PLISKA
STUDIA MATHEMATICA
BULGARICA

ПЛИСКА
БЪЛГАРСКИ
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СТУДИИ

ON A NEW CHARACTERISTIC OF FUNCTIONS. II. DIRECT AND CONVERSE THEOREMS FOR THE BEST ALGEBRAIC APPROXIMATION IN $C[-1, 1]$ AND $L_p[-1, 1]$

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1. Introduction. Let D be the set H_n (T_n) of all algebraic (trigonometrical) polynomials of a degree at most n. The best L_p approximation with the weight w of the function $f \in L_p[a, b]$ by means of elements of D is

$$E(D, w; f)_{L_p[a,b]} = \inf\{\| w(f-Q)\|_p : Q \in D\}.$$

Here $L_\infty[a, b]$ denotes the set of all measurable and bounded in $[a, b]$ functions equipped with the uniform norm. We shall denote this norm with $\|\cdot\|_\infty$. Therefore $C[a, b] \subset L_\infty[a, b]$.

Let $\omega_k(f; t)_p = \sup\{\|\Delta_h^k f(\cdot)\|_{p[a, b_1]} : 0 \leqq h \leqq t\}$ be the k-th modulus of L_p continuity of f, where $b_1 = b$, if f is $(b-a)$ periodic, and $b_1 = b - kh$, if f is defined in $[a, b]$.

As it is well known (see e. g. Timan [12]) there is the following connection between the best trigonometrical approximation of a function and its moduli of continuity:

$$(1.1)\qquad E(T_n, 1; f)_{L_p[0, 2\pi]} = O(n^{-\alpha}) \Leftrightarrow \omega_k(f; t)_p = O(t^\alpha)\ (t \to 0)\ \text{for}\ 0 < \alpha < k.$$

We can interpret the equivalence (1.1) in two ways:

A) A characterization of the best approximations in terms of the moduli of continuity.

B) A characterization of the moduli of continuity (or of the classical Lipschitz spaces) in terms of the best approximations.

These two interpretations are not one and the same as we shall see in the best algebraic approximation case.

We shall consider only the approximations on the interval $[-1,1]$. The algebraic approximations on the other interval are evidently connected with those on $[-1, 1]$. We set $\Delta(d, x) = d\sqrt{1-x^2} + d^2$, $\Delta_n(x) = \Delta(n^{-1}, x)$.

The following equivalence is established as a result of the researches of Timan [13], Dzjadik [3], Freud [18] and Brudnij [1]:

$$(1.2)\quad E(H_n, (\Delta_n)^{-\alpha}; f)_{L_\infty[-1,1]} = O(1) \Leftrightarrow \omega_k(f; t)_\infty = O(t^\alpha)\ (t \to 0\ \text{for}\ 0 < \alpha < k).$$

This is an algebraic analog of B) for $p = \infty$. Motornij [8] and De Vore [17] show that the direct replacement of ∞ with p ($1 \leqq p < \infty$) in (1.2) is impossible.

PLISKA Studia mathematica bulgarica. Vol. 5, 1983, p. 151—163.

There are many papers on the problem of characterization of the best algebraic approximations. We shall mention only few of them. Potapov [10], Dzafarov [2], Butzer, Stens and Wehrens [16, 23] defined a variety of moduli for characterizing the best algebraic approximation. Their researches are based on modified translation concept. In [14] Fuksman gives a structural characteristic of the best uniform algebraic approximations in terms of the local modulus of continuity.

In this paper we show that new moduli defined as norms of local moduli can be successfully used for characterization of the best algebraic and trigonometrical approximation of functions in C and L_p. These moduli are introduced for the first time in [19]. Their properties are proved in [20]. The results of this paper were announced at the Conference on *Functions, Series, Operators* in Budapest, 1980.

2. Definitions and auxiliary results. Everywhere c is an absolute constant, $c(A, B, \dots)$ is a constant depending only on the marked parameters. These constants may differ at each occurrence.

We shall use the following moduli of f (see [19; 20]):

$$\tau_k(f, w; \delta)_{p', p[a,b]} = \| w(\cdot)\omega_k(f, \cdot; \delta(\cdot))_{p'} \|_{p[a,b]}, \tag{2.1}$$

where

$$\omega_k(f, x, \delta(x))_{p'} = \{ \frac{1}{2\delta(x)} \int_{-\delta(x)}^{\delta(x)} |\Delta_v^k f(x)|^{p'} dv \}^{1/p'} \text{ for } 1 \leq p' < \infty \text{ and}$$

$$\omega_k(f, x, \delta(x))_\infty = \sup\{ |\Delta_v^k f(x)| : |v| \leq \delta(x) \}, \tag{2.2}$$

where the finite difference $\Delta_v^k f(x) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} f(x+iv)$ is defined as 0, if x or $x+kv$ are not in $[a, b]$. In (2.1) δ is an arbitrary positive function of x and the weight w is a non-negative function belonging to $L_\infty[a, b]$, $1 \leq p \leq \infty$, $1 \leq p' \leq \infty$.

As far as we know the local $L_{p'}$ modulus of continuity (2.2) is used for $p' < \infty$ for the first time in this problem. The local modulus concept in the definition of global moduli is used for the following structural characteristic

$$\tau_k(f; \delta)_p = \| \omega_k(f, \cdot; \delta) \|_p \quad (\delta = \text{const}),$$

where $\omega_k(f, x; \delta) = \sup\{ |\Delta_h^k f(t)| : t, t+kh \in [x-k\delta/2, x+k\delta/2] \cap [a, b] \}$. This modulus is used for the first time by Sendov [11] and Korovkin [7] and it already has some important applications (see [4, 15, 21, 22]).

In both following theorems we give some properties of $\tau_k(f, w; \delta)_{p', p}$ and its connections with $\omega_k(f; \delta)_p$ and $\tau_k(f; \delta)_p$.

