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RELATIONS BETWEEN THE INTEGRAL AND HAUSDORFF DISTANCE WITH APPLICATIONS TO DIFFERENTIAL EQUATIONS

SVETOSLAV M. MARKOV

Bl. Sendov and V. Popov (1966) estimated the integral distance between two functions from above by means of the Hausdorff metric. In this note a similar lower bound is established. These estimates are used for studying the uniform distance between the solutions of two first order ordinary differential equations by means of the Hausdorff distance between the right-hand sides of the equations.

1. Introduction. We shall denote by $\mathbf{B}_1$ the class of all measurable on the interval $I=[a, b]$ functions, bounded by the constant $B>0$, so that $\sup_{t\in I}|\varphi(t)|\leqq B$.

The modulus of non-monotonicity of $\varphi\in\mathbf{B}_1$ is defined by (see for instance [1]):

$$\mu_\varphi(\delta)=\frac{1}{2}\sup_{|x-y|\leqq\delta}\{\sup_{x\leqq z\leqq y}[|\varphi(x)-\varphi(z)|+|\varphi(z)-\varphi(y)|]-|\varphi(x)-\varphi(y)|\}.$$

This module is a convenient tool for characterizing periodic functions. When functions on a finite interval $[a, b]$ are considered it is more convinient to use

$$\tilde{\mu}_\varphi(\delta)=\max\{\mu_\varphi(\delta),\ \sup_{\substack{|x-y|\leqq\delta\\ |x-a|\leqq\delta}}|f(x)-f(y)|,\ \sup_{\substack{|x-y|\leqq\delta\\ |b-x|\leqq\delta}}|f(x)-f(y)|\},$$

which we call the corrected modulus of non-monotonicity of φ.

In the second part of the present note functions of two variables are involved. We shall denote by $\mathbf{B}_2$ the class of all measurable functions defined on $D=[a, b;\ c, d]\in R^2$ and bounded by the constant $B>0$, i. e. $\sup_{(t,x)\in D}|f(x, t)|\leqq B$.

Given $f\in\mathbf{B}_2$ and $x\in[c, d]$ denote φ_x the function defined on $[a, b]$ by $\varphi_x(t)=f(x, t)$. Then the modulus of non-monotonicity $\bar{\mu}_f$ of a bounded function f of two variables is defined by $\bar{\mu}_f(\delta)=\sup_{c\leqq x\leqq d}\mu_{\varphi_x}(\delta)$.

We need next the definition of Hausdorff metric for functions of one or two variables. It is equally easy to formulate it in the n-dimensional case (for more detail see [2]). First the notion of complete graph is introduced. For a bounded real function of n variables $f=f(x_1, x_2,\dots, x_n)$ its complete graph $\bar{f}$ is defined as the intersection of all closed subsets F in R^{n+1}, having the following properties:

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1) F contains the graph of f;

2) if the points $(x_1, x_2, \ldots, x_n, x_{n+1})$ and $(x_1, x_2, \ldots, x_n, \bar{x}_{n+1})$ belong to F and $x_{n+1} < \tilde{x}_{n+1} < \bar{x}_{n+1}$, then $(x_1, x_2, \ldots, x_n, \tilde{x}_{n+1})$ also belongs to F.

Now for two bounded functions f and g of n variables the Hausdorff distance between them is given by

$$r(f, g) = \max\{\max_{A \in \bar{f}} \min_{B \in \bar{g}} \rho(A, B), \max_{B \in \bar{g}} \min_{A \in \bar{f}} \rho(A, B)\},$$

wherein $\bar{f}$ and $\bar{g}$ are the complete graphs of f and g as point sets in the space R^{n+1}, normed by $\|X\| = \|(x_1, x_2, \ldots, x_{n+1})\| = \max\{|x_1|, |x_2|, \ldots, |x_{n+1}|\}$, so that $\rho(X, Y) = \|X - Y\| = \max_{i=1, \ldots, n+1}\{|x_i - y_i|\}$.

