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ПЛИСКА
МАТЕМАТИЧЕСКИ СТУДИИ

INVERSE SCATTERING BY OBSTACLES AND SANTALO'S FORMULA

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We consider some problems related to recovering information about an obstacle K in an Euclidean space from certain measurements of lengths of generalized geodesics in the exterior of the obstacle – e.g. sojourn times of scattering rays in the exterior of the obstacle, or simply, travelling times of geodesics within a certain large ball containing the obstacle. It is well-known in scattering theory that this scattering data is related to the singularities of the scattering kernel of the scattering operator for the wave equation in the exterior of K with Dirichlet boundary condition on the boundary. For some classes of obstacles, K can be completely recovered from the scattering data. On the other hand, for some obstacles the set of trapped points is too large and this makes it impossible to recover them from scattering data. We discuss also a certain stability property of the trapping set, which is obtained from a generalisation of Santalo's formula to integrals over billiard trajectories in the exterior of an obstacle. Some other applications of this formula to scattering by obstacles are discussed as well.

1. Sojourn times and travelling times

An *obstacle* is a compact subset of $\mathbb{R}^n$ ($n \geq 2$) with smooth (C^k for some $k \geq 2$) boundary ∂K and a connected complement

$$\Omega_K = \overline{\mathbb{R}^n \setminus K}.$$

By a *scattering trajectory* (or simply a *trajectory*) in Ω_K we mean the trajectory of
a point moving with unit speed in the interior of Ω and reflecting at ∂K following
the usual law of geometrical optics.

Remark. Strictly speaking, one has to consider more general trajectories in Ω_K,
namely the so called *generalised geodesics* of Melrose and Sjöstrand [16], [17],
which combine billiard trajectories in Ω_K with geodesics on the boundary $\partial \Omega_K =
\partial K$. However almost all generalised geodesics are standard billiard trajectories
(assuming the boundary ∂K is not too bad – see below).

Shifting along generalized geodesics (billiard trajectories) defines the *gener-
alized geodesic (billiard) flow*

$$\mathcal{F}_t^{(K)} : \dot{T}^*(\Omega_K) = T^*(\Omega_K) \setminus \{0\} \longrightarrow \dot{T}^*(\Omega_K),$$

which preserves the *unit cosphere bundle* $\mathbb{S}^*(\Omega_K)$.

We will now describe a (generic) family of obstacles in $\mathbb{R}^n$ ($n \geq 2$) to which
the results presented in this paper apply.

Denote by $\mathcal{K}$ be the *class of obstacles* such that for each $(x, \xi) \in \dot{T}^*(\partial K)$ if
the curvature of ∂K at x vanishes of infinite order in direction ξ, then all points
(y, η) sufficiently close to (x, ξ) are *diffractive* (see e.g. Ch. 1 in [22]; roughly
speaking, this means that ∂K is convex at y in the direction of η). It is known
that for $K \in \mathcal{K}$ the flow $\mathcal{F}_t^{(K)}$ is well-defined and continuous ([17]).

Next, let $\mathbb{S}_0$ be a "large" sphere in $\mathbb{R}^n$ with centre 0 and let M be the *open
ball* with boundary $\mathbb{S}_0$. We will consider obstacles contained in M. Set

$$\mathbb{S}_+^*(\mathbb{S}_0) = \{(p, \xi) : p \in \mathbb{S}_0, \xi \in \mathbb{S}^{n-1}, \langle \xi, \nu(p) \rangle \geq 0\},$$

where $\nu(p)$ is the *inward unit normal* to $\mathbb{S}_0$ at p.

Given $x = (p, \xi) \in \mathbb{S}_+^*(\mathbb{S}_0)$, let $\gamma^+(x)$ be the *scattering trajectory in the
exterior* of K issued from p in direction ξ. Set $t(x) = \infty$ if $\gamma^+(x)$ is bounded
(i.e. doesn't intersect $\mathbb{S}_0$). If $\gamma^+(x)$ intersects $\mathbb{S}_0$ at some $q \in \mathbb{S}_0$, denote by $t(x)$
the length of $\gamma^+(x)$ from p to q along the trajectory. The number $t(x) = t_K(x)$
defined in this way is called the *travelling time* of $\gamma^+(x)$.