Theorem 1 [20]. *If* $1 \leq p, p_1, p_2, p', p_1', p_2', p'' \leq \infty$, $f, g \in L_{\max\{p, p'\}}$ $\delta > 0$, $d = \text{const}$, $w \geq 0$, $w \in C[-1, 1]$, *then*

1) $\tau_k(f+g, w; \delta)_{p', p} \leq \tau_k(f, w; \delta)_{p', p} + \tau_k(g, w; \delta)_{p', p}$;

2) $\tau_k(\alpha f, w; \delta)_{p', p} = |\alpha| \tau_k(f, w; \delta)_{p', p}$ *for* $\alpha \in \mathbf{R}$;

3) $\tau_k(f, w_1; \delta)_{p', p} \leq \tau_k(f, w_2; \delta)_{p', p}$ *for* $0 \leq w_1 \leq w_2$;

4) $\tau_k(f, w; \delta)_{p', p_1} \leq (b-a)^{1/p_1 - 1/p_2} \tau_k(f, w; \delta)_{p', p_2}$ *for* $p_1 \leq p_2$;

5) $\tau_k(f, w; \delta)_{p_1', p} \leq \tau_k(f, w; \delta)_{p_2', p}$ *for* $p_1' \leq p_2'$;

6) $\tau_1(f, 1; nd)_{1,p} \leqq n\tau_1(f, 1; d)_{1,p}$ *for each* $n \in \mathbf{N}$ *and* $\tau_1(f, 1; \lambda d)_{1,p} \leqq ([\lambda]+3)\tau_1(f, 1; d)_{1,p}$ *for* $\lambda \in \mathbf{R}$;

7) $\tau_k(f, 1; d)_{p',p} \leqq c(k)\tau_{k-1}(f', d; d)_{p'',p}$ *for every* p', p''; $k \geqq 2$; $f' \in L_{\max\{p, p''\}}$;

8) $\tau_k(f, 1; d)_{p',p} \leqq \omega_k(f; d)_p \leqq c(k)\tau_k(f, 1; d)_{p',p}$ *for each* $p' \in [1, p]$;

9) $1/2\tau_1(f, 1; d)_{\infty,p} \leqq \tau_1(f; d)_p \leqq 2\tau_1(f, 1; d)_{\infty,p}$ *for each* $f \in L_\infty[a, b]$;

10) $\tau_k(f, 1; d)_{\infty,p} \leqq 2\tau_k(f; d)_p$ *for each* $f \in L_\infty[a, b]$; $k \geqq 2$.

We shall impose now one condition on the weight w:

(2.3) $$w(x) \leqq L(\lambda)w(t) \text{ for each } x, t \in [-1, 1], |x-t| \leqq \lambda\Delta(d, x),$$

where $\lambda \geqq 1$ and d is a fixed number, $0 < d \leqq 1$. We have to pay attention to the behaviour of the constant $L(\lambda) = L(w, \lambda)$ and so we introduce a special denotation for it.

Remark 1. The weights $w(x) = \Delta^\mu(d, x)$ (real μ) satisfy (2.3) with a constant $L(\lambda) \leqq (4\lambda+2)^{|\mu|}$ (see [20] or (2.4) and (2.5)).

Theorem 2 [20]. *If* w *satisfies* (2.3), $d = \text{const}$, $1 \leqq p, p', p'' \leqq \infty$, $f \in L_{\max\{p, p'\}}$ *we have*

1) $\tau_k(f, w; \Delta(d))_{p',p} \leqq c(k, L(k)) \| wf \|_p$ *for* $p' \leqq p$;

2) $\tau_k(f, w; \Delta(d))_{p'',p} \leqq c(k, L(k))\tau_{k-1}(f', w\Delta(d); \Delta((4k+2)d))_{p',p}$ *for* $k \geqq 2$ $f' \in L_{\max\{p', p\}}$;

3) $\tau_k(f, w; \Delta(d))_{p',p} \leqq c(k, L(k)) \| wf^{(k)}\Delta^k(d) \|_p$ *for* $k \geqq 1$, $f^{(k)} \in L_p[a, b]$;

4) $\tau_1(f, w; A\Delta(d))_{p',p} \leqq c(A, L(1))\tau_1(f, w; \Delta(d))_{p',p}$ *for* $A = \text{const} \geqq 1$, $d \leqq (2A)^{-1}$;

5) $\tau_k(f, w; \Delta(d))_{p',p} \leqq \tau_k(f, w; \Delta(d))_{p'',p} \leqq c(k, L(k))\tau_k(f, w; \Delta(d))_{p',p}$ *for* $p' \leqq p'' \leqq p$.

Let us note that Theorem 1, property 8) permits to replace $\omega_k(f; t)_p$ in (1.1) with $\tau_k(f, 1; t)_{p',p}$ for arbitrary $p' \in [1, p]$, i. e. $\tau_k(f, 1; t)_{p',p}$ is a structural characteristic of the best trigonometrical approximations.

We denote $J(x, \psi(x)) = [x-\psi(x), x+\psi(x)] \cap [-1, 1]$.

Let $\lambda \geqq 1$ and $0 < d \leqq (2\lambda)^{-1}$. We set (see [20])

$$a_\lambda(d, x) = a(d, x) = a(x) = \begin{cases} \Delta(d, x), & \text{if } |x| \leqq \sqrt{1-4\lambda^2 d^2}; \\ (2\lambda+1)d^2, & \text{if } |x| > \sqrt{1-4\lambda^2 d^2}. \end{cases}$$

It is easy to prove (see [20]) that

(2.4) $$\Delta(d, x) \leqq a_\lambda(d, x) \leqq (2\lambda+1)\Delta(d, x) \text{ for } |x| \leqq 1;$$

(2.5) $$\| a'_\lambda \|_\infty \leqq (2\lambda)^{-1}.$$

We denote $t_i^{(n)} = t_i = (2i-1)\pi/2n$; $x_i^{(n)} = x_i = \cos t_i$; $I'_i = [(i-1)\pi/n, i\pi/n]$ for each integer i; $I_i^{(n)} = I_i = [\cos i\pi/n, \cos(i-1)\pi/n]$ for $i = 1, 2, \ldots, n$, $I_i = I_{2n+1-i}$ for $i = n+1, n+2, \ldots, 2n$ and if $r = i \pmod{2n}$ for some $i = 1, 2, \ldots, 2n$, then $I_r = I_i$.

Lemma 1. *For every* $x \in I_j^{(n)}$, $y \in I_{j-1} \cup I_j \cup I_{j+1}$ *we have* $\Delta_n(y) \leqq 13\Delta_n(x)$ *and* $|x-(y \pm \Delta_n(y))| \leqq 33\Delta_n(x)$.

