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ПЛИСКА
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REPRESENTATIVE DOMAINS OF COMPLEX MANIFOLDS

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In this paper we consider the Bergman kernel function of a complex manifold with respect to its volume element, define the representative domain of a complex manifold and study the representative domains.

According to the Riemann mapping theorem every two simply connected plane domains different from **C** are biholomorphically equivalent, i. e. there exists a biholomorphic mapping from one of the domains onto the other. This is not true in the space of several complex variables; it is well-known that the unit polydisk and the unit ball in $\mathbf{C}^n$, $n \geqq 2$, are not biholomorphically equivalent. Hence, there is no "canonical" domain in the class of simply connected domains in $\mathbf{C}^n$ different from $\mathbf{C}^n$, $n \geqq 2$. To determine, therefore, a "canonical" domain for some class of domains is very worthwhile.

We can regard the so called representative domains, introduced by S. Bergman, as representatives of some classes of domains (cf. [8, p. 289, Corollary 2]). For a bounded domain M in $\mathbf{C}^n$, Bergman defined a mapping F (the representative mapping) on M by use of some extremum problem (see [4; 5] or the identity (2.6), point 2, in this paper). Bergman called the image set of M in $\mathbf{C}^n$ under F a representative domain of M. But F is, in general, meromorphic on M and the representative domain of M is no longer in $\mathbf{C}^n$. Moreover, we do not know whether the representative domain is a domain at all. In this paper, we refuse to consider the representative mapping on the whole domain M. We consider F on a suitable subdomain $M \setminus S$ of M. The mapping F is locally biholomorphic on $M \setminus S$. Using this fact, we construct a suitable domain (Δ, π) over $\mathbf{C}^n$ and a biholomorphic mapping $\tilde{F}: M \setminus S \to \Delta$ such that $\pi \circ \tilde{F} = F$ on $M \setminus S$. We call (Δ, π) a representative domain of M. If $S = \varnothing$ and F is biholomorphic on M, we can identify (Δ, π) with $F(M)$. Then (Δ, π) is the Bergman representative domain of M.

The fact that the representative domain (Δ, π) is a domain over $\mathbf{C}^n$ suggests considering domains over $\mathbf{C}^n$. But many definitions and propositions can be formulated for complex manifolds. In particular, one can define the representative domain of a complex manifold.

In the present paper, we consider the Bergman kernel function of a complex manifold with respect to its volume element in point 1, define the representative domain of a complex manifold in point 2 and study the representative domains in point 3 (some results are announced in [7]).

Preliminaries. The convention in force throughout this paper is that all complex manifolds are Hausdorff, connected and second countable.

We shall consider every complex manifold M with its canonical orientation; if $(z^1, \ldots, z^n)$, $n=\dim M$, is a coordinate system of M then the form $(i^{n^2}/2^n)dz^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^1\wedge\cdots\wedge d\bar{z}^n$ is positive under this orientation of M. A positive real-value C^∞-differential $2n$-form on M is called a volume element of M. There exists at least one volume element of M since M is paracompact.

We shall denote the holomorphic tangent vector space of a complex manifold M at its point p by M_p. We shall identify M_p, $p\in M$, with $\mathbf{C}^n$ for every domain M in $\mathbf{C}^n$.

For a complex manifold M, we shall denote the conjugate complex manifold of M by $\bar{M}$ ($\bar{M}=M$ as topological space and if $\{(U_\alpha, z_\alpha)\}$ is an atlas of M, then $\{(U_\alpha, \bar{z}_\alpha)\}$ is an atlas of $\bar{M}$).

Let M and N be complex manifolds. We shall identify $(M\times N)_{(p,q)}$, $(p, q)\in M\times N$, with the complex vector space $M_p\times N_q$. Let $f: M\to N$ be a holomorphic mapping. By f_{*p}, we shall denote the linear tangent mapping of f at $p\in M$. For a differential form ω on N, we shall denote the pull-back image of ω under f by $f^*\omega$.

Let M be a connected Hausdorff topological space. Let $\pi: M\to\mathbf{C}^n$ be a local homeomorphism. Then M is second countable by the Poincaré-Volterra theorem and M carries a natural structure of a complex manifold under which π is a locally biholomorphic mapping. We shall consider M with this structure. The couple (M, π) is called a domain over $\mathbf{C}^n$. If $\pi=(\pi^1, \ldots, \pi^n)$, then the form $(i^{n^2}/2^n)d\pi^1\wedge\cdots\wedge d\pi^n\wedge d\bar{\pi}^1\wedge\cdots\wedge d\bar{\pi}^n$ is a volume element of M. It is called the Eucledean volume element of M. If π is an injective mapping then (M,π) is called univalent.

Let (M, π) and $(\tilde{M}, \tilde{\pi})$ be domains over $\mathbf{C}^n$ and $\mathbf{C}^m$, respectively; let $\pi=(\pi^1, \ldots, \pi^n)$ and $\tilde{\pi}=(\tilde{\pi}^1, \ldots, \tilde{\pi}^m)$. We say that a mapping $f: M\to\tilde{M}$ is linear if there are complex constants a^j_k, $j=1, \ldots, n$, $k=1, \ldots, m$, such that $\tilde{\pi}^k\circ f=\sum_{j=1}^n a^j_k\pi^j$ for $k=1, \ldots, m$. Obviously, every linear mapping f is holomorphic and the constants a^j_k are uniquely determined.

Let M be an n-dimensional complex manifold. By (H_M, p_M) we shall denote the sheaf of germs of holomorphic mappings from open subsets of $\mathbf{C}^n$ into M. H_M is a Hausdorff topological space and $p_M: H_M\to\mathbf{C}^n$ is a local homeomorphism. Hence if Δ is an open and connected subset of H_M and $\pi=p_M|\Delta$ (the restriction of p_M on Δ), then (Δ, π) is a domain over $\mathbf{C}^n$.

For a matrix $[a_{j,k}]$, $a_{j,k}\in\mathbf{C}$, $[a_{j,k}]'$ and $[a_{j,k}]^*$ will denote the transpose matrix of $[a_{j,k}]$ and the matrix $[\bar{a}_{j,k}]'$ respectively. The determinant of a square matrix $[a_{j,k}]$ we shall denote by $\det[a_{j,k}]$.

1. The Bergman function of a complex manifold with respect to its volume element. Let M be an n-dimensional complex manifold and v_M a volume element of M. By $L^2H(v_M)$ we denote the complex vector space of all holomorphic functions f on M such that the integral $\int_M|f|^2v_M$ is convergent. We define a scalar product in $L^2H(v_M)$ by the formula $(f, g)=\int_M f\bar{g}v_M$; $f, g\in L^2H(v_M)$. We set $\|f\|=(f, f)^{1/2}$ for $f\in L^2H(v_M)$.

Lemma 1.1. *If $p\in M$, then there is a compact neighbourhood U of p and a constant $C>0$ such that:*

(i) $|f(p)|^2\leq C\int_U|f|^2v_M$ *for every holomorphic function f on M;*

(ii) $|f(x)|^2\leq C\|f\|^2$ *for every $x\in U$ and $f\in L^2H(v_M)$.*

Proof. Let (W, z), $z=(z^1, \ldots, z^n)$, be a chart of M at p such that $z(W)$ is the unit polydisk in $\mathbf{C}^n$ and $z(p)=0$. We have $v_M | W=\varphi (i^{n^2}/2^n)dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$, where φ is a positive C^∞-function on W. Put:

$$V=\{x \in W: |z^k(x)| \leq 1/2, \ k=1, \ldots, n\}, \quad U=\{x \in W: |z^k(x)| \leq 1/4, \ k=1, \ldots, n\}$$

and $m=\min_V \varphi$; obviously, $m>0$. Let f be a holomorphic function on M. We may expand f on W in the Taylor series $f|W=\Sigma_{\nu_1,\ldots,\nu_n} a_{\nu_1,\ldots,\nu_n} (z^1)^{\nu_1} \ldots (z^n)^{\nu_n}$. The above series converges absolutely and uniformly on V.

(i) (see e. g. [2, p. 78, Lemma]) We have

$$\int_U |f|^2 v_M=(i^{n^2}/2^n)\int_U |f|^2 \varphi dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n$$

$$\geq m(i^{n^2}/2^n)\int_U |f|^2 dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$$

$$=(m\pi^n/16^n) \sum_{\nu_1,\ldots,\nu_n} |a_{\nu_1,\ldots,\nu_n}|^2 4^{-2(\nu_1+\ldots+\nu_n)} (\nu_1+1)^{-1} \ldots (\nu_n+1)^{-1}$$

$$\geq (m\pi^n/16^n) |a_{0, \ldots, 0}|^2=(m\pi^n/16^n) |f(p)|^2.$$

(ii) (cf. [1, p. 66, Proposition 5]). If $f \in L^2H(v_M)$, then $\|f\|^2 \geq \int_V |f|^2 v_M \geq (m\pi^n/4^n)\Sigma_{\nu_1,\ldots,\nu_n} |a_{\nu_1,\ldots,\nu_n}|^2 2^{-2(\nu_1+\ldots+\nu_n)} (\nu_1+1)^{-1} \ldots (\nu_n+1)^{-1}$. Let $x \in U$. We have by Schwarthz's inequality

$$|f(x)|^2 \leq (\sum_{\nu_1,\ldots,\nu_n} |a_{\nu_1,\ldots,\nu_n}| 4^{-(\nu_1+\ldots+\nu_n)})^2$$

$$\leq (1-1/4)^{-2n} \sum_{\nu_1,\ldots,\nu_n} |a_{\nu_1,\ldots,\nu_n}|^2 2^{-2(\nu_1+\ldots+\nu_n)} (\nu_1+1)^{-1} \ldots (\nu_n+1)^{-1}.$$

Hence $|f(x)|^2 \leq (4^n/m\pi^n) . (3/4)^{-2n} \|f\|^2$.

We obtain from Lemma 1.1 (ii):

Corollary 1.1 (cf. [1, p. 67, Corollary]). *Let $p \in M$. Then, there is a compact neighbourhood U of p and a constant $C>0$ such that if $f_1, \ldots, f_k$ is a finite orthonormal system tn $L^2H(v_M)$, we have $\Sigma_{\nu=1}^k |f_\nu|^2 \leq C$ on U.*

By means of Lemma 1.1 (ii) and Corollary 1.1, one can prove:

Proposition 1.1 (see [1, p. 68, Proposition 6 and p. 69, Corollary]). *$L^2H(v_M)$ is a separable complex Hilbert space.*

Let $t \in M$. The linear functional $l_t(f)=f(t)$, $f \in L^2H(v_M)$, is continuous according to Lemma 1.1 (ii). Hence, by Riesz's theorem, there is a unique element $K_t \in L^2H(v_M)$ such that $l_t(f)=(f, K_t)$, i. e. $f(t)=(f, K_t)=\int_M f\bar{K}_t v_M$ for $f \in L^2H(v_M)$. We have $K_t(t)=(K_t, K_t) \geq 0$. The following is well-known:

Proposition 1.2 *Let $t \in M$. Then, $K_t(t)>0$ iff there is a function $f \in L^2H(v_M)$ such that $f(t) \neq 0$.*

Proof. If $K_t(t)>0$, we put $f=K_t$. If $K_t(t)=0$, then $0=K_t(t)=(K_t, K_t)$, hence $K_t \equiv 0$. Therefore $f(t)=(f, K_t)=0$ for every $f \in L^2H(v_M)$.

Let $\varphi_0, \varphi_1, \ldots$ be an orthonormal basis in $L^2H(v_M)$. If $f \in L^2H(v_M)$ and $a_\nu=(f, \varphi_\nu)$, then the series $\Sigma_\nu a_\nu \varphi_\nu$ converges to f uniformly on compact subsets of M by Lemma 1.1 (ii). We have $(K_t, \varphi_\nu)=\overline{(\varphi_\nu, K_t)}=\overline{\varphi_\nu(t)}$, hence

$$K_t(x)=\sum_\nu \overline{\varphi_\nu(t)}\varphi_\nu(x); \qquad x, t \in M. \tag{1.1}$$

The above series converges absolutely and uniformly on compact subsets of $M\times\overline{M}$ by Corollary 1.1 and [1, p. 62, Lemma].

Definition 1.1. *We call* $K(x, t)=K_t(x)$, $(x, t)\in M\times\overline{M}$, *the Bergman function of* $L^2H(v_M)$.

The Bergman function K of $L^2H(v_M)$ is holomorhic on $M\times\overline{M}$. The function $k(x)=K(x, x)=\Sigma_\nu|\varphi_\nu(x)|^2$, $x\in M$, is real-value and analytic; the latter series converges uniformly on compact subsets of M.

Example 1.1. Let M be a domain over $\mathbf{C}^n$ and e_M the Euclidean volume element of M. The Bergman function of $L^2H(e_M)$ is the ordinary Bergman kernel function. If v_M is a volume element of M, we have $v_M=\varphi e_M$ where φ is a positive C^∞-function on M. Then the Bergman function of $L^2H(v_M)$ is the weighted Bergman kernel function with "weight" φ.

