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SWITCH-TIME DISTRIBUTIONS AND PROCESSES

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A family of probability distributions and related to them processes known as Switch Time distributions and processes (ST distributions and processes) are presented and simulated. Here by switch time we denote the time of sharp change of value of a stochastic process (jump) or sharp change of some characteristics of a stochastic process (regime switch).

1. Introduction – additive processes, characterization and possible generalizations

With a given filtered probability space (Ω, F, F_t, P), satisfying the usual conditions, an (one- or d-dimensional) additive process is defined as a stochastic process $\{X(t);\ 0 \leq t < \infty\}$ which is càdlàg and satisfies the following conditions:

1. $X(0) = 0$;
2. The process has independent increments, i. e. for every sequence of numbers $0 < t_1 < t_2 < \cdots < t_n$ the random variables

$$X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent;

3. The process is stochastically continuous (continuous in probability), i. e.

$$\lim_{t \to s} X(t) = X(s),$$

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where the limit is taken in probability.

The additive processes are introduced by Lévy [3,4]. Their properties are studied by Sato [5]. In Sato [5], the following theorem is presented:

Theorem (Sato)**.** *Let $\{X(t);\ t \geq 0\}$ be an additive process with values in R^d. Then:*

1. *For every $t \geq 0$ the random variable $X(t)$ has infinitely divisible distribution.*

2. *The distribution of $\{X(t);\ t \geq 0\}$ is uniquely identified by its point characteristics $\{(A(t), \mu(t), \Gamma(t));\ t \geq 0\}$, which are related to the characteristic function in the following way:*

$$Ee^{iuX(t)} = e^{\psi(u,t)}$$

and

$$\psi(u,t) = -\frac{1}{2}uA(t)u + iu\Gamma(t) + \int_{R^d} \mu(t,dx)(e^{iux} - 1 - iux1_{|x|\leq 1}).$$

The point characteristics of the additive process satisfy the properties:

2.1. *For every $t \geq 0$, the matrix $A(t)$ is positive definite matrix with dimension $d \times d$, $\Gamma(t)$ is a function with values which are vectors in R^d and $\mu(t,dx)$ is a positive measure in R^d, satisfying the conditions $\mu(t,0) = 0$ and $\int_{R^d} min(|x|^2, 1)\mu(t,dx) < \infty$.*

2.2. *$A(0) = 0, \mu(0,dx) = 0, \Gamma(0) = 0$ and for all $t \geq s \geq 0$ the matrix $A(t) - A(s)$ is a positive definite matrix with dimension $d \times d$ and $\mu(s,B) \leq \mu(t,B)$ for all measurable sets $B \in B(R^d)$.*

2.3. *If $s \to t$, then $A(s) \to A(t), \Gamma(s) \to \Gamma(t), \mu(s,B) \to \mu(t,B)$ for all measurable sets $B \in B(R^d)$, for which $B \subset \{x : |x| \geq \epsilon\}$ for some $\epsilon > 0$.*

Conversely, for the family triples $\{(A(t), \mu(t), \Gamma(t)), t \geq 0\}$, which satisfies 2.1.*,* 2.2. *and* 2.3. *there is an additive process $\{X(t);\ t \geq 0\}$ with the corresponding point characteristics.*

Important special cases of the additive processes are the processes for which there exists the following parametric representation of the point characteristics:

1. $A(t) = \int_0^t \sigma^2(s)ds$, where for every $t \geq 0$ the matrix $\sigma(t)$ is a real matrix with dimension $d \times n$ for which $\int_0^T \sigma^2(t)dt < \infty$ for fixed $T > 0$, such that $\sigma(t)$ is defined in the interval $[0,T]$.

2. $\mu(t,B) = \int_0^t \nu(s,B)ds$, for all measurable sets $B \in B(R^d)$, where $\{\nu(t),\ t \in [0,T]\}$ is a family of Lévy measures satisfying the condition

$$\int_0^T \int_{R^d} min(|x|^2, 1)\nu(t,dx)dt < \infty.$$

3. $\Gamma(t) = \int_0^t \gamma(s)ds$, where $\gamma : [0,T] \to R$ is a deterministic function with values in the set R of real numbers and finite variation.