We shall also denote shortly $\min_{B \in \bar{g}} \rho(A, B) = d(A, g)$. Finally we shall use the notation $f(a)$ for the intersection of the point set $\bar{f}$ with the line $x = a$ (in the n-dimensional case the line $\{x_i = a_i, i = 1, \ldots, n\}$).

2. Relation between the integral and the Hausdorff metric. The problem of estimating $\int_a^b |f(x) - g(x)| dx$ by means of Hausdorff metric is considered in [1]. It is established that for every two functions $f, g \in \mathbf{B}_1$ the inequality

$$(1) \qquad \int_a^b |f(x) - g(x)|\, dx \leq (b-a)\{\inf_{0<\delta<b-a} [4Br(\delta+1)/\delta + 2\mu(\delta)] + r\}$$

holds, where $r = r(f, g)$ and μ is the modulus of non-monotonicity of f (or g).

It is also shown in [1] that if $f, g \in \mathbf{B}_1$ with $\mu_f(t) \leq Kt^q$, where $K \geq 0$, $q \geq 0$, then

$$(2) \qquad \int_a^b |f(x) - g(x)|\, dx \leq c(b-a) r^{q/(1+q)},$$

where c may depend only on the choice of q.

Our purpose in this section is to estimate the integral distance from below. We shall make use of the following result which is an extension of lemma 8 from [3].

Lemma 1. *For every bounded in $[a, b]$ function f, if the point (x_0, y_0) belongs to $\bar{f}$, $x_0 \in (a, b)$ and $0 < t \leq \min\{b - x_0, x_0 - a\}$, then*

i) *the inequality*

$$(3) \qquad f(x) \geq y_0 - \mu_f(2t)$$

holds either for every $x \in [x_0 - t, x_0)$, or for every $x \in (x_0, x_0 + t]$;

ii) *the inequality $f(x) \leq y_0 + \mu_f(2t)$ holds either for every $x \in [x_0 - t, x_0)$ or for every $x \in (x_0, x_0 + t]$.*

Proof. We prove conclusion i); conclusion ii) is treated analogously. Notice that in view of lemma 8 from [3] the inequality

$$(4) \qquad f(x) \leq f(x_0) - \mu_f(2t)$$

holds either for every $x \in [x_0 - t, x_0)$ or for every $x \in (x_0, x_0 + t]$. To obtain conclusion i) it is enough to prove that the inequality

$$(5) \qquad f(x) \geq \eta - \mu_f(2t), \quad \eta = \limsup_{x \to x_0} f(x),$$

is true either for every $x \in [x_0 - t, x_0)$ or for every $x \in (x_0, x_0 + t)$.

Assume (5) is false, i. e. there are $x_1 \in [x_0-t, x_0)$ and $x_2 \in (x_0, x_0+t]$, such that $f(x_1)<\eta-\mu_f(2t)$ and $f(x_2)<\eta-\mu_f(2t)$. Choose $\varepsilon>0$ so that $f(x_1)+\varepsilon<\eta-\mu_f(2t)$ and $f(x_2)+\varepsilon<\eta-\mu_f(2t)$ simultaneously hold, and a point $x_3 \in (x_1, x_2)$ such that $|f(x_3)-\eta|<\varepsilon$, i. e. $\eta-\varepsilon<f(x_3)<\eta+\varepsilon$. Such a point does exist because there is a sequence $x_k \to x_0$, for which $f(x_k)\to\eta$, and $x_1<x_0<x_2$. From $\eta-f(x_1)>\mu_f(2t)+\varepsilon$ and $\eta-f(x_3)<\varepsilon$ we get $\eta-f(x_1)>\mu_f(2t)+\varepsilon>\mu_f(2t)+\eta-f(x_3)$, i. e. $f(x_3)-f(x_1)>\mu_f(2t)>0$. Similarly is obtained that $f(x_3)-f(x_2)>\mu_f(2t)>0$. Using the last two inequalities and the inequalities $|x_1-x_2|<2t$, $x_1<x_3<x_2$ we obtain

$$2\mu_f(2t)\geqq |f(x_1)-f(x_3)|+|f(x_2)-f(x_3)|-|f(x_1)-f(x_2)|$$
$$=f(x_3)-f(x_1)+f(x_3)-f(x_2)-|f(x_1)-f(x_2)|$$
$$=\begin{cases} 2(f(x_3)-f(x_1)), & \text{when } f(x_1)\geqq f(x_2) \\ 2(f(x_3)-f(x_2)), & \text{when } f(x_1)<f(x_2)\end{cases} > 2\mu_f(2t).$$