One can prove the following:

Proposition 1.3. *Let* $t\in M$. *Assume that* $K(t, t)\neq 0$. *Then*:

(i) $(K(t, t))^{-1}=\inf\{\|f\|^2: f\in L^2H(v_M),\ f(t)=1\}$

(see e. g. [2, p. 88, Theorem 4.6]);

(ii) *There is a unique function* $\mu_t\in L^2H(v_M)$ *such that*

$$\mu_t(t)=1 \text{ and } \|\mu_t\|^2=\inf\{\|f\|^2: f\in L^2H(v_M),\ f(t)=1\}.$$

We have $\mu_t(x)=K(x, t)/K(t, t)$, $x\in M$. *(see e. g.* [2, p. 89, Corollary]).

Let $x_0\in M$ and let $K(x_0, x_0)\neq 0$. There is a chart (U, z), $z=(z^1, \dots, z^n)$ of M at x_0 such that $k(x)=K(x, x)>0$ for every $x\in U$. The form Ω given locally by the formula

$$\Omega\,|\,U=i\sum_{r,s=1}^{n}\frac{\partial^2\ln k}{\partial z^r\,\partial\overline{z}^s}\,dz^r\wedge d\overline{z}^s \tag{1.2}$$

is well-defined on the open set $\{x\in M: K(x, x)\neq 0\}$ and real-value.

Definition 1.2. *We call* Ω *the Bergman form of* $L^2H(v_M)$.

Let ω be a real-value differential (1, 1)-form on a complex manifold N. Let $x\in N$. One can associate with ω a Hermitian form H_x on $N_x\times N_x$ in an invariant way (see [1, pp. 14, 15, 43, 54]). Let (U, z), $z=(z^1, \dots, z^m)$, be a chart of N at x. If

$\omega\,|\,U=i/2\Sigma_{\alpha,\beta=1}^{m} h_{\alpha,\beta}\,dz^\beta\wedge d\overline{z}^\alpha$, then $H_x=\Sigma_{\alpha,\beta=1}^{m} h_{\alpha,\beta}(x)\,d\overline{z}^\alpha(x)\otimes dz^\beta(x)$. We say that ω is semi-definite (definite) at x if H_x is semi-definite (resp. definite) (see [1, pp. 43, 44]).

We have:

Proposition 1.4. *Let* $x\in M$ *and* $K(x, x)\neq 0$. *Then:*

(i) *Ω is positive semi-definite at* x;

(ii) *Let* $f_0\in L^2H(v_M)$ *be such that* $f_0(x)\neq 0$. *Ω is positive definite at x iff there are n functions* $f_1, \dots, f_n$ *in* $L^2H(v_M)$ *such that* $(f_1/f_0, \dots, f_n/f_0)$ *is a local coordinate system of M in a neighbourhood of x.*

The statement (i) follows from Corollary 1.1 and [1, p. 64, Proposition]. The proof of (ii) is analogous to [1, p. 71, Corollary 3].

Let $(x_0, t_0)\in M\times\overline{M}$. Let (U, ξ), $\xi=(\xi^1, \dots, \xi^n)$, and (V, η), $\eta=(\eta^1, \dots, \eta^n)$, be charts of M at x_0 and at t_0, respectively. Let $(x, t)\in U\times\overline{V}$ and $K(x, t)\neq 0$, We put:

$$(1.3)\qquad T_{r,\bar{s}}(v_M;\xi;\bar{\eta};U;\bar{V})(x,t)=(K(x,t))^{-2}\Big(K(x,t)\frac{\partial^2 K}{\partial\xi^r\,\partial\bar{\eta}^s}(x,t)$$

$$-\frac{\partial K}{\partial\xi^r}(x,t)\frac{\partial K}{\partial\bar{\eta}^s}(x,t)\Big);\quad r,s=1,\dots,n.$$

If $K(x,t)\neq 0$ for every $(x,t)\in U\times\bar{V}$, the functions $T_{r,\bar{s}}(v_M;\xi;\bar{\eta};U;\bar{V})$ are holomorphic on $U\times\bar{V}$. The form T given locally by the formula:

$$(1.4)\qquad T\,|\,U\times\bar{V}=\sum_{r,s=1}^{n}T_{r,\bar{s}}(v_M;\xi;\bar{\eta};U;\bar{V})d\xi^r\wedge d\bar{\eta}^s$$

is well-defined on the open set $\{(x,t)\in M\times\bar{M}:K(x,t)\neq 0\}$.

Definition 1.3. *We call T the Tsuboi form of $L^2H(v_M)$.*

Remark. Let $K(x_0,t_0)\neq 0$ and let U, and V be such that $|K(x,t)-K(x_0,t_0)|<|K(x_0,t_0)|$ for every $(x,t)\in U\times\bar{V}$. We can pick out a single-value holomorphic branch $\log w$ of the logarithm in the disk $\{w\in\mathbf{C}:|w|<|K(x_0,t_0)|\}$. Obviously, $T_{r,\bar{s}}(v_M;\xi;\bar{\eta};U;\bar{V})=\partial^2\log K/\partial\xi^r\partial\bar{\eta}^s$. T. Tsuboi introduced the matrix $[(\partial^2\log K/\partial\xi^r\partial\bar{\eta}^s)_{r,s=1}^{n}]$ in the case when M is a bounded domain in $\mathbf{C}^n$ and v_M is the Euclidean volume element of M, and the derivatives are taken with respect to the standard coordinate system of M (see [14, p. 141]).

Let $K(x_0,x_0)\neq 0$ for some $x_0\in M$. Let (U,ξ), $\xi=(\xi^1,\dots,\xi^n)$, be a chart of M at x_0 such that $K(x,x)\neq 0$ for every $x\in U$. We have $T_{r,\bar{s}}(v_M;\xi;\bar{\xi};U;\bar{U})(x,x)=(\partial^2\ln k/\partial\xi^r\partial\bar{\xi}^s)(x)$ for $x\in U$. Hence $\operatorname{rank}T(x,x)=\operatorname{rank}\Omega(x)$ for $x\in U$.

Proposition 1.5. *Let M_1 and M_2 be complex manifolds. Let v_{M_s} be a volume element of M_s, $s=1,2$. Let $p_s:M_1\times M_2\to M_s$ be the natural projection, $s=1,2$. Then $v_{M_1\times M_2}=p_1^*v_{M_1}\wedge p_2^*v_{M_2}$ is a volume element of $M_1\times M_2$ and if K_{M_s}, $K_{M_1\times M_2}$, Ω_{M_s}, $\Omega_{M_1\times M_2}$, T_{M_s}, $T_{M_1\times M_2}$ are the Bergman functions, the Bergman forms and the Tsuboi forms of $L^2H(v_{M_s})$ and $L^2H(v_{M_1\times M_2})$, respectively, $s=1,2$, we have:*

$$(1.5)\qquad K_{M_1\times M_2}((x_1,x_2),(t_1,t_2))=K_{M_1}(x_1,t_1)\,K_{M_2}(x_2,t_2)$$

or $x_1,t_1\in M_1$; $x_2,t_2\in M_2$.

$$(1.6)\qquad \Omega_{M_1\times M_2}=p_1^*\Omega_{M_1}+p_2^*\Omega_{M_2},\quad T_{M_1\times M_2}=(p_1\times p_1)^*T_{M_1}+(p_2\times p_2)^*T_{M_2}.$$

Proof. It is obvious that $v_{M_1\times M_2}$ is a volume element of $M_1\times M_2$. We can prove the identity (1.5) in the same manner as the Bremermann theorem using Lemma 1.1 (i) (see e. g. [2, p. 91, Theorem 4.8]). The identities (1.6) follow at once from (1.5), (1.2) and (1.4).

Proposition 1.6. *Let M and N be complex manifolds. Let v_M and v_N be volume elements of M and N, respectively. Let $f:M\to N$ be a biholomorphic mapping onto N. Assume that*

$$(1.7)\qquad f^*v_N=|\Phi|^2v_M,$$

where Φ is a holomorphic function on M. If K_M, K_N, Ω_M Ω_N, T_M, T_N are the Bergman functions, the Bergman forms and the Tsuboi forms of $L^2H(v_M)$ and $L^2H(v_N)$, respectively, then

(1.8) $$K_M(x, t) = K_N(f(x), f(t))\, \Phi(x)\overline{\Phi(t)}; \qquad x, t \in M;$$

(1.9) $$\Omega_M = f^*\Omega_N, \quad T_M = (f\times f)^* T_N.$$

Proof. Let $\{\psi_\nu\}$ be an orthonormal basis in $L^2H(v_N)$. Then $\{(\psi_\nu \circ f)\Phi\}$ is an orthonormal basis in $L^2H(v_M)$. Hance, (1.8) holds by (1.1). The identities (1.9) follow at once from (1.8), (1.2) and (1.4).

Exemples 1.2. (a) Let V be an n-dimensional complex manifold and $B(V)$ the separable complex Hilbert space of all holomorphic n-forms α such that $i^{n^2}\int_V \alpha \wedge \overline{\alpha} < +\infty$ (the scalar product in $B(V)$ is given by $(\alpha, \beta) = i^{n^2}\int_V \alpha\wedge\overline{\beta}$). Let $\alpha_0, \alpha_1, \ldots$ be an orthonormal basis in $B(V)$. The $2n$-form θ_V defined by $\theta_V = i^{n^2}\Sigma_\nu \alpha_\nu \wedge \overline{\alpha}_\nu$ is independent of the choice of the orthonormal basis $\alpha_0, \alpha_1, \ldots$. θ_V is invariant under biholomorphic mappings. If for every $x \in V$ there is an $\alpha \in B(V)$ such that $\alpha(x) \neq 0$, then θ_V is a volume element of V (see [1, pp. 67—75]).

If $v_M = \theta_M$ and $v_N = \theta_N$ in Proposition 1.6, then (1.7) holds with $\Phi \equiv 1$.

(b) If $M = (M, \pi)$ and $N = (N, p)$ are domains over $\mathbf{C}^n$ and v_M, and v_N are the Euclidean volume elements of M and N, respectively, then (1.7) holds and Φ is the Jacobi determinant of f with respect to π and p.

(c) Let $M = N$ be a complex Lie group. Let $v_M = v_N$ be a left (right) invariant volume element. If f is a left (resp. right) translation of M, then (1.7) holds with $\Phi \equiv 1$.

Examples 1.3. It is possible $L^2H(v_M)$ to be trivial, for example when $M = \mathbf{C}^n$ and v_M is the Euclidean volume element of M. Then the Bergman function K of $L^2H(v_M)$ is identically zero. If M has a finite volume with respect to v_M, i. e. $\int_M v_M < +\infty$, then $L^2H(v_M)$ contains all constants, hence $K(x, x) > 0$ for every $x \in M$ by Proposition 1.1. But the Bergman form Ω of $L^2H(v_M)$ can be degenerated at some point of M. For example, if M is compact, then $0 < V = \int_M v_M < +\infty$, $K(x, t) = V^{-1}$ for every $(x, t) \in M \times \overline{M}$ and $\Omega \equiv 0$ on M.

(a) Let M be a bounded domain in $\mathbf{C}^n$ and v_M the Euclidean volume element of M. Then $K(x, x) > 0$ and $\operatorname{rank} \Omega(x) = n$ for every $x \in M$ by Propositions 1.1 and 1.4.

(b) Let M be an n-dimensional complex manifold which admits a holomorphic immersion $\varphi = (\varphi_1, \ldots, \varphi_m): M \to \mathbf{C}^m$ such that $\varphi(M)$ is a bounded subset of $\mathbf{C}^m$. Let v_M be a volume element of M. If $K(x, x) > 0$ for some $x \in M$, then $\operatorname{rank} \Omega(x) = n$. In fact, there is a function $f_0 \in L^2H(v_M)$ such that $f_0(x) \neq 0$. Obviously, $f_0 \circ \varphi_\nu \in L^2H(v_M)$, $\nu = 1, \ldots, m$, since φ_ν is bounded and $f_0 \in L^2H(v_M)$. We can find n functions $\varphi_{\nu_1}, \ldots, \varphi_{\nu_n}$ which make a coordinate system of M around x since φ is a holomorphic immersion. Hence $\operatorname{rank} \Omega(x) = n$ by Proposition 1.4.

(c) Let N be a complex manifold and v_N a volume element of N. Let M be an open connected and relatively compact subset of N. Put $v_M = v_N | M$. Let $x \in M$. If there is a holomorphic mapping $z_x: N \to \mathbf{C}^n$, $n = \dim N$, which is a local coordinate system of N in a neighbourhood of x, then $K_M(x, x) > 0$ and $\operatorname{rank} \Omega_M(x) = n$. In particular, if N is a Stein manifold or if N is a domain over $\mathbf{C}^n$, then $K_M(x, x) > 0$ and $\operatorname{rank} \Omega_M(x) = n$ for every $x \in M$.

(d) Let v_M and v'_M be two volume elements of M. v'_M/v_M is a positive C^∞-function. Let v'_M/v_M is bounded above. Then $L^2H(v_M) \subset L^2H(v'_M)$. Hence if $K(x, x) > 0$ for some $x \in M$, then $K'(x, x) > 0$ and if rank $\Omega(x) = \dim M$, then $\operatorname{rank} \Omega'(x) = \dim M$; here K, K', Ω, Ω' are the Bergman functions and the Bergman forms of $L^2H(v_M)$ and $L^2H(v'_M)$, respectively.