For the triple $(\sigma^2(t), \nu(t), \gamma(t))$ we say that it defines the local characteristics of the additive process.
The additive processes with local characteristics are semimartingales.
From now on, we will consider one-dimentional processes, i.e processes with values in R – the set of real numbers.
There are different possibilities for generalization of the additive processes. One way of generalization is to introduce jumps in deterministic moments, as proposed by Kallsen [2]. In this way, the process is not already stochastically continuous. To characterize such kind of processes, to the point and local characteristics there is a need to add new characteristics identifying the position and the size of these deterministic-time jumps. The point characteristics become

$$\{(\Theta, K(t), A(t), \mu(t), \Gamma(t)),\ t \geq 0\},$$

where Θ is a discrete set giving the times of the jumps and $K(t)$ is conditional distribution of the jumps for which

$$K(t,G) = \mu(t,G) + \epsilon_0(G)(1 - \mu(t,R)),\ t \in \Theta$$

where R is the set of real numbers and

$$K(t,G) = 0,\ t \notin \Theta.$$

Here $\epsilon_0(G)$ is the Dirac measure in the point zero. In the case when local characteristics exist we have

$$\begin{aligned} (1) \qquad \{(\Theta, K(t), A(t) &= \int_0^t \sigma^2(s)ds, \mu(t,G) = \\ &= \int_0^t \nu(s,G)ds + \sum_{s\in\Theta\cap[0,t]} \mu(s,G) = \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \nu(s,G)ds + \sum_{s\in\Theta\cap[0,t]} K(s,G\backslash\{0\}),\\
\Gamma(t) &= \int_0^t \gamma(s)ds + \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(s) =\\
&= \int_0^t \gamma(s)ds + \sum_{s\in\Theta\cap[0,t]} \int xK(s,dx)),\ t\geq 0\},
\end{aligned}
$$

and the local characteristics are:

$$\{(\Theta, K(t), \sigma^2(t), \nu(t), \gamma(t)),\ t \geq 0\}.$$

A next step of generalization is to allow stochastic characteristics - drift, volatility and jump intensity. The corresponding processes are considered for example by Grigelionis [1]. In this case, the point characteristics of the process are

$$\{(A(\omega,t), \mu(\omega,t,G), \Gamma(\omega,t)),\ t\geq 0\}$$

and the local characteristics (when they exist) are

$$\{(\sigma^2(\omega,t), \nu(\omega,t,G), \gamma(\omega,t)),\ t\geq 0\}.$$

The obtained characterization allows to specify also some specific jumps in random times and to characterize them separately from the remaining jumps of the process. In this case, the point characteristics can be presented as:

$$\{(\Theta(\omega,t), K(\omega,t), A(\omega,t), \mu(\omega,t,B), \Gamma(\omega,t)),\ t\geq 0\}$$

where the random times of the specific jumps are represented by the point process $\{\Theta(\omega,t),\ t\geq 0\}$.

It is also possible to combine specific jumps in specific random times with jumps in deterministic moments. In this case, the parameters of the process (in the case when local characteristics exist) are:

$$
\begin{aligned}
(2)\qquad & \{(\Theta_1, \Theta_2(\omega,t), K_1(\omega,t), K_2(\omega,t), A(\omega,t) =\\
&= \int_0^t \sigma^2(\omega,s)ds, \mu(\omega,t,B) =\\
&= \int_0^t \nu(\omega,s,B)ds + \sum_{s\in\Theta\cap[0,t]} K_1(\omega,\theta_i,B\backslash\{0\}) +
\end{aligned}
$$

$$+\sum_{\theta_i<t} K_2(\omega,\theta_i,B\backslash\{0\}),\Gamma(\omega,t)=$$

$$=\int_0^t \gamma(\omega,s)ds+\sum_{s\in\Theta\cap[0,t]}\int xK_1(\omega,\theta_i,dx)+\sum_{\theta_i<t}\int xK_2(\omega,\theta_i,dx)),\ t\geq 0\}.$$

Here $K_1(\omega,t)$ and $K_2(\omega,t)$ are the conditional distributions of the jumps in deterministic and specific random times correspondingly.

