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HENSEL MODULES

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In [8] and [9] I noticed that the study of minimal compact representations of finitely generated free modules, over Dedekind domain R in which each non-trivial ideal is of finite index, involves p-adic technique. A kind of compact modules appeared episodically in [9]. They coincide with p-adic completions, if the latter are compact, i. e. each non-trivial ideal of R is of finite index, and otherwise these modules are something distinct. There are reasons to call them Hensel modules.

It is well-known that the p-adic completions R_p of a given ring R play an important role in the study of R. This is especially true if R_p are compact. It turns out that when the compactness of R_p does not take place, sometimes the rôle of R_p could be performed by Hensel modules. Here I consider only one application of that type, but I hope the number of the examples could be enlarged. In this way we find an extension of p-adic methods.

In Section 1 the definition and general properties of Hensel modules are presented. Attention is directed mainly to the cases of Noetherian and Dedekind domains.

Section 2 contains a study of Hensel modules of the field of fractions Q of the ring R considered as R-modules. There are useful connections between them and the maximal Hensel modules of R. The main results concern Dedekind domains.

Section 3 contains a description of minimal compact representations of torsion-free modules over a Dedekind domain, by means of Hensel modules.

In Section 4 the results of Section 3 are specialized when natural additional conditions are fulfilled.

Section 5 is devoted to a topological characterization of Hensel modules in terms of minimal compact representations.

In general, the notations and terminology follow [9].

1. Definition and General Properties of Hensel Modules. Here we shall study Hensel modules. By definition, they are naturally connected with minimal compact representations of modules. The main tool used to establish their properties is Pontryagin's duality. It turns out that each Hensel module is associated with a maximal ideal p of the ring R. The set of the Hensel modules associated with p has minimal and maximal elements. If the ring R is Noetherian, all they are topological modules over the p-adic completion R_p. In case R is a Dedekind domain, Hensel modules have rather special properties.

1.A. Definition. *Let R be a commutative ring (with unit) endowed with the discrete topology. A Hensel R-module is a compact topological R-module Ξ such that there exists a closed submodule $\Xi_1 \neq \Xi$ of Ξ with $M \subset \Xi_1$ for each proper closed submodule M of Ξ. Ξ_1 is called the maximal submodule of Ξ.*

The following theorem shows that Hensel modules may be useful when minimal compact representations are concerned (see [9, Corollary 7, p. 519]).

1.1. Theorem. *Let R be a commutative ring endowed with the discrete topology. Then every non-zero compact R-module contains a Hensel R-module as a topological submodule.*

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Proof. Let C be a non-zero compact R-module. Denote by $x: C \to \mathrm{T}$ an arbitrary non-zero continuous character of the compact group C. Let U be a neighbourhood of the zero element of T which does not contain non-zero subgroups of T. Denote by $\mathcal{M}$ the set of all closed submodules M of C with $\chi(M) \cap \mathrm{T} \setminus U \neq \emptyset$. By compactness arguments, Zorn's lemma implies that there are minimal elements Ξ of $\mathcal{M}$. Clearly $\Xi \neq 0$. Let M be a closed submodule of Ξ with $M \neq \Xi$. Since Ξ is minimal, $\chi(M) \subset U$, and by the choice of U, we have $\chi(M)=0$. Now it is clear that, if M_0 is the additive subgroup of Ξ generated by the union of all closed submodules M of Ξ with $M \neq \Xi$, then $\chi(M_0)=0$. On the other hand, M_0 is a submodule of Ξ. Let $\Xi_1 = \bar{M}_0$. Then Ξ_1 is a closed submodule of Ξ, Ξ_1 contains all closed submodules M of Ξ with $M \neq \Xi$, and

$$\chi(\Xi_1) = \chi(\bar{M}_0) = \overline{\chi(M_0)} = \{0\} \subset U.$$

Hence $\Xi_1 \neq \Xi$, and the theorem is proved..

