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INTERVAL METHODS OF NEWTON TYPE FOR NONLINEAR EQUATIONS

NELI S. DIMITROVA, SVETOSLAV M. MARKOV

Two interval iteration methods of Newton type for finding real zeros of nonlinear equations are formulated and their convergence is investigated. A realization of the methods in computer arithmetic is obtained and some computer experiments are presented.

1. Introduction. We consider two interval iteration methods of Newton type producing two-sided approximations of real roots of nonlinear equations of the form $f(x)=0$. The first method, considered in section 3, is of linear convergence and assumes continuity and monotonicity of f. A second interval method of Newton type, formulated in [8], is discussed in detail in section 4. A proof of quadratic convergence of the method is given in the situation when f possesses continuos second derivative (Theorem 2). In Theorem 3 we prove quadratic convergence for rather relaxed assumptions on f. Thus, it may be asserted that our method never fails in the sence of [10], and produces always two quadratically convergent sequences. A detailed computer realization of both methods is presented. Since our methods are formulated in a nonstandard interval arithmetic [8], a new computer realization technique is used, which is discussed in some detail in section 2. Some computer experiments with the methods are reported.

2. Preliminaries. A. Computer realization of two-sided algorithms. We shall first consider the computer realization of two-sided algorithms which are not formulated in interval-arithmetic form. Let $\mathcal{R}$ be the set of real numbers and let $\mathcal{R}_M \subset \mathcal{R}$ be a finite set of machine numbers (see [1], p. 47). For $a \in \mathcal{R}$ we denote by $\downarrow a$ the largest machine number $\downarrow a \in \mathcal{R}_M$ such that $\downarrow a \leq a$, and by $\uparrow a$ the smallest machine number $\uparrow a \in \mathcal{R}_M$ such that $\uparrow a \geq a$. We thus obtain two mappings $\downarrow, \uparrow$ from $\mathcal{R}$ onto $\mathcal{R}_M$ called monotone directed (downwardly and upwardly, resp.) roundings in $\mathcal{R}$ [6, 7]. By means of these roundings we define the following computer arithmetic: if $a, b \in \mathcal{R}_M$ and $* \in \{+, -, \times, :\}$, then $a \,\check{*}\, b = \downarrow (a * b)$, $a \,\hat{*}\, b = \uparrow (a * b)$. We thus obtain 8 computer arithmetic operations which may be used for the computer realization of arbitrary two-sided algorithm.

If a, b are real, then the computation of $\downarrow (a * b)$ and $\uparrow (a * b)$ on a computer is not possible in general (a, b may not be machine numbers). In this situation we compute upper and lower bounds of $a * b$ using the following inequalities:

$$\downarrow a \,\check{+}\, \downarrow b \leq a + b \leq \uparrow a \,\hat{+}\, \uparrow b;$$

$$\downarrow a \,\check{-}\, \uparrow b \leq a - b \leq \uparrow a \,\hat{-}\, \downarrow b;$$

$$\downarrow a \,\check{\times}\, \downarrow b \leq a \times b \leq \uparrow a \,\hat{\times}\, \uparrow b, \quad \text{if} \quad a > 0, b > 0,$$

$$\uparrow a \check{\times} \downarrow b \leq a \times b \leq \downarrow a \hat{\times} \uparrow b, \quad \text{if} \quad a>0, b<0,$$

(1)
$$\downarrow a \check{\times} \uparrow b \leq a \times b \leq \uparrow a \hat{\times} \downarrow b, \quad \text{if} \quad a<0, b>0,$$

$$\uparrow a \check{\times} \uparrow b \leq a \times b \leq \downarrow a \hat{\times} \downarrow b, \quad \text{if} \quad a<0, b<0;$$

$$\downarrow a \,\check{:}\, \uparrow b \leq a : b \leq \uparrow a \,\hat{:}\, \downarrow b, \quad \text{if} \quad a>0, b>0,$$

$$\uparrow a \,\check{:}\, \uparrow b \leq a : b \leq \downarrow a \,\hat{:}\, \downarrow b, \quad \text{if} \quad a>0, b<0,$$

$$\downarrow a \,\check{:}\, \downarrow b \leq a : b \leq \uparrow a \,\hat{:}\, \uparrow b, \quad \text{if} \quad a<0, b>0,$$

$$\uparrow a \,\check{:}\, \downarrow b \leq a : b \leq \downarrow a \,\hat{:}\, \uparrow b, \quad \text{if} \quad a<0, b<0.$$

In order to demonstrate the utilization of formulas (1) consider the following example:

Example. Assume that $f \in C_2[a, b]$ and the derivatives f' and f'' have constant signs on $[a, b]$. Assume also that there is a unique real (unknown) zero x^* of f in the (known) interval $X_0 = [\underline{x}_0, \overline{x}_0] \subset [a, b]$, that is $f(x^*) = 0$ and $x^* \in X_0$. In this situation the solution x^* may be sought by means of the well-known two-sided (interval) method

(2)
$$\underline{x}_{k+1} = \underline{x}_k - f(\underline{x}_k)/f'(\underline{x}_k),$$
$$\overline{x}_{k+1} = \overline{x}_k - f(\overline{x}_k)(\overline{x}_k - \underline{x}_k)/(f(\overline{x}_k) - f(\underline{x}_k)), \quad k = 0, 1, \dots,$$

in case that $f(\underline{x}_0) f''(\underline{x}_0) > 0$ and

(2′)
$$\underline{x}_{k+1} = \underline{x}_k - f(\underline{x}_k)(\overline{x}_k - \underline{x}_k)/(f(\overline{x}_k) - f(\underline{x}_k)),$$
$$\overline{x}_{k+1} = \overline{x}_k - f(\overline{x}_k)/f'(\overline{x}_k), \quad k = 0, 1, \dots,$$

in case that $f(\underline{x}_0) f''(\underline{x}_0) < 0 \; (f(\overline{x}_0) f''(\overline{x}_0) > 0)$.

The method (2)–(2′) delivers two monotone sequences $\underline{x}_0 \leq \underline{x}_1 \leq \cdots$ and $\overline{x}_0 \geq \overline{x}_1 \geq \cdots$ converging to x^*.