2. Additive processes with returns to zeros

An example for a process with specific jumps in specific random times are the so called additive processes with returns to zero (or to a fixed level). They are processes $X(t)=G(t)+J(t)$, where $G(t)$ is an additive process and $J(t)$ is a pure jump process giving the returns to zero (or other fixed level).

The return to zero are restrictive conditions for the specific jumps leading to more precise characterization.

Theorem 1. *Let $X(t)=G(t)+J(t)$ be an additive process with returns to zero. Then we have*

$$K(t,B)=\begin{cases}\epsilon_{-X(t-)}(B),\ t\in\Theta\\ 0,\ t\notin\Theta,\end{cases}$$

$$\mu(t,B)=\mu_G(t,B)+\sum_{s\in\Theta\cap[0,t]}\epsilon_{-X(s-)}(s,B\backslash\{0\}),$$

$$\Gamma(t)=\Gamma_G(t)-\sum_{s\in\Theta\cap[0,t]}X(s-),$$

where Θ is the set of the times at which the process $J(t)$ has jumps, $\mu_G(t,B)$ and $\Gamma_G(t)$ are the corresponding characteristics of the additive process $G(t)$.

P r o o f. We have $K(t,B)=P_{\Delta X(t)|F_{t-}(B)}$, i. e. $K(t,B)$ is the conditional distribution of the jumps. For the jumps leading to zero we have:

$$\Delta X=X(t)-X(t-)=\begin{cases}0-X(t-)=-X(t-),\ t\in\Theta\\ G(t)-G(t-),\ t\notin\Theta.\end{cases}$$

Then

$$K(t,B)=\begin{cases}\epsilon_{-X(t-)}(B),\ t\in\Theta\\ 0,\ t\notin\Theta.\end{cases}$$

At the points of Θ for processes with returns to zero we have:

$$\sum_{s\in\Theta\cap[0,t]} K(s,B\backslash\{0\}) = \sum_{s\in\Theta\cap[0,t]} \epsilon_{-X(s-)}(s,B\backslash\{0\}).$$

Then

$$\begin{aligned}(3) \qquad \mu(t,B) &= \mu_G(t,B) + \sum_{s\in\Theta\cap[0,t]} K(s,B\backslash\{0\}) = \\ &= \mu_G(t,B) + \sum_{s\in\Theta\cap[0,t]} \epsilon_{-X(s-)}(s,B\backslash\{0\}).\end{aligned}$$

We have also:

$$\int xK(s,dx) = \int x\epsilon_{-X(s-)}(dx) = -X(s-).$$

So:

$$\begin{aligned}(4) \quad \Gamma(t) &= \Gamma_G(t) + \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(s) = \\ &= \Gamma_G(t) + \sum_{s\in\Theta\cap[0,t]} \int xK(s,dx) = \Gamma_G(t) - \sum_{s\in\Theta\cap[0,t]} X(s-). \qquad \square\end{aligned}$$

When the processes with return to zero admit local characteristics, we have

$$\begin{aligned}(5) \qquad \{(\Theta, K(t,B), A(t) &= \int_0^t \sigma^2(s)ds, \mu(t,B) = \int_0^t \nu(s,B)ds + \\ + \sum_{s\in\Theta\cap[0,t]} \mu(s,B) &= \int_0^t \nu(s,B)ds + \sum_{s\in\Theta\cap[0,t]} K(s,B\backslash\{0\}) = \\ &= \int_0^t \nu(s,B)ds + \sum_{s\in\Theta\cap[0,t]} \epsilon_{-X(s-)}(B\backslash\{0\}), \\ \Gamma(t) &= \int_0^t \gamma(s)ds + \sum_{s\in\Theta\cap[0,t]} \Delta\Gamma(s) = \\ &= \int_0^t \gamma(s)ds - \sum_{s\in\Theta\cap[0,t]} X(s-)),\ t\geq 0\}\end{aligned}$$

and the corresponding local characteristics are:

$$\{(\Theta, K(t,B), \sigma^2(t), \nu(t), \gamma(t)),\ t\geq 0\}.$$

Additive processes with returns to zero at deterministic times in the specific case of Brownian motion are considered for example by Stoynov [6,7]. In this article, we consider an example with a modified Poisson process with specific jumps at specific random times leading to returns to non-negative integer levels (zero, one etc.).