1.2. Corollary. *Let R be a commutative ring endowed with the discrete topology, and X be a topological R-module. Then an injective compact representation $r: X \to C$ of X is minimal, iff $r(X)$ contains non-zero elements of each Hensel submodule of C.*

1.B. Pontryagin's duality. There is a trivial extension of Pontryagin's duality for modules over a given commutative topological ring R. Let $\mathcal{D}$ be one of the categories of discrete or compact topological R-modules, and $\mathcal{E}$ be the other of them. For arbitrary object X of $\mathcal{D}$, consider the topological group X (discrete or compact), and denote by X^* the conjugate (compact or discrete) group of X in the sense of Pontryagin's duality for groups (see [1]). Endow X^* with the multiplication $R \times X^* \to X^*$ defined by

$$(s\chi)(x) = \chi(sx)$$

for arbitrary $s \in R$, $\chi \in X^*$, and $x \in X$. It is trivial that X^* becomes a topological R-module, and $X^* \in \mathcal{E}$. Moreover, X^{**} is naturally isomorphic with X. If $h: X \to Y$ is a morphism in $\mathcal{D}$ (continuous module homomorphism), define the morphism $h^*: Y^* \to X^*$ in $\mathcal{E}$, by

$$(h^*(\chi))(x) = \chi(h(x))$$

for arbitrary $\chi \in Y^*$, and $x \in X$. It is again trivial that h^* is a morphism in $\mathcal{E}$, and h^{**} is naturally isomorphic with h. We shall use in the sequel the above extension of Pontryagin's duality without references.

1.C. Associated maximal ideal. Let R be a commutative ring. Clearly, a compact R-module Ξ is a Hensel R-module, iff the conjugate module Ξ^* has a minimal non-zero submodule. The latter coincides with

$$\Xi_1^{\perp} = \{\chi \in \Xi^* : \chi \mid \Xi_1 = 0\}.$$

We shall call the ideal

$$(1.1) \qquad p = \{s \in R: \ s\Xi \subset \Xi_1\} = \{s \in R; \ s\Xi_1^{\perp} = 0\}$$

associated with Ξ.

1.3. Proposition. *Let R be a commutative ring, and Ξ be a Hensel R-module. Then the associated ideal (1.1) is maximal.*

Proof. Let $s \in p$, and x be an element of $\Xi_1^{\perp} \setminus \{0\}$ with $sx \neq 0$. Since $\Xi_1^{\perp}$ is minimal, there exists $t \in R$ with $tsx = x$. Therefore, $1 - ts \in p$, and so p is maximal.

1.D. Minimal Hensel modules. Let p be a maximal ideal in R, and Ξ be a Hensel R-module associated with p. We shall call Ξ minimal, if Ξ is an epimorphic continuous image of every Hensel module associated with p.

1.4. Proposition. *Let p be a maximal ideal in R. Then $(R/p)^*$ is a minimal Hensel module associated with p. If Ξ is a Hensel module associated with p, each continuous epimorphism $h: (R/p)^* \to \Xi$ is an isomorphism.*

The proof is straightforward.

1.5 Corollary. *Let p be a maximal ideal in R. Then the minimal Hensel modules associated with p are topologically isomorphic.*

1.E. Maximal Hensel modules. Eckmann and Schopf [4] proved that for every R-module M there exists an injective R-module I, and essential embedding $M \to I$, i. e., each non-zero submodule of I contains non-zero elements of M. I is called injective hull of M.

Let p be a maximal ideal in R. Denote by I_p, the injective hull of the R-module R/p. If Ξ is a Hensel module associated with p, we shall call Ξ maximal, if every Hensel module associated with p is an epimorphic continuous mage of Ξ.

1.6. Proposition. *Let p be a maximal ideal in R. Then I_p^* is a maximal Hensel module associated with p. If Ξ is a Hensel module associated with p, each continuous epimorphism $h: \Xi \to I_p^*$ is an isomorphism.*

Proof. Let M be a non-zero submodule of I_p. Since the embedding $R/p \to I_p$ is essential, we have $R/p \cap M \neq \{0\}$. On the other hand, each non-zero submodule of R/p coincides with R/p. Hence $R/p \subset M$. Therefore I_p has minimal non-zero submodule, and so I_p^* is a Hensel R-module. It is straightforward that the associated ideal of I_p^* is p.

Let now Ξ be a Hensel module associated with p. Then the submodule $\Xi_1^{\perp}$ of Ξ^* is isomorphic with R/p. Since I_p is injective, there exists a homomorphism $h: \Xi^* \to I_p$ such that $h \mid \Xi_1^{\perp}$ is a monomorphism. We prove that h is also monomorphism. Indeed $\operatorname{Ker} h \neq \{0\}$ implies $\operatorname{Ker} h \supset \Xi_1^{\perp}$, and that is a contradiction. By Pontryagin's duality, the conjugate homomorphism $h^*: I_p^* \to \Xi$ is an epimorphism. Therefore, I_p^* is a maximal Hensel module associated with p.