Consider the cross-sectional map $\mathcal{P}_K : \mathbb{S}_+^*(\mathbb{S}_0) \setminus \mathrm{Trap}_K^+(\mathbb{S}_0) \longrightarrow \mathbb{S}^*(\mathbb{S}_0)$ defined
by the shift along the flow $\mathcal{F}_t^{(K)}$. Here $\mathrm{Trap}_K^+(\mathbb{S}_0)$ is the set of points $x \in \mathbb{S}_+^*(\mathbb{S}_0)$
with $t(x) = \infty$ (see Sect. 2 below). Let $\gamma = \gamma^+(p_0, \xi_0)$ be a scattering trajectory
in Ω_K intersecting $\mathbb{S}_0$ at some point q_0 (before going to infinity) which is a simply
reflecting (no tangent points to ∂K). We will say that γ is *regular* if the differential

of the map $\mathbb{S}^{n-1} \ni \xi \mapsto \mathcal{P}_K(p_0, \xi) \in \mathbb{S}_0$ is a submersion at $\xi = \xi_0$, i.e. its differential at that point has rank $n-1$. Denote by $\mathcal{L}_0$ the *class of all obstacles* $K \in \mathcal{K}$ such that ∂K does not contain non-trivial open flat subsets and $\gamma_K(x, u)$ is a regular simply reflecting ray for almost all $(x, u) \in \mathbb{S}^*_+(\mathbb{S}_0)$ such that $\gamma(x, u) \cap \partial K \neq \emptyset$. Using an argument from Ch. 3 in [22] one can show that $\mathcal{L}_0$ is of second Baire category in $\mathcal{K}$ with respect to the C^∞ Whitney topology in $\mathcal{K}$. That is, generic obstacles $K \in \mathcal{K}$ belong to the class $\mathcal{L}_0$.

A slightly different way of measuring lengths of trajectories produces the so called *scattering lengths spectrum*. Let again M be a large ball in $\mathbb{R}^n$ containing the obstacle K in its interior and let $\mathbb{S}_0 = \partial M$. Given $\xi \in \mathbb{S}^{n-1}$ denote by Z_ξ the hyperplane in $\mathbb{R}^n$ orthogonal to ξ and tangent to $\mathbb{S}_0$ such that M is contained in the half-space R_ξ determined by Z_ξ and having ξ as an inner normal. For an (ω, θ)-ray γ in Ω_K, the *sojourn time* T_γ of γ is defined by $T_\gamma = T'_\gamma - 2a$, where T'_γ is the length of that part of γ which is contained in $R_\omega \cap R_{-\theta}$ and a is the radius of the ball M. It is known that this definition does not depend on the choice of the ball M (see [9] for the case of classical geodesics and Ch. 2 and 5 in [22] for the general case of generalized geodesics).

The *scattering length spectrum* of K is defined to be the family of sets of real numbers $SL_K = \{SL_K(\omega, \theta)\}_{(\omega, \theta)}$ where (ω, θ) runs over $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and $SL_K(\omega, \theta)$ is the set of sojourn times T_γ of all (ω, θ)-rays γ in Ω_K. It is known (see Theorem 11.1.1 in [22]) that for $n \geq 3$, n odd, and generic[1] obstacles K with C^∞ boundary ∂K there exists a subset $\mathcal{R}$ of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ of full Lebesgue measure such that

$$\text{sing supp } s_K(t, \theta, \omega) = \{-T_\gamma : \gamma \text{ is an } (\omega, \theta)\text{-ray in } \Omega_K\} = -SL_K(\omega, \theta)$$

for all $(\omega, \theta) \in \mathcal{R}$. Here s_K is the *scattering kernel* related to the scattering operator for the wave equation in $\mathbb{R} \times \Omega_K$ with Dirichlet boundary condition on $\mathbb{R} \times \partial \Omega_K$ (cf. [12], [15], [22]). Following [28], we will say that two obstacles K and L have *almost the same SLS* if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that $SL_K(\omega, \theta) = SL_L(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R}$.

In the present paper we are dealing with the following

General Inverse Problem. Obtain information about the obstacle K from measurements of travelling times $t(x)$ of scattering trajectories $\gamma^+(x)$, $x \in \mathbb{S}^*_+(\mathbb{S}_0)$, or from the scattering length spectrum $\{T_\gamma\}$.