This contradiction proves (5). In order to obtain (3) it is sufficient to notice that $y_0 \leqq \max\{f(x_0), \limsup_{x\to x_0} f(x)\}$. Then (3) follows from (4) and (5).

We now can state

Theorem 1. *For every two* $f, g \in \mathbf{B}_1$ *with* $r=r(f, g)\geqq\min\{\lim_{t\to 0}\tilde\mu_f(t), \lim_{t\to 0}\tilde\mu_g(t)\}$ *the inequality*

$$(6)\qquad \int_a^b |f(x)-g(x)|\,dx \geqq \max\{m_1 r-\int_0^{m_1}\tilde\mu_f(2t)\,dt,\ m_2 r-\int_0^{m_2}\tilde\mu_g(2t)dt\}$$

holds, wherein

$$m_1=\min\{r, \inf\{t\,|\,\tilde\mu_f(2t)\geqq r\}\},$$

$$m_2=\min\{r, \inf\{t\,|\,\tilde\mu_g(2t)\geqq r\}\}.$$

Proof. First we assume that the Hausdorff distance between f and g is attained from f to g at some point $x \in [a, b]$, i. e. $r(f, g)=\max_{A\in\bar f(x)} d(A, g)=d((x, y^*), g)$. Then according to the definition of $r(f, g)$, for every t, such that $|t|\leqq r=r(f, g)$ and $x+t \in [a, b]$ holds either $g(x+t)\leqq y^*-r$ or $g(x+t)\geqq y^*+r$.

Case 1. Suppose that $g(x+t)\leqq y^*-r$ for every t, such that $|t|\leqq r$, $x+t \in [a, b]$. Assume also that

Case 1a. $a+r\leqq x\leqq b-r$. In this case $x \in (a, b)$ and according to lemma 1 for every $t \in (0, m_1]$ we have either $f(x-t)\geqq y^*-\mu_f(2t)$ or $f(x+t)\geqq y^*-\mu_f(2t)$. Denote by S_1 the set of points t' belonging to $(0, m_1]$, which satisfy the first inequality and by S_2 the set of points, satisfying the second inequality. Obviously $S_1\cup S_2=[0, m_1]$ and S_1, S_2 are measurable sets. Then

$$\int_a^b |f(x)-g(x)|\,dx\geqq\int_{x-r}^{x+r}|f(\xi)-g(\xi)|\,d\xi$$

$$=\int_0^r |f(x+t)-g(x+t)|\,dt+\int_0^r |f(x-t)-g(x-t)|\,dt$$

$$\geqq \int_{S_1} |f(x+t)-g(x+t)|dt+\int_{S_2} |f(x-t)-g(x-t)|dt$$

$$\geqq \int_{S_1} [y^*-\mu_f(2t)]-(y^*-r)dt+\int_{S_2} [y^*-\mu_f(2t)]-(y^*-r)dt$$

$$= \int_{S_1 \cup S_2} [r-\mu_f(2t)]dt = \int_0^{m_1} [r-\mu_f(2t)]dt = r_{m_1} - \int_0^{m_1} \mu_f(2t)dt$$

and hence, in view of the fact that $\tilde{\mu}_f(t) \geqq \mu_f(t)$,

$$\int_a^b |f(x)-g(x)|dx \geqq rm_1 - \int_0^{m_1} \tilde{\mu}_f(2t)dt \tag{7}$$

holds, as well. Consider now

Case 1b. $x \in [a, a+r)$ or $x \in (b-r, b]$. Assume that $x \in [a, a+r)$. Notice that if $m_1 \leqq x-a$, then the above considerations hold and (7) is true. Suppose $m > x-a$. Then we can write