Let Ξ be a Hensel module associated with p, and $h: \Xi \to I_p^*$ be a continuous epimorphism. Then $h^*: I_p \to \Xi^*$ is a monomorphism. Since I_p is injective, there is a homomorphism $\varphi: \Xi^* \to I_p$ with

$$\varphi(h^*(x)) = x \tag{1.2}$$

for each $x \in I_p$. We show that h^* is an epimorphism. Let $y \in \Xi^* \setminus h^*(I_p)$. Then $y - h^*(\varphi(y)) \neq 0$, and

$$\varphi(y - h^*(\varphi(y))) = \varphi(y) - \varphi(h^*(\varphi(y))) = 0.$$

Therefore, $\operatorname{Ker} \varphi \neq \{0\}$, and hence $\operatorname{Ker} \varphi \supset \Xi_1^{\perp}$. But $\Xi_1^{\perp} \subset h^*(I_p)$ in contradiction with (1.2). In this way, we have proved that h^* is isomorphism. Hence h is also isomorphism, which proves the proposition.

1.7. Corollary. *Let p be a maximal ideal in R. Then the maximal Hensel modules associated with p are topologically isomorphic.*

It follows from Corollary 1.7 that up to topological isomorphism, Hensel R-modules constitute a set (not a class). This and Corollary 1.2 give a principal simplification of study of minimal compact representations of R-modules.

1.F. Coincidence of minimal and maximal Hensel modules. Here we consider conditions under which Hensel modules coincide.

1.8. Proposition. *Let p be a maximal ideal in R. Then the Hensel modules associated with p are topologically isomorphic, iff for each $s \in p$ we have $s \in sp$.*

Proof. Suppose first that the Hensel modules associated with p are isomorphic. Then the same is true for $(R/p)^*$ and I_p^*, and so R/p and I_p are isomorphic. Hence the R-module R/p is injective. Assume now that there exists $s \in p$ with $s \bar{\in} sp$. Then there exists a maximal among the ideals a in R with $s \bar{\in} a$ and $sp \subset a$. Consider the R-module R/a. Since a is maximal, each submodule of R/a contains the submodule (s) generated by s. On the other hand, $ps = 0$ in R/a, and hence $(s) = R/p$. At the same time R/p is an injective module, and there exists a homomorphism $h: R/a \to (s)$ such that $h/(s)$ is the identity of (s). Since (s) is the minimal non-zero submodule of R/a, h is monic. Therefore $h(1) \neq 0$, and so $h(1) = s_1 s = h(s_1 s)$. Now it follows $1 - s_1 s \in a$, which requires

$$s = s(1 - s_1 s) + s_1 s^2 \in a,$$

and that is a contraduction. This proves the necessity.

Now for each $s \in p$ let us have $s \in sp$. We prove that $R/p = I_p$. Otherwise there exists $x \in I_p \setminus R/p$. Let

$$a_x = \{s \in R : sx = 0\}.$$

Clearly a_x is ideal in R. Let $y \in R/p \setminus \{0\}$. Then there is $s_0 \in R$ with $s_0 x = y$ For each $s \in a_x$ we have $sy = s_0 sx = 0$. Therefore, $a_x \subset p$. At the same time $s_0 \bar{\in} a_x$. We shall prove that $s_0 \in p$. Otherwise there would exist $y_1 \in R/p$ with $s_0 y_1 = y$. Then $s_0(x - y_1) = 0$, and hence $s_0 \in a_{x-y_1}$. But $x - y_1 \neq 0$ implies $a_{x-y_1} \subset p$, which is a contradiction, since $s_0 \bar{\in} p$. Therefore, $x = y$, which is impossible, since $x \bar{\in} R/p$. Hence $s_0 \in p$. At the same time $ps_0 x = py = 0$, which requires $ps_0 \subset a_x$. In this way we see that $s_0 \bar{\in} ps_0$, and the proposition is proved.

1.9. Corollary. *If p is a maximal ideal in R such that the Hensel modules associated with p are isomorphic, and $1 + p$ does not contain zero divisors, then R is a field.*

1.10. Corollary. *Let R be a commutative ring. The Hensel R-modules are topologically isomorphic, iff R is a field.*

1.G. Extension of scalars. It turns out that, if R is a Noetherian ring, Hensel modules associated with a given maximal ideal p have a natural structure of R_p-module. Here R_p is the p-adic completion of R.