P r o o f. Let $t\in[0, \pi]$ be such that $x=\cos t$. Then

$$(2.6)\quad |x-y|\leq\max\{|\cos j\pi/n-\cos(j-2)\pi/n|;\ |\cos(j+1)\pi/n-\cos(j-1)\pi/n|\}$$
$$=2\sin(\pi/n)\max\{\sin(j-1)\pi/n,\ \sin j\pi/n\}$$
$$\leq 2(\pi/n)\max\{\sin(((j-1)\pi/n-t)+t);\ \sin((j\pi/n-t)+t)\}$$
$$=(2\pi/n)\max\{\sin((j-1)\pi/n-t)\cos t+\cos((j-1)\pi/n-t)\sin t;$$
$$\sin(j\pi/n-t)\cos t+\cos(j\pi/n-t)\sin t\}$$
$$\leq(2\pi/n)((\pi/n)x+\sqrt{1-x^2})\leq 2\pi^2\Delta_n(x)<20\Delta_n(x).$$

Using (2.4), (2.5) and (2.6) we have

$$\Delta_n(y)\leq a_1(n^{-1}, y)\leq|a_1(n^{-1}, y)-a_1(n^{-1}, x)|+a_1(n^{-1}, x)$$
$$\leq 1/2|x-y|+3\Delta_n(x)\leq 13\Delta_n(x).$$

This inequality and (2.6) give

$$|x-(y\pm\Delta_n(y))|\leq|x-y|+\Delta_n(y)\leq 33\Delta_n(x).$$

L e m m a 2. *For every integers i, ν, $\nu\neq 0$ we have* $\operatorname{mes}(I_i)\leq 5|\nu|\operatorname{mes}(I_{i+\nu})$.

P r o o f.

$$\frac{\operatorname{mes}(I_i)}{\operatorname{mes}(I_{i+\nu})}=\frac{|\cos(i-1)\pi/n-\cos i\pi/n|}{|\cos(i+\nu-1)\pi/n-\cos(i+\nu)\pi/n|}=\frac{|\sin(2i-1)2\pi/2n|}{|\sin(2(i+\nu)-1)\pi/n|}.$$

Let k be such integer that $(-n+1)/2\leq i+\nu+kn<(n+1)/2$. Then $-\pi/2\leq[2(i+\nu+kn)-1]\pi/2n\leq\pi/2$ and

$$\frac{\operatorname{mes}(I_i)}{\operatorname{mes}(I_{i+\nu})}=\frac{|\sin(2(i+kn)-1)\pi/2n|}{|\sin(2(i+\nu+kn)-1)\pi/2n|}\leq\frac{|(2(i+kn)-1)\pi/2n|}{(2/\pi|(2(i+\nu+kn)-1)\pi/2n|}$$

$$=\frac{\pi}{2}\,\frac{|2(i+kn)-1|}{|2(i+\nu+kn)-1|}=\frac{\pi}{2}\left(\frac{|2(i+kn)-1|-|2(i+\nu+kn)-1|}{|2(i+\nu+kn)-1|}+1\right)$$

$$\leq\frac{\pi}{2}\left(\frac{2|\nu|}{1}+1\right)\leq 5|\nu|.$$

L e m m a 3. *If $x\in I_i$, $y\in I_j$, $i\neq j$, then* $|x-y|\leq 246\,(i-j)^2\Delta_n(x)$.

P r o o f. We shall consider only the case $i<j$, because the case $i>j$ is similar. Using Lemma 2, we get

$$|x-y|\leq\sum_{k=i}^{j}\operatorname{mes}(I_k)\leq\left(1+\sum_{k=1}^{j-i}5\nu\right)\operatorname{mes}(I_i)$$

$$=(1+5(j-i)(j-i+1)/2)\operatorname{mes}(I_i)\leq(1+5(i-j)^2)\operatorname{mes}(I_i)\leq 6(j-i)^2\operatorname{mes}(I_i).$$

Using Lemma 1, we have

$$\operatorname{mes}(I_i)=|\cos(i-1)\pi/n-\cos i\pi/n|=2\sin(\pi/2n)|\sin(2i-1)\pi/2n|$$

$$\leq(\pi/n)|\sin t_i|\leq\pi\Delta_n(x_i)\leq 41\,\Delta_n(x).$$

This proves the lemma.

3. Jackson's type theorems for the best algebraic approximations.

T h e o r e m 3 [19]. *Let w satisfies* (2.3) *with $d=m^{-1}$ and* $L(\lambda)\leq c(s)\lambda^s$ $(\lambda\geq 1)$ *for some $s>0$. Then for each $F\in L_p[-1, 1]$* $(1\leq p\leq\infty)$ *we have*

(3.1) $$E(H_m, w; F)_{L_p[-1,1]} \leq c(s)\tau_1(F, w; \Delta_m)_{1,p}.$$

Proof. We set $r=-[-s]$ and $n=[m/(r+2)]+1$. Then

$$E(H_m, w; F)_p \leq E(H_{(r+2)(n-1)}, w; F)_p \text{ and}$$

$(r+2)^{-2}\Delta_n(x) \leq \Delta_m(x) \leq \Delta_n(x)$. In view of Theorem 2.4), (2.1) and (2.2) it is enough to prove that

$$E(H_{(r+2)(n-1)}, w; F)_p \leq c(s)\tau_1(F, w; c(s)\Delta_n)_{1,p},$$

because the weight w will satisfy (2.3) for $d=n^{-1}$ with a constant $L(\lambda) \leq c(s)((r+2)^2\lambda)^s = c(s)\lambda^s$.