We consider the computer realization of this method. To this end the following four possibilities can be distinguished:

i) $f' < 0, f'' > 0$ on X_0. In this situation we have $f(\underline{x}_0) f''(\underline{x}_0) > 0$ and therefore we should use (2). Consider the first equality of (2). The ideal situation would be $\underline{x}_{k+1} = \downarrow (\underline{x}_k - f(\underline{x}_k)/f'(\underline{x}_k))$ but this cannot be computed in general. Using the fact that $\underline{x}_k$ is a machine number (and hence $\downarrow \underline{x}_k = \underline{x}_k$) and that $f(\underline{x}_k) > 0$, $f(\overline{x}_k) < 0$, by means of (1) we obtain

$$\downarrow (\overline{x}_k - f(\underline{x}_k)/f'(\underline{x}_k)) \geq \underline{x}_k \,\check{-}\, \uparrow (f(\underline{x}_k) : f'(\underline{x}_k)) \geq \underline{x}_k \,\check{-}\, (\downarrow f(\underline{x}_k) \,\hat{:}\, \downarrow f'(\underline{x}_k)).$$

The last bound is already suitable for the computer realization of $\underline{x}_{k+1}$ (unless the computer realization of $\downarrow f(\underline{x}_k)$ and $\downarrow f'(\underline{x}_k)$ remains to be done, which depends on the particular expression for f and can be done again on the basis of formulas (1)). Thus we may set $\underline{x}_{k+1} = \underline{x}_k \,\check{-}\, (\downarrow f(\underline{x}_k) \,\hat{:}\, \downarrow f'(\underline{x}_k))$.

For the sake of brevity we shall write down the above arguments in the following conditional form:

$$\underline{x}_{k+1} = \downarrow(\underline{x}_k - f(\underline{x}_k)/f'(\underline{x}_k)) = \underline{x}_k \check{-} \uparrow(f(\underline{x}_k)/f'(\underline{x}_k)) = \underline{x}_k \check{-} (\downarrow f(\underline{x}_k) \,\hat{:}\, \downarrow f'(\underline{x}_k)).$$

The reader should note the nonstandard meaning of the equality notations used above. Such notations are used throughout the paper, where machine realization of algorithms is involved.

Similarly, for the machine realization of the second sequence $\overline{x}_{k+1}$ of (2) we have

$$\begin{aligned}
\overline{x}_{k+1} &= \overline{x}_k \hat{-} \downarrow((f(\overline{x}_k) : (f(\overline{x}_k) - f(\underline{x}_k))) \times (\overline{x}_k - \underline{x}_k)) \\
&= \overline{x}_k \hat{-} (\downarrow(f(\overline{x}_k) : (f(\overline{x}_k) - f(\underline{x}_k))) \check{\times} \downarrow(\overline{x}_k - \underline{x}_k)) \\
&= \overline{x}_k \hat{-} ((\uparrow f(\overline{x}_k) \,\check{:}\, \downarrow(f(\overline{x}_k) - f(\underline{x}_k))) \check{\times} (\overline{x}_k \check{-} \underline{x}_k)) \\
&= \overline{x}_k \hat{-} ((\uparrow f(\overline{x}_k) \,\check{:}\, (\downarrow f(\overline{x}_k) \check{-} \uparrow f(\underline{x}_k))) \check{\times} (\overline{x}_k \check{-} \underline{x}_k)).
\end{aligned}$$

Here again the equality sign is used in the sense mentioned above. The calculations are given in detail in order to demonstrate once more the computer realization technique based upon inequalities (1). In what follows we shall omit such detailed calculations.

ii) $f'>0$, $f''<0$ on X_0. In this situation using the technique described above we obtain from (2)

$$\underline{x}_{k+1} = \underline{x}_k \check{-} (\uparrow f(\underline{x}_k) \,\hat{:}\, \uparrow f'(\underline{x}_k)),$$

$$\overline{x}_{k+1} = \overline{x}_k \hat{-} ((\downarrow f(\overline{x}_k) \,\check{:}\, (\uparrow f(\overline{x}_k) \hat{-} \downarrow f(\underline{x}_k))) \check{\times} (\overline{x}_k \check{-} \underline{x}_k)).$$

iii) $f'>0$, $f''>0$ on X_0. In this case we have from (2′)

$$\underline{x}_{k+1} = \underline{x}_k \check{-} ((\uparrow f(\underline{x}_k) \,\hat{:}\, (\uparrow f(\overline{x}_k) \hat{-} \downarrow f(\underline{x}_k))) \hat{\times} (\overline{x}_k \check{-} \underline{x}_k)),$$

$$\overline{x}_{k+1} = \overline{x}_k \hat{-} (\downarrow f(\overline{x}_k) \,\check{:}\, \uparrow f'(\overline{x}_k)).$$

iv) $f'<0$, $f''<0$ on X_0. Using (2′) we obtain

$$\underline{x}_{k+1} = \underline{x}_k \check{-} ((\downarrow f(\underline{x}_k) \,\hat{:}\, (\downarrow f(\overline{x}_k) \check{-} \uparrow f(\underline{x}_k))) \hat{\times} (\overline{x}_k \check{-} \underline{x}_k)),$$

$$\overline{x}_{k+1} = \overline{x}_k \hat{-} (\uparrow f(\overline{x}_k) \,\check{:}\, \downarrow f'(\overline{x}_k)).$$

This accomplishes the computer realization of (2)—(2′).

It is often convenient to use two additional types of roundings: rounding towards zero

$$\Box a = \begin{cases} \downarrow a, & a \geqq 0, \\ \uparrow a, & a < \;, \end{cases}$$

and rounding away of zero

$$\Box_0 a = \begin{cases} \uparrow a, & a \geqq 0, \\ \downarrow a, & a < 0. \end{cases}$$

These roundings generate the following computer arithmetic: $a[*]b = \Box(a*b)$, $a[*]_0 b = \Box_0(a*b)$, where $* \in \{+, -, \times, :\}$.

The part of formulas (1) concerning the product and the quotient of two real numbers can be written in terms of the operations $[*]$, $[*]_0$ as follows:

$$(1') \qquad \begin{aligned} &\Box a[\times]\Box b \leq a\times b \leq \Box_0 a[\times]_0\Box_0 b, \text{ if } ab>0,\\ &\Box_0 a[\times]_0\Box_0 b \leq a\times b \leq \Box a[\times]\Box b, \text{ if } ab<0;\\ &\Box a[:]\Box_0 b \leq a:b \leq \Box_0 a[:]_0\Box b, \text{ if } ab>0,\\ &\Box_0 a[:]_0\Box b \leq a:b \leq \Box a[:]\Box_0 b, \text{ if } ab<0. \end{aligned}$$

Using the roundings $\downarrow, \uparrow, \Box$ and $\Box_0$ the computer realization of the method (2)–(2′) obtains the following compact form:

$$(2C) \qquad \begin{aligned} &\underline{x}_{k+1} = \underline{x}_k \stackrel{\smile}{-} (\Box f(\underline{x}_k)[:]\Box_0 f'(\underline{x}_k)),\\ &\overline{x}_{k+1} = \overline{x}_k \stackrel{\frown}{-} (\Box f(\overline{x}_k)[:](\Box_0 f(\overline{x}_k)[-]_0\Box_0 f(\underline{x}_k))[\times](\overline{x}_k \stackrel{\smile}{-} \underline{x}_k)) \end{aligned}$$

in the situation when $f(\underline{x}_0)f''(\underline{x}_0)>0$, and

$$(2'C) \qquad \begin{aligned} &\underline{x}_{k+1} = \underline{x}_k \stackrel{\smile}{-} (\Box f(\underline{x}_k)[:](\Box_0 f(\overline{x}_k)[-]_0\Box_0 f(\underline{x}_k))[\times](\overline{x}_k \stackrel{\smile}{-} \underline{x}_k)),\\ &\overline{x}_{k+1} = \overline{x}_k \stackrel{\frown}{-} (\Box f(\overline{x}_k)[:]\Box_0 f'(\overline{x}_k)) \end{aligned}$$

in the situation when $f(\underline{x}_0)f''(\underline{x}_0)<0$.