1.11 Theorem. *Let R be a commutative Noetherian ring, p be a maximal ideal in R, and Ξ be a Hensel R-module associated with p. If $i: R \to R_p$ is the canonical mapping, and $R \times \Xi \to \Xi$ is the multiplication with scalars in Ξ, there exists unique continuous multiplication*

$$R_p \times \Xi \longrightarrow \Xi \tag{1.3}$$

such that the diagram

(1.4)
$$\begin{array}{ccc} R\times\Xi & & \\ {\scriptstyle i\times 1}\downarrow & \searrow & \\ R_p\times\Xi & \longrightarrow & \Xi \end{array}$$

is commutative.

Proof. Consider the discrete module Ξ^*. By Pontryagin's duality, the statement will be proved, if we show that the multiplication

(1.5) $$R\times\Xi^*\longrightarrow\Xi^*$$

has a unique continuous extension

(1.6) $$R_p\times\Xi^*\longrightarrow\Xi^*.$$

Since $i(R)$ is dense in R_p, the uniqueness of (1.6) is clear. It is straightforward that the existence will be established, if we prove that (1.5) is continuous when R is endowed with the p-adic topology. By Proposition 1.6, Ξ is an epimorphic image of I_p^*. Hence Ξ^* is a submodule of I_p. But in [7] Matlis has proved that the multiplication $R\times I_p\longrightarrow I_p$ is continuous, when R is endowed with the p-adic, and I_p with the discrete topology. This completes the proof of the theorem.

1.12. Corollary. *Let R be commutative Noetherian ring, and p be a maximal ideal in R with finite index. Then the maximal Hensel module associated with p coincides with the R-module R_p.*

Indeed, since p is of finite index, R_p is compact. Let Ξ_1 be the maximal submodule of an arbitrary Hensel module Ξ associated with p and $\xi\in\Xi\setminus\Xi_1$. Then $R_p\xi$ is a dense and compact submodule of Ξ, and so $R_p\xi=\Xi$. Therefore, Ξ is an epimorphic image of R_p. On the other hand, R_p is clearly Hensel module associated with p, and the statement is proved.

Theorem 1.11 shows that if R is a Noetherian ring, then the Hensel modules associated with a given maximal ideal p coincide with the Hensel modules for R_p. Since $i(R)\subset R_{(p)}\subset R_p$, Hensel modules associated with p coincide also with the Hensel modules of the localization $R_{(p)}$ of R at p.

1.H. The scalars as monomorphisms and epimorphisms.

1.13. Proposition. *Let R be a commutative ring with unit, p be a maximal ideal in R, Ξ be the maximal Hensel module associated with p, and $s\in R$. Then the condition $s\xi\neq 0$ is fulfilled for each $\xi\in\Xi\setminus\{0\}$ iff for each $x\in R$ with $sx=0$ we have $x\in xp$.*

Proof. Suppose first that for each zero divisor x for s we have $x\in xp$. Consider arbitrary element y of I_p, and the ideal $a=\{\sigma\in R:\ \sigma y=0\}$. We shall prove that $x\in a$. Otherwise we would have $xy\neq 0$. Therefore there would exist element $s_0\in R$ with $s_0xy\in R/p$ and $s_0xy\neq 0$. Then $s_0x\,\bar{\in}\,a$ and $s_0xp\subset a$. Therefore $s_0x\,\bar{\in}\,s_0xp$, in contradiction with $x\in xp$.

Hence for each zero divisor x of s we have $xy=0$. Now we can consider the homomorphism $h:(s)\rightarrow I_p$ defined by $h(zs)=zy$ for arbitrary $z\in R$. Since I_p is injective, there exists an extension $h_1:\ R\rightarrow I_p$ of h. Clearly $sh_1(1)=h_1(1.s)=y$, and therefore $s:\ I_p\rightarrow I_p$ is epimorphic. Then the conjugate homomorphism $s:\ \Xi\rightarrow\Xi$ is monomorphic.

