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FINITE NONSOLVABLE GROUPS HAVING A MAXIMAL SUBGROUP OF ORDER $2p$

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The object of this paper are finite nonsolvable groups G having a maximal subgroup of order $2p$, p prime. By using group theoretic, character theoretic, and elementary arithmetical arguments, the following result is proved: If the order of G is divisible by at most four distinct primes, then G is isomorphic to $PSL(2, q)$ or $Sz(2^q)$ for an appropriate value of q.

1. Introduction. In this paper we are interested in finite nonsolvable groups G with the following property:

(∗) G has a maximal subgroup H of order $2p$, p prime.

W. Feit and J. Thompson have proved [6] that if $p=3$ then G is isomorphic to PSL(2, 5) and G. Higman has shown [10] that G is isomorphic to PSL(2, 5) if $p=5$. The more general situation of finite groups having a dihedral maximal subgroup of twice odd order has been investigated by several authors, see [9; 10; 12; 13].

The only known simple (∗)-groups are as follows:

$PSL(2, p\pm 1)$, where p is a Mersenne or Fermat prime $\geqq 3$;

$PSL(2, 2p\pm 1)$, where $2p\pm 1$ is a prime power $\geqq 5$;

$Sz(p+1)$, where p is a Mersenne prime $\geqq 7$.

Here we prove the following result.

Theorem. *Let G be a finite nonsolvable* (∗)*-group. Suppose that the order of G is divisible by at most four distinct primes. Then G is isomorphic to one of the groups listed above.*

The proof depends upon a method essentially due to R. Brauer and consisting in analysis of the possible degrees of the irreducible characters in the principal p-block of G. This method has been developed and applied by L. Alex in a similar context ([1—3]).

The notation is standard. However, $\pi(G)$ denotes the set of all distinct prime divisors of $|G|$, the order of G. $B(p)$ is the principal p-block of G and χ_n means an irreducible ordinary complex character of degree n. If r is a prime, the notation x_r is used only for an element of order r in the center of a Sylow r-subgroup of G, and $C(x_r)$ for the centralizer of x_r in G. The symbol $\#(x, x, y)$ denotes the class algebra coefficient, that is the number of distinct ways in which the group element y can be represented as a product of two conjugate to x elements.

2. Preliminary results. The following information concerning the principal p-block of a simple (∗)-group is crucial for the proof ([1—5, 9, 10]).

2.1. $B(p)$ consists of the principal character, 1_G, a character χ, and $(p-1)/2$ (exceptional) characters $\chi^{(i)}$, $i=1, \ldots, (p-1)/2$.

There is a sign $\delta = \pm 1$ so that for $i=1, \ldots, (p-1)/2$

(2.1.1) $$\chi(1) \equiv \delta, \quad \chi^{(i)}(1) \equiv 2\delta \pmod p.$$

For every p-regular element x of G

(2.1.2) $$\chi^{(i)}(x) = \chi(x) + \delta.$$

In particular,

(2.1.3) $$\chi^{(i)}(1) = \chi(1) + \delta,$$

the so called degree equation for $B(p)$.

2.2. If the degree equation is $r^c = \pm 1 + u$, where r is a prime, then r^c is the order of a Sylow r-subgroup of G. Hence a character of degree r^c vanishes on all r-singular elements.

2.3. For every p-regular element x and $1_G \neq \chi_n \in B(p)$,

(2.3.1) $$\#(x, x, x_p) = |G|(n - \chi_n(x))^2 / n(n+1) |C(x)|^2.$$

In particular,

(2.3.2) $$\#(x_2, x_2, x_p) = p.$$

2.4. All the involutions in G are contained in a single conjugate class.

Finally, we need a simple arithmetical result.

2.5. Lemma. *The solutions to the Diophantine equation* (.) $1+x=y$, *where* x *and* y *are of the form* $2^\alpha 3^\beta r^\gamma$, $r=13$ *or* 17, $x>25$, *are* $x=26$, $y=27$ *and* $x=288$, $y=289$.