3. Switch time family of distributions $STF(n, \beta)$

We say that a random variable ξ with probability mass function $f_\xi(x)$ has distribution of $ST(n, \beta)$ family and denote this fact $\xi \in ST(n, \beta)$ if the probability mass function of ξ is given by the formula:

$$f_\xi(x) = \begin{cases} \sum_{k=1}^{n+1} f_{D^n,\xi}(k, x) = \sum_{k=1}^{n+1} P(D^n = k) f_\xi(x|D^n = k) = & \\ \qquad = \sum_{k=1}^{n+1} P(D^n = k) f_{G^k}(x), & t \geq 0 \\ 0, & x < 0 \end{cases}$$

where G^k are random variables with probability mass function $f_{G^k}(x) = f(k, \beta)$ and D^n are discrete random variables.

Different choices of the variables G^k and D^n lead to different kinds of switch time distributions.

In this article, we consider switch time distributions of first kind. For these distributions,

$$G^k \in \Gamma(k, \frac{1}{\beta}) \equiv Erlang\left(k, \frac{1}{\beta}\right) \tag{6}$$

i.e.

$$\xi|D^n \equiv Erlang\left(D^n, \frac{1}{\beta}\right).$$

Correspondingly, for D^n we have:

$$P(D^n = k) = \frac{C(n, \beta) n!}{\beta^k (n - k + 1)!}, \ k = 1, 2, \ldots, (n + 1) \tag{7}$$

where the coefficients $C(n, \beta)$ are given by the formulas:

$$C(n, \beta) \quad = \quad \frac{1}{I(n, \beta)}, \tag{8}$$

$$\begin{aligned} I(0,\beta) &= \frac{1}{\beta}, \\ I(n,\beta) &= \frac{1}{\beta} + \frac{n}{\beta} I(n-1,\beta),\ n = 1,2,\ldots \end{aligned}$$

Also, variables $\tilde{D}^n = D^n - 1$ can be introduced for which:

$$P(\tilde{D}^n = k) = \frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!},\ k = 0,1,\ldots,n.$$

Then the probability mass function $f_\xi(x)$ of ξ may be presented also by the formula:

$$f_\xi(x) = \begin{cases} \sum_{k=0}^{n} P(\tilde{D}^n = k) f_\xi(x|\tilde{D}^n = k) = \sum_{k=0}^{n} P(\tilde{D}^n = k) f_{G^{k+1}}(x), & t \geq 0 \\ 0, & x < 0. \end{cases}$$

The class of switch time distributions of first kind we will denote $ST1(n,\beta)$.

Theorem 2. *Let $\xi \in ST1(n,\beta)$. Then its probability mass function can be presented as*

$$f_\xi(x) = \begin{cases} C(n,\beta)e^{-\beta x}(1+x)^n,\ x \geq 0 \\ 0,\ x < 0. \end{cases}$$

P r o o f. We have:

$$\begin{aligned} C(n,\beta)e^{-\beta x}(1+x)^n &= C(n,\beta)e^{-\beta x} \sum_{k=0}^{n} \binom{n}{k} x^k = \\ &= \sum_{k=0}^{n} \frac{C(n,\beta)n!}{\beta^{k+1}(n-k)!} \frac{\beta^{k+1}x^k e^{-\beta x}}{k!} = \\ &= \sum_{k=0}^{n} P(D^n = k+1) f_{G^{k+1}}(x) = \\ &= \sum_{k=1}^{n+1} \frac{C(n,\beta)n!}{\beta^k(n-k+1)!} \frac{\beta^k x^{k-1} e^{-\beta x}}{(k-1)!} = \\ &= \sum_{k=1}^{n+1} P(D^n = k) f_{G^k}(x) = f_\xi(x). \qquad \square \end{aligned} \tag{9}$$

Special cases of $ST1(n,\beta)$ distribution are $ST1(0,\beta) \equiv Exp(\beta)$ and $ST1(1,\beta) \equiv Lindley(\beta)$. The case $n = 2$ is introduced by Stoynov [8] and further generalized in some other publications of the author, for example [9], where Moment Generating Function of $ST1(2,\beta)$ is derived.