Let n be a fixed real number. From Lemma 3 and from the condition on $L(\lambda)$ we have the inequality

(3.2) $$w(x) \leq c(s)|i-j|^{2s} w(y) \text{ for every } x \in I_i^{(n)},\ y \in I_j^{(n)},\ i \neq j.$$

We set $T_{n,r}(u)=T(u)=\left(\frac{\sin nu/2}{n \sin u/2}\right)^{2r+4} \in T_{(r+2)(n-1)}$; $\gamma_n^{(r)}=\gamma=\frac{1}{\pi}\int_{-\pi}^{\pi} T(u)du$. The numbers $\gamma_n^{(r)}$ satisfy the inequality (see [12, p. 236])

(3.3) $$n\gamma_n^{(r)} \geq c(r).$$

Let $x \in [-1, 1]$ and let $y \in [0, \pi]$ be such that $x=\cos y$. We consider the function

$$T(f; y)=\frac{1}{\pi\gamma_n^{(r)}}\int_{-\pi}^{\pi} f(\cos(y+t))T_{n,r}(t)dt,$$

where f is a continuous function on $[-1, 1]$. We have

$$T(f, y)=\frac{1}{\pi\gamma}\int_{y-\pi}^{y+\pi} f(\cos u)T_{n,r}(u-y)du=\frac{1}{\pi\gamma}\int_{-\pi}^{\pi} f(\cos u)T_{n,r}(u-y)du,$$

because $f_0 \cos$ and $T_{n,r}$ are 2π-periodic. Therefore $T(f) \in T_{(r+2)(n-1)}$. Also we have

$$T(f; -y)=\frac{1}{\pi\gamma}\int_{-\pi}^{\pi} f(\cos(-y+t))T_{n,r}(t)dt=\frac{1}{\pi\gamma}\int_{-\pi}^{\pi} f(\cos(-y-v))T_{n,r}(-v)dv$$

$$=\frac{1}{\pi\gamma}\int_{-\pi}^{\pi} f(\cos(y+v))T_{n,r}(v)dv=T(f; y),$$

because $T_{n,r}$ is an even function. Hence $T(f)$ is an even trigonometrical polynomial and there is $Q(f) \in H_{(r+2)(n-1)}$ such that $Q(f; \cos z)=T(f; z)$ for each z. We shall estimate $A=|f(x)-Q(f; x)|$. We have

$$A=\left|f(x)\frac{1}{\pi\gamma}\int_{-\pi}^{\pi} T(t)dt-\frac{1}{\pi\gamma}\int_{-\pi}^{\pi} f(\cos(y+t))T(t)dt\right|$$

$$\leq \frac{1}{\pi\gamma}\int_{-\pi}^{\pi} |f(\cos y)-f(\cos(y+t))|\, T(t)dt.$$

Let t_j be such a point in $\{t_1^{(n)}, t_2^{(n)}, \dots, t_n^{(n)}\}$, that

(3.4) $$|t_j-y|\leqq\pi/2n.$$

Then we have

(3.5) $$A\leqq\frac{1}{\pi\gamma}\int_{-\pi}^{\pi}\{|f(\cos y)-f(\cos t_j)|+|f(\cos t_j)-f(\cos(y+t))|\}T(t)dt$$

$$=|f(\cos y)-f(\cos t_j)|+\frac{1}{\pi\gamma}\sum_{i=-n+1}^{n}\int_{I'_i}|f(\cos t_j)-f(\cos(y+t))|\,T(t)dt.$$

We set $f(x)=f(\cos y)=\bar{f}$, $f(\cos t_j)=f_j$. Using (3.4), for $t\in I'_i$ we have

$$|t_{j+i}-(y+t)|=|t_j+t_i+\pi/2n-y-t|\leqq|t_j-y|+|t_i-t|+\pi/2n\leqq3\pi/2n.$$

Therefore

(3.6) $$\frac{1}{\pi\gamma}\int_{I'_n}|f(\cos t_j)-f(\cos(y+t))|\,T(t)dt$$

$$\leqq\frac{1}{\pi\gamma}\int_{I'_n}\{|f_j-f_{j+i}|+\omega_1(f_0\cos,\ t_{j+i};\ \frac{3\pi}{2n})_\infty\}T(t)dt.$$

Let $i=0,\ 1$ ($I'_0=[-\pi/n,\ 0]$, $I'_1=[0,\ \pi/n]$). Then from (3.3) we have

(3.7) $$\frac{1}{\pi\gamma}\int_{I'_i}(\frac{\sin nv/2}{n\sin v/2})^{2r+4}dv\leqq\frac{1}{\pi\gamma}\int_{I'_i}1\,.\,dv=\frac{\pi}{n}\ \frac{1}{\pi\gamma}\leqq c(r).$$

Using (3.3), for $i\geqq2$ we obtain

(3.8) $$\frac{1}{\pi\gamma}\int_{I'_i}(\frac{\sin nv/2}{n\sin v/2})^{2r+4}dv\leqq\frac{1}{\pi\gamma}\int_{(i-1)\pi/n}^{i\pi/n}\frac{dv}{(n\sin v/2)^{2r+4}}$$

$$\leqq\frac{1}{\pi\gamma}\cdot\frac{\pi}{n}\cdot\frac{1}{(\frac{2}{\pi}n\frac{i-1}{2n}\pi)^{2r+4}}\leqq c(r)\ (i-1)^{-2r-4}.$$

Similary for $i\leqq-1$ we have

(3.9) $$\frac{1}{\pi\gamma}\int_{I'_i}(\frac{\sin nv/2}{n\sin v/2})^{2r+4}dv\leqq c(r)\ |i|^{-2r-4}.$$

Now inequalities (3.5), (3.6), (3.7), (3.8) and (3.9) give

(3.10) $$A\leqq|\bar{f}-f_j|+c(r)\{\omega_1(f_0\cos,\ t_j;\ \frac{3\pi}{2n})_\infty+|f_j-f_{j+1}|$$

$$+\omega_1(f_0\cos,\ t_{j+1};\ \frac{3\pi}{2n})_\infty+\sum_{i=2}^{n}(i-1)^{-2r-4}[|f_j-f_{j+i}|+\omega_1(f_0\cos,\ t_{j+i};\ \frac{3\pi}{2n})_\infty]$$

$$+\sum_{i=1-n}^{-1}|i|^{-2r-4}[|f_j-f_{j+i}|+\omega_1(f_0\cos,\ t_{i+j};\ \frac{3\pi}{2n})_\infty]\}$$

$$=|\bar{f}-f_j|+c(r)\{\omega_1(f_0\cos,\ t_j;\ \frac{3\pi}{2n})_\infty+\sum_{i=1-n}^{-1}{}' |i|^{-2r-4}[|f_j-f_{j+i}|$$

$$+\omega_1(f_0\cos,\ t_{j+i};\ \frac{3\pi}{2n})_\infty)]\},$$

where we miss the term for $i=0$ in Σ_i'.