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[1] $K \in \mathcal{K}$ would be enough.

Similar inverse problems concerning metric rigidity (involving lengths of geodesics) have been studied for a very long time in Riemannian geometry. Without going into details and without providing too many precise references, let us just mention a (very incomplete) list of authors with significant contributions in this area: J. P. Otal 1990, R. Michel 1981, C. Croke 1991, C. Croke – Kleiner 1994, P. Stefanov – G. Uhlmann [24], P. Stefanov – G. Uhlmann – A. Vasy [25], C. Guillarmou [8], S. Dyatlov – C. Guilarmou [6]. We refer the reader to [24] and [25] for a more comprehensive information (see also the references in these two papers).

Some classes of obstacles K are uniquely 'recoverable' from travelling times. For example, convex obstacles are easily recoverable by using back-scatter rays (Lax-Phillips [12], Majda [14] in the 70's).
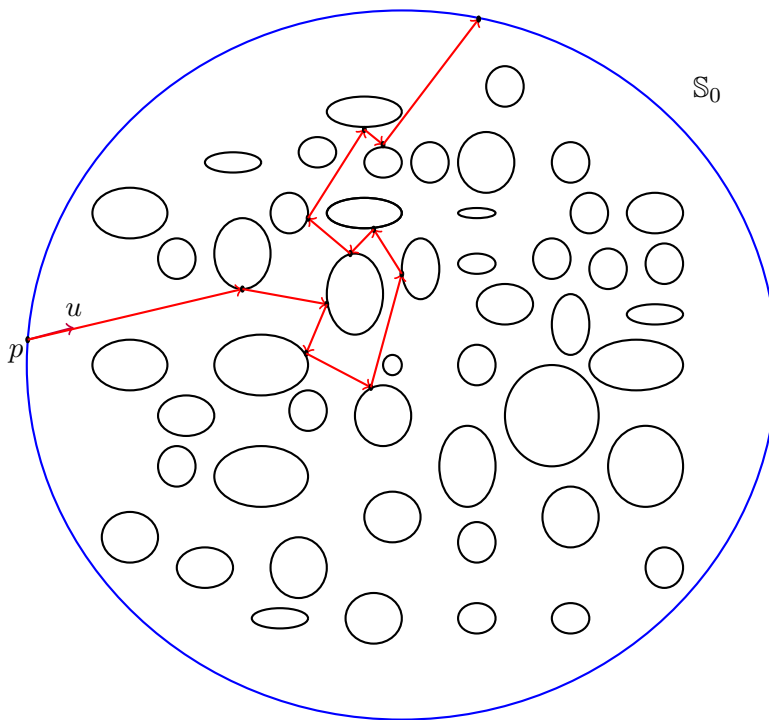

Figure 1

More generally, star-shaped obstacles are uniquely recoverable, too (see [28] or Ch. 10 in [22]).

A much more difficult positive result concerns obstacles K that are finite disjoint unions of strictly convex domains with smooth boundaries (Figure 1).

Theorem 1. ([28], [18], [21]) *Assume that K and L are obstacles in $\mathbb{R}^n$ ($n \geq 2$) and each of them is a finite disjoint union of strictly convex domains with C^3 boundaries. If K and L have almost the same travelling times, i.e. $t_K(x) = t_L(x)$ for almost all $x \in \mathbb{S}^*_+(\mathbb{S}_0)$, or almost the same scattering length spectrum, then $K = L$.*

The-real analytic case of the above (in the case of the scattering length spectrum) was dealt with in [27].

However recovering K in practice from travelling times appears to be rather difficult. See [19] where an algorithm is described for the recovery of an obstacle which is a disjoint union of two strictly convex domains in the plane with smooth boundaries from travelling times. Even this seemingly simple case is non-trivial.

2. Conjugate flows

As one can easily observe, trajectories in the vicinity of an obstacle that never escape to infinity present difficulties in the study of inverse scattering problems, especially if there is a large set of points in the phase space that generate such trajectories.