Let now $s: \Xi \to \Xi$ be monomorphism. Then $s: I_p \to I_p$ is epimorphism and if x is a zero divisor of s, then $xy=0$ for each $y \in I_p$. Assume that $x \in xp$. Then there would exist a maximal ideal a in R with $x \in a$ and $xp \subset a$. Thus we can construct a monomorphism $h: R/a \to I_p$. It is clear that $0 = xh(1) = h(x)$, in contradiction with $x \in a$. This completes the proof.

1.14. Corollary. *Let R be a Noetherian commutative ring, p be a maximal ideal in R, Ξ be the maximal Hensel module associated with p, and $\sigma \in R_p$. Then $\sigma: \Xi \to \Xi$ is a monomorphism iff σ is not a zero divisor in R_p.*

1.15. Proposition. *Let R be a commutative ring with unit, p be a maximal ideal in R, Ξ be the maximal Hensel module associated with p, and $s \in R$. Then $s: \Xi \to \Xi$ is epimorphism iff $s \bar{\in} p$ and then it is isomorphism.*

Proof. Let $s: \Xi \to \Xi$ be epimorphism. Then the conjugate homomorphism $s: I_p \to I_p$ is monomorphism. Since $R/p \subset I_p$, we have $s \bar{\in} p$. Conversely, if $s \bar{\in} p$, then $i(s) \in R_p \setminus pR_p$. But pR_p is the unique maximal ideal of R_p, and hence $i(s)$ has an inverse in R_p. By Theorem 1.11, $s: \Xi \to \Xi$ is isomorphism, which proves the proposition.

1.I. Hensel modules over Dedekind domains. In the general case of arbitrary Noetherian ring R, the structure of I_p is rather complicated, and the detail study of Hensel modules seems hopeless. If R is a Dedekind domain, however, the description of all Hensel R-modules is simple.

Let R be a Dedekind domain, and p be a maximal ideal in R. Let $R(p^\infty) = Q/R_{(p)}$, where Q is the field of fractions of the ring R, and $R_{(p)}$ — the localization of R at p. It is well-known that the submodules of $R(p^\infty)$ are $R(p^k) = R/p^k$ $(k=1, 2, \ldots)$, and that the embedding $R/p \subset R(p^\infty)$ is essential. On the other hand, $R(p^\infty)$ is divisible R-module, and by [3], $R(p^\infty)$ is injective. Therefore, $I_p = R(p^\infty)$.

1.16. Proposition. *Let R be a Dedekind domain which is not a field and p be a maximal ideal in R. Then the Hensel modules associated with p are*

$$\Xi_{p^k} = (R(p^k))^*, \quad k=1, 2, \ldots, \infty.$$

The proof is straightforward.

Clearly the minimal Hensel module associated with p is Ξ_p and the maximal — Ξ_{p^∞}. If the ideals p or the powers k are different, the corresponding Hensel modules are also different. Proposition 1.16 applies when R is not a field. If R is a field, by Corollary 1.10, there is unique Hensel module $\Xi = R^*$. In this case Ξ is a compact linear space over the discrete field R. Since Ξ is the minimal Hensel module associated with the ideal 0, the unique closed linear subspaces of Ξ are 0 and Ξ.

1.17. Proposition. *Let R be a Dedekind domain which is not a field, and p be a maximal ideal in R. If $t \in p \setminus p^2 \subset R_p$, the unique closed non-zero submodules of Ξ_{p^∞}* are

$$t^k \Xi_{p^\infty}, \quad k=0, 1, 2, \ldots$$

The proof is straightforward.

If R is a Dedekind domain, the p-adic completion R_p has no zero divisors. By Corollary 1.14, the maximal Hensel module Ξ_{p^∞} is torsion-free R_p-module. The following theorem gives the rank of Ξ_{p^∞} as R_p-module.

1.18. Theorem. *Let R be a Dedekind domain, and p be a maximal ideal in R. If the ideal p is of finite index, the R_p-modules R_p and Ξ_{p^∞} are topologically isomorphic. If the index of p is infinite, then*

$$\text{card}\ \Xi_{p^\infty} = \text{rank}\ \Xi_{p^\infty} = 2^{\text{card}\ R/p}. \tag{1.7}$$

Proof. If the ideal p is of finite index, the statement follows from Corollary 1.12.