B. Computer realization of algorithms in interval-arithmetic form. We shall next consider some concepts and relations connected with the computer realization of algorithms written in interval-arithmetic form. Thereby we shall use the interval arithmetic as formulated in [8]. We first give a short review of this arithmetic.

We shall denote the compact intervals by capital letters $A, B, \ldots$ throughout the paper. We adopt the notation $[\alpha\vee\beta]$ for the compact interval with end-points α and β (α not necessarily $\leq\beta$). The end-points of the interval X are denoted by $\underline{x}, \overline{x}$ so that $X=[\underline{x}, \overline{x}]$. If $X\not\ni 0$ we shall also write $X=[x_c, x_d]$, if $X>0$ and $X=[x_d, x_c]$, if $X<0$; x_c meaning always the closest to zero end-point of X and x_d the other end-point. We also define $w(X)=\overline{x}-\underline{x}$ and for $X\neq[0,0]$:

$$\chi(X)=\begin{cases} \underline{x}/\overline{x}, & \text{if } |\underline{x}|\leq|\overline{x}|,\\ \overline{x}/\underline{x}, & \text{otherwise.}\end{cases}$$

We shall make use of the following interval-arithmetic operations:

$$A+B=[\underline{a}+\underline{b}, \overline{a}+\overline{b}];$$

$$A-B=[(\underline{a}-\underline{b})\vee(\overline{a}-\overline{b})];$$

$$\alpha A=[(\alpha\underline{a})\vee(\alpha\overline{a})],\ \alpha\in\mathcal{R} \text{ (in particular } -A=[-\overline{a}, -\underline{a}]);$$

$$\left.\begin{aligned} AB&=[(a_c b_c)\vee(a_d b_d)]\\ A/B&=[(a_c/b_c)\vee(a_d/b_d)]\end{aligned}\right\}, \text{ when } A\not\ni 0, B\not\ni 0;$$

$$\left.\begin{array}{l} AB=b_d A \\ A/B=b_d^{-1}A \end{array}\right\}, \quad \text{when} \quad A\ni 0,\ B\not\ni 0;$$

$$AB=[\min\{\underline{a}\overline{b},\ \overline{a}\underline{b}\},\ \max\{\underline{a}\,\underline{b},\ \overline{a}\,\overline{b}\}], \quad \text{when} \quad A\ni 0,\ B\ni 0.$$

The following notations are introduced for brevity:

$$A\oplus B=A-(-B)=[(\underline{a}+\overline{b})\vee(\overline{a}+\underline{b})];$$

$$A\ominus B=A+(-B)=[\underline{a}-\overline{b},\ \overline{a}-\underline{b}];$$

$$\left.\begin{array}{l} A\otimes B=A/(1/B)=[(a_c b_d)\vee(a_d b_c)] \\ A\odot B=A\,(1/B)=[(a_c/b_d)\vee(a_d/b_c)] \end{array}\right\}, \quad \text{when} \quad A\not\ni 0,\ B\not\ni 0;$$

$$\left.\begin{array}{l} A\otimes B=A/(1/B)=b_c A \\ A\odot B=A\,(1/B)=b_c^{-1}A \end{array}\right\}, \quad \text{when} \quad A\ni 0,\ B\not\ni 0;$$

$$A\otimes B=[\max\{\underline{a}\overline{b},\ \overline{a}\underline{b}\},\ \min\{\underline{a}\,\underline{b},\ \overline{a}\,\overline{b}\}], \quad \text{when} \quad A\ni 0,\ B\ni 0.$$

The following two kinds of roundings are introduced in the set of intervals

$$\Diamond\, A=[\downarrow\underline{a},\ \uparrow\overline{a}];\quad \bigcirc\, A=[\uparrow\overline{a},\ \downarrow\underline{a}],\ \text{when}\ w(A)\neq 0,$$

which generate the computer interval-arithmetic operations:

$$A\langle * \rangle B=\Diamond\,(A*B);\quad A(*)B=\bigcirc\,(A*B),$$

where $*$ can be any of the interval-arithmetic operations defined above and A, B are machine intervals (that is intervals, whose end-points are machine numbers).

The rounding $\Diamond$ is examined in [1, 6, 7]. The rounding $\bigcirc$ may lead to a decrease of the width of the resulting intervals (contraction of intervals), which may be used for additional control of the machine calculations. This fact is discussed in some detail in [2].

If A, B are nondegenerate intervals with real end-points, then the following inclusions hold true (by the assumption that the intervals in the left-hand sides of the inclusions can be computed):

$$\begin{array}{ll} & \bigcirc\, A(+)\bigcirc\, B\subset A+B\subset\Diamond\, A\langle +\rangle\Diamond\, B; \\ & \bigcirc\, A(-)\,\Diamond\, B\subset A-B\subset\Diamond\, A\langle -\rangle\bigcirc B,\ \text{when}\ w(A)\geq w(B), \\ & \Diamond\, A(-)\bigcirc\, B\subset A-B\subset\bigcirc\, A\langle -\rangle\,\Diamond\, B,\ \text{when}\ w(A)<w(B); \\ (3) & \bigcirc\, A(\times)\bigcirc\, B\subset A\times B\subset\Diamond\, A\langle\times\rangle\,\Diamond\, B; \\ & \bigcirc\, A(:)\,\Diamond\, B\subset A/B\subset\Diamond\, A\langle :\rangle\bigcirc\, B,\ \text{when}\ \varkappa(A)\leq\varkappa(B), \\ & \Diamond\, A(:)\bigcirc\, B\subset A/B\subset\bigcirc\, A\langle :\rangle\Diamond\, B,\ \text{when}\ \varkappa(A)>\varkappa(B); \end{array}$$

and, correspondingly,

$$\bigcirc\, A(\oplus)\Diamond\, B\subset A\oplus B\subset\Diamond\, A\langle\oplus\rangle\bigcirc\, B,\ \text{when}\ w(A)\geq w(B),$$

$$\Diamond\, A(\oplus)\bigcirc\, B\subset A\oplus B\subset\bigcirc\, A\langle\oplus\rangle\,\Diamond\, B,\ \text{when}\ w(A)<w(B);$$

(3′) $\bigcirc A(\ominus)\bigcirc B\subset A\ominus B\subset\Diamond A\langle\ominus\rangle\Diamond B$;

$\bigcirc A(\otimes)\Diamond B\subset A\otimes B\subset\Diamond A\langle\otimes\rangle\bigcirc B$, when $\varkappa(A)\leqq\varkappa(B)$,

$\Diamond A(\otimes)\bigcirc B\subset A\otimes B\subset\bigcirc A\langle\otimes\rangle\Diamond B$ when $\varkappa(A)>\varkappa(B)$;

$\bigcirc A(\odot)\bigcirc B\subset A\odot B\subset\Diamond A\langle\odot\rangle\Diamond B$,

by the assumption that the intervals in the denominators (in the inclusions involving division) does not contain zero.

