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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON THE BOUNDEDNESS OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE ON THE REAL AXIS

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Let $f(z)$ be an entire function of exponential type σ, $0<\sigma<\pi$, i. e., $|f(z)|\leqq Ae^{\sigma|z|}$ for every complex z. Let there be an integer s, a constant $C>0$ and integers $m_r\geqq 0$, $r=0, 1, 2, \dots, s-1$, $\sum_{r=0}^{s-1} m_r=s$ such that

$$|f(sk+r)|\leqq C,\ |f'(sk+r)|\leqq C, \dots, |f^{(m_r-1)}(sk+r)|\leqq C;\ k=0, \pm 1, \pm 2, \dots, r=0, 1, 2, \dots, s-1.$$

Then $f(z)$ is bounded on the real axis.

An entire function $f(z)$ is said to be of exponential type σ if there is a constant A such that $|f(z)|\leqq Ae^{\sigma|z|}$ for every complex z. According to a well known theorem of Cartwright [1], if an entire function of exponential type σ, $0<\sigma<\pi$, is bounded at the integer points $n=0, \pm 1, \pm 2, \dots$, then it is bounded on the entire real axis. There are a lot of generalizations of this theorem. For instance Duffin ang Schaeffer [2] replaced the integer points by points λ_n, which satisfy

$$|\lambda_n-n|\leqq \Gamma,\ n>\Gamma,\ \Gamma=\text{const},\ |\lambda_n-\lambda_m|\geqq\gamma>0,\ n\neq m,\ \gamma=\text{const}.$$

Korevaar [3] extended Cartwright's theorem in another direction.

Let $f(z)$ be a function of exponential type $s\sigma$, where $0<\sigma<\pi$ and $s\geqq 1$ is an integer. Let there be a constant C such that

$$|f(n)|\leqq C,\ |f'(n)|\leqq C, \dots, |f^{(s-1)}(n)|\leqq C,\ n=0, \pm 1, \pm 2, \dots$$

Then $|f(x)|<M$, $M=M(s, \sigma, C)$, $-\infty<x<\infty$.

Thus, Korevaar increased the type of the function, but required boundedness at the integer points not only for the function, but also for some of its derivatives.

It will be more convenient for our considerations to formulate the last theorem in the following (equivalent) way:

Let $f(z)$ be an entire function which satisfies the conditions:

$$|f(z)|\leqq Ae^{\sigma|z|},\ 0<\sigma<\pi,\ A=\text{const},\ |f(sn)|\leqq C,\ |f'(sn)|\leqq C, \dots, |f^{(s-1)}(sn)|\leqq C,$$

where $s\geqq 1$ is an integer. Then $|f(x)|<M$, $M=$const, $-\infty<x<\infty$.

We shall generalize Korevaar's theorem, showing that some freedom in the choice of the points and derivatives is admissible. Beforehand we need a definition.

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Let $r>0$ is an integer. We shall say that an entire function $f(z)$ is multiple bounded of order r at the points of the sequence $\{z_n\}$, if there is a constant C such that $|f(z_n)|\leq C$, $|f'(z_n)|\leq C, \ldots, |f^{(r-1)}(z_n)|\leq C$ for all z_n of the sequence.

Theorem 1. *Let $f(z)$ be an entire function and $|f(z)|\leq Ae^{\sigma|z|}$, $0<\sigma<\pi$, $A=\text{const}$. Let all integer points $n=0, \pm 1, \pm 2, \ldots$ be distributed into s sequences: $n=sk+r$, $k=0, \pm 1, \pm 2, \ldots$, $r=0, 1, 2, \ldots, s-1$.*

Further let the function $f(z)$ be multiple bounded of order $m_r\geq 0$ at the points of the sequence $\{sk+r\}_{-\infty}^{\infty}$, $r=0, 1, 2, \ldots, s-1$ and let $m_0+m_1+\cdots+m_{s-1}=s$.

Then $|f(x)|\leq K$, $-\infty<x<\infty$, where K is a constant which depends on σ, s, C but not on the concrete function $f(z)$,

Remark. We set $m_r=0$ when no conditions are imposed on $f(z)$ at the points of the sequence $\{sk+r\}$.

To prove this theorem we shall follow, just as in our paper [4] (see also [5]) the elegant method of Duffin and Schaeffer from [2].

Theorem 1 is a consequence of a similar theorem concerning functions regular in the half-plane $\operatorname{Re} z\geq 0$.

Theorem 2. *Let $f(z)$ be regular in the half-plane $\operatorname{Re} z\geq 0$ and let it satisfy the conditions*

(1) $$|f(z)|\leq e^{\sigma|z|},\quad 0<\sigma<\pi,\quad \operatorname{Re} z\geq 0;$$

(2) *$f(z)$ is multiple bounded of order m_r, $m_r\geq 0$, by the constant $C=1$ at the points $\{sk+r\}_{k=0}^{\infty}$, $r=0, 1, \ldots, s-1$, and let $\Sigma_{r=0}^{s-1}m_r=s$.*

Then $|f(z)|\leq Me^{\sigma|y|}$, $M=M(\sigma, s)$, $y=\operatorname{Im} z$; i. e. $f(z)$ is bounded on the ray $0\leq x<\infty$.