Definition 1. *A point $x = (q, v) \in \mathbb{S}^*(\Omega_K) = \Omega_K \times \mathbb{S}^{n-1}$ is called trapped if the forward trajectory $\gamma^+(x) = \gamma^+_K(x)$ in Ω_K issued from q in the direction of v is bounded.*

A point $x = (q, v) \in \mathbb{S}^*(\Omega_K)$ is called completely trapped if both (q, v) and $(q, -v)$ are trapped, i.e. both the forward trajectory $\gamma^+(x)$ and the backward trajectory $\gamma^-(x)$ in Ω_K issued from q in the direction of $-v$ are bounded.

Denote by $\mathrm{Trap}^+_K(\mathbb{S}_0)$ *the set of all trapped points $x = (q, v) \in \mathbb{S}^*_+(\mathbb{S}_0)$* (trapping considered with respect to the obstacle K), and by $\mathrm{Trap}(\Omega_K)$ *the set of all completely trapped points $x = (q, v) \in \mathbb{S}^*(\Omega_K)$.*

Definition 2. *Let K, L be two obstacles in $\mathbb{R}^n$. We will say that the domains Ω_K and Ω_L have conjugate flows if there exists a homeomorphism*

$$\Phi : \dot{T}^*(\Omega_K) \setminus \mathrm{Trap}(\Omega_K) \longrightarrow \dot{T}^*(\Omega_L) \setminus \mathrm{Trap}(\Omega_L)$$

which defines a smooth map on an open dense subset of $\dot{T}^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$, and the following conditions are satisfied:

(a) Φ *maps* $\mathbb{S}^*(\Omega_K) \setminus \text{Trap}(\Omega_K)$ *onto* $\mathbb{S}^*(\Omega_L) \setminus \text{Trap}(\Omega_L)$,

(b) $\mathcal{F}_t^{(L)} \circ \Phi = \Phi \circ \mathcal{F}_t^{(K)}$ *for all* $t \in \mathbb{R}$,

(c) $\Phi = id$ *on* $\dot{T}^*(\mathbb{R}^n \setminus M) \setminus \text{Trap}(\Omega_K) = \dot{T}^*(\mathbb{R}^n \setminus M) \setminus \text{Trap}(\Omega_L)$.

It was established in [28] that if two obstacles K and L have (almost) the same scattering length spectrum, then Ω_K and Ω_L have conjugate flows. A similar result holds for travelling times:

Theorem 2. ([28], [19]) *If the obstacles* $K, L \in \mathcal{K}_0$ *have almost the same scattering length spectrum or almost the same travelling times, then* Ω_K *and* Ω_L *have conjugate flows.*

Some interesting consequences of this have been derived, too.

Corollary 1. ([28], [19]) *Let the obstacles* $K, L \in \mathcal{L}_0$ *have almost the same scattering length spectrum or almost the same travelling times. Then:*

(a) *If the sets of trapped points of both K and L have Lebesgue measure zero, then $Vol(K) = Vol(L)$.*

(b) *If K is star-shaped, then $L = K$.*

(c) *Let $\text{Trap}^{(n)}(\partial K)$ be the set of those $x \in \partial K$ such that $(x, \nu_K(x)) \in \text{Trap}(\Omega_K)$, where $\nu_K(x)$ is the outward unit normal to ∂K at x. There exists a homeomorphism $\varphi : \partial K \setminus \text{Trap}^{(n)}(\partial K) \longrightarrow \partial L \setminus \text{Trap}^{(n)}(\partial L)$ such that $\varphi(x) = y$ whenever $\Phi(x, \nu_K(x)) = (y, \nu_L(y))$, where Φ is the homeomorphism from Definition 2.*

(d) *Let $n \geq 3$, let $\dim(\text{Trap}^{(n)}(\partial K)) < n - 2$ and let $\dim(\text{Trap}^{(n)}(\partial L)) < n - 2$. Then K and L have the same number of connected components.*

3. Trapped trajectories

As we have already mentioned, the size of the set $\text{Trap}(\Omega_K)$ of trapped points is rather important for the inverse scattering problem that we are considering in this paper.

Let us mention that there are relatively few points that are trapped in only one direction. This was observed by Lax and Phillips in their very well-known monograph [12]; a rigorous proof was given later in [28].