$$\int_a^b |f(x)-g(x)|dx \geqq \int_a^{x+r} |f(\xi)-g(\xi)|d\xi \geqq \int_a^{2x-a} |f(\xi)-g(\xi)|d\xi$$

$$+\int_{2x-a}^{x+m_1} |f(\xi)+g(\xi)|d\xi$$

and try to estimate the two integrals. For every $t \in (0, x-a]$ we have: either $f(x+t) \geqq y^*-\tilde{\mu}_f(2t)$ for t belonging to some $S_1 \subset (0, x-a]$, or $f(x-t) \geqq y^*-\tilde{\mu}_f(2t)$ for t belonging to $S_2 \subset (0, x-a]$, $S_1 \cup S_2 = (0, x-a]$.

Using the assumption that $g(x+t) \leqq y^*-r$ for $|t| \leqq x-a<r$ we obtain

$$\int_a^{2x-a} |f(\xi)-g(\xi)|d\xi = \int_0^a |f(x-t)-g(x-t)|dt+\int_0^a |f(x+t)-g(x+t)|dt$$

$$\geqq \int_{S_2} |f(x-t)-g(x-t)|dt+\int_{S_1} |f(x+t)-g(x+t)|dt$$

$$\geqq \int_{S_2} [y^*-\tilde{\mu}_f(2t)-(y^*-r)]dt+\int_{S_1} [y^*-\tilde{\mu}_f(2t)-(y^*-r)]dt$$

$$= \int_0^a [r-\tilde{\mu}_f(2t)]dt.$$

In order to obtain a bound for the second integral, wherein t takes values $t \geqq x-a$, notice that

$$\tilde{\mu}_f(2t) \geqq \sup_{\substack{|\xi-\eta| \leqq 2t \\ |\xi-a| \leqq 2t}} |f(\xi)-f(\eta)| \geqq \sup_{|x+t-\eta| \leqq 2t} |f(x+t)-f(\eta)| . \tag{8}$$

In the second inequality above we have chosen $\xi = x+t$, which is allowed because of $t \geqq x-a$, i. e. $|x+t-a| \leqq 2t$. The number η satisfies the inequalities $x-t \leqq \eta \leqq x+3t$ and hence there is a sequence $\eta_{k} \to x$, for which $\{f(\eta_{k})\}$ tends to $\limsup_{t\to x} f(t)$. Passing to the limit in (8) yields

$$\tilde{\mu}_f(2t) \geqq \limsup_{t\to x} f(t) - f(x+t) ,$$

which implies

(9) $$f(x+t)\geqq\limsup_{t\to x} f(t)-\tilde{\mu}_f(2t),\quad t\geqq x-a.$$

Stilll easier we obtain that for $t\geqq x-a$

(10) $$f(x+t)\geqq(x)-\tilde{\mu}_f(2t).$$

Indeed,

$$\tilde{\mu}_f(2t)\geqq\tilde{\mu}_f(t)\geqq\sup_{\substack{|h|\leq t\\ |\xi-a|\leq t}}|f(\xi+h)-f(\xi)|$$

$$\geqq\sup_{|\xi-a|\leq t}|f(\xi+t)-f(\xi)|\geqq|f(x+t)-f(x)|.$$

Combining (9) and (10) yields for any y_0, $y_0\leqq\max\{f(x), \limsup_{t\to x} f(t)\}$

$$f(x+t)\geqq v_0-\tilde{\mu}_f(2t),$$

which is enough to conclude that $f(x+t)\geqq y^*-\tilde{\mu}_f(2t)$, $t\geqq x-a$ (recall that y^* satisfies $(x, y^*)\in f(x)$).