Let $i>0$. Then $|f_j-f_{j+i}|\leq\Sigma_{\nu=1}^i|f_{j+\nu-1}-f_{j+\nu}|$ and

(3.11)
$$\sum_{i=1}^n i^{-2r-4}|f_i-f_{j+i}|\leq\sum_{i=1}^n i^{-2r-4}\sum_{\nu=1}^i|f_{j+\nu-1}-f_{j+\nu}|$$

$$=\sum_{\nu=1}^n|f_{j+\nu-1}-f_{j+\nu}|\sum_{i=\nu}^n i^{-2r-4}\leq 2\sum_{\nu=1}^n\nu^{-2r-3}|f_{j+\nu-1}-f_{j+\nu}|,$$

because $\Sigma_{i=\nu}^n i^{-2r-4}<\nu^{-2r-4}+\int_\nu^\infty t^{-2r-4}dt<2\nu^{-2r-3}$. Similarly we have

(3.12)
$$\sum_{=1-n}^{-1} i^{-2r-4}|f_j-f_{j+i}|\leq 2\sum_{\nu=1-n}^{-1}|\nu|^{-2r-3}|f_{j+\nu}-f_{j+\nu+1}|.$$

We set $\Omega_i=\omega_1(f_0\cos,\ t_i;\ \frac{3\pi}{2n})_\infty$. Using (3.11) and (3.12) in (3.10), we obtain

(3.13)
$$A\leq|\bar{f}-f_j|+c(r)\{\Omega_j+\sum_{\nu=1-n}^{n}{}'\nu^{-2r-4}\Omega_{j+\nu}$$

$$+2\sum_{\nu=1}^n\nu^{-2r-3}|f_{j+\nu}-f_{j+\nu-1}|+2\sum_{\nu=1-n}^{-1}|\nu|^{-2r-3}|f_{j+\nu}-f_{j+\nu+1}|\}.$$

We denote

$$\varphi_\nu(x)=\Omega_{j+\nu}\quad\text{for}\quad x\in I_j,\ \nu=-n+1,\ -n+2,\dots,n;$$

$$\psi_\nu(x)=|f_{j+\nu}-f_{j+\nu-1}|\quad\text{for}\quad x\in I_j,\ \nu=1,\ 2,\dots,n;$$

(3.14)
$$\psi_\nu(x)=|f_{j+\nu}-f_{j+\nu+1}|\quad\text{for}\quad x\in I_j,\ \nu=-n+1,\ -n+2,\dots,-1;$$

$$\psi(x)=|\bar{f}-f_j|\quad\text{for}\quad x\in I_j$$

(if $x=\cos j\pi/n$, $j=1,2,\dots,n-1$, then we set

$\varphi_\nu(x)=\max\{\Omega_{j+\nu},\ \Omega_{j+1+\nu}\}$, $\psi(x)=\max\{|\bar{f}-f_j|,\ |\bar{f}-f_{j+1}|\}$ and so on).

Using these denotations in (3.13), we obtain

$$|f(x)-Q(f;\ x)|\leq\psi(x)+c(r)\{\varphi_0(x)+\sum_{\nu=1-n}^{n}{}'(\nu^{-2r-4}\varphi_\nu(x)+2|\nu|^{-2r-3}\psi_\nu(x))\}$$

for each $x\in[-1,\ 1]$ and

(3.15)
$$\|w(f-Q(f))\|_{L_p[-1,1]}\leq\|w\psi\|_p+c(r)\{\|w\varphi_0\|_p+\sum_{\nu=1-n}^{n}{}'[\frac{\|w\varphi_\nu\|_p}{\nu^{2r+4}}+\frac{2\|w\psi_\nu\|_p}{|\nu|^{2r+3}}]\}.$$

Using (3.14), (3.2) and Lemma 2, we get ($\nu\neq 0$)

(3.16)
$$\|w\varphi_\nu\|_p=[\sum_{j=1}^n\int_{I_j}w^p(x)\Omega_{j+\nu}^p dx]^{1/p}$$

$$\leq [\sum_{j=1}^{n} \operatorname{mes}(I_j)(c(s)|\nu|^{2s})^p \inf_{x \in I_{j+\nu}} w^p(x)\Omega_{j+\nu}^p]^{1/p}$$

$$\leq c(s)|\nu|^{2s}[\sum_{j=1}^{n} 5|\nu| \operatorname{mes}(I_{j+\nu}) \inf_{x \in I_{j+\nu}} w^p(x)\Omega_{j+\nu}^p]^{1/p}$$

$$\leq c(s)|\nu|^{2s+1}[\sum_{j=\nu+1}^{n+\nu} \int_{I_j} w^p(x)\Omega_j^p dx]^{1/p}$$

$$\leq c(s)|\nu|^{2s+1}[2\int_{-1}^{1} w^p(x)\varphi_0^p(x)dx]^{1/p} = c(s)|\nu|^{2s+1}\| w\varphi_0\|_p$$

(this is the case, when $p<\infty$. If $p=\infty$, then (3.16) is a direct consequence of (3.2)). Similarly we get

$$\| w\psi_\nu\|_p \leq c(s)(\nu-1)^{2s+1}\| w\psi_1\|_p \quad \text{for} \quad \nu\geq 2;$$

(3.17)

$$\| w\psi_\nu\|_p \leq c(s)|\nu+1|^{2s+1}\| w\psi_{-1}\|_p \quad \text{for} \quad \nu\leq -2.$$

(3.15), (3.16) and (3.17) give

(3.18) $$\| w(f-Q(f))\|_p \leq \| w\psi\|_p + c(r)\{\| w\varphi_0\|_p(1+\sum_{\nu=1-n}^{n}{}' \frac{c(s)|\nu|^{2s+1}}{\nu^{2r+4}})$$

$$+\| w\psi_1\|_p(1+\sum_{\nu=2}^{n}\frac{c(s)(\nu-1)^{2s+1}}{\nu^{2r+3}}) + \| w\psi_{-1}\|_p(1+\sum_{\nu=2}^{n-1}\frac{c(s)(\nu-1)^{2s+1}}{\nu^{2r+3}})$$

$$\leq \| w\psi\|_p + c(s)\{\| w\varphi_0\|_p + \| w\psi_1\|_p + \| w\psi_{-1}\|_p\},$$

because $2r+3-(2s+1)\geq 2$. Now on $[-1, 1]$ we define the function f as

(3.19) $$f(x) = \frac{1}{\operatorname{mes}(J(x, \Delta_n(x)))} \int_{J(x, \Delta_n(x))} F(t)dt.$$