Suppose p has no finite index. Then $F=R/p$ is infinite field. According to Kakutani [6],

$$\text{card}\ F^* = 2^{\text{card}\ F}.$$

Hence

$$\dim F^* = 2^{\text{card}\ F}, \tag{1.8}$$

where $\dim F^*$ denotes the dimension of the linear space F^* over F. Let $\{\chi_\alpha\}_{\alpha \in A}$ be an F-basis of F^*. From (1.8) it follows

$$\text{card}\ A = 2^{\text{card}\ F}.$$

Since $F=R/p$ is canonically isomorphic with a submodule of $R(p^\infty)$, the characters $\chi_\alpha : F \to \mathrm{T}$ can be extended to characters $\chi'_\alpha : R(p^\infty) \to \mathrm{T}$. In this way we find a family $\{\chi'_\alpha\}_{\alpha \in A}$ of elements of Ξ_{p^∞}. It may be proved that $\{\chi'_\alpha\}_{\alpha \in A}$ is independent over R_p. Therefore, rank $\Xi_{p^\infty} \geqq 2^{\text{card}\ R/p}$.

Now it is clear that (1.7) will be proved, if we show that card $\Xi_{p^\infty} \leqq 2^{\text{card}\ R/p}$. It is straightforward that card $R(p^k) =$ card F. Since $R(p^\infty) = \bigcup_{k=1}^{\infty} R(p^k)$, then card $R(p^\infty) =$ card F. From [6] it follows that card $\Xi_{p^\infty} = 2^{\text{card}\ R/p}$, and the theorem is proved.

1.19. Proposition. *Let R be a Dedekind domain which has at least two maximal ideals, p be a maximal ideal in R, and C be a compact R-module such that each Hensel submodule of C is p-adic. Then there exists unique continuous multiplication $R_p \times C \to C$, which extends the multiplication $R \times C \to C$.*

Proof. Let q be a maximal ideal in R with $q \neq p$, and x be an element of C^* with $x \neq 0$. We state that $x \in qx$. Otherwise there would exist a maximal submodule A of C^* with $x \notin A$ and $qx \subset A$. Then $C^*/A \neq 0$, and C^*/A has a minimal non-zero submodule (x). Since $q(x)=0$, $(C^*/A)^*$ is a q-adic Hensel submodule of C, and that is a contradiction. Therefore, $x \in qx$. Hence there exists $s_0 \in q$ with $x = s_0 x$.

In this way we have proved that C^* is periodic. Let $a_x = \{s \in R : sx = 0\}$. Since $1 - s_0 \in a_x$, $q \neq p$ implies $a_x \not\subset q$. Let $a_x = p^n q_1^{n_1} \ldots q_l^{n_l}$ where $p, q_1, \ldots, q_l$ are different maximal ideals in R. If $n_\lambda \neq 0$, then $a_x \subset q_\lambda$, which is impossible. Hence $a_x = p^n$. Now it is clear that the multiplication $R \times C^* \to C^*$ is continuous, if R is endowed with the p-adic, and C^* with the discrete topology. Now the statement follows from Pontryagin's duality.

2. Q-adic Modules. In Section 3 we shall study minimal compact representations of torsion-free modules over Dedekind domains. Along with the Hensel modules of R, we shall need for the purpose the Hensel module of the field of fractions Q. We shall study it in the present section.

2.A. Definition. *Let R be a commutative ring without zero divisors, and Q be the field of fractions of R. By Corollary* 1.10 *the ring Q has only one Hensel module $\Xi = Q^*$. Of course Ξ is a compact linear space over the discrete field Q. Therefore, Ξ is R-module without periodic elements. We call that R-module Q-adic module of R.*

From Kakutani [6] it follows that if R is infinite, the dimension of Ξ over Q is $2^{\operatorname{card} R}$. Hence Ξ algebraically is a direct sum of $2^{\operatorname{card} R}$ copies of Q.

2.B. Closed submodules of the Q-adic modules. If we consider Ξ as Q-module, the unique closed submodules of Ξ are 0 and Ξ. But as R-module Ξ has more closed submodules.

Let p be a maximal ideal of R. Then the localization $R_{(p)}$ of R at p is a submodule of Q. Hence we may form the module

$$R_{(p)}^{\perp} = \{\xi \in \Xi : \xi \mid R_{(p)} = 0\}. \tag{2.1}$$

The following algebraic lemma permits to prove an important property of the closed submodules (2.1) of Ξ.