Proof. Since x and y clearly are mutually prime, at least one of them must be a prime power. We next recall a simple fact. If r is an odd prime, and $r^\gamma = 2^\alpha \pm 1$, then $\gamma \leqq 1$ unless $\gamma = 2$ and $r=3$. Using this it is easily seen in each of the cases below that there is no solution to (.) with $x>25$ when any of the exponents α, β, and γ is zero. Thus we may assume $\alpha\beta\gamma \neq 0$. The cases $r=13$ and $r=17$ will be considered separately.

1) $r=13$. We have three possibilities:

1.1) x (or y) $=13^\gamma$ and (.) reads $13^\gamma = 2^\alpha 3^\beta \pm 1$. Now [12] (Lemma 2) implies $\gamma \leqq 2$ which gives no solutions.

1.2) x (or y) $=3^\beta$ and (.) becomes $3^\beta = 2^\alpha 13^\gamma \pm 1$. Since now $3^\beta \equiv \pm 1 \pmod{13}$, 3 divides β. If β is even, then $3^3 \equiv -1 \pmod 7$ yields $3^\beta \equiv 1 \pmod 7$ whence $2^\alpha 13^\gamma \equiv 0 \pmod 7$, a contradiction. Thus β is odd. $\alpha>1$ implies that $3^\beta \equiv 1 \pmod 4$ while $3^3 \equiv -1 \pmod 4$ shows that $3^\beta \equiv -1 \pmod 4$. Therefore $\alpha=1$ and we can write the equation in the form $2.13^\gamma = (u-1)(u^2+u+1)$, where $u=3^{\beta/3}$. Since $u-1$ and u^2+u+1 are mutually prime, we must have $u-1=2$ which implies that $x=26$, $y=27$.

1.3) x (or y) $=2^\alpha$, that is $2^\alpha = 3^\beta 13^\gamma \pm 1$. $2^\alpha \equiv \pm 1 \pmod{13}$ yields α is divisible by 6 whence $2^\alpha \equiv 1 \pmod 7$. Now we have $2^\alpha + 1 = 3^\beta 13^\gamma$ so that $2 \equiv 0 \pmod 3$, an impossibility.

2) $r=17$. We have again three possibilities:

2.1) x (or y) $=17^\gamma$ and (.) becomes $17^\gamma = 2^\alpha 3^\beta \pm 1$. Again [12] (Lemma 2) implies $\gamma \leqq 2$ which forces $x=288$, $y=289$.

2.2) x (or y)$=3^\beta$ in which case $3^\beta=2^a17^\gamma\pm1$. If $a=1$ or 2, then $\beta\geqq2$ and we are led to $2(-1)^\gamma\pm1\equiv0 \pmod 9$ or $4(-1)^\gamma\pm1\equiv0 \pmod 9$, both these congruences being impossible. Let now $a\geqq3$. Then $3^\beta\equiv\pm1 \pmod 8$ so that we have $3^\beta=2^a17^\gamma+1$. Now $3^\beta\equiv1 \pmod{17}$ shows that 16 divides β and then $3^4\equiv1 \pmod 5$ produces $2^a17^\gamma+1\equiv3^\beta\equiv1 \pmod 5$, a contradiction.

2.3) x (or y)$=2^a$ and (.) reads $2^a=3^\beta17^\gamma\pm1$. Now $2^a\equiv\pm1 \pmod{51}$ whence 8 divides a and then $2^8\equiv1 \pmod 5$ yields $3^\beta17^\gamma\equiv0 \pmod 5$, again an absurd. This proves the lemma.