Using the inequality $\Delta_n(x) \leq \operatorname{mes}(J(x, \Delta_n(x))) \leq 2\Delta_n(x)$, we have

(3.20) $$|F(x)-f(x)| \leq \frac{1}{\operatorname{mes}(J(x, \Delta_n(x)))} \int_{J(x, \Delta_n(x))} |F(x)-F(t)|dt \leq 2\omega_1(F, x; \Delta_n(x))_1.$$

Let $x' \in I_{j-1}\cup I_j \cup I_{j+1}$. Using Lemma 1 and (3.19), for each $x \in I_j$ we have

(3.21) $$|f(x_j)-f(x')| \leq |f(x_j)-F(x)| + |F(x)-f(x')|$$

$$\leq \frac{1}{\operatorname{mes}(J(x_j, \Delta_n(x_j)))} \int_{J(x_j, \Delta_n(x_j))} |F(t)-F(x)|dt$$

$$+\frac{1}{\operatorname{mes}(J(x', \Delta_n(x')))} \int_{J(x', \Delta_n(x'))} |F(x)-F(t)|dt$$

$$\leq \frac{2.13}{\Delta_n(x)} \int_{J(x, 33\Delta_n(x))} |F(x)-F(t)|dt = c\omega_1(F, x; 33\Delta_n(x))_1.$$

Therefore for each $x \in I_j$ we have

$$\omega_1(f_0 \cos, t_j; 3\pi/2n)_\infty = \sup\{|f(x_j)-f(x')| : x' \in I_{j-1}\cup I_j\cup I_{j+1}\}$$

$$\leq c\omega_1(F, x; 33\Delta_n(x))_1$$

and

(3.22) $$\| w\varphi_0 \|_p \leq c \| w(x)\omega_1(F, x;\ 33\Delta_n(x))_1 \|_p = c\tau_1(F, w;\ 33\Delta_n)_{1,p}.$$

From (3.14) and (3.21) with $x' = x \in I_j$ we get

(3.23) $$\| w+ \|_p \leq c\tau_1(F, w;\ 33\Delta_n)_{1,p}.$$

Similarly we get

(3.24) $$\| w\psi_{\pm 1} \|_p \leq c\tau_1(F, w;\ 33\Delta_n)_{1,p}.$$

(3.18), (3.20), (2.1), (3.22), (3.23) and (3.24) give

$$E(H_{(r+2)(n-1)},\ m;\ F)_p \leq \| w(F-Q(f)) \|_p \leq \| w(F-f) \|_p + \| w(f-Q(f)) \|_p$$

$$\leq 2\tau_1(F, w;\ \Delta_n)_{1,p} + c(s)\tau_1(F, w;\ 33\Delta_n)_{1,p} \leq c(s)\tau_1(F, w;\ 33\,\Delta_n)_{1,p}.$$

This completes the proof of Theorem 3.

Theorem 3 and Remark 1 give

Corollary 1. *For each* $F \in L_p[-1, 1]$, $1 \leq p \leq \infty$, *real* μ, *we have*

$$E(H_n, (\Delta_n)^\mu;\ F)_{L_p[-1,1]} \leq c(\mu)\tau_1(F, (\Delta_n)^\mu;\ \Delta_n)_{1,p}$$

and in particular $E(H_n, 1;\ F)_{L_p[-1,1]} \leq c\tau_1(F, 1;\ \Delta_n)_{1,p}$.

Theorem 4. *If* w *satisfies* (2.3) *for* $d = n^{-1}$ *and* $L(\lambda) \leq c(s)\lambda^s (\lambda \geq 1)$ *for some* $s > 0$ *and* $f' \in L_p[-1, 1]$, *then*

$$E(H_{n+1}, w;\ f)_{L_p[-1,1]} \leq c(s)E(H_n, w\Delta_n;\ f)_{L_p[-1,1]}.$$

Proof. We can only consider the case $f(x) = \int_{-1}^{x} f'(t)dt$, because $E(H_{n+1}, w;\ f)_p = E(H_{n+1}, w, f - f(-1))_p$. Let $P \in H_n$ be such that $E(H_n, w\Delta_n;\ f')_p = \| w\Delta_n(f' - P) \|_p$. We set $F(x) = \int_{-1}^{x} (f'(t) - P(t))dt$. Using Theorem 3, we obtain a polynomial $Q \in H_n$ such that

(3.25) $$\| w(F-Q) \|_p \leq c(s)\tau_1(F, w;\ \Delta_n)_{1,p}.$$

If we set $R(x) = Q(x) + \int_{-1}^{x} P(t)dt \in H_{n+1}$, then

(3.26) $$\| w(F-Q) \|_p = \| w(f-R) \|_p \geq E(H_{n+1}, w;\ f)_p.$$

Theorem 2 property 3) gives

(3.27) $$\tau_1(F, w;\ \Delta_n)_{1,p} \leq c(s) \| w\Delta_n F' \|_p = c(s) \| w\Delta_n(f' - P) \|_p = c(s)E(H_n, w\Delta_n, f')_p.$$

Now the inequalities (3.25), (3.26) and (3.27) prove the theorem.

Corollary 2. *If* w *satisfies* (2.3) *for* $d = n^{-1}$ *with* $L(\lambda) \leq c(s)\lambda^s (\lambda \geq 1)$ *form some* $s > 0$ *and* $f^{(k)} \in L_p[-1, 1]$, *then*

$$E(H_{n+k}, w;\ f)_{L_p[-1,1]} \leq c(k, s)E(H_n, w(\Delta_n)^k;\ f^{(k)})_{L_p[-1,1]}.$$

Proof. In view of Remark 1 the weights $w(\Delta_n)^\mu$ $(\mu = 1, 2, \ldots, k)$ satisfy (2.3) with $L(w(\Delta_n)^\mu, \lambda) \leq L(w, \lambda).(4\lambda + 2)^\mu \leq c(s, \mu)\lambda^{s+\mu}$. Applying k times Theorem 4, we complete the proof.

