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ON THE DECAY OF SOLUTIONS TO A CLASS OF HARTREE EQUATIONS

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We consider the Cauchy problem for the Hartree equation

$$
\begin{cases} iu_t + \Delta u - c_{d,\gamma}[|x|^{-\gamma} * |u|^2]u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^d), \end{cases} \tag{1}
$$

with $d \geq 1$, $c_{d,\gamma} > 0$ and where $0 < \gamma < \min(4, d)$. Then, we prove that the global solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ to (3) enjoys the following decay property:

$$
\lim_{t \to \pm\infty} \|u(t,x)\|_{L^q(\mathbb{R}^d)} = 0, \quad 2 < q < \frac{2d}{d-2}. \tag{2}
$$

1. Introduction

Consider the Cauchy problem associated to nonlinear defocusing Schrödinger equations with Hartree-type nonlinearity (NLHS), for $d \geq 1$:

$$
\begin{cases} i\partial_t u + \Delta_x u - vu = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (-\Delta)^{\alpha/2} v = |u|^2 \\ u(0,\cdot) = u_0 \in H^1(\mathbb{R}^d). \end{cases} \tag{3}
$$

Here, $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, and $0 < \alpha < d$. Let us solve the elliptic equation,

$$
v = (-\Delta)^{-\alpha/2}[|u|^2] = I_\alpha[|u|^2] = I_\alpha(\cdot) * |u|^2,
$$

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with
$$I_\alpha(x) = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{d/2}2^\alpha|x|^{d-\alpha}}.$$
We obtain the system

$$\left\{\begin{array}{l} iu_t + \Delta_x u - c_{d,\gamma}[|\cdot|^{-\gamma} * |u|^2]u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{array}\right. \tag{4}$$

where we have introduced the parameter $\gamma := d - \alpha$ and $c_{d,\gamma} > 0$ defined by
$$c_{d,\gamma} = \frac{\Gamma(\frac{d-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{d/2}2^\alpha}.$$
We denote
$$H^1(\mathbb{R}^d) = (1 - \Delta_x)^{-\frac{1}{2}} L^2(\mathbb{R}^d),$$
indicating by $f \in L^q(\mathbb{R}^n)$, for $1 \leq q < \infty$, if
$$\|f\|^q_{L^q(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|^q \, dx < +\infty.$$
The solution u associated to (4) satisfies two conservation laws:

$$\|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}, \tag{5}$$
$$E(u(t)) = E(u_0), \tag{6}$$

with
$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx + \frac{c_{d,\gamma}}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)|^2|u(y)|^2}{|x-y|^\gamma} dxdy.$$
It is possible to investigate some relevant mathematical questions as:

a) Local and global existence as well as the persistence of regularity for the map data-solution $u_0 \to u(t, \cdot)$, assuming the initial data in the space $H^1(\mathbb{R}^d)$. This forces to $0 < \gamma < \min(4, d)$.

b) The long-time behavior of the solutions to (4) in the space $L^q(\mathbb{R}^d)$ (which leads to the scattering in the energy space $H^1(\mathbb{R}^d)$).

With regard to the point a), that is the study of the global well-posedness, we remand to [7] (and references therein) for a comprehensive overlook on the argument. We concentrate our attention on the point b). Thus we state our main results about the decay of solution to (4) (problem b)), that is

Theorem 1. *Let* $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ *be a global solution to* (4) *with* $0 < \gamma < \min(4,d)$*, with* $d \geq 1$*. Then:*

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L^q(\mathbb{R}^d)} = 0, \tag{7}$$

provided that $2 < q < \dfrac{2d}{d-2}$.

We are motivated also by the Physics. In fact equations of type (4) play an important role in describing the quantum mechanics of a polaron at rest, the electron trapped in its own hole, the self-gravitating matter and the self consistency effect of many electron systems in the context of Bohr theory.