3. Proof of the Theorem. Let G be a finite nonsolvable $(*)$-group and $|\pi(G)|\leqq4$. Then it follows from [14] that G is simple and H is nonabelian so that H is the dihedral subgroup of order $2p$, $p>2$. If $p\leqq5$, G is PSL(2,5) ([6] and [10]). So we may suppose $p\geqq7$. Let P be the subgroup of order p in H. Since P is normal in H, $N_G(P)=H$ and hence P is a Sylow p-subgroup of G. Furthermore, $C_G(P)=C_H(P)=P$. It is clear, because of the maximality of H, that P cannot lie in two distinct subgroups of order $2p$ and therefore G has exactly one conjugate class of subgroups of order $2p$.

If every proper subgroup of G is solvable Thompson's work [15] implies that G is one of the groups PSL$(2, 2^q)$, PSL$(2, 3^q)$, PSL$(2, q)$ or Sz(2^q) for suitable primes q, or PSL(3, 3). The latter one is not a $(*)$-group, since the normalizer of a Sylow 13-subgroup is not of order 26. Each of the remaining groups possesses, as mentioned above, the property $(*)$ for some q.

Now we can suppose that G contains nonsolvable proper subgroups. Let K be a minimal nonsolvable subgroup of G. We shall first show that p does not divide $|K|$. For in the contrary case we may assume $P\subset K$. Then $N_K(P)=K\cap H=P=C_K(P)$ and Burnside's well known theorem [7] implies that K has a normal p-complement A. Obviously P acts on A as a fixed point free automorphism of prime order and by another result of Thompson [7] A is nilpotent. Hence K is solvable. This contradiction proves the assertion. Let now L be a maximal normal subgroup of K. Then K/L is a simple group whose all proper subgroups are solvable by the minimal choice of K. Furthermore, it is clear that $|\pi(K/L)|=3$. Now the above list of the minimal simple groups yields $\pi(G)=\{2, 3, r, p\}$, where $r=5$, 7, 13 or 17.

If $r=5$ it follows from [11] that $G\cong$PSL$(2, q)$ for some q. If $r=7$, [1] implies that $G\cong$PSL(3, 4) or PSL $(2, q)$ for some q. The former one is not a $(*)$-group, since the normalizer of a Sylow 7-subgroup is not of order 14, and the normalizer of a Sylow 5-subgroup, of order 10, is not maximal in PSL(3, 4) which contains a subgroup isomorphic to PSL(2, 5).

Hereafter we shall suppose that $r=13$ or 17. Consider now the principal p-block $B(p)$. If $B(p)$ contains a nonidentity character of degree $\leqq25$, then [3] can be applied to obtain G is PSL$(2, q)$ for some q. Thus, we may assume that $\chi(1)>25$, $\chi^{(i)}(1)>25$ in the degree equation. Since these character degrees are also relatively prime to p, Lemma 2.5 yields two possibilities for the degree equation; $1+26=27$ and $1+288=289$.

Consider first the degree equation $1+26=27$. Now 2.2 implies that $|G|=2^a3^313^cp$, and (2.1.1) yields the unique choice $p=7$. Furthermore, $\#(x_2, x_2, x_7)=7$ from (2.3.2) and then (2.3.1) becomes $7.2.13.3^3\,|C(x_2)|^2=|G|(26-\chi_{26}(x_2))^2$. If 2^α, $\alpha\geqq0$, is the highest power of 2 dividing $26-\chi_{26}(x_2)$, then this equality shows that $a=2\alpha-1$ is odd. Since $0<26-\chi_{26}(x_2)<52$, $\alpha\leqq5$ whence $a\leqq9$., The same equality yields c odd and a similar consideration of $\#(x_{13}, x_{13}, x_7)$ forces $c\leqq1$ so that $c=1$. The number of Sylow 7-subgroups in G is $2^{a-1}3^313$

and is $\equiv 1 \pmod 7$ which yields $2^{a-1} \equiv 1 \pmod 7$. It follows $a=1$ or 7. But $a=1$ is impossible since the order of a simple nonabelian group is divisible by 4, and, therefore, $|G|=2^7 3^3 13 . 7 = 314{,}496$. Now [8] shows that there is no simple group of this order, a contradiction.