Theorem 3. ([12], [28]) *The set*

$$\mathrm{Trap}_K^+(\mathbb{S}_0) = \mathrm{Trap}(\Omega_K) \cap \mathbb{S}_+^*(\mathbb{S}_0)$$

has Lebesgue measure zero in $\mathbb{S}_+^*(\mathbb{S}_0)$.

It is easy to create trajectories that are trapped in one direction. For example, let K be the union of two disjoined strictly convex compact domains in $\mathbb{R}^n$, and let $[x_0, y_0]$ be the shortest distance between them. The trajectory γ_0 issued from x_0 in direction $v_0 = \dfrac{y_0 - x_0}{\|y_0 - x_0\|}$ is periodic (bouncing from ∂K at x_0 and y_0 repeatedly). Let $W_\epsilon^s(x_0, v_0))$ be the local stable manifold of (x_0, v_0). Then for every $(x, v) \in W_\epsilon^s(x_0, v_0)$ the trajectory $\gamma^+(x, v)$ has infinitely many reflections at ∂K getting closer and closer to γ_0. So, any $(x, v) \in W_\epsilon^s(x_0, v_0)$ is a trapped point. However the backward trajectory $\gamma^-(x, v)$ escapes to infinity after finitely many reflections at ∂K, so (x, v) is not completely trapped.

However there are cases when the set of completely trapped trajectories is very large. *Livshits' Example* (see Figure 2 adapted from [15]) describes one such case.
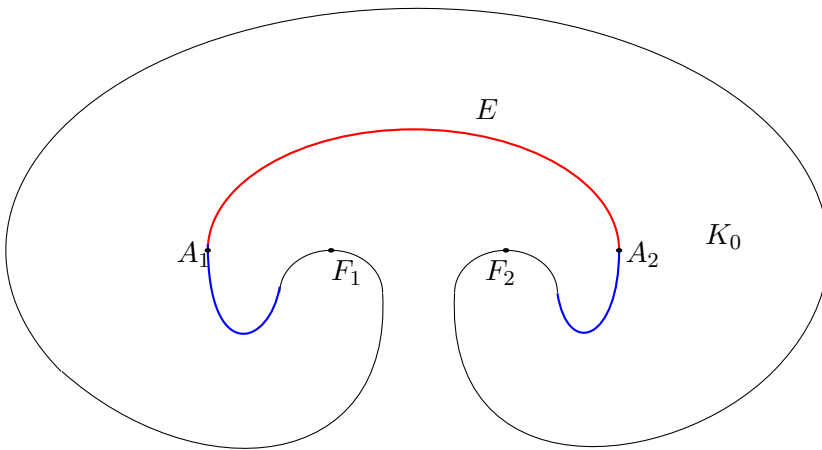
Figure 2: Livshits' Example

The main features of this example are the following:

- The closed curve in Figure 2 determines an obstacle K_0 in $\mathbb{R}^2$.

- The part of the boundary E is half an ellipse with end points A_1 and A_2,

- F_1 and F_2 are the foci of the ellipse.

- According to a well-known property of the ellipse, any straight-line ray entering the area inside the ellipse between the foci F_1 and F_2, after reflection at E, will go out intersecting the segment $[F_1, F_2]$.

- No scattering ray 'coming from infinity' can have a common point with the parts of the boundary ∂K from A_1 to F_1 and from A_2 to F_2. That is, no information about these parts of ∂K can be obtained from travelling times.

- Clearly for this obstacle K_0 the set of trapped points in $\mathbb{S}^*(\Omega_{K_0})$ has a non-empty interior, and so a positive measure.

Thus, the obstacle K in Livshits' Example is not recoverable from travelling times. Using this example one can create similar examples in higher dimensions.

Theorem 4. ([20]) *For any integer $n \geq 2$ there exist obstacles K in $\mathbb{R}^n$ such that $\mathrm{Trap}(\Omega_K)$ has a non-empty interior in $\mathbb{S}^*(\Omega_K)$, and therefore a positive Lebesgue measure in $\mathbb{S}^*(\Omega_K)$.*

It is natural to ask whether the set of trapped points possesses some kind of stability.