For the second integral we obtain then

$$\int_{2x-a}^{x+m_1}|f(\xi)-g(\xi)|\,d\xi=\int_{x-a}^{m_1}|f(x+t)-g(x+t)|\,dt\geqq\int_{x-a}^{m_1}[(y^*-\tilde{\mu}_f(2t))-(y^*-r)]\,dt$$

$$=\int_{x-a}^{m_1}[r-\tilde{\mu}_f(2t)]\,dt,$$

so that eventually we have

$$\int_a^b|f(x)-g(x)|\,dx\geqq\int_0^{x-a}[r-\tilde{\mu}_f(2t)]dt+\int_{x-a}^{m_1}[r-\tilde{\mu}_f(2t)]dt$$

$$=\int_0^{m_1}[r-\tilde{\mu}_f(2t)]dt=rm_1-\int_0^{m_1}\tilde{\mu}_f(2t)dt,$$

which is again (7). When $x\in[b-r, b]$ we proceed similarly.

Case 2. $g(x+t)\geqq y^*+r$, $|t|\leqq r$. By proceeding as in case 1 and using conclusion ii) of Lemma 1, one obtains again (7).

Thus inequality (7) is proved by the assumption that the Hausdorff distance $r(f, g)$ is attained from f to g. In case that it is attained from g to f, i. e. $r(f, g)=\max_{A\in\bar{g}(x)}d(A, f)$, we obtain similarly

$$\int_a^b|f(x)-g(x)|\,dx\geqq rm_2-\int_0^{m_2}\tilde{\mu}_g(2t)dt.$$

This inequality and (7) imply (6), which completes the proof.

Corollary 1. *For every two functions $f, g\in\mathbf{B}_1$, with $\mu_f(2t)=\mu_g(2t)=t^q$, $q\geqq0$, the inequality*

$$\int_a^b|f(x)-g(x)|\,dx\geqq Cr^\beta$$

holds, where $\beta=\{2$, *when* $q\geq 1$; $(1+q)/q$, *when* $q\leq 1\}$, *and* C *depends only on* q.

Indeed we have $m_1=m_2=\{r$, when $q\geq 1$; $r^{1/q}$, when $q\leq 1\}$, which substituted in (6) yields

$$\int_a^b |f(x)g-g(x)|\geq \begin{cases} r^2-[2^q/(q+1)]r^{1+q}\geq r^2, & q\geq 1,\\ r^{(1+q)/q}-[2^q/(1+q)]r^{(1+q)/q}=C(q).r^{(1+q)/q}, & q\leq 1.\end{cases}$$

R e m a r k 1. Results (1) and (6) suggest the question: are there functions, for which both the bounds are of the same order with respect to r. For $\mu_f(t)=\mu_g(t)=t^q$, $q\geq 1$ and $b-a=\text{const}.r$ in view of (2) and Corollary 1 we have

$$r^2\leq\int_a^b |f(x)-g(x)|\,dx\leq Cr^{(1+2q)/(1+q)}.$$

Because of $\lim_{q\to\infty}(1+2q)/(1+q)=2$, we might expect that the expression in the right-hand side will attain the order $O(r^2)$ perhaps only for monotone functions. Indeed it is easy to check that for monotone functions ($\mu(t)=0$), defined on an interval $[a, b]$, such that $b-a=\text{const}.r$, the inequalities

$$r^2\leq\int_a^b |f(x)-g(x)|\,dx\leq Cr^2$$

hold, showing that the estimates (1) and (6) are exact by order in $\mathbf{B}_1$.

3. An application in the theory of ordinary differential equations. We shall use the results (1) and (6) for bounding the uniform distance between the solutions of two ordinary differential equations by means of the Hausdorff distance between the right-hand sides of the equations.

To clarify this, suppose y and u satisfy in $[a, b]$ the differential equations $y'=f(x, y)$, resp. $u'=g(x, u)$, and for the sake of simplicity suppose that $y(a)=u(a)$. Assume that $f, g\in\mathbf{B}_2$ satisfy a Lipschitz condition (with respect to the second variable) in D. It is easy to show (for more detail see for instance [4, ch. 1, §2]) that if $\max_{(x,y)\in D}|f(x, y)-g(x, y)|\leq\varepsilon$, then

$$\|y-u\|_C=\sup_{a\leq x\leq b}|y(x)-u(x))|\leq\int_a^b |f(x, y(x))-g(x, u(x))|\,dx$$

$$\leq\int_a^b |f(x, y(x))-f(x, u(x))|\,dx+\int_a^b |f(x, u(x))-g(x, u(x))|\,dx$$

$$\leq K(b-a)\|y-u\|_C+\varepsilon(b-a),$$

where K is the Lipschitz constant of f. Hence we have

$$\|y-u\|_C\leq\varepsilon(b-a)/[1-K(b-a)],$$

assuming that $K(b-a)<1$.