A fundamental tool to study the properties like (7) for solutions to (4) is the Morawetz multiplier technique and the resulting estimates. These were obtained for the first time by K. Morawetz in [12] for the NLKG equation with a general nonlinearity and were successively used for proving the asymptotic completeness by J. Lin and W. Strauss [9], for the cubic NLS in $\mathbb{R}^3$ and by Ginibre ad Velo [7] (and references therein) for the NLS with pure power nonlinearity for $2/d < p < 2/(d-2)$. Recently, a new approach simplified the proof of scattering. It consists on the bilinear Morawetz inequalities (or interaction). We refer for instance to the papers of Colliander, M. Grillakis, N. Tzirakis [2] and Ginibre, Velo [7] and Planchon, Vega [14]. As far as concerns NLHS, Ginibre and Velo in [6] derived the associated Morawetz inequality and extracted a useful Birman–Solomjak type estimate to obtain the asymptotic completeness in the energy space. Nakanishi in [13] improved the results by a new Morawetz estimate. For the critical case, Miao et al. in [11] took advantage of a new kind of the localized Morawetz estimate to rule out the possibility of the energy concentration at origin and established the scattering results in the energy space for the radial data in dimension $d \geq 5$. We remand also to the paper [11] of Miao et al. for the case of general data in the same space dimension framework of the previous one. We present here a peculiar decay property to the solution of (4), mandatory to shed light on scattering states, which arises from a combination of a localization trick, the nonlinear interaction Morawetz estimate and interpolation. We underline that our version of the proof strongly simplifies the various results already cited, moreover it guarantees to treat in an unified manner all the space dimensions, namely $d \geq 1$, allowed by the condition $0 < \gamma < \min(4,d)$.

2. Morawetz identities

We start with the first tool to prove the main Theorem 1: Morawetz equalities. We introduce the following further notations: given a function $f \in H^1(\mathbb{R}^d, \mathbb{C})$, we denote by

$$m_f(x) := |f(x)|^2, \qquad j_f(x) := \Im\left[\overline{f}(x)\nabla_x f(x)\right] \in \mathbb{C}^d. \tag{8}$$

Then our main contribution is the following

Proposition 1. *Let $d \geq 1$ and $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^d))$ be a global solution to (4), let $\phi = \phi(x) : \mathbb{R}^d \to \mathbb{R}$ be a sufficiently regular and decaying function, and denote by*

$$V(t) := \int_{R^d} \phi(x)\, m_u(x)\, dx.$$

The following identities hold:

$$\dot{V}(t) = \int_{\mathbb{R}^d} \phi(x)\dot{m}_u(x)\, dx = 2\int_{\mathbb{R}^d} j_u(x) \cdot \nabla_x \phi(x)\, dx, \tag{9}$$

$$\begin{aligned}\ddot{V}(t) = \int_{\mathbb{R}^d} \phi(x)\ddot{m}_{u_\mu}(x)\, dx = -\int_{\mathbb{R}^d} m_u(x)\Delta_x^2\phi(x)\, dx \\ +4\int_{\mathbb{R}^d} \nabla_x u(x) D_x^2\phi(x) \cdot \nabla_x \overline{u}(x)\, dx \\ -c_{d,\gamma}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x \bar{I}_\gamma(x-y)|u(x)|^2|u(y)|^2\nabla_x\phi(x)\, dxdy,\end{aligned} \tag{10}$$

with

$$\bar{I}_\gamma(x) = \frac{1}{|x|^\gamma}, \tag{11}$$

for $d \geq 1$, $c_{d,\gamma} > 0$, with $0 < \gamma < \min(4, d)$ and where $D_x^2\phi \in \mathcal{M}_{n\times n}(\mathbb{R}^d)$ is the hessian matrix of ϕ, and $\Delta_x^2\phi = \Delta_x(\Delta_x\phi)$ the bi-laplacian operator.

Proof. We prove the identities for a smooth rapidly decreasing solution $u = u(t,x)$, letting the general case $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^d))$ to a final standard density argument (see for instance [7], Appendix 4) We give the details for obtaining (10) and from now on we shall drop the variable t for simplicity. An integration by parts and (4), gives

$$2\partial_t \int_{\mathbb{R}^d} j_{u(x)} \cdot \nabla_x \phi(x)\, dx \tag{12}$$

$$= -2\Im \int_{\mathbb{R}^d} \partial_t u(x) \left(\Delta_x \phi(x) \bar{u}(x) + 2\nabla_x \phi(x) \cdot \nabla_x \bar{u}(x)\right) \, dx$$

$$= 2\Re \int_{\mathbb{R}^d} i\partial_t u(x) \left(\Delta_x \phi(x) \bar{u}(x) + 2\nabla_x \phi(x) \cdot \nabla_x \bar{u}(x)\right) \, dx$$

$$= 2\Re \int_{\mathbb{R}^d} \left(-\Delta_x u(x) + c_{d,\gamma}[|x|^{-\gamma} * |u(x)|^2] u(x)\right) \cdot \left(\Delta_x \phi(x) \bar{u}(x) + 2\nabla_x \phi(x) \cdot \nabla_x \bar{u}(x)\right) \, dx.$$