2. Definition of the representative domain of a complex manifold. Let M be an n-dimensional complex manifold and v_M a volume element of M. Let $u \in M$ and let $l: M_u \to \mathbf{C}$ be a $\mathbf{C}$-linear mapping. Using Lemma 1.1 (ii), one can prove the following:

Lemma 2.1. *Denote* $Q(u, v_M, l) = \{f \in L^2H(v_M) : f(u) = 0, f_{*u} = l\}$. *If* $Q(u, v_M, l) \neq \emptyset$, *then there is a unique function* $G \in Q(u, v_M, l)$ *such that* $\|G\|^2 = \inf\{\|f\|^2 : f \in Q(u, v_M, l)\}$ (*see e. g.* [3, p. 552, Theorem 1]).

S. Kobayashi proved the following assertion ([9, p. 269, the proof of Theorem 2.2]; see and [8, p. 279]):

Lemma 2.2. *Let* $x \in M$. *There is an orthonormal basis* $\varphi_0, \varphi_1, \ldots$ *in* $L^2H(v_M)$ *such that* $\varphi_0(x) \geq 0$ *and* $\varphi_v(x) = 0$ *for* $v \geq 1$.

Let K and Ω be the Bergman function and the Bergman form of $L^2H(v_M)$, respectively. Put $k(x) = K(x, x)$ for $x \in M$. Let $Q(u, v_M, l) \neq \emptyset$ and let $G \in Q(u, v_M, l)$ be the minimizing function defined by Lemma 2.1. Assume that $K(u, u) \neq 0$ and rank $\Omega(u) = n$. We shall express G by l and the Bergman function K. Let (V, ζ), $\zeta = (\zeta^1, \ldots, \zeta^n)$, be a chart of M at u such that $K(x, x) \neq 0$ for every $x \in V$. We put $l_k = l(\frac{\partial}{\partial \zeta^k}(u))$, $k = 1, \ldots, n$. Let $\varphi_0, \varphi_1, \ldots$ by an orthonormal basis in $L^2H(v_M)$ such that $\varphi_0(u) \geq 0$ and $\varphi_\nu(u) = 0$ for $\nu \geq 1$. We have $\varphi_0(u) > 0$ since $(\varphi_0(u))^2 = K(u, u) \neq 0$. If $f \in L^2H(v_M)$ and $a_\nu = (f, \varphi_\nu)$, then the series $\Sigma_{\nu \geq 0}\, a_\nu \varphi_\nu$ converges to f uniformly on compact subsets of M by Lemma 1.1 (ii). The conditions $f(u) = 0$ and $f_{*u} = l$ become:

$$a_0 = 0 \text{ and } \sum_{\nu \geq 1} a_\nu (\partial \varphi_\nu / \partial \zeta^k)(u) = l_k \quad \text{for } k = 1, \ldots, n. \tag{2.1}$$

We have $\|f\|^2 = \Sigma_{\nu \geq 0} |a_\nu|^2$ for $f = \Sigma_{\nu \geq 0}\, a_\nu \varphi_\nu \in L^2H(v_M)$. By the well-known rule of finding an extremum under auxiliary conditions, we set the derivatives of the expression

$$\sum_{\nu \geq 0} |a_\nu|^2 - \lambda_0 a_0 - \overline{\lambda_0}\, \overline{a_0} - \sum_{j=1}^{n} \lambda_j \Big(\sum_{\nu \geq 1} a_\nu \frac{\partial \varphi_\nu}{\partial \zeta^j}(u) - l_j\Big) - \sum_{j=1}^{n} \overline{\lambda_j} \Big(\sum_{\nu \geq 1} \overline{a_\nu}\, \overline{\frac{\partial \varphi_\nu}{\partial \zeta^j}}(u) - \overline{l_j}\Big), \quad \lambda_0, \lambda_j \in \mathbf{C},$$

with respect to each of the variables $\mathrm{Re}(a_\nu)$ and $\mathrm{Im}(a_\nu)$, $\nu = 0, 1, \ldots$, equal to zero. We obtain $a_0 = \overline{\lambda_0}$ and

$$a_\nu = [\overline{\lambda_1}, \ldots, \overline{\lambda_n}] \Big[\frac{\partial \varphi_\nu}{\partial \zeta^1}(u), \ \ldots, \frac{\partial \varphi_\nu}{\partial \zeta^n}(u)\Big]^* \quad \text{for } \nu \geq 1. \tag{2.2}$$

Now, by (2.1), we have $\lambda_0 = 0$ and

$$\sum_{j=1}^{n} \lambda_j \sum_{\nu \geq 1} \overline{\frac{\partial \varphi_\nu}{\partial \zeta^j}}(u) \frac{\partial \varphi_\nu}{\partial \zeta^k}(u) = l_k \quad \text{for } k = 1, \ldots, n. \tag{2.3}$$

We have:

$$\frac{\partial^2 \ln k}{\partial \zeta^k \partial \overline{\zeta^j}}(u) = (k(u))^{-2} \Big(k(u) \frac{\partial^2 k}{\partial \zeta^k \partial \overline{\zeta^j}}(u) - \frac{\partial k}{\partial \zeta^k}(u) \frac{\partial k}{\partial \overline{\zeta^j}}(u)\Big) \tag{2.4}$$

$$= (\varphi_0(u))^{-2} \sum_{\nu \geq 1} \overline{\frac{\partial \varphi_\nu}{\partial \zeta^j}}(u) \frac{\partial \varphi_\nu}{\partial \zeta^k}(u).$$

The determinant det $[(\partial^2 \ln k / \partial \zeta^k \partial \overline{\zeta^j})(u)]$ is defferent from zero by the assumption that rank $\Omega(u) = n$. We obtain from (2.3) and (2.4):

$$[\bar{\lambda}_1, \ldots, \bar{\lambda}_n] = (K(u, u))^{-1}[l_1, \ldots, l_n]\left[\left(\frac{\partial^2 \ln k}{\partial\zeta^k\, \partial\bar{\zeta}^j}(u)\right)_{j,k=1}^{n}\right]^{-1}.$$

The latter identity and (2.2) imply

$$(2.5)\qquad G(x) = (K(u, u))^{-1}[l_1, \ldots, l_n]\left[\left(\frac{\partial^2 \ln k}{\partial\zeta^k\, \partial\bar{\zeta}^j}(u)\right)_{j,k=1}^{n}\right]^{-1}$$

$$\cdot \sum_{\nu\geq 1}\left[\frac{\partial\varphi_\nu}{\partial\zeta^1}(u), \ldots, \frac{\partial\varphi_\nu}{\partial\zeta^n}(u)\right]^*\varphi_\nu(x)$$

$$= (K(u, u))^{-1}[l_1, \ldots, l_n]\left[\left(\frac{\partial^2 \ln k}{\partial\zeta^k\, \partial\bar{\zeta}^j}(u)\right)_{j,k=1}^{n}\right]^{-1}$$

$$\cdot\left[\frac{\partial K}{\partial\bar{\zeta}^1}(x, u) - \frac{\partial k}{\partial\bar{\zeta}^1}(u)\frac{K(x, u)}{K(u, u)}, \ldots, \frac{\partial K}{\partial\bar{\zeta}^n}(x, u) - \frac{\partial k}{\partial\bar{\zeta}^n}(u)\frac{K(x, u)}{K(u, u)}\right]'$$

for $x \in M$.

P r o p o s i t i o n 2.1. *Let M be an n-dimensional complex manifold and v_M a volume element of M. Let K, Ω and T be the Bergman function, the Bergman form and the Tsuboi form of $L^2H(v_M)$, respectively. Let (H_M, P_M) be the sheaf of germs of holomorphic mappings from open subsets of $\mathbf{C}^n$ into M. Let $u \in M$ and let $L = (L^1, \ldots, L^n): M_u \to \mathbf{C}^n$ be a $\mathbf{C}$-linear mapping. Put:*

$$Q(u, v_M, L^s) = \{f \in L^2H(v_M): f(u) = 0,\ f_{*u} = L^s\}, \qquad s = 1, \ldots, n,$$

$$S^{(1)} \equiv S^{(1)}(u, v_M) = \{x \in M: K(x, u) = 0\},$$

$$S^{(2)} \equiv S^{(2)}(u, v_M) = \{x \in M\setminus S^{(1)}: \operatorname{rank} T(x, u) < n\},$$

$$S \equiv S(u, v_M) = S^{(1)} \cup S^{(2)}.$$

Assume that: (a) $K(u, u) \neq 0$ *and* $\operatorname{rank}\Omega(u) = n$; (b) *$L$ is non-degenerated;* (c) $Q(u, v_M, L^s) \neq \emptyset$ *for each* $s = 1, \ldots, n$.

Let $G^s \in Q(u, v_M, L^s)$ be the minimizing function defined by Lemma 2.1; $s = 1, \ldots, n$. *Put:* $G \equiv G(u, v_M, L) = (G^1, \ldots, G^n)$.

Then:

(i) *The mapping $F \equiv F(u, v_M, L)$ defined by*

$$(2.6)\qquad F(x) = (K(u, u)/K(x, u))G(x)$$

for $x \in M\setminus S^{(1)}$ is locally biholomorphic on $M\setminus S$;

(ii) *There is a natural holomorphic imbedding $\widetilde{F}: M\setminus S \to H_M$ such that $P_M \circ \widetilde{F} = F \mid (M\setminus S)$.*

P r o o f. Note that $M\setminus S$ is an open and connected subset of $M\setminus S^{(1)}$ since $S^{(2)}$ is an analytic subset of $M\setminus S^{(1)}$. The point u lies in $M\setminus S$ by assumption (a). Let (V, ζ), $\zeta = (\zeta^1, \ldots, \zeta^n)$, be a chart of M at u such that $K(x, x) \neq 0$ for every $x \in V$. Put $l_{k,s} = L^s\left(\frac{\partial}{\partial\zeta^k}(u)\right)$; $k, s = 1, \ldots, n$. Denote the matrix $[l_{k,s}]$ by $L(\zeta)$. Let $\varphi_0, \varphi_1, \ldots$ be an orthonormal basis in $L^2H(v_M)$ such that $\varphi_0(u) \geq 0$ and $\varphi_\nu(u) = 0$ for $\nu \geq 1$. Let $F = (F^1, \ldots, F^n)$. We have by (2.6) and (2.5)

$$(2.7)\qquad F^s(x) = [l_{1,s}, \ldots, l_{n,s}]\left[\left(\frac{\partial^2 \ln k}{\partial\zeta^k\, \partial\bar{\zeta}^j}(u)\right)_{j,k=1}^{n}\right]^{-1}$$

$$\cdot \sum_{\nu\geq 1} [\frac{\partial\varphi_\nu}{\partial\zeta^1}(u), \ldots, \frac{\partial\varphi_\nu}{\partial\zeta^n}(u)]^* \frac{\varphi_\nu(x)}{K(x, u)}$$

for $x \in M\setminus S^{(1)}$. The above series converges uniformly on compact subset of $M\setminus S^{(1)}$. Let $x_0 \in M\setminus S^{(1)}$ and let (U, z), $z=(z^1, \ldots, z^n)$, be a chart of $M\setminus S^{(1)}$ at x_0. We have $K(x, u)=\varphi_0(u)\varphi_0(x)$ and

$$\frac{\partial}{\partial z^k}\Big(\sum_{\nu\geq 1}\overline{\frac{\partial\varphi_\nu}{\partial\zeta^j}(u)}\frac{\varphi_\nu}{K(\,.\,, u)}\Big)(x) \tag{2.8}$$

$$=(K(x, u))^{-2}\sum_{\nu\geq 1}\Big(\overline{\frac{\partial\varphi_\nu}{\partial\zeta^j}(u)}\frac{\partial\varphi_\nu}{\partial z^k}(x)\overline{\varphi_0(u)}\,\varphi_0(x)-\frac{\partial\varphi_0}{\partial z^k}(x)\overline{\frac{\partial\varphi_\nu}{\partial\zeta^j}(u)}\varphi_\nu(x)\overline{\varphi_0(u)}\Big)$$

$$=(K(x, u))^{-2}\sum_{\nu\geq 0}\Big(\overline{\frac{\partial\varphi_\nu}{\partial\zeta^j}(u)}\frac{\partial\varphi_\nu}{\partial z^k}(u)\overline{\varphi_0(u)}\varphi_0(x)-\overline{\varphi_0(u)}\frac{\partial\varphi_0}{\partial z^k}(u)\overline{\frac{\partial\varphi_\nu}{\partial\zeta^j}(u)}\varphi_\nu(x)\Big)$$

$$=(K(x, u))^{-2}\Big(K(x, u)\frac{\partial^2 K}{\partial z^k\,\partial\bar{\zeta}^j}(x, u)-\frac{\partial K}{\partial z^k}(x, u)\frac{\partial K}{\partial\bar{\zeta}^j}(x, u)\Big)$$

for $x \in u$ and $j=1, \ldots, n$. We obtain form (2.7) and (2.8):

$$[(\frac{\partial F^s}{\partial x^k}(x))_{s,k=1}^n]=L(\zeta)'[(\frac{\partial^2 \ln k}{\partial\zeta^k\,\partial\bar{\zeta}^j}(u))_{j,k=1}^n]^{-1}.[T_{r,\bar{s}}(v_M; z; \bar{\zeta}; U; \bar{V})(x, u)]' \tag{2.9}$$

for $x \in U$. Therefore, $\det[\partial F^s/\partial z^k](x)\neq 0$ for $x \in U\setminus S$ by the assumptions (a) and (b). This proves (i). The statement (ii) follows from (i) and the following:

Lemma 2.3. *Let M and N be complex manifolds. Let $f: M\to N$ be a locally biholomorphic mapping. Denote the sheaf of germs of holomorphic mappings from open subsets of N into M by (H, p). Then, there is a natural holomorphic imbedding $\tilde{f}: M\to H$ such that $p\circ\tilde{f}=f$.*

Proof. Let $m \in M$ and let U be an open neighbourhood of m such that $f(U)$ is open in N and $f_U=f|U: U\to f(U)$ is a biholomorphic mapping. The germ $(f_U^{-1})_{f(m)}$ of f_U^{-1} at $f(m)$ does not depend on the choice of U. In fact, if U' is an open neighbourhood of m such that $f_{U'}=f/U'$ is a biholomorphic mapping, then $D=f(U\cap U')$ is a neighbourhood of $f(m)$, $D\subset f(U)\cap f(U')$ and $f_{U'}^{-1}|D=f_U^{-1}|D$. We put $\tilde{f}(m)=(f_U^{-1})_{f(m)}$. Let A be a neighbourhood of $\tilde{f}(m)$ in H. There is an open neighbourhood W of $f(m)$ such that $W\subset f(U)$ and the set $\{(f_U^{-1})_x : x \in W\}$ is contained in A. The set $V=f_U^{-1}(W)$ is a neighbourhood of m and $\tilde{f}(V)\subset A$. Therefore, $\tilde{f}$ is a continuous mapping. Obviously, $p\circ\tilde{f}=f$. Hence $\tilde{f}$ is holomorphic. Let m_1 and m_2 be points of M and let $\tilde{f}(m_1)=\tilde{f}(m_2)$. Let U_s be an open neighbourhood of m_s such that $f(U_s)$ is open in N and the mapping $f_{U_s}=f|U_s : U_s\to f(U_s)$ is biholomorphic, $s=1, 2$. We have $f(m_1)=f(m_2)$ and $f_{U_1}^{-1}=f_{U_2}^{-1}$ in a neighbourhood of the point $f(m_1)=f(m_2)$ since $(f_{U_1}^{-1})_{f(m_1)}=(f_{U_2}^{-1})_{f(m_2)}$. In particular, $m_1=m_2$. Hence $\tilde{f}$ is an injective mapping. Therefore, $\tilde{f}$ is a holomorphic imbedding.

Let the assumption of Proposition 2.1 be fulfilled. Put $\Delta=\tilde{F}(M\setminus S)$ and $\pi=P_M|\Delta$. Then Δ is an open and connected subset of H_M, hence (Δ, π) is a domain over $\mathbf{C}^n$. $\tilde{F}$ is a biholomorphic mapping of $M\setminus S$ onto Δ.

Definition 2.1. *We shall say that M possesses a representative domain with respect to u, v_M and L if the assumptions of Proposition 2.1 are fulfilled. We call (Δ, π) a representative domain of M with respect to u, v_M and L. We call $\widetilde{F}$ a representative mapping of M with respect to u, v_M and L.*

We shall often denote (Δ, π) and $\widetilde{F}$ by $(\Delta, \pi; u, v_M, L)$ and $\widetilde{F}(u, v_M, L)$, respectively. Note that $F(u)=0$ and $F_{*u}=L$.

Examples 2.1. Every bounded domain M in $\mathbf{C}^n$ possesses a representative domain with respect to $u \in M$, the Euclidean volume element of M and $id_{\mathbf{C}^n}$ (cf. example 1.3 (a)).

2.2. Let M be an n-dimensional complex manifold which admits a holomorphic immersion $\varphi=(\varphi_1, \ldots, \varphi_m): M \to \mathbf{C}^m$ such that $\varphi(M)$ is a bounded subset of $\mathbf{C}^m$. Let v_M be a volume element of M such that $\int_M v_M < +\infty$. Let $u \in M$. There exist n functions $\varphi_{\nu_1}, \ldots, \varphi_{\nu_n}$ $(1 \leq \nu_s \leq m,\ s=1, \ldots, n)$ such that if we set $\widetilde{\varphi}_s=\varphi_{\nu_s}-\varphi_{\nu_s}(u)$, $s=1, \ldots, n$, and $\widetilde{\varphi}=(\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$, then $\widetilde{\varphi}(u)=0$, rank $\widetilde{\varphi}_{*u}=n$, and $\widetilde{\varphi}_s \in L^2H(v_M)$, $s=1, \ldots, n$. Hence M possesses a representative domain with respect to u, v_M and $\widetilde{\varphi}_{*u}$ (cf. example 1.3 (b)).

2.3. Let N be a complex manifold and v_N a volume element of N. Let M be an open connected and relatively compact subset of N. Put $v_M=v_N|N$. Let $u \in M$. Assume that there is a holomorphic mapping $z_u: N \to \mathbf{C}^n$, $n=\dim N$, which is a local coordinate system of N in a neighbourhood of u. We set $h=z_u-z_u(u)$. Then M possesses a representative domain with respect to u, v_M and h_{*u} (cf. exemple 1.3 (c)).

2.4. Let M be a complex manifold possessing a representative domain with respect to u, v_M and L. Let v'_M be a volume element of M such that the function v'_M/v_M is bounded above. Then M possesses a representative domain with respect to u, v'_M and L (cf. example 1.3 (d)).

Remark. Let M be a bounded domain in $\mathbf{C}^n$ and e_M the Euclidean volume element of M. Let $u \in M$. S. Bergman [4; 5, pp. 105, 188, 189] called the image set of M in $\mathbf{C}^n$ under $F=F(u, e_M, id_{\mathbf{C}^n})$ a representative domain of M (with centre at u). But the mapping F is, in general, meromorphic on M and the representative domain of M is no longer in $\mathbf{C}^n$. Moreover, we do not know whether F is locally one-to-one. We also do not know whether the image set of M under F is a domain at all (cf. [10, p. 293]). If $S^{(1)}=\varnothing$ and F is biholomorphic on M, then $S^{(2)}=\varnothing$ and $\pi(\Delta)=F(M)$. The domain (Δ, π) is univalent and we may identify Δ with $\pi(\Delta)$. Then $\Delta=\pi(\Delta)=F(M)$ is the Bergman representative domain of M with centre at u.

3. Properties of the representative domains. Let M be an n-dimensional complex manifold and v_M a volume element of M. Let $u \in M$ and let $L: M_u \to \mathbf{C}^n$ be a linear mapping. By $Q(u, v_M, L)$ denote the set of all holomorphic mappings $h=(h_1, \ldots, h_n): M \to \mathbf{C}^n$ such that $h(u)=0$, $h_{*u}=L$ and $h_s \in L^2H(v_M)$, $s=1, \ldots, n$.

Proposition 3.1. *Let M and N be n-dimensional complex manifolds. Let v_M and v_N be volume elements of M and N, respectively. Let $f: M \to N$ be a biholomorphic mapping such that $f^* v_N=|\Phi|^2 v_M$, where Φ is a holomorphic function on M. Assume that M possesses a representative domain $(\Delta_M, \pi_M; u, v_M, L)$. Then:*

(i) *N possesses a representative domain (Δ_N, π_N) with respect to $w=f(u)$ v_N and $P=L\circ f_{*w}^{-1}$;*

(ii) $f(S^{(r)}(u, v_M))=S^{(r)}(w, v_N)$, $r=1, 2$; $f(S(u, v_M))=S(w, v_N)$;

(iii) $G_M=(G_N\circ f).(\Phi/\Phi(u))$, $F_M=F_N\circ f$, *the diagram*

$$\begin{array}{ccc}
M\setminus S_M & \xrightarrow{\quad f \quad} & N\setminus S_N \\
\tilde{F}_M \downarrow & & \downarrow \tilde{F}_N \\
\Delta_M & \xrightarrow{\tilde{F}_N\circ f\circ \tilde{F}_M^{-1}} & \Delta_N \\
& \pi_M \searrow \quad \swarrow \pi_N & \\
& \mathbf{C}^n &
\end{array}$$

is commutative and the mapping $\tilde{F}_N\circ f\circ\tilde{F}_M^{-1}:\Delta_M\to\Delta_N$ is linear. Here $G_M=G(u, v_M, L)$, $G_N=G(w, v_N, P)$, $F_M=F(u, v_M, L)$, $F_N=F(w, v_N, P)$, $S_M=S(u, v_M)$, $S_N=S(w, v_N)$.

Proof. (i) Let K_N and Ω_N be the Bergman function and the Bergman form of $L^2H(v_N)$, respectively. We have $K_N(w, w)\neq 0$ and rank $\Omega_N(w)=n$ by Proposition 1.6. Obviously, P is non-degenerated. Let $h\in Q(u, v_M, L)$. The function Φ has not zeroes on M, hence the mapping $\Psi=\Phi(u)(h/\Phi)\circ f^{-1}$ is holomorphic on N. We have $\psi(u)=0$. Let $h=(h_1, \ldots, h_n)$, $\psi=(\psi_1, \ldots, \psi_n)$. Then:

$$\int_N |\psi_s|^2 v_N=|\Phi(u)|^2\int_N |(h_s/\Phi)\circ f^{-1}|^2 \; |\Phi\circ f^{-1}|^2 \; (f^{-1})^* v_M=|\Phi(u)|^2\int_M |h_s|^2 v_M<+\infty,$$

i. e. $\psi_s\in L^2H(v_N)$, $s=1, \ldots, n$. Let (U, z), $z=(z^1, \ldots, z^n)$, and (V, ζ), $\zeta=(\zeta^1, \ldots, \zeta^n)$ be charts of M at u and of N at w, resp., such that $f(U)=V$. We have:

$$\frac{\partial\psi_s}{\partial\zeta^k}(w)=\Phi(u)\,\frac{\partial(h_s\circ f^{-1}/\Phi\circ f^{-1})}{\partial\zeta^k}(w)=\frac{\partial(h_s\circ f^{-1})}{\partial\zeta^k}(w)$$

$$=\sum_{j=1}^{n}\frac{\partial h_s}{\partial z^j}(u)\;\frac{\partial(z^j\circ f^{-1})}{\partial\zeta^k}(w),$$

$s=1, \ldots, n$, since $h_s(u)=0$. Hence $\Psi_{*w}=h_{*u}\circ f_{*w}^{-1}=L\circ f_{*w}^{-1}=P$. Therefore $\psi\in Q(w, v_N, P)$.

(ii) The identities $f(S_M^{(r)})=S_N^{(r)}$, $r=1, 2$, follow at once from Proposition 1.6. Hence $f(S_M)=S_N$.

(iii) Let $G_M=(G_M^1, \ldots, G_M^n)$ and $G_N=(G_N^1, \ldots, G_N^n)$. We have $(G_N\circ f)\cdot(\Phi/\Phi(u))\in Q(u, v_M, L)$ since $G_N\in Q(w, v_N, P)$. Then:

$$\| G_M^s\|^2\leq\|(G_N^s\circ f)\cdot(\Phi/\Phi(u))\|^2=|\Phi(u)|^{-2}\| G_N^s\|^2, \qquad s=1, \ldots, n.$$

On the other hand $\Phi(u)\,(G_M/\Phi)\circ f^{-1}\in Q(w, v_N, P)$, hence

$$\| G_M^s\|^2=|\Phi(u)|^{-2}\|\Phi(u)(G_N^s/\Phi)\circ f^{-1}|^2\geq|\Phi(u)|^{-2}\| G_N^s\|^2, \quad s=1, \ldots, n.$$

It follows that $G_M^s=(G_N^s\circ f)\,(\Phi/\Phi(u))$, $s=1, \ldots, n$, by the uniqueness of the minimizing function. Now, we obtain $F_M=F_N\circ f$ by (2.6) and (1.8). Hence $\pi_N\circ(\tilde{F}_N\circ f\circ\tilde{F}_M^{-1})=(F_N\circ f)\circ\tilde{F}_M^{-1}=F_M\circ\tilde{F}_M^{-1}=\pi_M$. Proposition 3.11 which we shall further prove implies that $\tilde{F}_N\circ f\circ\tilde{F}_M^{-1}$ is a linear mapping.

R e m a r k. If F_M is biholomorphic on $M\setminus S_M$, then F_N is biholomorphic on $N\setminus S_N$ and Δ_M, and Δ_N are univalent domains; we may identify Δ_M and Δ_N with $\pi_M(\Delta_M)=\pi_N(\Delta_N)$. Thus, Proposition 3.1 generalizes the following well-known proposition: If M and N are bounded domain in $\mathbf{C}^n$, u and w are points of M and of N, respectively, and if $f: M\to N$ is a biholomorphic mapping such that $f(u)=w$ and $f_{*u}=id_{\mathbf{C}^n}$, then the Bergman representative domains of M and of N with centres at u and at w, respectively, coincide (see e. g. [8, p. 289, Corollary 2]).

P r o p o s i t i o n 3.2. *Let M be an n-dimensional complex manifold possessing a representative domain* $(\Delta, \pi; u, v_M, L)$.