4. Switch time family processes

We say that a process $X(t)$ is switch time process of first kind or $ST1(n, \beta)$ process and denote this fact $X(t) \in ST1(t; n, \beta)$, if:

1. $X(0) = 0$.

2. $X(t)$ is pure jump process with jumps at times T_i, $i = 1, 2, \dots$ and jump sizes $\Delta X(T_i) = 1$.

3. The intervals between two jumps are

$$\tau_i = T_i - T_{i-1} \in ST1(n-1, \beta),\ i = 0, 1, \dots,\ T_0 = 0.$$

We say that $X(t)$ is compound $ST1(n, \beta)$ process, if we condition 3. by condition:

3'. $X(t)$ is pure jump process with jumps at times T_i, $i = 1, 2, \dots$ and jump sizes $\Delta X(T_i) = Y_i$, where Y_i are independent and identically distributed random variables.

Switch time processes of first kind can be used to model different phenomena in nature and society. Let us for example consider a company which can switch its policy (for example income or expense rate). Those switches can happen during meetings of board of managers. At every meeting, there is a decision to keep the policy (variable, characteristic of the activity) as it is or to switch its value (jump of the characteristic). If the time intervals between the meetings are exponentially distributed independent variables and the switch occurs after D^{n-1} meetings, the characteristic, related to the policy, follows an $ST1(n, \beta)$ process.

Theorem 3. *Let $N(t)$ be a homogeneous Poisson process with intensity β and jumps at times Z_i, $i = 1, 2, \dots$. Let n be a positive integer and D_j^{n-1}, $j = 1, 2, \dots$ be a sequence of independent random variables with the distribution of D^{n-1} (as in (7)). Let $N^R(t)$ be the corresponding Poisson process with returns to zero at times Z_i, $i = 1, 2, \dots, D_1^{n-1} - 1$, returns to level one at times Z_i, $i = D_1^{n-1} + 1, D_1^{n-1} + 2, \dots, D_1^{n-1} + D_2^{n-1} - 1$, and so on, returns to level k at times Z_i, $i = \sum_{j=1}^{k} D_j^{n-1} + 1, \sum_{j=1}^{k} D_j^{n-1} + 2, \dots, \sum_{j=1}^{k+1} D_j^{n-1} - 1$, and so on. Then $N^R(t)$ is a $ST1(n, \beta)$ process.*

P r o o f. We have by definition that $N^R(0) = 0$. As the defined returns to levels $0, 1, \dots$ eliminate part of the jumps of $N(t)$, the only remaining jumps of $N^R(t)$ are at times Z_i, $i = D_1^{n-1} = S_1, D_1^{n-1} + D_2^{n-1} = S_2, \dots, \sum_{j=1}^{m} D_j^{n-1} = S_m, \dots$. We denote $Z_{S_m} = T_m$. The sizes of the jumps at times T_m are $\Delta N^R(T_m) = 1$.

The intervals between two consecutive jumps of $N^R(t)$ are $\tau_m = T_m - T_{m-1} = Z_{S_m} - Z_{S_{m-1}} = \sum_{r=1}^{D_m^{n-1}} \xi_r^m$ where ξ_r^m, $r = 1, \dots, D_m^{n-1}$ are independent exponentially distributed random variables with intensity β. Then $\tau_m = T_m - T_{m-1} \in Erlang(D_m^{n-1}, \frac{1}{\beta}) \equiv ST1(n-1, \beta)$. This finalizes the proof. □

Theorem 4. *Let $N(t)$ be a compund Poisson process with intensity β and jumps with independent identically distributed sizes Y_i, $i = 1, 2, \dots$ at times Z_i, $i = 1, 2, \dots$. Let n be a positive integer and D_j^{n-1}, $j = 1, 2, \dots$ be a sequence of independent random variables with the distribution of D^{n-1} (as in (7)). Let $N^R(t)$ be the corresponding compound Poisson process with returns to zero at times Z_i, $i = 1, 2, \dots, D_1^{n-1} - 1$, returns to level one at times Z_i, $i = D_1^{n-1} + 1, D_1^{n-1} + 2, \dots, D_1^{n-1} + D_2^{n-1} - 1$, and so on, returns to level k at times Z_i, $i = \sum_{j=1}^{k} D_j^{n-1} + 1, \sum_{j=1}^{k} D_j^{n-1} + 2, \dots, \sum_{j=1}^{k+1} D_j^{n-1} - 1$, and so on. Then $N^R(t)$ is a compound $ST1(n, \beta)$ process.*