We have the following identity

$$2\Re \int_{\mathbb{R}^d} -\Delta_x u(x) \left(\Delta_x \phi(x) \bar{u}(x) + 2\nabla_x \phi(x) \cdot \nabla_x \bar{u}(x)\right) \, dx = -\int_{\mathbb{R}^d} \Delta_x^2 \phi(x) \, |u(x)|^2 \, dx + 4\nabla_x u(x) D_x^2 \phi(x) \nabla_x \bar{u}(x) \, dx. \tag{13}$$

Moreover, one gets

$$c_{d,\gamma}\Re \int_{\mathbb{R}^d} [|x|^{-\gamma} * |u(x)|^2] u(x) \Delta_x \phi(x) \bar{u}(x) \, dx + 2c_{d,\gamma}\Re \int_{\mathbb{R}^d} [|x|^{-\gamma} * |u(x)|^2] u(x) \nabla_x \phi(x) \cdot \nabla_x \bar{u}(x) \, dx$$
$$= c_{d,\gamma}\Re \int_{\mathbb{R}^d} [|x|^{-\gamma} * |u(x)|^2] u(x) \Delta_x \phi(x) \bar{u}(x) \, dx + c_{d,\gamma}\Re \int_{\mathbb{R}^d} [|x|^{-\gamma} * |u(x)|^2] \nabla_x \phi(x) \cdot \nabla_x |u(x)|^2 \, dx.$$

Then we can write

$$2c_{d,\gamma}\Re \int_{\mathbb{R}^d} [|x|^{-\gamma} * |u(x)|^2] u(x) \Delta_x \phi(x) \bar{u}(x) \, dx + 4c_{d,\gamma}\Re \int_{\mathbb{R}^d} [|x|^{-\gamma} * |u(x)|^2] u(x) \nabla_x \phi(x) \cdot \nabla_x \bar{u}(x) \, dx = -2c_{d,\gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla_x \bar{I}_\gamma(x-y) |u(x)|^2 |u(y)|^2 \nabla_x \phi(x) \, dx dy. \tag{14}$$

Combining now the identities (13) and (14), we arrive finally at (10). □

3. Interaction Morawetz identities and inequalities

This section is devoted to prove the bilinear Morawetz identities and inequalities. Namely we have the following proposition.

Proposition 2. *Let $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^d))$ be a global solution to system (4), let $\phi = \phi(|x|) : \mathbb{R}^d \to \mathbb{R}$ be convex radial function, regular and decaying enough, let be $\psi(x,y) := \phi(|x-y|) : \mathbb{R}^{2d} \to \mathbb{R}$ and*

$$I(t) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(x,y) m_u(x) m_u(y)\, dxdy.$$

Then the following holds:

$$\dot{I}(t) = 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_u(x,y) \cdot \nabla_x \psi(x,y)\, m_u(y)\, dxdy, \tag{15}$$

$$\ddot{I}(t) \geq 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x \psi(x,y) \nabla_x m_u(x) \cdot \nabla_y m_u(y)\, dxdy + N_{(d,\gamma,\psi)}, \tag{16}$$

with

$$N_{(d,\gamma,\psi)} = 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \bar{I}_{\gamma+2}(x-y) m_u(x) m_u(y)(x-y)(k(x)-k(y))\, dxdy \tag{17}$$

and

$$k(\cdot) = -c_{d,\gamma} \int_{\mathbb{R}^d} \Delta_z \psi(\cdot, z) m_u(z)\, dz. \tag{18}$$

P r o o f. As for the previous proposition, we prove the identities for a smooth solution u, treating the general case $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^d))$ by a standard density argument. First one has

$$\dot{I}(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \dot{m}_u(x) m_u(y) \psi(x,y)\, dxdy + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_u(x) \dot{m}_u(y) \psi(x,y)\, dxdy. \tag{19}$$

Then, due to the symmetry of $\psi(x,y) = \phi(|x-y|)$, we obtain

$$\dot{I}(t) = 2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \dot{m}_u(x) m_u(y) \psi(x,y)\, dxdy.$$

So, (15) follows by (9) and the Fubini's Theorem. Analogously, we can differentiate again and get the identity

$$\begin{aligned}\ddot{I}(t) = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \ddot{m}_u(x)m_u(y)\psi(x,y)\,dxdy\\ &+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)\ddot{m}_u(y)\psi(x,y)\,dxdy\\ &+2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \dot{m}_u(x)\dot{m}_u(y)\psi(x,y)\,dxdy.\end{aligned} \tag{20}$$