The above inclusions can be used for the machine realization of interval-arithmetric algorithms in the same manner as relations (1) were used for the machine realization of (2)–(2′). This is demonstrated in the computer realization of the algorithms considered in sections 3 and 4.

C. Convergence of interval sequences. Denote $|A|=\max\{|\underline{a}|, |\bar{a}|\}$ for any interval $A=[\underline{a}, \bar{a}]$.

Definition 1. *The interval sequence $A_1, A_2, \ldots, A_n, \ldots$ converges to the interval A, if* $\lim_{n\to\infty}|A_n-A|=0$.

Definition 2 [5]. *The interval sequence $\{A_n\}_{n=1}^{\infty}$ is inclusion monotone decreasing, if $A_1\supset A_2\supset A_3\supset\cdots$.*

Proposition [1]. *Every inclusion monotone decreasing interval sequence $\{A_n\}_{n=1}^{\infty}$ is convergent and* $\lim_{n\to\infty}A_n=\bigcap_{n=1}^{\infty}A_n$.

In what follows $\{A_n\}_{n=1}^{\infty}$ denotes an inclusion monotone decreasing interval sequence. The following definitions are taken from [5].

Definition 3. *$\{A_n\}_{n=1}^{\infty}$ is point-wise convergent, if* $\lim_{n\to\infty}w(A_n)=0$.

Definition 4. *$\{A_n\}_{n=1}^{\infty}$ is linearly convergent, if there exists $0<q<1$, such that $w(A_{n+1})\leqq qw(A_n)$ for $n=1, 2, \ldots$.*

Definition 5. *$\{A_n\}_{n=1}^{\infty}$ is superlinearly convergent, if $w(A_{n+1})\leqq q_n w(A)$,* where $\lim_{n\to\infty}q_n=0$.

Definition 6. *Let $\{A_n\}_{n=1}^{\infty}$ be point-wise convergent. $\{A_n\}_{n=1}^{\infty}$ is quadratically convergent, if there exists $c>0$, such that $w(A_{n+1})\leqq c(w(A_n))^2$ for $n=1, 2, \ldots$.*

Obviously, if the inclusion monotone decreasing sequence $\{A_n\}_{n=1}^{\infty}$ is point-wise convergent and $a\in A_n$ for all $n=1, 2, \ldots$, then $\{A_n\}_{n=1}^{\infty}$ is convergent to a.

3. An interval method of Newton type with linear convergence. Let f be a continuous function on $[a, b]\subset\mathcal{R}$. Assume that there exists a unique solution $x^*\in[a, b]$ of the equation $f(x)=0$. Assume also that the interval $X_0=[\underline{x}_0, \bar{x}_0]\subset[a, b]$ is such that $x^*\in X_0$ and one of the conditions

(4) $$0<\underline{m}\leqq(f(x)-f(y))/(x-y)\leqq\bar{m}<\infty \text{ for all } x, y\in X_0,\ x\neq y,$$

or

(4′) $$-\infty<\underline{m}\leqq(f(x)-f(y))/(x-y)\leqq\bar{m}<0 \text{ for all } x, y\in X_0,\ x\neq y,$$

holds true.

Denote by $f(X)$ the interval extension of f on $X\subset[a, b]$, that is $f(X)=\{f(x) : x\in X\}$ and consider the interval iteration process

(5) $$X_{k+1}=X_k-f(X_k)/M, \quad k=0, 1, 2, \ldots,$$

where $M=[m, \overline{m}]$.

Theorem 1. *By the above assumptions on f and X_0 the interval sequence defined by* (5) *converges linearly to x^* and $x^* \in X_k$ for $k=0, 1, 2, \ldots$.*

Proof. In order to prove the theorem we shall consecutively prove the following assertions:

i) $\{X_k\}_{k=0}^{\infty}$ is inclusion monotone decreasing, that is $X_0 \supset X_1 \supset \cdots$;

ii) $x^* \in X_k$ for $k=0, 1, 2, \ldots$;

iii) there exists $0<q<1$ such that $w(X_{k+1}) \leq q w(X_k)$, $k=0, 1, 2, \ldots$.

Let (4) hold true; the case (4′) is treated analogously.

We shall first prove simultaneously i) and ii) by induction.

The assumption (4) implies that f is monotone increasing in $X_0=[\underline{x}_0, \overline{x}_0]$, so that $f(X_0)=[f(\underline{x}_0), f(\overline{x}_0)]$, $f(\underline{x}_0)<0<f(\overline{x}_0)$. From the definitions of the interval-arithmetic operations we have $f(X_0)/M=[f(\underline{x}_0), f(\overline{x}_0)]/M=[f(\underline{x}_0)/\overline{m}, f(\overline{x}_0)/\overline{m}]$, and hence $w(f(X_0)/M)=(f(\overline{x}_0)-f(\underline{x}_0))/\overline{m}$,

According to (4) $(f(\overline{x}_0)-f(\underline{x}_0))/(\overline{x}_0-\underline{x}_0) \leq \overline{m}$, $\overline{m}>0$, and therefore $(f(\overline{x}_0)-f(\underline{x}_0))/\overline{m} \leq \overline{x}_0-\underline{x}_0=w(X_0)$, so that $w(f(X_0)/M) \leq w(X_0)$. This implies that (5) for $k=0$ can be written by means of end-point as follows: $\underline{x}_1=\underline{x}_0-f(\underline{x}_0)/\overline{m}$, $\overline{x}_1=\overline{x}_0-f(\overline{x}_0)/\overline{m}$.

The inequalities $\underline{x}_1-\underline{x}_0=-f(\underline{x}_0)/\overline{m}>0$, $\overline{x}_1-\overline{x}_0=-f(\overline{x}_0)/\overline{m}<0$ imply that $\underline{x}_0<\underline{x}_1$, $\overline{x}_1<\overline{x}_0$ and hence $X_0 \supset X_1$. We recall that $x^* \in X_0$. Assume now that $X_0 \supset X_1 \supset \cdots \supset X_{k-1} \supset X_k$ and $x^* \in X_k$ for some $k>0$. We shall prove that $X_k \supset X_{k+1}$ and $x^* \in X_{k+1}$.