2.1. Lemma. *Let R be a commutative ring without zero divisors, and M be an R-submodule of Q such that for each maximal ideal p in R*

$$M + R_{(p)} = Q. \tag{2.2}$$

Then if R is not a field, $M = Q$.

Proof. Let $s, t \in R$ and $t \neq 0$. We show that $s/t \in M$. For the purpose it is enough to prove that $s \in tM$. Assume the contrary. Then $tM \cap R$ is ideal in R, different from R. Therefore there exists a maximal ideal p in R with $tM \cap R \subset p$. Since R is not a field, it follows from (2.2) that $M \neq 0$, and hence there exists $\tau \in M \cap R$ with $\tau \neq 0$. Let $\sigma \in R \setminus p$. By (2.2), there are $\tau \in M$, $\tau' \in R \setminus p$, and $\sigma' \in R$ with $\sigma/\tau t = r + \sigma'/\tau'$. Then

$$\sigma\tau' = r\tau t\tau' + \sigma'\tau t.$$

Therefore, $\sigma\tau' \in tM \cap R \subset p$, and since $\tau' \bar{\in} p$, then $\sigma \in p$, which is a contradiction. This completes the proof of Lemma 2.1.

2.2. Proposition. *Let R be a commutative ring without zero divisors, and C be a non-zero closed R-submodule of Ξ. If R is not a field, then there exists a maximal ideal p in R with $C \cap R_{(p)}^{\perp} \neq \{0\}$.*

Proof. The proposition follows directly from Lemma 2.1 and Pontryagin's duality.

If R is a Dedekind ring, and p is a maximal ideal in R, the sequence

$$0 \longrightarrow R_{(p)} \xrightarrow{i_p} Q \xrightarrow{j_p} R(p^{\infty}) \longrightarrow 0$$

is exact. So the sequence

$$0 \longleftarrow R_{(p)}^{*} \xleftarrow{i_p^{*}} \Xi \xleftarrow{j_p^{*}} \Xi_{p^{\infty}} \longleftarrow 0$$

is also exact. Hence

$$R_{(p)}^{\perp} = \operatorname{Ker} i_p^{*} = j_p^{*}(\Xi_{p^{\infty}}). \tag{2.3}$$

Since j_p^{*} is a monomorphism, $R_{(p)}^{\perp}$ is a copy of the maximal Hensel module $\Xi_{p^{\infty}}$. We shall call $R_{(p)}^{\perp}$ the standard copy of $\Xi_{p^{\infty}}$ in Ξ.

Let us remind that the elements of the p-adic field Q_p of R are of the type σ/t^k, where $\sigma \in R_p$, t is fixed element of $p \setminus p^2$, and $k=0, 1, 2, \ldots$ Since Ξ is a linear space over Q, it is clear that for each $\xi \in R^{\perp}_{(p)}$, and for each $\sigma \in Q_p$ we may form the product $\sigma\xi$. So we can consider continuous homomorphisms $h: R^{\perp}_{(p)} \to \Xi$ defined by $h(\xi)=\sigma\xi$. The next proposition describes the set of all homomorphisms $R^{\perp}_{(p)} \to \Xi$.

2.3. Proposition. *Let R be a Dedekind ring, and p be a maximal ideal in R. Then for each continuous homomorphism $h: R^{\perp}_{(p)} \to \Xi$ there exists $\sigma \in Q_p$ with*

$$h(\xi)=\sigma\xi \tag{2.4}$$

for each $\xi \in R^{\perp}_{(p)}$.

Proof. We consider only the non-trivial case $h \neq 0$. Then h is a monomorphism. Indeed, if $\operatorname{Ker} h \neq 0$, then $\operatorname{Ker} h = t^k R^{\perp}_{(p)}$, since $R^{\perp}_{(p)}$ is isomorphic with Ξ_{p^∞}. Therefore, $h(R^{\perp}_{(p)})$ would be periodic, and at the same time Ξ is torsion-free. So h is monomorphism.

Let $H_p = h(R^{\perp}_{(p)})$. By Proposition 2.2. H_p contains non-zero elements of $R^{\perp}_{(q)}$. Clearly $q=p$. Since Ξ is torsion-free, it follows from Proposition 1.17 that $R^{\perp}_{(p)} \subset H_p$ or $H_p \subset R^{\perp}_{(p)}$. In the second case h is of the type $h: \Xi_{p^\infty} \to \Xi_{p^\infty}$. Consider the conjugate homomorphism $h^*: R(p^\infty) \to R(p^\infty)$. It is not difficult to see that there exists $\sigma \in R_p$ with $h^*(x)=\sigma x$ for each $x \in R(p^\infty)$. Now by Pontryagin's duality, $h(\xi)=\sigma\xi$ for each $\xi \in \Xi_{p^\infty}$. It is easy to reduce the first case to the second, and the proposition is proved.