Then we write $\ddot{I}(t) := A + B$. By (10), an application of the Fubini's Theorem and using once again the symmetry of $\psi(x,y)$ we are allowed to set

$$\begin{aligned}A = &-2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)m_u(y)\Delta_x^2\psi(x,y)\;dxdy\\ &+4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{I}_{\gamma+2}(x-y)m_u(x)m_u(y)(x-y)(k(x)-k(y))\,dxdy,\end{aligned} \tag{21}$$

with the function k defined as in (18). We reshape the linear term in the previous identity (21) as follows

$$\begin{aligned}&-2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)m_u(y)\Delta_x^2\psi(x,y)\,dxdy\\ &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)m_u(y)\partial_x\partial_y\Delta_x\psi(x,y)\;dxdy\\ &= 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_x m_u(x)\partial_y m_u(y)\Delta_x\psi(x,y)\;dxdy,\end{aligned} \tag{22}$$

by applying integration by parts (with no boundary terms) and using the property $\partial_x\psi = -\partial_y\psi$. In conclusion, we get

$$A = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x\psi(x,y)\nabla_x m_u(x)\cdot\nabla_y m_u(y)\,dxdy + N_{(d,\gamma,\psi)}. \tag{23}$$

Moreover by (9), (10) and the Fubini's Theorem we introduce

$$B = 4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x u(x)D_x^2\psi(x,y)\nabla_x\overline{u}(x)m_u(y)\,dxdy$$

$$+4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)\nabla_y u(x) D_y^2\psi(x,y)\nabla_y\overline{u}(y)\,dxdy$$
$$+8\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} j_u(x) D_{xy}^2\psi(x,y)\cdot j_u(y)\,dxdy,$$

where in the previous identity we took the advantage of the symmetry of $D^2\psi$ to eliminate the real part condition in the first two summands of the equality above. At this point, by using some rearrangements, integration by part and dispersion properties of u we find out that

$$B = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D_x^2\phi(|x-y|)\left[C(x,y)\overline{C}(x,y) + D(x,y)\overline{D}(x,y)\right]\,dxdy, \tag{24}$$

with $C(x,y), D(x,y)$ defined as follows:

$$C(x,y) := u(x)\nabla_y\bar{u}(y) + \bar{u}(y)\nabla_x u(x)$$

and

$$D(x,y) := u(x)\nabla_y u(y) - u(y)\nabla_x u(x).$$

Therefore the fact that that ϕ is a convex function give $B \geq 0$. This argument implies, in combination with (23), (24), the proof of (16). □

Remark. One notice that $N_{(d,\gamma,\psi)} \geq 0$. In fact from (17) we have that

$$k(x) - k(y) = \int_{\mathbb{R}^d} m_u(z)\left(\frac{x-z}{|x-z|} - \frac{y-z}{|y-z|}\right)\,dz.$$

Then one observes that $(x-y)(k(x)-k(y)) \geq \|u\|_{L^2}^2$, because of the elementary inequality

$$(x-y)\cdot\left(\frac{x-z}{|x-z|} - \frac{y-z}{|y-z|}\right) \geq 0.$$

This lead to the following

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \bar{I}_{\gamma+2}(x-y) m_u(x) m_u(y)(x-y)(k(x)-k(y))\,dxdy \geq 0. \tag{25}$$

At this point we need the following lemma, that is a straightforward consequence of the inequality (16).

Lemma 1. *Let $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^d))$ be a global solution to (4), for $d \geq 1$. Then one has*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2|u(t,z)|^2}{|x-y|^{\gamma+2}}\,dx\,dy\,dy\,dt < \infty, \tag{26}$$

for any $0 < \gamma < \min(4,d)$.

P r o o f. Integrating (16) to time variable one obtains, by (15) and the above remark, the following

$$(27)\quad \begin{aligned} &2\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} j_u(t,x)\cdot\nabla_x\psi(x,y)m_u(t,y)\,dxdy\right]_{t=S}^{t=T} \\ &\geq 2\int_S^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x\psi(x,y)\nabla_x m_u(x)\cdot\nabla_y m_u(y)\,dxdydt \\ &+2\int_S^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\bar{I}_{\gamma+2}(x-y)m_u(x)m_u(y)(x-y)(k(x)-k(y))\,dxdydt \\ &=2\int_S^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x\psi(x,y)\nabla_x m_u(x)\cdot\nabla_y m_u(y)\,dxdydt \\ &+2\int_S^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|u(t,x)|^2|u(t,y)|^2|u(t,z)|^2}{|x-y|^{\gamma+2}}\,dxdydzdt. \end{aligned}$$