From $X_0 \supset X_k$, $x^* \in X_k$ and (4) it follows that $f(\underline{x}_k)<0<f(\overline{x}_k)$ and therefore

$$f(X_k)/M=[f(\underline{x}_k), f(\overline{x}_k)]/M=[f(\underline{x}_k)/\overline{m}, f(\overline{x}_k)/\overline{m}],$$

$$w(f(X_k)/M)=(f(\overline{x}_k)-f(\underline{x}_k))/m.$$

But $\underline{x}_k, \overline{x}_k \in X_0$ so that according to (4) $(f(\overline{x}_k)-f(\underline{x}_k))/(\overline{x}_k-\underline{x}_k) \leq \overline{m}$, $\overline{m}>0$, which implies $(f(\overline{x}_k)-f(\underline{x}_k))/\overline{m} \leq \overline{x}_k-\underline{x}_k=w(X_k)$, and hence $w(f(X_k)/M) \leq w(X_k)$.

Thus (5) can be written by means of end-points as follows:

(5E) $$\underline{x}_{k+1}=\underline{x}_k-f(\underline{x}_k)/\overline{m}, \qquad \overline{x}_{k+1}=\overline{x}_k-f(\overline{x}_k)/\overline{m}.$$

The inequalities $\underline{x}_{k+1}-\underline{x}_k=-f(\underline{x}_k)/\overline{m}>0$, $\overline{x}_{k+1}-\overline{x}_k=-f(\overline{x}_k)/\overline{m}<0$ imply that $X_k \supset X_{k+1}$. Further we have

$$\underline{x}_{k+1}-x^*=\underline{x}_k-x^*-f(\underline{x}_k)/\overline{m}=\underline{x}_k-x^*-(f(\underline{x}_k)-f(x^*))/\overline{m}.$$

From (4) for $\underline{x}_k, x^* \in X_0$ we have $(f(\underline{x}_k)-f(x^*))/(\underline{x}_k-x^*) \leq \overline{m}$ and using the inductive assumption $\underline{x}_k<x^*$ we obtain $(f(\underline{x}_k)-f(x^*))/\overline{m} \geq \underline{x}_k-x^*$. Thus

$$\underline{x}_{k+1}-x^*=\underline{x}_k-x^*-(f(\underline{x}_k)-f(x^*))/\overline{m}\leqq 0. \tag{6}$$

Similarly, $(f(\overline{x}_k)-f(\underline{x}^*))/(\overline{x}_k-x^*)\leqq\overline{m}$ and the inductive assumption $\overline{x}_k>x^*$ imply $(f(\overline{x}_k)-f(x^*))/\overline{m}\leqq\overline{x}_k-x^*$, so that

$$\overline{x}_{k+1}-x^*=\overline{x}_k-x^*-f(\overline{x}_k)/\overline{m}=\overline{x}_k-x^*-(f(\overline{x}_k)-f(x^*))/\overline{m}\geqq 0.$$

This, together with (6), implies $x^*\in X_{k+1}$.

We proved that i) and ii) hold true. We shall now prove iii). We have

$$w(X_{k+1})=w(X_k)-w(f(X_k)/M)=w(X_k)-(f(\overline{x}_k)-f(\underline{x}_k))/\overline{m}$$

$$=w(X_k)(1-((f(\overline{x}_k)-f(\underline{x}_k))/(\overline{x}_k-\underline{x}_k))/\overline{m})\leqq(1-\underline{m}/\overline{m})w(X_k),$$

in view of $(f(\overline{x}_k)-f(\underline{x}_k))/(\overline{x}_k-\underline{x}_k)\geqq\underline{m}$. For $q=1-\underline{m}/\overline{m}$ we obviously have $0<q<1$, which completes the proof of iii).

The situation when (4′) holds true is considered similarly. We note that in this case the end-points-form of the interval method (5) is

$$\begin{aligned}\underline{x}_{k+1}&=\underline{x}_k-f(\underline{x}_k)/\underline{m},\\ \overline{x}_{k+1}&=\overline{x}_k-f(\overline{x}_k)/\underline{m}.\end{aligned} \tag{5E′}$$

This completes the proof of Theorem 1.

Computer realization of the method. Using inclusions (3) we obtain the following simple formula for the computer realization of (5):

$$X_{k+1}=X_k\langle -\rangle(\bigcirc f(X_k)(:)\overline{\Diamond}M). \tag{5C}$$

We shall also give the machine realization of the end-point presentations of (5). The machine realization of (5E) is

$$\underline{x}_{k+1}=\underline{x}_k\overset{\smile}{-}\uparrow(f(\underline{x}_k)/\overline{m})=\underline{x}_k\overset{\smile}{-}(\uparrow f(\underline{x}_k)\overset{\frown}{:}\downarrow\overline{m}),$$

$$\overline{x}_{k+1}=\overline{x}_k\overset{\frown}{-}\downarrow(f(\overline{x}_k)/\overline{m})=\overline{x}_k\overset{\frown}{-}(\downarrow f(\overline{x}_k)\overset{\smile}{:}\uparrow\overline{m}),$$

and the machine realization of (5E′) is

$$\underline{x}_{k+1}=\underline{x}_k\overset{\smile}{-}\uparrow(f(\underline{x}_k)/\underline{m})=\underline{x}_k\overset{\smile}{-}(\downarrow f(\underline{x}_k)\overset{\frown}{:}\uparrow\underline{m}),$$

$$\overline{x}_{k+1}=\overline{x}_k\overset{\frown}{-}\downarrow(f(\overline{x}_k)/\underline{m})=\overline{x}_k\overset{\frown}{-}(\uparrow f(\overline{x}_k)\overset{\smile}{:}\downarrow\underline{m}).$$

Using the notation

$$m_c=\begin{cases}\overline{m}, & \text{if } |\underline{m}|\leqq|\overline{m}|,\\ \underline{m}, & \text{if } |\underline{m}|>|\overline{m}|,\end{cases}$$

we can unify the both cases, corresponding to assumptions (4) and (4′) resp., in the following common formula

$$\begin{aligned}\underline{x}_{k+1}&=\underline{x}_k\overset{\smile}{-}(\square f\underline{x}_k)[:]\square_0 m_c),\\ \overline{x}_{k+1}&=\overline{x}_k\overset{\frown}{-}(\square f(\overline{x}_k)[:]\square_0 m_c).\end{aligned} \tag{5EC}$$

We note the interesting fact that the interval $f(X_k)$ should be computed with contraction, as it is clearly seen from formula (5C). This provides us with additional control over the computations, since in reality $\bigcirc f(X_n)$ cannot be computed, when $w(X_n)$ is very small and the precision of the calculations is not sufficiently high (moreover it is necessary that $\bigcirc f(X_n)\ni 0$).