Our purpose in this section is to obtain an upper bound for $\|y-u\|_C$ by means of the Hausdorff distance $r(f, g)$ instead of the uniform distance ε.

In order to use inequality (1) for estimating the integral $\int_a^b |f(x, u(x)) - g(x, u(x))| dx$ we need some preliminary results.

Lemma 2. *Suppose that* $f(t, u)$ *satisfies a Lipschitz condition (with respect to* u*) with constant* K *in* $D=[a, b; c, d]$. *Suppose* $y=y(t)$ *satisfies a Lipschitz condition with constant* k *in* $[a, b]$ *and let* $y(t) \in [c, d]$ *for every* $t \in [a, b]$. *Then for the modulus of non-monotonicity of the function* φ *defined on* $[a, b]$ *by* $\varphi(t)=f(t, y(t))$ *the inequality*

$$\mu_\varphi(\delta) \leqq \mu_f(\delta) + 4Kk\delta \tag{11}$$

holds.

Proof. For $t_1, t_2, t \in [a, b]$ let us put $y_1=y(t_1)$, $y_2=y(t_2)$, $y=y(t)$. We have $|f(t, y_2)-f(t, y_1)| \leqq K|y_2-y_1| \leqq Kk|t_2-t_1|$. Obviously

$$\begin{aligned} m &\overset{\mathrm{Def}}{=} \sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_1)-f(t, y)| + |f(t_2, y_2)-f(t, y)|] - |f(t_2, y_2)-f(t_1, y_1)| \\ &\leqq \sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_1)-f(t, y_1)| + |f(t, y_1)-f(t, y)| + |f(t_2, y_2)-f(t, y_2)| \\ &\qquad + |f(t, y_2)-f(t, y)|] - |f(t_2, y_2)-f(t_1, y_1)| \\ &\leqq \sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_1)-f(t, y_1)| + Kk|t_1-t| + |f(t_2, y_2)-f(t, y_2)| + Kk|t_2-t|] \\ &\qquad - |f(t_2, y_2)-f(t_1, y_1)| \\ &\leqq \sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_1)-f(t, y_1)| + |f(t_2, y_2)-f(t, y_2)|] - |f(t_2, y_2)-f(t_1, y_1)| \\ &\qquad + Kk|t_2-t_1|. \end{aligned}$$

But

$$\begin{aligned} |f(t_1, y_1)-f(t, y_1)| &\leqq |f(t_1, y_1)-f(t_1, y_2)| + |f(t_1, y_2)-f(t, y_2)| + |f(t, y_2)-f(t, y_1)| \\ &\leqq Kk|t_2-t_1| + |f(t_1, y_2)-f(t, y_2)| + Kk|t_2-t_1| = |f(t_1, y_2)-f(t, y_2)| + 2Kk|t_2-t_1| \end{aligned}$$

and hence

$$\begin{aligned} m &\leqq \sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_2)-f(t, y_2)| + |f(t_2, y_2)-f(t, y_2)|] - |f(t_2, y_2)-f(t_1, y_1)| \\ &\qquad + 3Kk|t_2-t_1| \\ &= \sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_2)-f(t, y_2)| + |f(t_2, y_2)-f(t, y_2)|] - |f(t_2, y_2)-f(t_1, y_2)| \\ &\qquad + A + 3Kk|t_2-t_1|, \end{aligned}$$

wherein $A = |f(t_2, y_2)-f(t_1, y_2)| - |f(t_2, y_2)-f(t_1, y_1)| \leqq |f(t_1, y_1)-f(t_1, y_2)| \leqq Kk|t_2-t_1|$. Therefore,

$$\begin{aligned} m &\leqq \sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_2)-f(t, y_2)| + |f(t_2, y_2)-f(t, y_2)|] - |f(t_2, y_2)-f(t_1, y_2)| \\ &\qquad + 4Kk|t_2-t_1|. \end{aligned}$$