Corollary 3. *Under the conditions of Corollary* 2 *for* $n > k$ *we have*

$$E(H_n, w;\ f)_{L_p[-1,1]} \leq c(s, k)\tau_1(f^{(k)}, w(\Delta_n)^k;\ \Delta_n)_{1,p}.$$

P r o o f. We use Corollary 2 with n instead of $n+k$, Theorem 3, the inequalities $\Delta_n(x)\leq\Delta_{n-k}(x)\leq(k+1)^2\Delta_n(x)$ for $n>k$, (2.1) and Theorem 2 property 4).

From Corollary 3 we obtain

C o r o l l a r y 4. *If* $f^{(k)}\in L_p[-1, 1]$, $1\leq p\leq\infty$, $\mu\in\mathbf{R}$, *then* $E(H_n, (\Delta_n)^\mu; f)_{L_p[-1,1]}\leq c(k, \mu)\tau_1(f^{(k)}, (\Delta_n)^{k+\mu}; \Delta_n)_{1,p}$ *and in particular* $E(H_n, 1; f)_{L_p[-1,1]}\leq c(k)\tau_1(f^{(k)}, (\Delta_n)^k; \Delta_n)_{1,p}$.

4. Converse theorems for the best algebraic approximations.

T h e o r e m 5 [19]. *If w satisfies the conditions* (2.3) *for* $d=n^{-1}$ *and*

$$\| w(n\Delta_n)^k Q^{(k)} \|_{p[-1,1]}\leq Mn_1^k \| wQ \|_{p[-1,1]} \tag{4.1}$$

for each $Q\in H_{n_1}$, $n_1\leq n$, *where M may depend on k, w, p. Then for every* $p'\in[1, p]$ *and* $f\in L_p[-1, 1]$ *we have*

$$\tau_k(f, w; \Delta_n)_{p',p}\leq\frac{c(k, L(k), M)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E(H_s, w; f)_p. \tag{4.2}$$

P r o o f. We apply the standard Salem — Stečkin's method. Let $P_\nu\in H_\nu$ be such that $\| w(P_\nu-f)\|_p=E(H_\nu, w; f)_p$ ($\nu=0, 1, \dots$). We set $m=[\ln n/\ln 2]$, i. e. $2^m\leq n<2^{m+1}$. Theorem 2 property 1) and Theorem 2 property 1) give

$$\begin{aligned}\tau_k(f, w; \Delta_n)_{p',p} &\leq\tau_k(f-P_{2^{m+1}}, w; \Delta_n)_{p',p}+\tau_k(P_{2^{m+1}}, w; \Delta_n)_{p',p}\\ &\leq c(k, L(k))\| w(f-P_{2^{m+1}})\|_p+\tau_k(P_{2^{m+1}}, w; \Delta_n)_{p',p}\\ &=c(k, L(k))E(H_{2^{m+1}}, w; f)_p+\tau_k(P_{2^{m+1}}, w; \Delta_n)_{p',p}.\end{aligned} \tag{4.3}$$

Using Theorem 2.3), we get

$$\begin{aligned}\tau_k(P_{2^{m+1}}, w; \Delta_n)_{p',p} &\leq c(k, L(k)\| w(\Delta_n)^k P^{(k)}_{2^{m+1}}\|_p\\ &\leq c(k, L(k))\{\| w(\Delta_n)^k P_1^{(k)}\|_p+\sum_{\nu=0}^{m}\| w(\Delta_n)^k(P^{(k)}_{2^{\nu+1}}-P^{(k)}_{2^\nu})\|_p\}.\end{aligned} \tag{4.4}$$

Applying (4.1), we obtain

$$\begin{aligned}\| w(\Delta_n)^k(P^{(k)}_{2^{\nu+1}}-P^{(k)}_{2^\nu})\|_p &=n^{-k}\| w(x)(\sqrt{1-x^2}+n^{-1})^k(P^{(k)}_{2^{\nu+1}}(x)-P^{(k)}_{2^\nu}(x))\|_p\\ &\leq Mn^{-k}2^{(\nu+1)k}\| w(P_{2^{\nu+1}}-P_{2^\nu})\|_p\leq 2Mn^{-k}2^{(\nu+1)k}E(H_{2^\nu}, w; f)_p\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\| w(\Delta_n)^k P_1^{(k)}\|_p &=\| w(\Delta_n)^k(P_1^{(k)}-P_0^{(k)})\|_p\\ &\leq Mn^{-k}\| w(P_1-P_0)\|_p\leq 2Mn^{-k}E(H_0, w; f)_p.\end{aligned} \tag{4.6}$$

(4.4), (4.5) and (4.6) give

$$\tau_k(P_{2^{m+1}}, w; \Delta_n)_{p',p}\leq\frac{c(k, L(k), M)}{n^k}\{E(H_0, w; f)_p+\sum_{\nu=0}^{m}2^{k(\nu+1)}E(H_{2^\nu}, w; f)_p\}.$$

But

But $$2^{k(\nu+1)}E(H_{2^\nu}, w; f)_p=2^{2k}\sum_{s=2^{\nu-1}+1}^{2^\nu}(2^{\nu-1})^{k-1}E(H_{2^\nu}, w; f)_p$$

$$\leqq 2^{2k}\sum_{s=2^{\nu-1}+1}^{2^\nu}(s+1)^{k-1}E(H_s, w; f)_p.$$

Therefore

$$(4.7)\qquad \tau_k(P_{2^{m+1}}, w; \Delta_n)_{p',p}\leqq\frac{c(k, L(k), M)}{n^k}\{E(H_0, w; f)_p+2^kE(H_1, w; f)_p$$

$$+2^{2k}\sum_{\nu=1}^{m}\sum_{s=2^{\nu-1}+1}^{2^\nu}(s+1)^{k-1}E(H_s, w; f)_p\}\leqq\frac{c(k, L(k), M)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E(H_s, w; f)_p.$$

We also have

$$(4.8)\qquad E(H_{2^{m+1}}, w; f)_p\leqq E(H_n, w; f)_p$$

$$\leqq\frac{c(k)}{n^k}\sum_{s=0}^{n-1}(s+1)^{k-1}E(H_n, w; f)_p\leqq\frac{c(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E(H_s, w; f)_p.$$

We get (4.2) from (4.3), (4.7) and (4.8).