Thus, the degree equation for $B(p)$ is $1+288=289$. The results of the preceding section yield $|G|=2^a 3^b 17^2 p$, where $p=7$, 29 or 41. Furthermore, $\#(x_2, x_2, x_p)$ and $\#(x_3, x_3, x_p)$ show that a is odd, $3 \leq a \leq 13$, and b is even, $b \leq 8$. Also a count of Sylow p-subgroups of G implies that $2^{a-1} 3^b 17^2 \equiv 1 \pmod p$. We shall consider the cases $p=7, 29$, and 41 separately. The following lemma provides a usefull test for the analysis of the various cases appearing below.

Lemma. *Let* S *be a Sylow* 17-*subgroup of* G. *Then*

(i) p *does not divide* $|N_G(S)|$;

(ii) $a \geq 5$, *and if* $a=5$ *or* $b \leq 4$, *the number of conjugates of* S *in* G *is congruent 1* $\pmod{17^2}$.

Proof. If p divides $|N_G(S)|$ then $C_G(P)=P$ implies that S has an automorphism of order p. But S, of order 17^2, is either cyclic or elementary Abelian and its automorphism group is of order $2^4 . 17$ or isomorphic to GL(2, 17), of order $2^9 3^2 . 17$, respectively. This contradiction proves (i).

Furthermore 2.2 yields $\chi_{289}(x_{17})=0$, and $\#(x_{17}, x_{17}, x_p)$ then yields $a \geq 5$ and $|C(x_{17})| = 17^2 2^r 3^s$, where $0 \leq r \leq 4$, $0 \leq s \leq 3$. If $a=5$ or $b \leq 4$, then $r=0$ or $s \leq 1$. Now a count of Sylow 17-subgroups of $C(x_{17})$ forces S is normal in $C(x_{17})$. Hence clearly S is a trivial intersection set in G. Consider the action (under conjugation) of S on the set π of the remaining Sylow 17-subgroups of G. Since no element $\neq 1$ of S fixes any point of π, the congruence in (ii) is obvious.

Case $p=7$. Now the congruence for the number of the Sylow p-subgroups of G becomes $2^a 3^b \equiv 1 \pmod 7$. This leads to the following possibilities for $|G|$: $2^3 3^6 17^2 7$, $2^5 3^2 17^2 7$, $2^5 3^8 17^2 7$, $2^7 3^4 17^2 7$, $2^9 3^6 17^2 7$, $2^{11} 3^2 17^2 7$, $2^{11} 3^8 17^2 7$, and $2^{13} 3^4 17^2 7$.

$|G|=2^3 3^6 17^2 7$ is impossible by the lemma.

If $|G|=2^5 3^2 17^2 7$ or $2^5 3^8 17^2 7$ [8] or the above lemma produces a contradiction. Similarly, if $|G|=2^7 3^4 17^2 7$, the lemma is contradicted.

If $|G|=2^9 3^6 17^2 7$, then 2^7 divides $288-\chi_{288}(x_2)$ and $0 < 288-\chi_{288}(x_2) < 576$, whence $288-\chi_{288}(x_2)=128$ or 128.3. Thus $\chi_{288}(x_2)=160$ or -96 and $|C(x_2)| = 2^9 3^2$ or $2^9 3^3$, respectively. In both cases we have $|C(x_2)| < (\chi_{288}(x_2)) + (\chi_{289}(x_2))^2 \leq \sum_{\chi} (\chi(x_2))^2$, where the sum is taken over all irreducible characters of G. But this is incompatible with the orthogonality relations for group characters.

The possibility $|G|=2^{11} 3^2 17^2 7$ is rejected by the lemma or the argument of the preceding paragraph, since now $B(7)$ contains three characters of degree 289.