(i) *Let v^0_M be a volume element of M such that $v^0_M=|\Phi|^2 v_M$, where, Φ is a holomorphic function on M. Then:*

(i1) (Δ, π) *is the representative domain of M with respect to* u, v^0_M, L;

(i2) $S^{(r)}(u, v^0_M)=S^{(r)}(u, v_M)$, $r=1, 2$, $S(u, v^0_M)=S(u, v_M)$, $G=G^0\circ(\Phi/\Phi(u))$, $F=F^0$; *here* $G=G(u, v_M, L)$, $G^0=G(u, v^0_M, L)$, $F=F(u, v_M, L)$, $F^0=F(u, v^0_M, L)$.

(ii) *Let $L': M_u\to\mathbf{C}^n$ be a non-degenerated linear mapping. Then:*

(ii1) *M possesses a representative domain (Δ', π') with respect to* u, v_M, L';

(ii2) $G'=L'\circ L\circ G$; $F'=L'\circ L^{-1}\circ F$ *and the diagram*

$$\begin{array}{ccccc} & & M\setminus S(u, v_M) & & \\ & \swarrow^{\tilde F} & & \searrow^{\tilde F'} & \\ \Delta & & \xrightarrow{\tilde F'\circ \tilde F^{-1}} & & \Delta' \\ \pi\downarrow & & & & \downarrow\pi' \\ \mathbf{C}^n & & \xrightarrow{L'\circ L^{-1}} & & \mathbf{C}^n \end{array}$$

is commutative; here $G'=G(u, v_M, L')$, $G=G(u, v_M, L)$, $F'=F(u, v_M, L')$ $F=F(u, v_M, L)$.

P r o o f. (i) M possesses a representative domain (Δ^0, π^0) with respect to u, v^0_M and L by Proposition 3.1 (i). The statement (i2) holds by Propositions 3.1 (ii) and 3.1 (iii). Hence $(\Delta^0, \pi^0)=(\Delta, \pi)$. (ii) The set $S(u, v_M)$ depends upon u and v_M only but it does not depend upon L. It is clear that $h\in Q(u, v_M, L)$ iff $L'\circ L^{-1}\circ h\in Q(u, v_M, L')$. Hence the statement (ii) holds.

C o r o l l a r y 3.1. *Let M be an n-dimensional complex manifold. Let $(\Delta', \pi'; u', v'_M, L')$ and $(\Delta'', \pi''; u'', v''_M, L'')$ be representative domains of M. Assume that there is an automorphism f of M ($\equiv$ biholomorphic onto self mapping) such that $f(u')=u''$* and *$f^* v''_M=|\Phi|^2 v'_M$, where Φ is a holomorphic function on M. Then the diagram*

$$\begin{array}{ccc} M\setminus S' & \xrightarrow{\quad f \quad} & M\setminus S'' \\ \tilde F'\downarrow & & \downarrow\tilde F'' \\ \Delta' & \xrightarrow{\tilde F''\circ f\circ \tilde F'^{-1}} & \Delta'' \\ \pi'\downarrow & & \downarrow\pi'' \\ \mathbf{C}^n & \xrightarrow{L''\circ f_{*u'}\circ (L')^{-1}} & \mathbf{C}^n \end{array}$$

is commutative; here $S'=S(u', v'_M)$, $S''=S(u'', v''_M)$, $F'=F(u', v'_M, L')$, $F''=F(u'', v''_M, L'')$.

Let M_1 and M_2 be complex manifolds. The mapping $i: H_{M_1} \times H_{M_2} \to H_{M_1 \times M_2}$ defined by $i(f_a, g_b) = (f \times g)_{(a,b)}$, $f_a \in H_{M_1}$, $g_b \in H_{M_2}$, is a holomorphic imbedding such that $p_{M_1} \times p_{M_2} = p_{M_1 \times M_2} \circ i$. We shall identify $H_{M_1} \times H_{M_2}$ with $i(H_{M_1} \times H_{M_2})$ and $p_{M_1} \times p_{M_2}$ with $p_{M_1 \times M_2} | i(H_{M_1} \times H_{M_2})$.

Proposition 3.3. *Let M_1 and M_2 be complex manifolds,* $\dim M_1 = n_1$, $\dim M_2 = n_2$. *Let v_r be a volume element of M_r, u_r a point of M_r and $L_r: (M_r)_{u_r} \to \mathbf{C}^{n_r}$ a non-degenerated linear mapping, $r=1, 2$. Let $p_r: M_1 \times M_2 \to M_r$ be the natural projection, $r=1, 2$. Put $v = p_1^* v_1 \wedge p_2^* v_2$, $u = (u_1, u_2)$ and $L = L_1 \times L_2$. We have:*

(i) $(M_1 \setminus S(u_1, v_1)) \times (M_2 \setminus S(u_2, v_2)) = M_1 \times M_2 \setminus S(u, v)$;

(ii) *If $M_1 \times M_2$ and M_r possess representative domains $(\Delta, \pi; u, v, L)$ and $(\Delta_r, \pi_r; u_r, v_r, L_r)$, respectively, for $r=1, 2$, then $F(u, v, L) = F(u_1, v_1, L_1) \times F(u_2, v_2, L_2)$; in particular, $\tilde{F}(u, v, L) = \tilde{F}(u_1, v_1, L_2) \times \tilde{F}(u_2, v_2, L_2)$, $\Delta = \Delta_1 \times \Delta_2$ and $\pi = \pi_1 \times \pi_2$;*

(iii) *If $M_1 \times M_2$ possesses a representative domain with respect to u, v and L, then M_r possesses a representative domain with respect to u_r, v_r* and L_r for each $r=1, 2$;

(iv) *If $\int_{M_r} v_r < +\infty$ and M_r possesses a representative domain with respect to u_r, v_r and L_r for each $r=1, 2$, then $M_1 \times M_2$ possesses a representative domain with respect to u, v and L.*

Proof. (i) follows at once from Proposition 1.5.

(ii) Let K_r, K, Ω_r, Ω be the Bergman functions and the Bergman forms of $L^2H(v_r)$ and $L^2H(v)$, respectively, $r=1, 2$. Denote: $F_r = F(u_r, v_r, L_r)$, $F_r = (F_r^1, \ldots, F_r^{n_r})$, $r=1, 2$, and $F = F(u, v, L)$, $F = (F^1, \ldots, F^{n_1+n_2})$. Let $\varphi_0^r, \varphi_1^r, \ldots$ be an orthonormal basis in $L^2H(v_r)$ such that $\varphi_0^r(u_r) > 0$ and $\varphi_{\nu_r}^r(u_r) = 0$ for $\nu_r \geq 1$, $r=1, 2$. Using Lemma 1.1 (i), one can prove that $\varphi_{\nu_1,\nu_2}(x_1, x_2) = \varphi_{\nu_1}^1(x_1)\varphi_{\nu_2}^2(x_2)$, $(x_1, x_2) \in M_1 \times M_2$, is an orthonormal basis in $L^2H(v)$ just as in the Bremermann theorem (see e. g. [2, p. 91, Theorem 4.8]). Let (V_r, ζ_r), $\zeta_r = (\zeta_r^1, \ldots, \zeta_r^{n_r})$, be a chart of M_r at u_r such that $K_r(x_r, x_r) > 0$ for every $x_r \in V_r$; $r=1, 2$. Let $L_r = (L_r^1, \ldots, L_r^{n_r})$ and $l_{m,s}^r = L_r^s(\frac{\partial}{\partial \zeta_r^m}(u_r))$, $r=1, 2$; $m, s = 1, \ldots, n_r$. We set $(\zeta_1^1, \ldots, \zeta_1^{n_1}, \zeta_2^1, \ldots, \zeta_2^{n_2}) = (\zeta^1, \ldots, \zeta^{n_1+n_2})$. Let $L = (L^1, \ldots, L^{n_1+n_2})$ and $l_{k,j} = L^j(\frac{\partial}{\partial \zeta^k}(u))$, $k, j = 1, \ldots, n_1+n_2$. Then:

$$l_{k,j} = \begin{cases} l_{k,j}^r & \text{if } (r-1)n_1 + 1 \leq k, j \leq n_1 + (r-1)n_2, \quad r=1, 2; \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Obviously, $\varphi_{0,0}(u) > 0$ and $\varphi_{\nu_1,\nu_2}(u) = 0$ if $\nu_1 + \nu_2 \geq 1$. Hence, by (2.7) we have

$$F^s(x) = [l_{1,s}, \ldots, l_{n_1+n_2,s}][(\frac{\partial^2 \ln k}{\partial \zeta^k \partial \bar{\zeta}^j}(u))_{j,k=1}^{n_1+n_2}]^{-1} \tag{3.2}$$

$$\cdot \sum_{\substack{\nu_1,\nu_2 \\ \nu_1+\nu_2 \geq 1}} [\frac{\partial \varphi_{\nu_1,\nu_2}}{\partial \zeta^1}(u), \ldots, \frac{\partial \varphi_{\nu_1,\nu_2}}{\partial \zeta^{n_1+n_2}}(u)]^* \frac{\varphi_{\nu_1,\nu_2}(x)}{K(x, u)}, \quad x \in M_1 \times M_2,$$

where $k(t) = K(t, t)$, $t \in M_1 \times M_2$. For every natural p_r, $(r-1)n_1 + 1 \leq p_r \leq n_1 + (r-1)n_2$, $r=1, 2$ and $x = (x_1, x_2) \in M_1 \times M_2$ we have:

$$(3.3)\qquad \sum_{\nu_1+\nu_2\geq 1} \overline{\frac{\partial \varphi_{\nu_1,\nu_2}}{\partial \zeta^p_r}(u)}\, \frac{\varphi_{\nu_1,\nu_2}(x)}{K(x, u)}$$

$$= \sum_{\nu_1+\nu_2\geq 1} \overline{\frac{\partial \varphi^r_{\nu_r}}{\partial \zeta^{p_r}_r}(u_r)\varphi^{3-r}_{\nu_{3-r}}(u_{3-r})}\, \frac{\varphi^1_{\nu_1}(x_1)\varphi^2_{\nu_2}(x_2)}{\varphi^1_0(x_1)\,\varphi^2_0(x_2)\overline{\varphi^1_0(u_1)\varphi^2_0(u_2)}}$$

$$= \sum_{\nu_r\geq 1} \overline{\frac{\partial \varphi^r_{\nu_r}}{\partial \zeta^{p_r}_r}(u_r)}\, \frac{\varphi^r_{\nu_r}(x_r)}{\varphi^r_0(x_r)\overline{\varphi^r_0(u_r)}} = \sum_{\nu_r\geq 1} \overline{\frac{\partial \varphi^r_{\nu_r}}{\partial \zeta^{p_r}_r}(u_r)}\, \frac{\varphi^r_{\nu_r}(x_r)}{K_r(x_r, u_r)}.$$

Obviously

$$(3.4)\qquad \frac{\partial^2 \ln k}{\partial \zeta^k \partial \bar{\zeta}^j}(u) = \begin{cases} \dfrac{\partial^2 \ln k_r}{\partial \zeta^k_r \partial \bar{\zeta}^j_r}(u_r) & \text{if } (r-1)n_1+1\leq k, j\leq n_1+(r-1)n_2, \\ & \qquad r=1,\ 2; \\ 0 & \text{otherwise,} \end{cases}$$

since $k(x)=k_1(x_1)k_2(x_2)$ for $x=(x_1, x_2)\in M_1\times M_2$; here $k_r(x_r)=K_r(x_r, x_r)$ for $x_r\in M_r$; $r=1, 2$. We obtain from the identities (3.1)—(3.4) and (2.7) that $F^s=F^s_1$ if $1\leq s\leq n_1$ and $F^s=F^{s-n_1}_2$ if $n_1+1\leq s\leq n_1+n_2$, i. e. $F=F_1\times F_2$.

(iii) We have $K_r(u_r, u_r)\neq 0$ by (1.5) and $\operatorname{rank}\Omega_r(u_r)=n_r$ by (1.6) (or (3.4)), $r=1, 2$. Let $h\in Q(u, v, L)$ and let $h=(h^1, \ldots, h^{n_1}, h^{n_1+1}, \ldots, h^{n_1+n_2})$. The functions $h^\alpha(x, u_2)$, $x\in M_1$, $\alpha=1, \ldots, n_1$ and $h^\beta(u_1, y)$, $y\in M_2$, $\beta=n_1+1, \ldots, n_1+n_2$, belong to $L^2H(v_1)$ and $L^2H(v_2)$, respectively, by Lemma 1.1 (i) (see e. g. [2, p. 91, Theorem 4.8]). Put $h_1(x)=(h^1(x, u_2), \ldots, h^{n_1}(x, u_2))$ for $x\in M_1$ and $h_2(y)=(h^{n_1+1}(u_1, y), \ldots, h^{n_1+n_2}(u_1, y))$ for $y\in M_2$. Obviously, $h_1(u_1)=0$ and $h_2(u_2)=0$. Under notations in the proof of (ii), we have

$$\left(\frac{\partial h^j(x, u_2)}{\partial \zeta^k_1}\right)_{x=u_1} = \frac{\partial h^j}{\partial \zeta^k}(u)=l_{k,j}=l^1_{k,j} \qquad \text{if } 1\leq k, j\leq n_1,$$

$$\left(\frac{\partial h^j(u_1, y)}{\partial \zeta^{k-n_1}_2}\right)_{y=u_2} = \frac{\partial h^j}{\partial \zeta^k}(u)=l_{k,j}=l^2_{k,j} \qquad \text{if } n_1+1\leq k, j\leq n_1+n_2.$$

Hence $h_r\in Q(u_r, v_r, L_r)$, $r=1, 2$.