P r o o f. The proof is analogous to the proof of Theorem 2 with the only difference that the sizes of the jumps at times T_m are $\Delta N^R(T_m) = Y_{T_m}$. □

The integral (point) characteristics of the process $N^R(t)$ in the case of Theorem 3 are:

$$
(10)\qquad \begin{aligned}
\{(\Theta(\omega, t), K(\omega, t), A(\omega, t) &= 0,\\
\mu(\omega, t, B) &= t\beta f(B) + \sum_{\theta_i < t} K(\omega, \theta_i, B \backslash \{0\}) =\\
&= t\beta f(B) + \sum_{\theta_i < t} \epsilon_{-1}(B \backslash \{0\}),\\
\Gamma(\omega, t) &= \sum_{\theta_i < t} \int x K(\omega, \theta_i, dx) = -\sum_{\theta_i < t} 1 =\\
&= -|\Theta \cap [0, t]|),\ t \geq 0\}
\end{aligned}
$$

where $\Theta(\omega, t) \equiv \{\theta_i,\ i = 1, 2 \dots\} \equiv \{Z_i,\ i = 1, 2 \dots\} \backslash \{T_m,\ m = 1, 2 \dots\}$ and

$$
K(\omega, t, B) = \begin{cases} \epsilon_{-1}(B), & t \in \Theta \\ 0, & t \notin \Theta. \end{cases}
$$

The corresponding local characteristics are:

$$
\{(\Theta(\omega, t), K(\omega, t, B), A(\omega) = 0, \nu(\omega, B) = 0, \gamma(\omega) = \beta f(B)),\ t \geq 0\}.
$$

The integral (point) characteristics of the process $N^R(t)$ in the case of Theorem 4 are:

$$
\begin{aligned}
(11)\ \{(\Theta(\omega,t),K(\omega,t),A(\omega,t) &= 0,\\
\mu(\omega,t,B) &= t\beta f(B)+\sum_{\theta_i<t} K(\omega,\theta_i,B\backslash\{0\}) =\\
&= t\beta f(B)+\sum_{\theta_i<t} \epsilon_{-Y_i}(B\backslash\{0\}),\\
\Gamma(\omega,t) &= \sum_{\theta_i<t}\int xK(\omega,\theta_i,dx) = -\sum_{\theta_i<t} Y_i),\ t\geq 0\}
\end{aligned}
$$

where $\Theta(\omega,t)\equiv\{\theta_i,\ i=1,2\ldots\}\equiv\{Z_i,\ i=1,2\ldots\}\backslash\{T_m,\ m=1,2\ldots\}$ and

$$K(\omega,t,B)=\begin{cases}\epsilon_{-Y_i}(B), & t\in\Theta\\ 0, & t\notin\Theta.\end{cases}$$

The corresponding local characteristics are:

$$\{(\Theta(\omega,t),K(\omega,t,B),A(\omega)=0,\nu(\omega,B)=0,\gamma(\omega)=\beta f(B)),\ t\geq 0\}.$$

Based on Theorem 3 and Theorem 4, it is possible to simulate $ST1(n,\beta)$ process.

5. Simulation of $ST1(n,\beta)$ process

The simulation of $ST1(n,\beta)$ distribution and corresponding PMF graphics for different values of the parameters n and β can be done by using R language. The graphics for the case $n=100$ and $\beta=2$ is given in Fig. 1. The graphics for the case $n=40$ and $\beta=2$ is given in Fig. 2.

To make simulation of $ST1(n,\beta)$ process, the following algorithm can be applied:

1. Define interval $[0,T]$ of the simulation.
2. $k=0$.
3. While $\sum_{i=1}^{k}\tau_i=\sum_{i=1}^{k}(T_i-T_{i-1})<T$ do:

   3.1. Set $k=k+1$.

   3.2. Generate $\tau_i=T_i-T_{i-1}\in ST(n-1,\beta)$.

   3.3. Set $Y_i=1$ for standard $ST1(n,\beta)$ process or simulate Y based on a given distribution f for compound $ST1(n,\beta)$ process.