4. Interval method of Newton type with quadratic convergence. Let $f\in C_2[a, b]$ and x^* be the unique solution of $f(x)=0$ in $[a, b]$. Assume that the derivatives f' and f'' have constant signs in $[a, b]$. Consider the interval iteration process

$$X_{k+1}=X_k-f(X_k)/f'(X_k), \tag{7}$$

where $x^*\in X_0\subset[a, b]$ and $f(X_k)$, $f'(X_k)$ are the interval extensions of f and f', resp., on X_k, that is $f(X_k)=\{f(x) : x\in X_k\}$, $f'(X_k)=\{f'(x) : x\in X_k\}$.

Theorem 2. [8] *By the above asumptions on f and X_0 the interval iteration process (7) delivers a sequence $\{X_k\}_{k=1}^{\infty}$ with $x^*\in X_k$ which is quadratic convergent to x^*.*

Proof. The proof of the theorem is sketched in [8]. Here we give a more detailed proof of the theorem.

We shall consider in detail the situation when $f'>0, f''>0$ on $[a, b]$ (the rest three situations are treated analogously).

In view of the monotonocity of f and f' we have $f(X_0)=[f(\underline{x}_0), f(\overline{x}_0)]\ni 0$ and $f'(X_0)=[f'(\underline{x}_0), f'(\overline{x}_0)]$.

We shall prove by induction that $\{X_k\}_{k=0}^{\infty}$ is inclusion monotone decreasing and $x^*\in X_k$ for $k=1, 2, \ldots$.

We have

$$w\left(\frac{f(X_0)}{f'(X_0)}\right)=w\left(\frac{[f(\underline{x}_0), f(\overline{x}_0)]}{f'(\overline{x}_0)}\right)=\frac{f(\overline{x}_0)-f(\underline{x}_0)}{f'(\overline{x}_0)}=(f'(\xi)/f'(\overline{x}_0))w(X_0), \text{ where } \underline{x}_0<\xi<\overline{x}_0.$$

f' is monotone increasing on X_0, so that $f'(\xi)<f'(\overline{x}_0)$, which implies that $w(f(X_0)/f'(X_0))<w(X_0)$. Using this, we obtain from (7) for $k=0$ $\underline{x}_1=\underline{x}_0-f(\underline{x}_0)/f'(\overline{x}_0)$, $\overline{x}_1=\overline{x}_0-f(\overline{x}_0)/f'(\overline{x}_0)$.

The inequalities $\underline{x}_1-\underline{x}_0=-f(\underline{x}_0)/f'(\overline{x}_0)>0$, $\overline{x}_1-\overline{x}_0=-f(\overline{x}_0)/f'(\overline{x}_0)<0$ imply that $X_0\supset X_1$. Recall that $x^*\in X_0$.

Assume now that $X_0\supset X_1\supset\cdots\supset X_{k-1}\supset X_k$ and $x^*\in X_k$ for some $k>0$. We shall prove that $X_k\supset X_{k+1}$ and $x^*\in X_{k+1}$.

Since $X_0\supset X_k$, we have $f(X_k)=[f(\underline{x}_k), f(\overline{x}_k)]\ni 0$ and $f'(X_k)=[f'(\underline{x}_k), f'(\overline{x}_k)]>0$. Using this, we obtain

$$w\left(\frac{f(X_k)}{f'(X_k)}\right)=\frac{f(\overline{x}_k)-f(\underline{x}_k)}{f'(\overline{x}_k)}=\frac{f'(\xi)}{f'(\overline{x}_k)}w(X_k), \quad \underline{x}_k<\xi<\overline{x}_k,$$

and, taking into account $f'(\xi)/f'(\overline{x}_k)<1$, we see that $w(f(X_k)/f'(X_k))<w(X_k)$. We thus obtain $w(X_{k+1})=w(X_k)-w(f(X_k)/f'(X_k))$.

Formula (7) can be written end-points-wise:

$$\underline{x}_{k+1}=\underline{x}_k-f(\underline{x}_k)/f'(\overline{x}_k), \quad \overline{x}_{k+1}=\overline{x}_k-f(\overline{x}_k)/f'(\overline{x}_k).$$

From the inequalities

$$\underline{x}_{k+1}-\underline{x}_k=-f(\underline{x}_k)/f'(\overline{x}_k)>0,\quad \overline{x}_{k+1}=\overline{x}_k-f(\overline{x}_k)/f'(\overline{x}_k)<0$$

we obtain that $X_k\supset X_{k+1}$. Further we have

$$\underline{x}_{k+1}-x^*=\underline{x}_k-x^*-f(\underline{x}_k)/f'(\overline{x}_k)$$

$$=\underline{x}_k-x^*-(f(\underline{x}_k)-f(x^*))/f'(\overline{x}_k)=(\underline{x}_k-x^*)(1-f'(\zeta)/f'(\overline{x}_k)),$$

wherein $\underline{x}_k<\zeta<x^*$. We have $\underline{x}_k-x^*<0$ by the inductive assumption and $f'(\zeta)/f'(\overline{x}_k)<1$ since $\zeta<\overline{x}_k$ and f' is monotone increasing. Thus we have $\underline{x}_{k+1}<x^*$. Similarly,

$$\overline{x}_{k+1}-x^*=\overline{x}_k-x^*-f(\overline{x}_k)/f'(\overline{x}_k)=(\overline{x}_k-x^*)(1-f'(\eta)/f'(\overline{x}))>0,\quad x^*<\eta<\overline{x}_k.$$

From $\underline{x}_{k+1}<x^*$ and $\overline{x}_{k+1}>x^*$ we conclude $x^*\in X_{k+1}$.
We proved by now that $\{X_k\}_{k=0}^{\infty}$ is inclusion monotone decreasing, that is $X_0\supset X_1\supset X_2\supset\cdots$, and $x^*\in X_k$ for $k=0, 1, 2, \ldots$. We shall now prove that $\{X_k\}$ is point-convergent. We have

$$w(X_{k+1})=w(X_k)-w(f(X_k)/f'(X_k))=w(X_k)-(f(\overline{x}_k)-f(\underline{x}_k))/f'(\overline{x}_k)$$

$$=w(X_k)-(f'(\xi)/f'(\overline{x}_k))w(X_k)=(1-f'(\xi)/f'(\overline{x}_k))w(X_k)\leq(1-f'(\underline{x}_0)/f'(\overline{x}_0))w(X_k),$$

where $\underline{x}_0<\underline{x}_k<\xi<\overline{x}_k<\overline{x}_0$. Since $q=1-f'(\underline{x}_0)/f'(\overline{x}_0)$ is such that $0<q<1$, we conclude from $w(X_{k+1})\leq qw(X_k)$ that $\{X_k\}$ is point-wise convergent.