Question: Starting with an example of an obstacle K with a 'massive' set $\mathrm{Trap}(\Omega_K)$ of trapped points (e.g. Livshits' example) and smoothly perturbing the boundary of the obstacle slightly, can we destroy the 'massive' set $\mathrm{Trap}(\Omega_K)$ to a very big extend, i.e. obtain an obstacle whose set of completely trapped points has zero measure (and empty interior)?

For part of the Question the answer is negative. We will now state some precise results in this direction.

Let $k \geq 3$ and let $C^k(\partial K, \mathbb{R}^n)$ be *the space of all smooth embeddings* $F : \partial K \longrightarrow \mathbb{R}^n$ endowed with the *Whitney C^k topology* (see [10]). We will not attempt here to provide the precise definition of this topology, let us just mention that $C^k(\partial K, \mathbb{R}^n)$ is a metric space with a metric d such that, roughly speaking,

two embeddings F, G of ∂K into $\mathbb{R}^n$ are ϵ-close if $\|F(x) - G(x)\| < \epsilon$ for all $x \in \partial K$, and also all derivatives of F and G of order up to k satisfy similar relationships.

Given $F \in C^k(\partial K, \mathbb{R}^n)$, let K_F be the obstacle in $\mathbb{R}^n$ with boundary $\partial K_F = F(\partial K)$ contained in the interior of the sphere $\mathbb{S}_0$, and let $\Omega_{K_F} = \overline{\mathbb{R}^n \setminus K_F}$.

Let λ be the *Lebesgue measure* on $\mathbb{S}^*(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{S}^{n-1}$.

Theorem 5. ([29]) *Let K be an obstacle in $\mathbb{R}^n$, $n \geq 2$. Assume that K belongs to the generic class $\mathcal{K}$ described in Sect. 1.*

(a) If $\lambda(\mathrm{Trap}(\Omega_K)) > 0$, then there exists an open neighbourhood U of id in $C^k(\partial K, \mathbb{R}^n)$ such that $\lambda(\mathrm{Trap}(\Omega_{K_F})) > 0$ for every $F \in U$.

(b) More generally, for every $\epsilon > 0$ there exists an open neighbourhood U of id in $C^k(\partial K, \mathbb{R}^n)$ such that

$$|\lambda(\mathrm{Trap}(\Omega_{K_F})) - \lambda(\mathrm{Trap}(\Omega_K))| < \epsilon$$

for every $F \in U$.

The proof of the above result is based on a consequence of the well-known Santalo's formula in Riemannian geometry which we discuss in the next section.

4. The Travelling Time Formula

Let K, Ω_K and $\mathbb{S}_0$ be as in Sect. 1. Set

$$\Omega = \Omega_K \cap M,$$

where as before M is the closed ball with boundary $\mathbb{S}_0$. Let $\varphi_t : \mathbb{S}^*(\Omega) \longrightarrow \mathbb{S}^*(\Omega)$ ($t \in \mathbb{R}$) be the *billiard flow* in Ω. It is well-known that it preserves the *Lebesgue measure* $d\lambda = dq\,dv$ on $\mathbb{S}^*(\Omega)$ ([1]).

For $q \in \partial\Omega$, let $\nu(q) \in \mathbb{S}^{n-1}$ be the *inward unit normal* to $\partial\Omega$. Set

$$\mathbb{S}^*_+(\partial\Omega) = \{x = (q, v) : q \in \partial\Omega\,,\, v \in \mathbb{S}^{n-1}\,,\, \langle v, \nu(q) \rangle \geq 0\}.$$

Given $x = (q, v) \in \mathbb{S}^*_+(\partial\Omega)$, define the *first return time* $\tau(x) \geq 0$ as the maximal number (or ∞) such that $\varphi_t(x) = (q + tv, v)$ is in the interior of Ω for all $0 < t < \tau(x)$. In the special case when $x = (q, v) \in \mathbb{S}^*_+(\partial\Omega)$ is such that $\langle \nu(q), v \rangle = 0$ set $\tau(x) = 0$.

