gradient estimation on Navier-Stokes equations. *Comm. Anal. Geom.* **7**, *(2)* (1999), 221–257.

[34] A. Vasseur. A new proof of partial regularity of solutions to Navier-Stokes equations. *NoDEA Nonlinear Differential Equations Appl.* **14**, *5–6* (2007), 753–785.

Jmmy Alfonso Mauro
Liceo Scientifico "Enrico Fermi"
Via Enrico Fermi, 5
I-81031 Aversa (CE), Italy
email: `jmmyalfonso.mauro@istruzione.it`