When $|G|=2^{11} 3^8 17^2 7$, $\#(x_2, x_2, x_7)$ yields $|C(x_2)|=2^{11} 3^3$. Since all the involutions are conjugate in G by 2.4, this implies that 2 does not divide $|C(x_{17})|$. Then $\#(x_{17}, x_{17}, x_7)$ yields $|C(x_{17})|=17^2 3^s$, $0 \leq s \leq 3$. Thus, a Sylow 17-subgroup is normal in $C(x_{17})$ and the argument in the proof of the lemma, (ii), shows that the number of Sylow 17-subgroups of G is $\equiv 1 \pmod{17^2}$. This congruence leads to an impossibility.

Finally, if $|G|=2^{13}3^4 17^2 7$, we must have $\chi_{288}(x_2)=-224$, $\chi_{289}(x_2)=-223$ and $|C(x_2)|=2^{13}3$. Now again $(\chi_{288}(x_2))^2+(\chi_{289}(x_2))^2>|C(x_2)|$, a contradiction.

Case $p=29$. Now we have $2^{a-1}3^b\equiv -1 \pmod{29}$, whence $|G|=2^3 3^8 17^2 29$, $2^5 3^2 17^2 29$ or $2^{13}3^6 17^2 29$.

The former possibility is rejected by the lemma.

If $|G|=2^5 3^2 17^2 29$, the lemma yields no choice for the number of Sylow 17-subgroups of G.

When $|G|=2^{13}3^6 17^2 29$, a computation by $\#(x_2, x_2, x_{29})$ yields $\chi_{288}(x_2)=-224$, $\chi_{289}(x_2)=-223$, and $|C(x_2)|=2^{13}3^2$. This leads to the contradiction $|C(x_2)|<(\chi_{288}(x_2))^2+(\chi_{289}(x_2))^2$.

Case $p=41$. Here $2^a 3^b\equiv 1 \pmod{41}$ which yields the unique possibility $|G|=2^5 3^2 17^2 41$. This is however impossible, since the congruence of the lemma fails even modulo 17.

This completes the proof of the theorem.

REFERENCES

1. L. Alex. On simple groups of order $2^a 3^b 7^c p$. *J. Algebra*, **25**, 1973, 113—124.
2. L. Alex. Simple groups of order $2^a 3^b 5^c 7^d p$. *Trans. Amer. Math. Soc.*, **173**, 1972, 389—399.
3. L. Alex. Index two simple groups. *J. Algebra*, **31**, 1974, 262—275.
4. R. Brauer. On groups whose order contains a prime number to the first power. I, II. *Amer. J. Math.*, **64**, 1942, 401—440.
5. R. Brauer, H. F. Tuan. On simple groups of finite order. I. *Bull. Amer. Math. Soc.*, **51**, 1945, 756—766.
6. W. Feit, J. Thompson. Finite groups which contain a self-centralizing subgroup of order 3. *Nagoya J. Math.*, **21**, 1962, 185—197.
7. D. Gorenstein. Finite groups. New York, 1968.
8. M. Hall. Simple groups of order less than one million. *J. Algebra*, **20**, 1972, 98—102.
9. K. Harada. A characterization of the groups LF(2, q). III. *J. Math.*, **11**, 1967, 647—659.
10. G. Higman. Odd characterizations of finite simple groups. Lecture notes, University of Michigan, 1968.
11. D. Mutchler. On simple groups of order $2^a 3^b 5^c p$ (unpublished; see [2]).
12. Н. Петров. Конечные группы с максимальной диэдральной подгруппой. *Плиска*, **2**, 1981, 23—29.
13. W. Stewart. Groups having strongly self-centralizing 3-centralizers. *Proc. London Math. Soc.*, **26**, 1973, 653—680.
14. К. Чакърян. Конечные группы с максимальной подгруппой порядков p^n и pq. *Плиска*, **2**, 1981, 116—118.
15. J. Thompson. Nonsolvable finite groups all of whose local subgroups are solvable. *Bull. Amer. Math. Soc.*, **74**, 1968, 383—437.

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