The *Liouville measure* μ on $\mathbb{S}^*_+(\partial\Omega)$ is defined by

$$d\mu = d\rho(q)d\omega_q \,|\,\langle\nu(q),v\rangle\,|$$

where ρ is the measure on $\partial\Omega$ determined by the Euclidean structure (surface area) and ω_q is the Lebesgue measure on the $(n-1)$-dimensional unit sphere $\mathbb{S}_q(\Omega) = \mathbb{S}^{n-1}$. It is well-known (see e.g. [1]) that the *billiard ball map*

$$B : \mathbb{S}^*_+(\partial\Omega) \longrightarrow \mathbb{S}^*_+(\partial\Omega)$$

preserves the Liouville measure μ, i.e.

$$\mu(B(A)) = \mu(B^{-1}(A)) = \mu(A)$$

for every measurable subset A of $\mathbb{S}^*_+(\partial\Omega)$.

In the situation considered here the well-known Santalo's formula in Riemannian geometry ([23]) has the following form.

Theorem 6. (Santalo's formula)

$$\int_{\mathbb{S}^*(\Omega)} f(x)\,d\lambda(x) = \int_{\mathbb{S}^*_+(\partial\Omega)} \left(\int_0^{\tau(x)} f(\varphi_t(x))\,dt\right) d\mu(x)$$

for every λ-integrable function $f : \mathbb{S}^*(\Omega) \longrightarrow \mathbb{C}$.

The original Santalo's formula has been used a lot in Riemannian geometry – see e.g. [4] and [5] and the references there. In the case of billiard flows this formula has also been applied by various authors – see e.g. [2] and [3].

Using Santalo's formula we proved in [29] a similar formula that involves travelling times of whole scattering trajectories in the exterior of the obstacle K contained in the ball M:

Theorem 7. ([29]) *Let K be an obstacle in $\mathbb{R}^n$, $n \geq 2$, that belongs to the class $\mathcal{K}$ (see Sect. 1). Then for every λ-measurable function*

$$f : \mathbb{S}^*(\Omega) \setminus \operatorname{Trap}(\Omega_K) \longrightarrow \mathbb{C}$$

such that $|f|$ is integrable we have

(1)
$$\int_{\mathbb{S}^*(\Omega)\setminus\operatorname{Trap}(\Omega)} f(x)\,d\lambda(x)$$

$$= \int_{\mathbb{S}^*_+(\mathbb{S}_0)\setminus\mathrm{Trap}^+_K(\mathbb{S}_0)} \left(\int_0^{t_K(x)} f(\varphi_t(x))\, dt \right) d\mu(x).$$

Formula (1) is what we call the *travelling time formula* in this paper.

Remark. If $\lambda(\mathrm{Trap}(\Omega_K)) = 0$, then (1) gives

$$\int_{\mathbb{S}^*(\Omega)} f(x)\, d\lambda(x) = \int_{\mathbb{S}^*_+(\mathbb{S}_0)} \left(\int_0^{t_K(x)} f(\varphi_t(x))\, dt \right) d\mu(x).$$

As established in [29], this is true for every integrable f; there is no need to assume integrability of $|f|$).

We should remark also that the main tool used in the proof in [29] of Theorem 5 above is the travelling time formula (1).

Using Theorem 7 with $f = 1$ yields the following.

Theorem 8. ([29]) *Under the assumptions in Theorem 7,*

$$\lambda(\mathrm{Trap}(\Omega_K)) = \lambda(\mathbb{S}^*(\Omega)) - \int_{\mathbb{S}^*_+(\mathbb{S}_0)\setminus\mathrm{Trap}^+_K(\mathbb{S}_0)} t_K(x)\, d\mu(x).$$

So, if we know the travelling time function $t_K(x)$ and have enough information about $\Omega = M \cap \Omega_K$ to determine its volume, then we can determine the measure of the set of the completely trapped points in Ω_K, as well.

Another consequence of Theorem 7 with $f = 1$ is the following

Corollary 2. ([29]) *If $\lambda(\mathrm{Trap}(\Omega_K)) = 0$, then*

$$(2) \qquad Vol_n(K) = Vol_n(M) - \frac{1}{Vol_{n-1}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^*_+(\mathbb{S}_0)} t_K(x)\, d\mu(x),$$

where $Vol_n(K)$ is the standard Riemann volume of K in $\mathbb{R}^n$ and $Vol_{n-1}(\mathbb{S}^{n-1})$ is the standard $(n-1)$-dimensional volume (surface area) of $\mathbb{S}^{n-1}$.