Thus we have

$$\begin{aligned} \mu_\varphi(\delta) &= \sup_{|t_1-t_2| \leqq \delta} m \leqq \sup_{c \leqq y_2 \leqq d} \sup_{|t_1-t_2| \leqq \delta} m \\ &\leqq \sup_{c \leqq y_2 \leqq d} \sup_{|t_1-t_2| \leqq \delta} \{\sup_{t_1 \leqq t \leqq t_2}[|f(t_1, y_2)-f(t, y_2)| + |f(t_2, y_2)-f(t, y_2)|] \\ &\qquad - |f(t_2, y_2)-f(t_1, y_2)|\} + 4Kk\delta = \mu_f(\delta) + 4Kk\delta, \end{aligned}$$

which was to be proved.

Lemma 3. *Suppose that f and g are bounded in $D=[a, b; c, d]$ and satisfy the Lipschitz condition (with respect to y) with constant K. Suppose $y=y(t)$ and $u=u(t)$ satisfy the Lipschitz condition in $[a, b]$ with constant k and $y(t)$, $u(t)\in[c, d]$ for every $t\in[a, b]$. Then*

$$(12)\qquad r(\varphi, \psi)\leq Mr+\|y-u\|_C,$$

where $\varphi(t)=f(t, y(t))$, $\psi(t)=g(t, u(t))$, $r=r(f, g)$ *and* $M=Kk+K+1$.

Proof. According to the definition of $r=r(f, g)$, to every $t\in[a, b]$ correspond $t_1\in[a, b]$ and $u_1\in[c, d]$ with the properties that $|t_1-t|\leq r$, $|y(t)-u_1|\leq r$ and $|f(t, y(t))-g(t_1, u_1)|\leq r$. Similarly to every $t\in[a, b]$ correspond $t_2\in[a, b]$ and $y_2\in[c, d]$, such that $|t_2-t|\leq r$, $|u(t)-y_2|\leq r$ and $|f(t_2, y_2)-g(t, u(t))|\leq r$. Hence

$$(13)\quad r(\varphi, \psi)=\max\{r, \max_{a\leq t\leq b}|f(t, y(t))-g(t_1, u(t_1))|, \max_{a\leq t\leq b}|f(t_2, y(t_2))-g(t, u(t))|\}.$$

But

$$|f(t, y(t))-g(t_1, u(t_1))|\leq|f(t, y(t))-g(t_1, u_1)|+|g(t_1, u_1)-g(t_1, u(t_1))|$$
$$\leq r+K|u_1-u(t_1)|\leq r+K\{|u_1-y(t)|+|y(t)-y(t_1)|+|y(t_1)-u(t_1)|\}$$
$$\leq r+K(r+k|t-t_1|+\|y-u\|_C)\leq r+K(r+kr+\|y-u\|_C)=Mr+\|y-u\|_C,$$

where $M=Kk+K+1$. A similar proceeding yields

$$|g(t, u(t))-f(t_2, y(t_2))|\leq Mr+\|y-u\|_C.$$

The substitution of these inequalities in (13) leads to the conclusion (12) of the lemma.

Particularly, assuming that $y(t)\equiv u(t)$ on $[a, b]$, we obtain

Corollary 2. *Suppose f and g are bounded on $D=[a, b; c, d]$ and satisfy the Lipschitz condition (with respect to y) with constant K. Suppose $y=y(t)$ satisfies the Lipschitz condition on $[a, b]$ with constant k and $y(t)\in[c, d]$ for all $t\in[a, b]$. Then the Hausdorff distance between $\varphi=f(t, y(t))$ and $\psi=g(t, y(t))$ satisfies*