Now we shall apply Theorem 5 for the weights $w(x)=w_{\mu,n}(x)=(\sqrt{1-x^2}+n^{-1})^\mu$ (real μ). Therefore it is necessary $w_{\mu,n}$ to satisfy (2.3) and (4.1). We shall use the following inequalities ($1\leqq p\leqq\infty$):

$$(4.9)\qquad \|Q'\|_{p[-1,1]}\leqq cn^2\|Q\|_{p[-1,1]}\ \text{for each}\ Q\in H_n;$$

$$(4.10)\qquad \|(\sqrt{1-x^2})^{k+\mu}Q^{(k)}(x)\|_{p[-1,1]}\leqq c(k,\mu)n^k\|(\sqrt{1-x^2})^\mu Q(x)\|_{p[-1,1]}$$

for every $Q\in H_n$, $\mu\geqq 0$ and

$$(4.11)\qquad \|(\sqrt{1-x^2}+n^{-1})^{k+\mu}Q^{(k)}(x)\|_{p[-1,1]}\leqq c(k,\mu)n^k\|(\sqrt{1-x^2}+n^{-1})^\mu Q(x)\|_{p[-1,1]}$$

for every $Q\in H_n$, $\mu\in\mathbf{R}$.

Inequality (4.9) is proved in [24]. We get (4.10) setting $\mu=2\sigma$, $\delta=1/2$, $\Delta=1+(k+\mu)/2$, $\rho=(k+\mu)/2$, $p=q$ in Theorem 1 [6]. Potapov proved (4.11) in [9] with a constant depending on p. But it is easy to see (e. g. [5]) that these constants are uniformly bounded for $1\leqq p\leqq\infty$.

Corollary 5. *If* $\mu\in\mathbf{R}$, $1\leqq p'\leqq p$, $k\in\mathbf{N}$, $k\geqq\mu$, *then*

$$\tau_k(f, w_{\mu,n}; \Delta_n)_{p',p}\leqq\frac{c(k,\mu)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E(H_s, w_{\mu,n}; f)_p$$

and in particular

$$\tau_k(f, 1; \Delta_n)_{p',p}\leqq\frac{c(k)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E_s, 1; f)_p.$$

Proof. Using Remark 1 for $|x-t|\leqq\lambda\Delta_n(x)$ we have

$$w_{\mu,n}(x)/w_{\mu,n}(t)=(\Delta_n(x))^\mu/(\Delta_n(t))^\mu\leqq(4\lambda+2)^{|\mu|},$$

i. e. $w_{\mu,n}$ satisfy (2.3) for $d=n^{-1}$ with a constant $L(\lambda)\leqq(4\lambda+2)^{|\mu|}$.

Let $Q\in H_{n_1}$, $n_1\le n$. If $\mu<0$, then (4.11) with $n=n_1$ gives

$$\|w_{\mu,n}(n\Delta_n)^kQ^{(k)}\|_p=\|(\sqrt{1-x^2}+n^{-1})^{k+\mu}Q^{(k)}(x)\|_p \tag{4.12}$$

$$\le\|(\sqrt{1-x^2}+n_1^{-1})^{k+\mu}Q^{(k)}(x)\|_p\le c(k,\mu)n_1^k\|(\sqrt{1-x^2}+n_1^{-1})^{\mu}Q(x)\|_p$$

$$\le c(k,\mu)n_1^k\|(\sqrt{1-x^2}+n^{-1})^{\mu}Q(x)\|_p=c(k,\mu)n_1^k\|w_{\mu,n}Q\|_p.$$

If $\mu\ge0$, then using (4.10), (4.11) and the inequality $(A+B)^\alpha\le2^\alpha(A^\alpha+B^\alpha)(\alpha\ge0)$, we have

$$\|w_{\mu,n}(n\Delta_n)^kQ^{(k)}\|_p=\|(\sqrt{1-x^2}+n^{-1})^{k+\mu}Q^{(k)}(x)\|_p$$

$$\le2^{k+\mu}\{\|(\sqrt{1-x^2})^{k+\mu}Q^{(k)}(x)\|_p+n^{-k-\mu}\|Q^{(k)}\|_p\}$$

$$\le c(k,\mu)\{n_1^k\|(\sqrt{1-x^2})^{\mu}Q(x)\|_p+n^{-k-\mu}n_1^{2k}\|Q\|_p\}$$

$$\le c(k,\mu)\{n_1^k\|(\sqrt{1-x^2}+n^{-1})^{\mu}Q(x)\|_p+n_1^k\|n^{-\mu}Q\|_p\}\le c(k,\mu)n_1^k\|w_{\mu,n}Q\|_p.$$

Now applying Theorem 5 we complete the proof.

We can improve a little Corollary 5 for $\mu<0$.

Corollary 6. *If* $\mu<0$, $1\le p'\le p$, $k\in\mathbf{N}$, $k\ge-\mu$, *then*

$$\tau_k(f,w_{\mu,n};\Delta_n)_{p',p}\le\frac{c(k,\mu)}{n^k}\sum_{s=0}^{n}(s+1)^{k-1}E(H_s,w_{\mu,s+1};f)_p.$$

Proof. It is proved in (4.12) that for each $Q\in H_{n_1}$, $n_1\le n$, we have $\|w_{\mu,n}(n\Delta_n)^k\|_p\le c(k,\mu)n_1^k\|w_{\mu,n_1}Q\|_p$. Now we can repeat the proof ot Theorem 5 using the above inequality instead of (4.1) and summing in (4.8) from $[n/2]$ to n.

Corollary 1 and Corollary 5 or Corollary 6 give the following characterization of the best algebraic approximations:

If $0<\alpha<1$, $1\le p'\le p$ and $\mu\ge-1$, then

$$E(H_n,w_{\mu,n};f)_{L_p[-1,1]}=O(n^{-\alpha})\Leftrightarrow\tau_1(f,w_{\mu,n});\Delta_n)_{p',p}=O(n^{-\alpha})(n\to\infty)$$

and in particular

$$E(H_n,1;f)_{L_p[-1,1]}=O(n^{-\alpha})\Leftrightarrow\tau_1(f,1;\Delta_n)_{p',p}=O(n^{-\alpha})(n\to\infty)$$

and cf. (1.2))

$$E(H_n,w_{-\alpha,n};f)_{L_p[-1,1]}=O(n^{-\alpha})\Leftrightarrow\tau_1(\tau_1(f,w_{-\alpha,n};\Delta_n)_{p',p}=O(n^{-\alpha})(n\to\infty).$$

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Received 2. 12. 1981