It remains to be shown that $\{X_k\}$ is quadratic convergent. We have

$$w(X_{k+1})=w(X_k)+(f(\underline{x}_k)-f(\overline{x}_k))/f'(\overline{x}_k)$$

$$=w(X_k)+\frac{1}{f'(\overline{x}_k)}(f'(\overline{x}_k)(\underline{x}_k-\overline{x}_k)+\frac{1}{2}f''(\xi)(\underline{x}_k-\overline{x}_k)^2)$$

$$=w(X_k)-w(X_k)+\frac{1}{2}(f''(\xi)/f'(\overline{x}_k))w^2(X_k)=(1/2)(f''(\xi)/f'(\overline{x}_k))w^2(X_k),$$

where $\underline{x}_k<\xi<\overline{x}_k$. The continuity of f'' on $[a, b]$ implies that f'' is bounded, $f''(\xi)\leq c_1$, $c_1>0$. Since $f'(\overline{x}_k)\geq f'(\underline{x}_0)$, we have

$$w(X_{k+1})\leq(c_1/2f'(\underline{x}_0))w^2(X_k)=cw^2(X_k),\quad c>0,$$

showing that $\{X_k\}$ is quadratically convergent to x^*. This proves the theorem.

From the above proof it becomes clear that the end-points form of (7) is

(7E)
$$\underline{x}_{k+1}=\underline{x}_k-f(\underline{x}_k)/f'(x_k),$$
$$\overline{x}_{k+1}=\overline{x}_k-f(\overline{x}_k)/f'(x_k),$$

where $x_k=\underline{x}_k$, if $f'f''<0$ and $x_k=\overline{x}_k$, if $f'f''>0$.

Computer realization of (7). Using (3), we obtain

(7C)
$$X_{k+1}=X_k\langle-\rangle(\bigcirc f(X_k)\,(:)\,\diamond f'(X_k)).$$

This can be also written as $X_{k+1}=X_k\langle-\rangle(\bigcirc f(X_k)\,(:)\,\Box_0 f'(x_k))$, where $x_k=\underline{x}_k$, if $f'f''<0$ and $x_k=\overline{x}_k$, if $f'f''>0$.

The computer realization of the end-poits form (7E) is

(7EC)
$$\underline{x}_{k+1}=\underline{x}_k\,\check{-}\,(\Box f(\underline{x}_k)\,[:]\,\Box_0 f'(x_k)),$$
$$\overline{x}_{k+1}=\overline{x}_k\,\hat{-}\,(\Box f(\overline{x}_k)\,[:]\,\Box_0 f'(x_k)),$$

where again $x_k=\underline{x}_k$, if $f'f''<0$ and $x_k=\overline{x}_k$, if $f'f''>0$.

We note that the interval $f(X_k)$ should be computed in (7C) by contraction, which supplies us with a tool for additional control over the round-off error in the computations.

In what follows we show that (7) is quadratically convergent by rather relaxed assumptions on f and X_0.

Theorem 3. *Let f possess a derivative f' on $[a, b]$ and the derivative f' be Lipschitzian on $[a, b]$ and f' have constant sign on $[a, b]$. Let x^* be a unique solution of $f(x)=0$ in $[a, b]$ and the interval $X_0\subset[a, b]$ contains x^*. Then the interval iteration process* (7): $X_{k+1}=X_k-f(X_k)/f'(X_k)$, $k=0, 1, \ldots$, *produces a sequence* $\{X_k\}$, $x^*\in X_k$ *for* $k=0, 1, \ldots$, *quadratic convergent to* x^*.

Proof We shall consider in detail the situation $f'>0$ on $[a, b]$; the case $f'<0$ is treated analogously. We shall prove by induction that $X_0\supset X_1\supset\cdots$ and $x^*\in X_k$. We have $x^*\in X_0$. Since $f'>0$, the function f is monotone increasing on X_0, so that $f(X_0)=[f(\underline{x}_0), f(\overline{x}_0)]\ni 0$. Denote $f'(X_0)=[\underline{m}_0, \overline{m}_0]>0$. We have $w(f(X_0)/f'(X_0))=(f(\overline{x}_0)-f(\underline{x}_0))/\overline{m}_0=(f'(\xi)/\overline{m}_0)w(X_0)$, $\underline{x}_0<\xi<\overline{x}_0$. But $f'(\xi)\in f'(X_0)=[\underline{m}_0, \overline{m}_0]$ implies $f'(\xi)\leq\overline{m}_0$ and therefore $w(f(X_0)/f'(X_0))\leq w(X_0)$. Thus the end-points form of (7) becomes for $k=0$:

$$\underline{x}_1=\underline{x}_0-f(\underline{x}_0)/\overline{m}_0,\quad \overline{x}_1=\overline{x}_0-f(\overline{x}_0)/\overline{m}_0.$$

The inequalities $\underline{x}_1-\underline{x}_0=-f(\underline{x}_0)/\overline{m}_0>0$, $\overline{x}_1-\overline{x}_0=-f(\overline{x}_0)/\overline{m}_0<0$ imply $X_0\supset X_1$. Assume now that $X_0\supset X_1\supset\cdots\supset X_k$ and $x^*\in X_k$ for some $k>0$. We shall prove that $X_k\supset X_{k+1}$ and $x^*\in X_{k+1}$. Denote $f'(X_k)=[\underline{m}_k, \overline{m}_k]>0$. Since $X_0\supset X_k$, we have $f(X_k)=[f(\underline{x}_k), f(\overline{x}_k)]$ where, in view of $x^*\in X_k$, $f(\underline{x}_k)<0<f(\overline{x}_k)$. We have

$$w\left(\frac{f(X_k)}{f'(X_k)}\right)=\frac{f(\overline{x}_k)-f(\underline{x}_k)}{\overline{m}_k}=\frac{f'(\xi_1)}{\overline{m}_k}w(X_k),$$

where $\underline{x}_k<\xi_1<\overline{x}_k$. Since $f'(\xi_1)\in f'(X_k)$, that is $f'(\xi_1)\leq\overline{m}_k$, we have $f'(\xi_1)/\overline{m}_k\leq 1$ and therefore $w(f(X_k)/f'(X_k))\leq w(X_k)$. Thus (7) can be written as

$$\underline{x}_{k+1}=\underline{x}_k-f(\underline{x}_k)/\overline{m}_k,\quad \overline{x}_{k+1}=\overline{x}_k-f(\overline{x}_k)/\overline{m}_k.$$

The inequalities $\underline{x}_{k+1}-\underline{x}_k=-f(\underline{x}_k)/\overline{m}_k>0$, $\overline{x}_{k+1}-\overline{x}_k=-f(\overline{x}_k)/\overline{m}_k<0$ show that $X_k\supset X_{k+1}$. Further we have

$$\underline{x}_{k+1}-x^*=\underline{x}_k-x^*-(f(\underline{x}_k)-f(x^*))/\overline{m}_k$$
$$=\underline{x}_k-x^*-(f'(\eta)/\overline{m}_k)(\underline{x}_k-x^*)=(\underline{x}_k-x^*)(1-f'(\eta)/\overline{m}_k),$$

where $\underline{x}_k<\eta<x^*$. From $f'(\eta)\in f'(X_k)$ we have $f'(\eta)\leqq\overline{m}_k$ and using that $\underline{x}_k\leqq x^*$ we obtain $\underline{x}_{k+1}-x^*\leqq 0$. Similarly,

$$\overline{x}_{k+1}-x^*=\overline{x}_k-x^*-(f(\overline{x}_k)-f(x^*))/\overline{m}_k=(\overline{x}_k-x^*)(1-f'(\eta_1)/\overline{m}_k)\geqq 0,\quad \underline{x}_k<\eta_1<x^*.$$