An important step in the proof of theorem 2 is the following:

Lemma 1. *Suppose the assumptions of theorem 2 hold and besides $f(z$ is bounded on the positive axis, i. e.), $f(x)=O(1)$; $x\to\infty$. Then $|f(x)|\leq M$; $M=M(s, \sigma)$, $x\geq 0$.*

So this lemma states that if σ and s are fixed, all the functions that satisfy the conditions of theorem 2 and in addition are bounded on the ray $0\leq x<\infty$, are bounded by a common constant.

Proof of lemma 1. Suppose lemma 1 is false. Then there exists a sequence $\{f_v(z)\}$ of functions which fulfil the conditions of the lemma, but at the same time $c_v=\sup|f_v(x)|\to\infty$ when $v\to\infty$. We may choose the sequence $\{f_v(z)\}$ such that $c_v>v$, $v=1, 2, \ldots$

From the relations $|f_v(x)|\leq c_v$ and $|f_v(z)|\leq e^{\sigma|z|}$, $0<\sigma<\pi$, applying the Phragmen-Lindelöf principle we infer

(3) $$|f_v(z)|\leq c_v e^{\sigma|y|},\qquad x\geq 0,\ 0<\sigma<\pi.$$

Let the real numbers x_v, $v=1, 2, \ldots$ be chosen such that

(4) $$|f_v(x_v)|\geq c_v(1-1/v).$$

It follows from (1) that $x_v\to\infty$ when $v\to\infty$. Let us set

(5) $$\psi_v(z)=f_v(z+[x_v])/c_v,\qquad v=1, 2, \ldots$$

(Here $[x_v]$ is the integer part of x_v.) The function $\psi_v(z)$ is regular in the half-plane $\operatorname{Re} z\geq -[x_v]$ and in view of (3) and (4) satisfies the inequalities

(6) $$|\psi_v(z)|\leq e^{\sigma|y|}, \qquad \operatorname{Re} z \geq -[x_v]$$

and

(7) $$\max_{0\leq x\leq 1} |\psi_v(x)| \geq 1-1/v.$$

Besides, the assumption (2) implies a corresponding condition concerning multiple boundedness of $\psi_v(z)$ at the sequence

$$\{sk+r-[x_v]\}_{k=0}^{\infty}, \ r=0, 1, 2, \ldots, s-1.$$

Let $[x_v]\equiv r_v \pmod s$, $0\leq r_v \leq s-1$, $v=1, 2, \ldots$ Since the integers r_v get only a finite number of different values, there are infinitely many of them r_{v_a}, $a=1, 2, \ldots$ which are equal to one and the same integer, say l. Further we shall consider the subsequence $\{r_{v_a}\}$ and respectively $\{\psi_{v_a}\}$ but for simplicity instead of r_{v_a}, ψ_{v_a} and so on, we shall write again r_v, ψ_v and so on. Thus, we have $r_v=l=\text{const}$, $0\leq l\leq s-1$, $v=1, 2, \ldots$

Now if r is fixed, $0\leq r\leq s-1$, we have $r-[x_v]\equiv r-l \pmod s$ and if we put $r-l\equiv r^* \pmod s$, $0\leq r^*\leq s-1$, the sequence $\{sk+r-[x_v]\}_{k=0}^{\infty}$ gets the form $\{sk_v+r^*\}$, $k_v=-p_v, -p_v+1, \ldots, 0, 1, \ldots$ where $p_v>0$, $p_v\to\infty$ when $v\to\infty$.

Clearly when r runs over the set $\{0, 1, 2, \ldots, s-1\}$, r^* does the same (eventually in another order). Thus, we have again s sequences $\{sk_v+r^*\}$, $r^*=0, 1, 2, \ldots, s-1$ at which the function $\psi_v(z)$, $v=1, 2, \ldots$, is multiple bounded respectively of order m_r by the constant $1/c_v$ (see (2) and (5)), i. e

(8) $$|\psi_v^{(q)}(sk_v+r^*)|\leq 1/c_v, \ q=0, 1, 2, \ldots, m_r-1\,;\ k_v=-p_v, -p_v+1, \ldots, 0, 1, \ldots$$

Moreover $m_0+m_1+\cdots+m_{s-1}=s$.

Because of (6) we can apply the compactness principle to the sequence $\psi_v(z)$ and find a subsequence $\psi_{v_k}(z)$, which tends uniformly in any bounded region to an entire function $\psi(z)$. The inequalities (6) and (7) show that $\psi(z)$ satisfies the conditions

(9) $$|\psi(z)|\leq e^{\sigma|y|}, \qquad 0<\sigma<\pi,$$

(10) $$\max_{0\leq x\leq 1} |\psi(x)|=1.$$

The relations (8) imply

(11) $$\psi(sk+r^*)=0,\ \psi'(sk+r^*)=0, \ldots, \psi^{(m_r-1)}(sk+r^*)=0,$$

where $k=0, \pm 1, \pm 2, \ldots$; $r^*=0, 1, 2, \ldots, s-1$ and $m_0+m_1+\cdots+m_{s-1}=s$.

By using Jensen's formula (see [4]) one can easily prove, that an entire function, which satisfies (9) and (11) is identically zero. Thus $\psi(z)\equiv 0$. But this contradicts (10). The lemma is proved.

From here on we may proceed exactly as in [4] in order to get theorem 2 and then — theorem 1.

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