(iv) We have $K(u, u)\neq 0$ and $\operatorname{rank}\Omega(u)=n_1+n_2$ by Proposition 1.5. Let $h_r=(h^1_r, \ldots, h^{n_r}_r)\in Q(u_r, v_r, L_r)$, $r=1, 2$. Put $h=h_1\times h_2$. Then $h(u)=0$ $h_{*u}=L$ and

$$\int_{M_1\times M_2} |h^{k_r}_r|^2 v_r = \|h^{k_r}_r\|^2 \int_{M_{3-r}} v_{3-r} < +\infty, \qquad k=1, \ldots, n_r,\ r=1, 2,$$

i. e. $h\in Q(u, v, L)$.

Lemma 3.1. *Let M be an n-dimensional complex manifold and v_M a volume element of M. Let A be an analytic subset of M, $\dim A\leq n-1$. Denote $v_{M\setminus A}=v_M\,|\,(M\setminus A)$.*

(i) *Every function $f\in L^2H(v_{M\setminus A})$ extends to a (unique) function $\tilde{f}\in L^2H(v_M)$;*

(ii) *If K_M, $K_{M\setminus A}$, Ω_M, $\Omega_{M\setminus A}$, T_M, $T_{M\setminus A}$ are the Bergman functions, the Bergman forms and the Tsuboi forms of $L^2H(v_M)$ and $L^2H(v_{M\setminus A})$, respectively, then $K_{M\setminus A}=K_M\,|\,(M\setminus A)$, $\Omega_{M\setminus A}=\Omega_M\,|\,(M\setminus A)$, $T_{M\setminus A}=T_M\,|\,((M\setminus A)\times(\overline{M\setminus A}))$.*

Proof. (i) Let $f \in L^2H(v_{M\setminus A})$. It is enough to show that for every point $a \in A$ there is its neighbourhood U in M such that the restriction $f|(U\setminus A)$ extends to a holomorphic function on U. Therefore, we may assume that: (a) M is the unit polydisk $M=\{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_k|<1,\ k=1, \ldots, n\}$; (b) $a=0$; (c) $A \subset \{(z_1, \ldots, z_n) \in M : z_1=0\}$ and $A \cap \{(z_1, \ldots, z_n) \in M : z_2=\cdots=z_n=0\}=\{0\}$; (d) $v_M=\varphi.(i^{n^2}/2^n)\, dz_1\wedge\ldots\wedge dz_n\wedge d\bar{z}_1\wedge\ldots\wedge d\bar{z}_n$, where φ is a positive C^∞-function on M such that $\varphi \geqq c=\mathrm{const}>0$. We have $(i^{n^2}/2^n)\int_{M\setminus A}|f|^2 dz_1\wedge\ldots\wedge dz_n \wedge d\bar{z}_1\wedge\ldots\wedge d\bar{z}_n \leqq c^{-1}\int_{M\setminus A}|f|^2 v_{M\setminus A}<+\infty$. This implies that f can be extended to a holomorphic function on M (see [9, p. 270, Theorem 2.4]);

(ii) follows at once from (i).

Let M be an n-dimensional complex manifold and N an open complex submanifold of M. Let $f_a \in H_N$ and let the holomorphic mapping $f: V \to N$, where V is an open neighbourhood of a in $\mathbf{C}^n$, be a representative of the germ f_a. The mapping $f': V \to M$ defined by $f'(z)=f(z)$, $z \in V$, is holomorphic and the germ f'_a does not depend on the choice of the representative f of f_a. The mapping $i': H_N \to H_M$ defined by $i'(f_a)=f'_a$ is a holomorphic imbedding such that $p_M \circ i'=p_N$. We shall identify H_N with $i'(H_N)$ and p_N with $p_M|i'(H_N)$.

Corollary 3.2. *Let M be an n-dimensional complex manifold and v_M a volume element of M. Let A be an analytic subset of M,* $\dim A \leqq n-1$. *Denote $v_{M\setminus A}=v_M|(M\setminus A)$. Let $u \in M\setminus A$. Then, M possesses a representative domain $(\Delta_M, \pi_M; u, v_M, L)$ iff $M\setminus A$ possesses a representative domain $(\Delta_{M\setminus A}, \pi_{M\setminus A}; u, v_{M\setminus A}, L)$.*

Moreover, we have

$$(3.5)\qquad S^{(r)}(u, v_{M\setminus A})=S^{(r)}(u, v_M)\setminus A,\ r=1, 2,\ S(u, v_{M\setminus A})=S(u, v_M)\setminus A,$$

$$G(u, v_{M\setminus A}, L)=G(u, v_M, L)|(M\setminus A),\ F(u, v_{M\setminus A}, L)=F(u, v_M, L)|(M\setminus A),$$

$$\widetilde{F}_{M\setminus A}=\widetilde{F}_M|((M\setminus A)\setminus S_{M\setminus A}),\ \Delta_{M\setminus A}\subset\Delta_M,\ \pi_{M\setminus A}=\pi_M/\Delta_{M\setminus A}.$$

Following C. Carathéodory, we give the definitions:

Definition 3.1. *Let X be a topological space. Let D and D_ν, $\nu=1, 2, \ldots$, be open and connected subsets of X. We say that D is a kernel of the sequence $\{D_\nu\}_{\nu=1}^\infty$ if:*

(a) *for every $x \in D$ there is its neighbourhood U_x in X and a natural number ν_x such that $U_x \subset D_\nu$ for each $\nu \geqq \nu_x$;*

(b) *D is a maximal, with respect to the inclusion, open and connected subset of X having the property* (a).

Definition 3.2. *We say that the sequence $\{D_\nu\}_{\nu=1}^\infty$ converges to D if D is a kernel of every subsequence of the sequence $\{D_\nu\}_{\nu=1}^\infty$.*

If X is locally connected, then $\{D_\nu\}$ has a kernel iff there is a natural number μ such that the interior of $\cap\{D_\nu : \nu \geqq \mu\}$ is non-empty. Note that $\{D_\nu\}$ can have more than one kernel. If $D_\nu \subset D$ $\nu=1, 2, \ldots$ and D has the property (a) from Definition 3.1, then $D=\cup_{\nu=1}^\infty D_\nu$ and D is the unique kernel of every subsequence of $\{D_\nu\}$; in particular, $\{D_\nu\}_{\nu=1}^\infty$ converges to D.

Proposition 3.4. *Let M be an n-dimensional complex manifold and v a volume element of M. Let M_ν be an open and connected subset of M and v_ν a volume element of M_ν, $\nu=1, 2, \ldots$ Let $u \in M$ and $u_\nu \in M_\nu$, $\nu=1, 2, \ldots$ Let K and K_ν be the Bergman functions of $L^2H(v)$ and $L^2H(v_\nu)$, respectively. Assume that:*

(a) $\lim_{\nu\to\infty} u_\nu = u$;

(b) *For every* $a \in M\times\overline{M}$, *there is its neighbourhood* W_a *and a natural number* ν_a *such that* $W_a \subset M_\nu \times \overline{M}_\nu$ *for* $\nu \geqq \nu_a$ *and the sequence* $\{K_\nu : \nu \geqq \nu_a\}$ *converges to* K *uniformly on* W_a;

(c) $K(u, u) \neq 0$ *and* rank $\Omega(u) = n$, *where* Ω *is the Bergman form for* $L^2H(v)$.

Then, $M \setminus S$ *is the unique kernel of every subsequence of* $\{M_\nu \setminus S_\nu\}_{\nu=1}^{\infty}$; *in particular,* $\{M_\nu \setminus S_\nu\}$ *converges to* $M\setminus S$; *here* $S = S(u, v)$, $S_\nu = S(u_\nu, v_\nu)$.

Proof. By T and T_ν denote the Tsuboi forms of $L^2H(v)$ and $L^2H(v_\nu)$, respectively.

The set $M\setminus S$ is non-empty by assumption (c). Let $x_0 \in M\setminus S$. There are charts (U, z) and (V, ζ) of M at x and at u, respectively, and a natural number μ such that $K(x, t) \neq 0$ for every $(x, t) \in U\times\overline{V}$, $U\times\overline{V} \subset M_\nu \times \overline{M}_\nu$ for $\nu \geqq \mu$ and the sequence $\{K_\nu : \nu \geqq \mu\}$ converges to K uniformly on $U\times\overline{V}$. Then, since $K(x_0, u) \neq 0$, there exist open neighbourhoods U_1 of x_0 and V_1 of u, and a natural number $\mu_1 \geqq \mu$ such that $U_1 \subset U$, $V_1 \subset V$ and $K_\nu(x, t) \neq 0$ for every $(x, t) \in U_1 \times \overline{V}_1$ and $\nu \geqq \mu_1$. The functions

$$T_{r,\bar{s}} = T_{r,\bar{s}}(v; z; \bar{\zeta}; U_1; \overline{V_1}) \text{ aud } T^{(\nu)}_{r,\bar{s}} = T_{r,\bar{s}}(v_\nu; z; \bar{\zeta}; U_1; \overline{V_1}) \ \nu \geqq \mu_1,$$

$$r, s = 1, \ldots, n,$$

(see (1.3)) are holomorphic on $U_1 \times \overline{V}_1$. The sequence $\{T^{(\nu)}_{r,\bar{s}} : \nu \geqq \mu_1\}$ converges to $T_{r,\bar{s}}$ uniformly on $U_1\times\overline{V}_1$. We have $\det[T_{r,\bar{s}}(x_0, u)] \neq 0$ since rank $T(x_0, u) = n$. Hence there are neighbourhoods U_0 of x_0 and V_0 of u, and a natural number $\mu_0 \geqq \mu_1$ such that $U_0 \subset U_1$, $V_0 \subset V_1$ and $\det[T^{(\nu)}_{r,\bar{s}}(x, t)] \neq 0$, i. e. rank $T_\nu(x_0, t) = n$ for every $(x, t) \in U_0 \times V_{,0}$ and $\nu \geqq \mu_0$. There is a natural number $\nu_0 \geqq \mu_0$ such that $u_\nu \in V_0$ for $\nu \geqq \nu_0$. Hence rank $T_\nu(x, u_\nu) = n$ for every $x \in U_0$ and $\nu \geqq \nu_0$. Thus, $U_0 \subset M_\nu \setminus S_\nu$ for $\nu \geqq \nu_0$.

Let $x_0 \in M$ and let there is an open neighbourhood U_0 of x_0 and a natural number ν_0 such that $U_0 \subset M_\nu \setminus S_\nu$ for $\nu \geqq \nu_0$. Then $K_\nu(x, u_\nu) \neq 0$ for every $x \in U_0$ and $\nu \geqq \nu_0$. In what follows, we shall denote the closure of a subset X of M by $[X]$. By assumption (b), there are open and relatively compact neighbourhoods U' of x_0 and V' of u, and a natural number μ' such that: U' is connected, $[U'] \subset U_0$, $[U'] \times [V'] \subset M_\nu \times M_\nu$ for $\nu \geqq \mu'$, the sequence $\{K_\nu : \nu \geqq \mu'\}$ converges to K uniformly on $[U'] \times [V']$ and $u_\nu \in V'$ for $\nu \geqq \mu'$. The function K is uniformly continuous on $[U'] \times [V']$. Hence the sequence $\{K_\nu(\cdot, u_\nu) : \nu \geqq \mu'\}$ converges to $K(\cdot, u)$ uniformly on $[U']$ since $\lim_{\nu\to\infty} u_\nu = u$. Then, since $K_\nu(x, u_\nu) \neq 0$ for every $x \in U'$ and $\nu \geqq \mu'$, either $K(x, u) \neq 0$ for every $x \in U'$ or $K(\cdot, u) \equiv 0$ on U' by the Hurwitz theorem. If $K(\cdot, u) \equiv 0$ on U' we would have $K(x, u) = 0$ for every $x \in M$ by the identity theorem, hence $K(u, u) = 0$ contradicting the assumption (c). Thus $x_0 \in U' \subset M \setminus S^{(1)}$ where $S^{(1)} = S^{(1)}(u, v)$. Since $K(x_0, u) \neq 0$ and $\{K_\nu : \nu \geqq \mu'\}$ converges to K uniformly on $U' \times V'$ there exist charts (U'', z) and (V'', ζ) of M at x_0, and at u, respectively, and a natural number $\mu'' \geqq \mu'$ such that: $U'' \subset U'$, $V'' \subset V'$, $K(x, t) \neq 0$ for each $(x, t) \in U'' \times V''$, $K_\nu(x, t) \neq 0$ for each $(x, t) \in U'' \times V''$ and $\nu \geqq \mu''$. Denote $Q_{r,\bar{s}} = T_{r,\bar{s}}(v; z; \bar{\zeta}; U''; \overline{V''})$ and $Q^{(\nu)}_{r,\bar{s}} = T_{r,\bar{s}}(v_\nu; z; \bar{\zeta}; U''; \overline{V''})$ for $\nu \geqq \mu''$, $r, s = 1, \ldots, n$. The functions $Q_{r,\bar{s}}$ and $Q^{(\nu)}_{r,\bar{s}}$, $\nu \geqq \mu''$, are holomorphic on $U'' \times \overline{V''}$. Let U, V and $\mu \geqq \mu''$ be open and relatively compact neighbourhoods of x_0 and of u, and a natural number, respectively, such that: U is connected, $[U] \subset U''$, $[V] \subset V''$ and $u_\nu \in V$ for $\nu \geqq \mu$. The sequence $\{Q^{(\nu)}_{r,\bar{s}} : \nu \geqq \mu\}$ converges to $Q_{r,\bar{s}}$ uniformly on $[U] \times [V]$ and $Q_{r,\bar{s}}$ is uniformly continuous on $[U] \times [V]$. Hence the sequence $\{Q^{(\nu)}_{r,\bar{s}}(\cdot, u_\nu) : \nu \geqq \mu\}$ converges to $Q_{r,\bar{s}}(\cdot, u)$ uniformly on $[U]$. We have $\det[Q^{(\nu)}_{r,\bar{s}}(x, u_\nu)] \neq 0$ for every $x \in U$ and $\nu \geqq \mu$ since $U \subset M \setminus S_\nu$ for $\nu \geqq \mu$. Then either $\det[Q_{r,\bar{s}}(x, u)] \neq 0$ for every $x \in U$ or $\det[Q_{r,\bar{s}}(x, u)] = 0$ for every $x \in U$ by the Hurwitz theorem. If $\det[Q_{r,\bar{s}}(\cdot, u)] \equiv 0$ on U, i. e. rank $T(x, u) < n$ for each $x \in U$, the point x_0 would be interior for $S^{(2)}$ where $S^{(2)} = S^{(2)}(u, v)$, hence $S^{(2)} = M \setminus S^{(1)}$ since $S^{(2)}$ is an analytic subset of the open and connected set $M \setminus S^{(1)}$. Then $u \in S^{(2)}$, contradicting the assumption (c). Therefore, $x_0 \in U \subset M \setminus S^{(2)}$. Thus, every subset of M having the property (a) from Definition 3.1 with $D_\nu = M_\nu \setminus S_\nu$ is contained in $M \setminus S$. The set $M \setminus S$ is non-empty, open, connected and has the property (a) from Definition 3.1 with $D_\nu = M_\nu \setminus S_\nu$ as we showed above. Hence $M \setminus S$ is the unique kernel of $\{M_\nu \setminus S_\nu\}$. Let $\{M_{\nu_k}\}_{k=1}^{\infty}$ be a subsequence of the sequence $\{M_\nu\}_{\nu=1}^{\infty}$. The assumptions of our proposition are fulfilled for M and M_{ν_k}, $k = 1, 2, \ldots$. Hence $M \setminus S$ is the unique kernel of $\{M_{\nu_k} \setminus S_{\nu k}\}_{k=1}^{\infty}$.