We thus obtained $x^*\in X_{k+1}$. Thereby we showed that the interval sequence $\{X_k\}$ is inclusion monotone decreasing and $x^*\in X_k$ for all $k=0, 1, 2, \ldots$ We shall prove next that $\{X_k\}$ is point-convergent. We have

$$w(X_{k+1})=w(X_k)-w(f(X_k)/f'(X_k))$$
$$=w(X_k)-(f(\overline{x}_k)-f(\underline{x}_k))/\overline{m}_k=w(X_k)(1-f'(\xi)/\overline{m}_k),\quad \underline{x}_k<\xi<\overline{x}_k.$$

From $f'(\xi)\in f'(X_k)\subset f'(X_0)$ we have $f'(\xi)\geqq\underline{m}_0$. On the other hand, $\overline{m}_k\leqq\overline{m}_0$ and therefore $w(X_{k+1})\leqq w(X_k)(1-\underline{m}_0/\overline{m}_0)$.

This inequality, in view of $0<1-\underline{m}_0/\overline{m}_0<1$, implies that $w(X_k)\to 0$, that is $\{X_k\}$ is point-convergent, and because of $x^*\in\bigcap_{k=0}^{\infty}X_k$ we have $\lim_{k\to\infty}X_k=x^*$.

It remains to be shown that $\{X_k\}$ is quadratic convergent to x^*. We have for $\underline{x}_k<\xi<\overline{x}_k$

$$w(X_{k+1})=w(X_k)(1-f'(\xi)/\overline{m}_k)$$
$$\leqq w(X_k)(1-\underline{m}_k/\overline{m}_k)=w(X_k)(\overline{m}_k-\underline{m}_k)/\overline{m}_k=w(X_k)w(f'(X_k))/\overline{m}_k.$$

Using that f' is Lipschitzian on $[a, b]$ and therefore $w(f'(X_k))\leqq\gamma w(X_k)$ where γ does not depend on k, we obtain $w(X_{k+1})\leqq(\gamma/\overline{m}_k)w^2(X_k)\leqq(\gamma/\underline{m}_0)w^2(X_k)=cw^2(X_k)$, $c=\gamma/\underline{m}_0>0$. This proves the theorem.

Ramark 1. Since f' is not assumed monotone, the end-points from (7E) is not valid by the assumption of Theorem 3. Rather we have

$$\underline{x}_{k+1}=\underline{x}_k-f(\underline{x}_k)/m_k,\quad \overline{x}_{k+1}=\overline{x}_k-f(\overline{x}_k)/m_k,$$

where $m_k=\begin{cases}\overline{m}_k, & \text{if } |\underline{m}_k|\leqq|\overline{m}_k|,\\ \underline{m}_k, & \text{if } |\underline{m}_k|>|\overline{m}_k|,\end{cases}$ and $\underline{m}_k, \overline{m}_k$ are defined by $f'(X_k)=[\underline{m}_k, \overline{m}_k]$.

Correspondingly, the computer realization of these formulas is

$$\underline{x}_{k+1}=\underline{x}_k\,\check{-}\,(\Box f(\underline{x}_k)\,[:]\,\Box_0 m_k),\quad \overline{x}_{k+1}=\overline{x}_k\,\hat{-}\,(\Box f(\overline{x}_k)\,[:]\,\Box_0 m_k).$$

Remark 2. It is noted in [10] that the interval Newton method suggested by Moore [9] is always convergent by the assumptions $f\in C_1[a, b], f'\neq 0$ on $[a, b]$. As Theorem 3 shows the same assertion holds true for the method (7) as well.

Remark 3. The interval method (7) was reported on the Conference on Computer-oriented Numerical Analysis in Berlin 1979 and was published in [8].

In this paper we answer the question about the computer realization of (7), posed by Professor U. Kulisch on that Conference.

Numerical experiments. The methods (5) and (7) were tested numerically on an IBM/360 computer. An interval-arithmetic FORTRAN precompiler [3] and a package for directed roundings [4] were used for this purpose.

Here we present the numerical results of the computation of the solution of the equation $x(x^9-1)-1=0$ with $X^{(0)}=[1, 1.5]$ by means of (7C). This example is taken from the book of G. Alefeld and J. Herzberger [1, p. 99].

$$X^{(0)}=[1.000000000000000,\ 1.500000000000000]$$
$$X^{(1)}=[1.002608013529070,\ 1.356128831793315]$$
$$X^{(2)}=[1.008941568406117,\ 1.234922296044664]$$
$$X^{(3)}=[1.022860766838954,\ 1.143520152776015]$$
$$X^{(4)}=[1.046577598453889,\ 1.091730230277842]$$
$$X^{(5)}=[1.068925528839930,\ 1.076824667785088]$$
$$X^{(6)}=[1.075501427047270,\ 1.075770989848197]$$
$$X^{(7)}=[1.075765745837712,\ 1.075766066193790]$$
$$X^{(8)}=[1.075766066086384,\ 1.075766066086338]$$
$$X^{(9)}=[1.075766066086837,\ 1.075766066086838]$$

The computations are performed in double precision interval arithmetic with corresponding directed roundings.

REFERENCES

1. G. Alefeld, J. Herzberger. Einführung in die Intervallrechnung. Mannheim, 1974.
2. B. A. Chartres. Automatic controlled precision calculations. *J. ACM*, **13**, 1966, 386—403.
3. N. Duschkov. V FORTRAN 1/0: An interval-arithmetic FORTRAN precompiler for IBM/360. User information. Laboratory of Computer Science, University of Sofia, 1981.
4. N. Duschkov. A package for directed roundings in floating-point arithmetic for IBM/360, User information. Laboratory of Computer Science, University of Sofia, 1981.
5. R. Krawczyk. Newton-Algorithmen zur Bestimmung von Nullstellen mit Fehlerschranken. *Computing*, **4**, 1969, 187—201.
6. U. Kulisch. Grundlagen des Numerischen Rechnens. Mannheim, 1976.
7. U. Kulisch, W. Miranker. Computer Arithmetic in Theory and Practice. Academic Press, 1980.
8. S. Markov. Some applications of extended interval arithmetic to interval iterations. *Computing Suppl.*, **2**, 1980 69—84.
9. R. E. Moore. Interval analysis. Englewood Cliffs. N. J., 1966.
10. K. Nickel. On the Newton method in interval analysis. Univ. of Wisconsin. Math. Research Center Report No. 1136, 1971.

Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

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