Now choose $\psi(x,y)=|x-y|$. For the l.h.s of the (27) we have the immediate bound

$$(28)\quad 2\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} j_u(t,x)\cdot\nabla_x\psi(x,y)m_u(t,y)\,dxdy\right]_{t=S}^{t=T}\leq C\|u_0\|^4_{H^1_x}<\infty,$$

for some $C>0$ and any $T,S\in\mathbb{R}$, since the H^1_x-norm is a conserved quantity. By this observation, we have

$$(29)\quad \Delta_x\psi=\begin{cases}\dfrac{n-1}{|x-y|} & \text{if}\quad d\geq 2,\\ 2\delta_{x=y}\quad (=D_x^2\psi) & \text{if}\quad d=1,\end{cases}$$

we easily achieve the proof of (26) by letting $T\to\infty, S\to-\infty$. □

4. Proof of main theorem

Finally we are in position to prove the main achievement of the paper by a combined use of the results obtained in the previous sections. Specifically we have the following

P r o o f o f T h e o r e m 1. It is sufficient to prove that the property (7) for a suitable $2<q<\dfrac{2d}{d-2}$, since the thesis for the general case can be then obtained by the conservation of mass (5), the kinetic energy in (6). and interpolation. More precisely it is sufficient to show that

$$(30)\quad \lim_{t\to\pm\infty}\|u(t,x)\|_{L_x^{2+4/d}}=0.$$

Then the decay of the L^q_x norm for all $2 < q < \frac{2d}{d-2}$ follows by combining (30) with the bound

$$\sup_{t\in\mathbb{R}} \|u(t,x)\|_{H^1_x} < \infty. \tag{31}$$

We write the following localized Gagliardo-Nirenberg inequality (see [1], for example)

$$\|\varphi\|^{\frac{2d+4}{d}}_{L_x^{\frac{2d+4}{d}}} \leq C \left(\sup_{x\in\mathbb{R}^3} \|\varphi\|_{L^2(Q_x)} \right)^{\frac{4}{d}} \|\varphi\|^2_{H^1_x}, \tag{32}$$

where Q_x is the unit cube in $\mathbb{R}^3$ centered in x. Next, assume by the absurd that (30) is false, then by (31) and by (32) we deduce the existence of a sequence $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d$ with $|t_n| \to \infty$ and $\epsilon_0 > 0$ such that

$$\inf_n \|u(t_n, x)\|_{L^2(Q_{x_n})} = \epsilon_0. \tag{33}$$

For simplicity we can assume that $t_n \to \infty$ (the case $t_n \to -\infty$ can be treated by a similar argument). Notice that by (9) in conjunction with (31) we get

$$\sup_{n,t} \Big|\frac{d}{dt} \int \chi(x - x_n)|u(t,x)|^2 dx\Big| < \infty,$$

where $\chi(x)$ is a smooth and non-negative cut-off function taking values in $[0,1]$ such that $\chi(x) = 1$ for $x \in Q^d(0,1)$ and $\chi(x) = 0$ for $x \notin \tilde{Q}_x$, where $\tilde{Q}_x$ denotes the cube in $\mathbb{R}^d$ of radius 2 centered in x. By combining this fact with (33), then we get the existence of $T > 0$ such that

$$\inf_n \big(\inf_{t\in(t_n,t_n+T)} \|u(t,x,y)\|_{L^2(\tilde{Q}_{x_n})} \big) \geq \epsilon_0/2. \tag{34}$$

Observe also that since $t_n \to \infty$ then we can assume (modulo subsequence) that the intervals $(t_n, t_n + T)$ are disjoint. In particular we have

$$\begin{aligned} &\sum_n T(\epsilon_0/2)^{12} \\ &\leq \sum_n \int_{t_n}^{t_n+T} \int_{\tilde{Q}_{x_n}} \int_{\tilde{Q}_{x_n}} \int_{\tilde{Q}_{x_n}} |u(t,x)|^2 |u(t,y)|^2 |u(t,z)|^2 dxdydzdt \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(t,x)|^2 |u(t,y)|^2 |u(t,z)|^2}{|x-y|^{\gamma+2}} \, dx \, dy \, dy \, dt \end{aligned}$$

and hence we get a contradiction since the left hand side is divergent and the right hand side is bounded as in (26). Then the proof is completed. □

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