Remarks: (a) Formula (2) shows that from travelling times data we can recover the volume of K. That is, without seeing K and without any preliminary information about K (assuming though that K is not too bad so that its set of completely trapped points is relatively small), just measuring the times that certain kind of signals spend in the interior of the sphere $\mathbb{S}_0$, we can compute the

volume of K. Apart from that, it appears that (2) could be useful in numerical approximations of the volume of K.

(b) In Theorem 8 we only used the trivial function $f = 1$. Naturally, one would expect that using Theorem 8 for a large family of functions f would bring much more significant information about the obstacle K.

It is already known from previous results that a certain amount of information about K is recoverable from travelling times. However by means of the travelling times formula (1) it might be possible to get such information in a more explicit way.

Example 1. Assume that K is a disjoint union of k balls of the same radius $r > 0$, where $k \geq 1$ is arbitrary (possibly a large number). Suppose that we know r from some preliminary information. Then measuring travelling times $t(x) = t_K(x)$ for a relatively large number of points $x = (q, v) \in \mathbb{S}^*(\mathbb{S}_0)$ we get an approximation of the integral

$$\int_{\mathbb{S}^*_+(\mathbb{S}_0)} t(x) \, d\mu(x),$$

and therefore an approximate value for the number k of connected components of K. The precise formula (assuming we can measure almost all travelling times) is:

$$
\begin{aligned}
k &= \frac{\mathrm{Vol}_n(K)}{\pi^{n/2} r^n / \Gamma(n/2 + 1)} \\
&= \frac{R^n}{r^n} - \frac{\Gamma(n/2) \, \Gamma(n/2 + 1)}{2\pi^n r^n} \int_{\mathbb{S}^*_+(\mathbb{S}_0)} t(x) \, d\mu(x),
\end{aligned}
$$

where R is the radius of $\mathbb{S}_0$ and Γ is *Euler's Gamma function*,

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \, dt, \quad a > 0.$$

5. Some recent results and open problems

Recently, generalising the method from [18] it was proved in [30] that for obstacles K in $\mathbb{R}^n$ satisfying some regularity conditions and such that the set $\mathbb{S}^*_+(\mathbb{S}_0) \setminus \mathrm{Trap}(\Omega_K)$ is connected, the scattering length spectrum, and also the travelling times' spectrum, uniquely determine K.

Generally speaking it appears that for $n \geq 3$ the set $\mathbb{S}^*_+(\mathbb{S}_0) \setminus \text{Trap}(\Omega_K)$ is connected "more often". As Antoine Gansemer ([7]) observed, when K is a disjoint union of two strictly convex domains with smooth boundaries in $\mathbb{R}^2$ the set $\mathbb{S}^*_+(\mathbb{S}_0) \setminus \text{Trap}(\Omega_K)$ is disconnected. More precisely, $\mathbb{S}^*_+(\mathbb{S}_0) \cap \text{Trap}(\Omega_K)$ contains a closed curve. The same happens for every obstacle K in $\mathbb{R}^2$ which is a disjoint union of several strictly convex compact domains.

Comments and Some Open Problems

- Is there any stability of other numerical characteristics of $\text{Trap}(\Omega_K)$, e.g. its Hausdorff dimension or fractal dimension, similar to what we have about the Lebesgue measure in Theorem 5?

- If $\text{Trap}(\Omega_K)$ has a non-empty interior, is it true that for all sufficiently close to id perturbations F of ∂K, $\text{Trap}(\Omega_{K_F})$ also has a non-empty interior?

- Is it possible that $\text{Trap}(\Omega_K)$ has positive Lebesgue measure and an empty interior?

- If $\text{Trap}(\Omega_K)$ has an empty interior (regardless whether its Lebesgue measure is positive or not), then its complement in $\mathbb{S}^*(\Omega_K)$ is open and dense. So, every point in $\mathbb{S}^*(\Omega_K)$ is arbitrarily close to non-trapped points. That is, for every $q \in \partial K$ there exist non-trapped scattering trajectories having reflection points $q' \in \partial K$ arbitrarily close to q. Thus, generally speaking every point on ∂K should be 'observable', and so perhaps the obstacle K can be uniquely recovered from travelling times. Whether this is the case is not known at present.

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