Let M and M_ν, $\nu = 1, 2, \ldots$, be bounded domains in $\mathbf{C}^n$. Let v and v_ν be the Euclidean volume elements of M and M_ν, respectively. Let K and K_ν be the Bergman functions of $L^2H(v)$ and $L^2H(v_\nu)$, respectively. If $M_\nu \subset M$, $\nu = 1$ $2, \ldots$, and $\{M_\nu\}$ converges to M, then the sequence $\{K_\nu\}$ converges to K locally uniformly on $M \times \overline{M}$, i. e. the assumption (b) of Proposition 3.4 is fulfilled, ([12, p. 761, Corollary 3]); the proof in [12] is formulated in the case when $M_\nu \subset M_{\nu+1} \subset M$ and $M = \cup_{\nu=1}^{\infty} M_\nu$ however, as M. Skwarczynski ([13, p. 309]) remarked, it is applicable to the general case with only minor changes. By analogy, we have:

Proposition 3.5. *Let M be a complex manifold and v a volume element of M. Let M_ν, $v = 1, 2, \ldots$, be open and connected subsets of M. Denote $v_\nu = v | M_\nu$, $\nu = 1, 2, \ldots$ Let K and K_ν be the Bergman functions of $L^2H(v)$ and $L^2H(v^\nu)$, respectively, $\nu = 1, 2, \ldots$. Assume that:*

(a) *$\{M_\nu\}$ converges to M;*

(b) *$K(t, t) \neq 0$ for each $t \in M$.*

Then the sequence $\{K_\nu\}$ converges to K locally uniformly on $M \times \overline{M}$.

Note that Proposition 1.3 holds for M by assumption (b). Since $M_\nu \subset M$, we have $K_\nu(t, t) \neq 0$ for every $t \in M_\nu$, $\nu = 1, 2, \ldots$, by Proposition 1.2 and the assumption (b). Hence Proposition 1.3 holds for M_ν, $\nu = 1, 2, \ldots$. Now, the proof of Proposition 3.4 is analogous to [12, p. 761, Corollary 3].

Proposition 3.6. *Let M be an n-dimensional complex manifold and v a volume element of M. Let A_ν, $\nu = 1, 2, \ldots$, be analytic subsets of M,* $\dim A_\nu \leqq n-1$. *Assume that:*

(a) *$\{M \setminus A_\nu\}_{\nu=1}^{\infty}$ converges to M;*

(b) *$K(x, x) \neq 0$ for every $x \in M$; where K is the Bergman function of $L^2H(v)$;*

(c) *M possesses a representative domain $(\Delta, \pi; u, v, L)$.*

Then there is a natural number ν_0 such that if $\nu \geqq \nu_0$, the complex manifold $M \setminus A_\nu$ possesses a representative domain (Δ_ν, π_ν) with respect to u, $v_\nu = v | (M \setminus A_\nu)$ and L, $\Delta_\nu \subset \Delta$ for $\nu \geqq \nu_0$ and $\{\Delta_\nu : \nu \geqq \nu_0\}$ converges to Δ.

Proof. Let ν_0 be a natural number such that $u \in M \setminus A_\nu$ for $\nu \geqq \nu_0$. By Corollary 3.2, if $\nu \geqq \nu_0$, the manifold $M_\nu = M \setminus A_\nu$ possesses a representative domain $(\Delta_\nu, \pi_\nu; u, v_\nu, L)$ and if we set $\widetilde{F} = \widetilde{F}(u, v, L)$, $\widetilde{F}_\nu = \widetilde{F}(u, v_\nu, L)$, $S = S(u, v)$, $S_\nu = S(u, v_\nu)$, then $\widetilde{F}_\nu = \widetilde{F} | (M_\nu \setminus S_\nu)$ and $\Delta_\nu \subset \Delta$. The sequence $\{M_\nu \setminus S_\nu : \nu \geqq \nu_0\}$ converges to $M \setminus S$ by Propositions 3.4 and 3.5. It follows that $\{\Delta_\nu : \nu \geqq \nu_0\}$ converges to Δ.

Proposition 3.7. *Let M be a complex manifold and $(\Delta, \pi; u, v_M, L)$ be a representative domain of M. Let $\widetilde{F}_M = \widetilde{F}(u, v_M, L)$ be the corresponding representative mapping of M. Then (Δ, π) possesses a representative domain $(\widetilde{\Delta}, \widetilde{\pi})$ with respect to $a = \widetilde{F}_M(u)$, $v_\Delta = (\widetilde{F}_M^{-1})^* v_M$ and π_{*a}, $S(a, v_\Delta) = \emptyset$ and $F_\Delta = \pi$ where $F_\Delta = F(a, v_\Delta, \pi_{*a})$.*

Proof. We have $\pi_{*a} = L \circ (\widetilde{F}_M^{-1})_{*a}$ by $\pi \circ \widetilde{F}_M = F_M$ and $(F_M)_{*u} = L$. Then, since $S_M = S(u, v_M)$ is an analytic subset of M, (Δ, π) possesses a representative domain $(\widetilde{\Delta}, \widetilde{\pi})$ with respect to a, v_Δ and π_{*a} by Corollary 3.2 and Proposition 3.1 (i). Denote $v_M | (M \setminus S_M)$ by $v_{M \setminus S_M}$. We have $S(u, v_{M \setminus S_M}) = \emptyset$ by (3.5), Corollary 3.2. By Propositions 3.1 (ii) and 3.1 (iii), $S(a, v_\Delta) = \emptyset$ and $F_\Delta = \widetilde{\pi} \circ \widetilde{F}_\Delta = \pi$ on Δ.

Let $X = (X, \pi)$ be a domain over $\mathbf{C}^n$. Let $\tau_X : X \to H_X$ be the natural holomorphic imbedding of X into the sheaf (H_X, P_X) (see Lemma 2.3). We may identify (X, π) with $(\tau_X(X), p_X | \tau_X(X))$. If (Δ, π), $(\widetilde{\Delta}, \widetilde{\pi})$ and F_Δ are as in Proposition 3.7, then $\tau_\Delta = \widetilde{F}_\Delta$ by $\pi = F_\Delta$. Hence if we identify (Δ, π) with $(\tau_\Delta(\Delta)$, $p_\Delta | \tau_\Delta(\Delta))$ then $(\widetilde{\Delta}, \widetilde{\pi}) = (\Delta, \pi)$ and $\widetilde{F}_\Delta = \mathrm{id}_\Delta$. Roughly speaking, the representative domain of a representative domain Δ coincides with Δ.

Definition 3.3. *Let (Δ, π) be a domain over $\mathbf{C}^n$ possessing a representative domain $(\widetilde{\Delta}, \widetilde{\pi}; a, v_\Delta, L)$. Let $\tau_\Delta : \Delta \to H_\Delta$ be the natural holomorphic imbedding of Δ into the sheaf (H_Δ, p_Δ). We shall say that (Δ, π) is a representative domain itself with respect to a, v_Δ, L if $\tau_\Delta(\Delta) = \widetilde{\Delta}$.*

Proposition 3.8. *Let (Δ, π) be a domain over $\mathbf{C}^n$ possessing a representative domain $(\widetilde{\Delta}, \widetilde{\pi}; a, v_\Delta, L)$. Put $F_\Delta = F(a, v_\Delta, L)$. The following conditions are equivalent:*

(i) *(Δ, π) is a representative domain itself with respect to u, v_Δ, L;*

(ii) *$S(a, v_\Delta) = \emptyset$ and $F_\Delta = \pi$ on Δ;*

(iii) *$S(a, v_\Delta) = \emptyset$ and $\widetilde{F}_\Delta = \tau_\Delta$ on Δ;*

(iv) *$S^{(1)}(a, v_\Delta) = \emptyset$ and $F_\Delta = \pi$ on $\Delta \setminus S(a, v_\Delta)$.*

Proof. (i) $\Longrightarrow$ (ii). Let $x \in \Delta$ and let U be an open neighbourhood of x such that $\pi(U)$ is open and $\pi_U = \pi | U : U \to \pi(U)$ is a biholomorphic mapping. Then $(\pi_U^{-1})_{\pi(x)} = \tau_\Delta(x) \in \tau_\Delta(\Delta) = \widetilde{\Delta}$. Put $y = \widetilde{F}_\Delta^{-1}(\tau_\Delta(x))$. We have $y \in \Delta \setminus S(a, v_\Delta)$. Let V be an open neighbourhood of y such that $F_V = F_\Delta | V$ is a biholomorphic mapping. Then $(F_V^{-1})_{F_\Delta(y)} = \widetilde{F}_\Delta(y) = \tau_\Delta(x) = (\pi_U^{-1})_{\pi(x)}$. Hence $F_\Delta(y) = \pi(x)$ and $F_V^{-1} = \pi_U^{-1}$ on some open neighbourhood $W \subset F_\Delta(V) \cap \pi(U)$ of the point $F_\Delta(y) = \pi(x)$. Then $y = F_V^{-1}(F_\Delta(y)) = \pi_U^{-1}(\pi(x)) = x$ and $F_\Delta = \pi$ on $\pi_U^{-1}(W)$. Hence $F_\Delta = \pi$ on $\Delta \setminus S^{(1)}(a, v_\Delta)$ by the identity theorem. Assume that $S(a, v_\Delta) \neq \emptyset$ and let

$x \in S(a, v_\Delta)$. The point $y = \widetilde{F}_\Delta^{-1}(\tau_\Delta(x))$ belongs to $\Delta \setminus S(a, v_\Delta)$. We see that $y = x$ as above. Hence $x \overline{\in} S(a, v_\Delta)$, a contradiction. Therefore, $S(a, v_\Delta) = \varnothing$.

(ii) $\longrightarrow$ (iii) obvious by the definitions of $\widetilde{F}_\Delta$ and τ_Δ.

(iii) $\longrightarrow$ (iv) $F_\Delta = \widetilde{\pi} \circ \widetilde{F} = \widetilde{\pi} \circ \tau_\Delta = \pi$.

(iv) $\Longrightarrow$ (i) $S(a, v_\Delta) = S^{(2)}(a, v_\Delta) = \varnothing$ since $F_\Delta = \pi$ is locally biholomorphic on Δ. Hence $\widetilde{F}_\Delta = \tau_\Delta$ on Δ and $\widetilde{\Delta} = \widetilde{F}_\Delta(\Delta) = \tau_\Delta(\Delta)$.

Example 3.1. Under notations of Proposition 3.7, (Δ, π) is a representative domain itself with respect to $a = \widetilde{F}_M(u)$, $v_\Delta = (\widetilde{F}_M^{-1})^* v_M$ and π_{*a}.