$$(14)\qquad r(\varphi, \psi)\leq(Kk+K+1)r(f, g).$$

Let us formulate also the following

Corollary 3. *Suppose f, g are bounded in $D=[a, b; c, d]$ by the constant $B>0$ and satisfy the Lipschitz condition (with respect to y) with constant K. Suppose $y=y(t)$ and $u=u(t)$ satisfy the differential equations $y'=f(t, y)$, $y(a)=u_0$; resp. $u'=g(t, u)$, $u(a)=u_0$. Then the inequalities*

$$\text{i)}\qquad r(y', u')\leq Mr+\|y-u\|_C$$

$$\text{ii)}\qquad \|y'-u'\|_C\leq(1+B)(Mr+\|y-u\|_C)$$

hold, wherein $M=KB+K+1$, $r=r(f, g)$.

Using some of the results one can verify the following

Theorem 2. *Suppose $f, g\in\mathbf{B}_2$ and satisfy the Lipschitz condition (with resp. to y) with constant K, such that $K<(b-a)^{-1}$. Then the solutions $y=y(x)$, $u=u(x)$ of the initial problems $y'=f(x, y)$, $y(a)=\lambda$; $u'=g(x, u)$, $u(a)=\lambda$, satisfy the inequality*

$$(15)\qquad \|y-u\|_C \leq \frac{b-a}{1-(b-a)K}\left\{\inf_{0\leq\delta\leq b-a}\left[4BM\bar{r}\frac{\delta+1}{\delta}+2\bar{\mu}_f(\delta)+8KB\delta\right]+Mr\right\},$$

wherein $M=KB+K+1$, $r=r(f, g)$.

Proof. Indeed it follows from $y(x)-u(x)=\int_a^x [f(t, y(t)) - g(t, u(t))]dt$ that

$$(16)\qquad \begin{aligned} \|y-u\|_C &\leq \int_a^b |f(t, y(t))-g(t, u(t))|\,dt \\ &\leq \int_a^b |f(t, y(t))-g(t, y(t))|\,dt+\int_a^b |g(t, y(t))-g(t, u(t))|\,dt = I_1+I_2. \end{aligned}$$

Relying on inequality (1) we have

$$I_1\leq(b-a)\{\inf_{0\leq\delta\leq b-a}[4B\bar{r}(\delta+1)/\delta+2\mu_\varphi(\delta)]+\bar{r}\},$$

where $r=r(f(t, y(t)), g(t, y(t))$ and μ_φ is the modulus of non-monotonicity of $\varphi(t)=f(t, y(t))$. Hence, in view of (11) and (14)

$$I_1\leq(b-a)\{\inf_{0\leq\delta\leq b-a}[4BMr(\delta+1)/\delta+2\mu_f(\delta)+8KB\delta]+Mr\},$$

where $M=KB+K+1$, $r=r(f, g)$ and μ_f is the modulus of non-monotonicity of f with respect to t.

On the other hand, $I_2\leq\int_a^b K\|y-u\|_C\,dt=K\|y-u\|_C(b-a)$, using the assumption that g satisfies a Lipschitz condition. Substituting the bounds for I_1 and I_2 in (16) yields

$$\|y-u\|_C\leq(b-a)\{\inf_{0\leq\delta\leq b-a}[4BMr(\delta+1)/\delta+2\bar{\mu}_f(\delta)+8KB\delta]+Mr\}$$
$$+K(b-a)\|y-u\|_C,$$

which in view of $K<(b-a)^{-1}$ leads to (15).

Remark 2. Recalling Remark 1 one can easily observe that the bound for I_1 is exact by order, which immediately implies that the bound (15) is also exact.

Remark 3. As we have shown, if we have bounded $\rho(y, u)=\|y-u\|_C$ by means of the integral distance between f and g, then using relation (1) we can bound $\rho(y, u)$ by means of the Hausdorff distance between f and g. Inversely, if we have bounded in some way $\rho(y, u)$ by means of $r(f, g)$, then applying (6) we can find a corresponding bound for $\rho(y, u)$ based on the integral distance between f and g.

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Centre for Research and Education in Mathematics and Mechanics
1000 Sofia P. O. Box 373

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