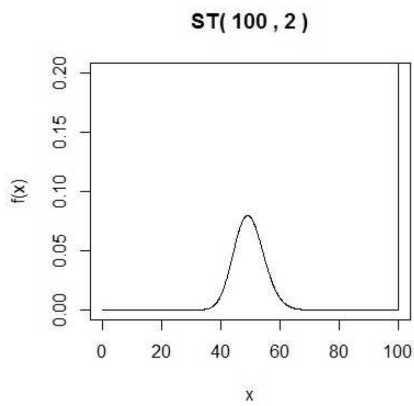

Figure 1: The graphics of $ST1(n, B)$ distribution for the case $n = 100$ and $B = 2$.

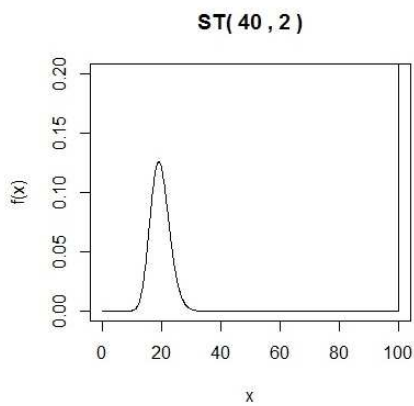

Figure 2: The graphics of $ST1(n, B)$ distribution for the case $n = 40$ and $B = 2$.

Then the trajectory of $X(t) \in ST(n, \beta)$ is given by the formula

$$X(t) = \sum_{i=1}^{N(t)} Y_i,$$

where

$$N(t) = \sum_i 1_{T_i < T}.$$

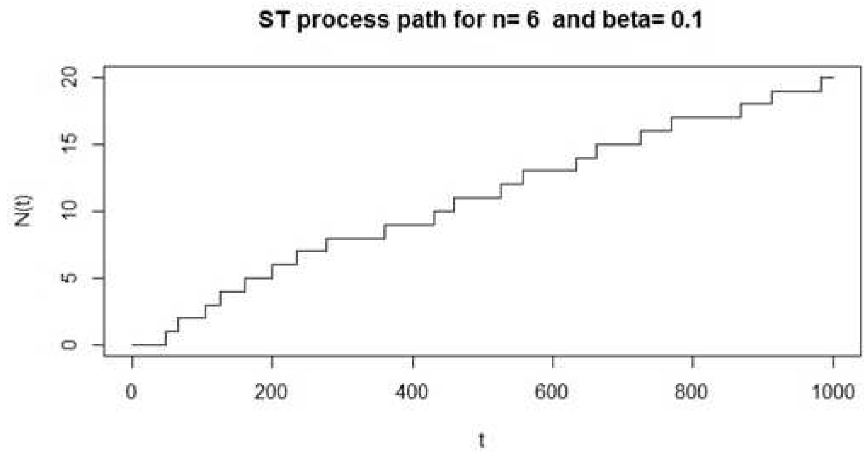

Figure 3: The graphics of a path of $ST1(n, B)$ process for the case $n = 6$ and $B = 0.1$.

The graphics of a trajectory of $ST1(n, \beta)$ process for the case $n = 6$ and $\beta = 0.1$ is given in Fig. 3.

6. Concluding remarks

The present work may be extended for example by studying other (than given in (6)) choices of G^k.

For example, we may choose $G^k \in NB(r, e^{-\beta})$. In this case, $\xi | D^n \equiv NB(D^n, e^{-\beta})$, and we obtain a distribution which we will call $ST2(n, \beta)$ distribution and denote $\xi \in ST2(n, \beta)$.

As another example, we may consider $G^k \equiv \delta_k(x)$ - random variable which takes value k with probability one. Then $\xi \equiv D^n$. In this case, we say that random variable ξ has $ST3(n, \beta)$ distribution and denote $\xi \in ST3(n, \beta)$. $ST1(n, \beta)$ processes can be used in economic models to present events which appear with some "larger" intervals between them compared to the intervals between the events counted by the Poisson processes. For example, they can be used as intervals between the last recovery and the start of the budget restriction in an oil company exploring oil fields [10]. Different applications of $ST(n, \beta)$ processes are also possible in financial and actuarial models.

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