Let (Δ, π), $\pi = (\pi^1, \ldots, \pi^n)$, be a domain over $\mathbf{C}^n$, v_Δ a volume element of Δ and K the Bergman function of $L^2H(v_\Delta)$. Let $(x, t) \in \Delta \times \bar{\Delta}$ and $K(x, t) \neq 0$. We put

$$(3.6) \qquad T_{r,\bar{s}}(v_\Delta)(x, t) = (K(x, t))^{-2}\Big(K(x, t)\, \frac{\partial^2 K}{\partial \pi^r \partial \bar{\pi}^s}(x, t) - \frac{\partial K}{\partial \pi^r}(x, t)\, \frac{\partial K}{\partial \bar{\pi}^s}(x, t)\Big);\ r, s = 1, \ldots, n.$$

We obtain at once from (2.9) and Proposition 3.8:

Proposition 3.9. *Let (Δ, π), $\pi = (\pi^1, \ldots, \pi^n)$ be a domain over $\mathbf{C}^n$ possessing a representative domain $(\widetilde{\Delta}, \widetilde{\pi};\ a,\ v_\Delta,\ L)$. Let $\pi(a) = 0$. Then (Δ, π) is a representative domain itself with respect to a, v_Δ and L iff $S^{(1)}(a, v_\Delta) = \varnothing$ and $[T_{r,\bar{s}}(v_\Delta)(x, a)] = L(\pi)^{-1}[T_{r,\bar{s}}(v_\Delta)(a, a)]$ for every $x \in \Delta \setminus S(a, v_\Delta)$; here $L(\pi) = [(L^s(\frac{\partial}{\partial \pi^k}(a)))^n_{k,s=1}]$; $(L^1, \ldots, L^n) = L$.*

Example 3.2. A domain Δ in $\mathbf{C}^n$ is called circular with centre at z_0 if $z_0 \in \Delta$ and $z_0 + (z - z_0)e^{i\theta}$ for every $z \in \Delta$ and $\theta \in \mathbf{R}$. Let Δ be a bounded circular domain with centre at the origin 0. Let e_Δ be the Euclidean volume element of Δ and K_Δ the Bergman function of $L^2H(e_\Delta)$. We have $K_\Delta(e^{i\theta} z, 0) = K_\Delta(z, 0)$ for every $z = (z_1, \ldots, z_n) \in \Delta$ and $\theta \in \mathbf{R}$ by Proposition 1.6. We obtain:

$$e^{i|k|\theta} \frac{\partial^{|k|} K_\Delta}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}(e^{i\theta} z, 0) = \frac{\partial^{|k|} K_\Delta}{\partial z_1^{k_1} \ldots \partial z_n^{k_n}}(z, 0)$$

for $z \in \Delta$, $\theta \in \mathbf{R}$, where $|k| = k_1 + \cdots + k_n$, $k_j \geq 0$ — integers, $j = 1, \ldots, n$. Hence $(\partial^{|k|} K_\Delta / \partial z_1^{k_1} \ldots \partial z_n^{k_n})(0, 0) = 0$ for $|k| \geq 1$. Therefore, $K_\Delta(z, 0) = K_\Delta(0, 0) \neq 0$ for every $z \in \Delta$, hence $S^{(1)}(0, e_\Delta) = \varnothing$. By analogy, we obtain $T_{r,\bar{s}}(e_\Delta)(z, 0) = T_{r,\bar{s}}(e_\Delta)(0, 0)$ for every $z \in \Delta$, $r, s = 1, \ldots, n$, from the identity

$$T_{r,\bar{s}}(e_\Delta)(e^{i\theta} z, 0) = T_{r,\bar{s}}(e_\Delta)(z, 0);\ z \in \Delta,\ \theta \in \mathbf{R}.$$

Hence Δ is a representative domain itself with respect to 0, e_Δ and $\mathrm{id}_{\mathbf{C}^n}$.

We obtain at once from Propositions 3.3 and 3.8:

Proposition 3.10. *Let (Δ_r, π_r) be a domain over $\mathbf{C}^{n_r}$ and v_r its volume element, $r = 1, 2$. Let $a_r \in \Delta_r$ and let $L_r : (\Delta_r)_{a_r} \to \mathbf{C}^{n_r}$ be a non-degenerated linear mapping, $r = 1, 2$. Put $\Delta = \Delta_1 \times \Delta_2$, $\pi = \pi_1 \times \pi_2$, $a = (a_1, a_2)$, $L = L_1 \times L_2$, $v = p_1^* v_1 \wedge p_2^* v_2$, where $p_r : \Delta_1 \times \Delta_2 \to \Delta_r$ $(r = 1, 2)$ is the natural projection. Assume that $\pi(a_r) = 0$ and $\int_{M_r} v_r < +\infty$ for $r = 1, 2$. Then (Δ, π) is a representative domain itself with respect to a, v and L iff (Δ_r, π_r) is a representative domain itself with respect to a_r, v_r and L_r for each $r = 1, 2$.*

Remark. The Propositions 3.9 and 3.10 were proved by Tsuboi ([14, p. 144, Theorem 2.2 and p. 145, Corollary 2.1]) for Bergman representative domains of bounded domains in $\mathbf{C}^n$.

Proposition 3.11. *Let M_1 and M_2 be n-dimensional complex manifolds. Let M_r possesses a representative domain $(\Delta_r, \pi_r; u_r, v_r, L_r)$ and let $\widetilde{F}_r = \widetilde{F}(u_r, v_r, L_r)$ be the corresponding representative mapping of M_r, $r=1, 2$. Let $f: \Delta_1 \to \Delta_2$ be a biholomorphic mapping such that:*

(a) $f(a_1)=f(a_2)$, *where* $a_r = \widetilde{F}_r(u_r)$, $r=1, 2$;

(b) $(\widetilde{F}_2^{-1} \circ f \circ \widetilde{F}_1)^* v_2 = |\Phi|^2 v_1$ *on* $M_1 \setminus S(u_1, v_1)$, *where* Φ *is a holomorphic function on* $M_1 \setminus S(u_1, v_1)$.

Then f is a linear mapping.

Proof. Put $v_{\Delta_r} = (\widetilde{F}_r^{-1})^* v_r, \pi_r = (\pi_r^1, \ldots, \pi_r^n)$; $r=1, 2$. We have $f^* v_{\Delta_2} = |\Phi \circ \widetilde{F}_1^{-1}|^2 v_{\Delta_1}$ by assumption (b). Hence, by Proposition 1.6, if K_{Δ_1} and K_{Δ_2} are the Bergman functions of $L^2H(v_{\Delta_1})$ and $L^2H(v_{\Delta_2})$, respectively, then

$$K_{\Delta_1}(x, t) = K_{\Delta_2}(f(x), f(t)) \, (\Phi \circ \widetilde{F}_1^{-1})(x) \, \overline{(\Phi \circ \widetilde{F}_1^{-1})(t)}$$

for every $(x, t) \in \Delta_1 \times \overline{\Delta}_1$. Therefore

$$[T_{k,\bar{s}}(v_{\Delta_1})(x, a_1)]' = [(\frac{\partial(\pi_2^l \circ f)}{\partial \pi_1^m}(a_1))_{l,m=1}^n]^*$$

$$\cdot [T_{k,\bar{s}}(v_{\Delta_2})(f(x), a_2)] \, [(\frac{\partial(\pi_2^l \circ f)}{\partial \pi_1^m}(x))_{l,m=1}^n]$$

for every $x \in \Delta_1$. We have $[T_{k,\bar{s}}(v_{\Delta_r})(x, a_r)] \equiv$ constant matrix, $x \in \Delta_r$, $r=1, 2$, by Propositions 3.7 and 3.9. Hence $[\partial(\pi_2^l \circ f)/\partial \pi_1^m] \equiv$ constant matrix. Therefore, $\pi_2^k \circ f = \Sigma_{j=1}^n a_k^j \pi_1^j + b_k$, $k=1, \ldots, n$, where a_k^j, $b_k \in \mathbf{C}$. But $\pi_2(f(a_1)) = \pi_2(a_2) = (\pi_2 \circ \widetilde{F}_2)(u_2) = F_2(u_2) = 0$ and $\pi_1(a_1) = (\pi_1 \circ \widetilde{F}_1)(u_1) = 0$. Hence $b_k = 0$, $k=1, \ldots, n$.

Remark. If M_1, M_2 are domains over $\mathbf{C}^n$ and v_1, v_2 are the Euclidean volume element of M_1 and M_2, respectively, then the assumption (b) of Proposition 3.11 is fulfilled. Proposition 3.11 was proved by Bergman ([5, p. 190, Theorem]) for the Bergman representative domains of bounded domains in $\mathbf{C}^n$ (see also [8, p. 301, Proposition 3], [14, p. 146, Corollary 2.2]).

Corollary 3.3. *Let M be a domain over $\mathbf{C}^n$ and e_M its Euclidean volume element. Assume that M possesses a representative domain $(\Delta, \pi; u, e_M, L)$ and let $\widetilde{F}_M = \widetilde{F}(u, e_M, L)$ be the corresponding representative mapping of M. Then every automorphism of Δ leaving the point $\widetilde{F}_M(u)$ fixed is a linear mapping.*

We obtain from Example 3.2 and Corollary 3.3:

Corollary 3.4 ([6, p. 30, Theorem VI]). *Let Δ be a bounded circular domain in $\mathbf{C}^n$ with centre at the origin 0. Then every automorphism of Δ leaving 0 fixed is a linear mapping.*

Corollary 3.5. *Let M be a complex manifold possessing a representative domain $(\Delta, \pi; u, v_M, L)$. Let φ be an automorphism of M such that:*

(a) $\varphi(u)=u$;

(b) $\varphi_{*u}=\mathrm{id}_{M_u}$;

(c) $\varphi^* v_M=|\Phi|^2 v_M$, *where* Φ *is a holomorphic function on* M.

Then $\varphi=\mathrm{id}_M$.

Proof. Put $\widetilde{F}=\widetilde{F}(u, v_M, L)$, $S=S(u, v_M)$, $a=\widetilde{F}(u)$. The mapping φ is an automorphism of $M\setminus S$ by Proposition 3.1. Set $f=\widetilde{F}\circ\varphi\circ\widetilde{F}^{-1}$. We have $f(a)=a$, $(\widetilde{F}^{-1}\circ f\circ\widetilde{F})^* v_M=\varphi^* v_M=|\Phi|^2 v_M$ and $f_{*a}=\widetilde{F}_{*u} \circ \varphi_{*u} \circ (\widetilde{F}^{-})_{*a}=\widetilde{F}^u\circ \mathrm{id}_{M^u} \circ(\widetilde{F}_{*u})^{-1}=\mathrm{id}$. Hence $\pi\circ f=\pi$ by Proposition 3.11. Therefore, $f=\mathrm{id}_\Delta$. Then $\varphi\,|\,(M\setminus S)=\mathrm{id}_{M\setminus S}$. Hence $\varphi=\mathrm{id}_M$.

Remark. If M is a bounded domain in $\mathbf{C}^n$ and v_M is the Euclidean volume element of M, then the assumption (c) of Corollary 3.5 is fulfilled and Corollary 3.5 coincides with a well-known result of H. Cartan ([6, p. 30, Theorem VII]) (see also [2, p. 368, Theorem 23.1], [8, p. 297, Theorem 16]).

Proposition 3.12. *Let* (M, p) *be a domain over* $\mathbf{C}^n$, v_M *a volume element of* M *and* u *a point of* M. *Put* $S=S(u, v_M)$. *We have:*

(i) *If* $S\neq M$ *and* (M, p) *is a domain of holomorphy, then* $(M\setminus S, p)$ *is a domain of holomorphy;*

(ii) *Let* (M, p) *possesses a representative domain* $(\Delta, \pi; u, v_M, L)$.

(ii1) *If* (M, p) *is a domain of holomorphy then* (Δ, π) *is a domain of holomorphy;*

(ii2) *If* (M, p) *is a bounded domain in* $\mathbf{C}^n$ *and* (Δ, π) *is a domain of holomorphy, then* M *is a domain of holomorphy.*

Proof. (i) It is enough to prove that if $\{x_k\}$ is a sequence of points of $M\setminus S$ which has no limit point in $M\setminus S$, then there is a holomorphic function f on $M\setminus S$ such that the sequence $\{f(x_k)\}$ is unbounded. If $\{x_k\}$ has no limit point in M, then such a function f exists since M is holomorphically convex. If $\{x_k\}$ has a limit point x_0 in M, then $x_0\in S$. Let K be the Bergman function of $L^2H(v_M)$. Put $T_{r,\bar{s}}=T_{r,\bar{s}}(v_M)$, $r, s=1,\ldots,n$. The function $f(x)=((K(x, u))^{2n+1}\det[T_{r,s}(x, u)])^{-1}$, $x\in M\setminus S$, is holomorphic on $M\setminus S$ and it converges to ∞ when x converges to some point of S. Hence $\{f(x_k)\}$ is unbounded.

(ii1) (Δ, π) is holomorphically convex since Δ is mapped biholomorphically onto $M\setminus S$ which is holomorphically convex by (i).

(ii2) $M\setminus S$ is a domain of holomorphy since $M\setminus S$ is mapped biholomorphically onto Δ. Hence, for every $z\in\partial(M\setminus S)$, there is an open neighbourhood U and a holomorphic function f on $(M\setminus S)\cap U$ such that f cannot be extended to a holomorphic function on U. In particular, this holds for every $z\in\partial M$ since $\partial M\subset\partial(M\setminus S)$. Hence M is a domain of holomorphy (see e. g. [3, p. 400, Theorem 4]).

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