2.C. Global Hensel modules. Let R be a Dedekind ring. Then R is a submodule of the R-module Q. So we may form the compact R-module $H=R^{\perp}=\{\xi \in \Xi : \xi \mid R=0\}$. We shall call H global Hensel module of R.

Since $\xi \mid R_{(p)}=0$ requires $\xi \mid R=0$, the standard copies of Ξ_{p^∞} in Ξ are submodules of H. The next theorem shows that they are the coordinate modules in a natural representation of H as a product.

2.4. Theorem. *Let R be a Dedekind ring, and p be a maximal ideal in R. Then there exists a unique module homomorphism*

$$h_p: H \longrightarrow R^{\perp}_{(p)} \tag{2.5}$$

such that

$$(h_p(\xi))(s/t^k)=\xi(s/t^k) \tag{2.6}$$

for arbitrary $\xi \in H$, $s \in R$, $t \in p \setminus p^2$ and $k=0, 1, 2, \ldots$ The homomorphisms (2.5) are continuous, and their product

$$\prod_{p \in \pi} h_p: H \longrightarrow \prod_{p \in \pi} R^{\perp}_{(p)}, \tag{2.7}$$

where π denotes the set of all maximal ideals in R, is a topological isomorphism. If $u_p: R^{\perp}_{(p)} \to H$ is the corresponding embedding, the composition $h_p u_p$ coincides with the identity ot $R^{\perp}_{(p)}$.

Proof. Since $h_p(\xi) \mid R_{(p)}=0$ ($\xi \in H$), (2.6) ensure the uniqueness. It is not difficult to see that the homomorphisms

$$\mu_p: R(p^\infty) \longrightarrow Q/R$$

defined by

(2.8) $$\mu_p(s/t^k + R_{(p)}) = s/t^k + R$$

for arbitrary $s \in R$, $t \in p \setminus p^2$, $k = 0, 1, 2, \ldots$, represent Q/R as a direct sum of $R(p^\infty)$. Hence the product

(2.9) $$\prod_{p \in \pi} \mu_p^*: (Q/R)^* \longrightarrow \prod_{p \in \pi} \Xi_{p^\infty}$$

is an isomorphism. Identify Ξ_{p^∞} and $R_{(p)}^\perp$, by (2.3). From Pontryagin's duality it follows that $(Q/R)^*$ and H are also naturally isomorphic. Define h_p by $h_p \approx \mu_p^*$. Now (2.6) follows from (2.8). Since (2.9) is isomorphism, the same is true for (2.7). From (2.8) it follows that $h_p \mid R_{(p)}^\perp$ is the identity of $R_{(p)}^\perp$. This completes the proof of the theorem.

2.D. Dense submodules of the Q-adic modules. To study the compact representations of the type $r: X \to \Xi$ we have to know the dense submodules of Ξ. They are described in the following theorem.

2.5. Theorem. *Let R be a Dedekind ring, and X be R-submodule of Ξ. Then X is dense iff one of the following conditions (a) or (b) holds:*

(a) *there is $\xi \in X$ such that $s\xi \notin H$ for each $s \in R \setminus \{0\}$;*

(b) *there is an infinite sequence $\{s_n\}_{n=1}^\infty$ of non-inversible elements of R, and a sequence $\{\xi_n\}_{n=1}^\infty$ of elements of X with*

(2.10) $$s_1 s_2 \ldots s_{n-1} \xi_n \notin H \text{ and } s_1 s_2 \ldots s_{n-1} s_n \xi_n \in H$$

for each positive integer n.

Proof. It is easy to see that X is dense in Ξ iff X separates the points of Q. Being a Dedekind ring, R has a good divisibility. Using that, we prove that X separates the points of Q iff one of the conditions (a) or (b) is fulfilled. We omit the details.

2.E. Copies of H in Ξ. It is trivial that the R-module Q/R endowed with the discrete topology is a topological module over the ring R endowed with the topology with a fundamental system of open sets — all non-zero ideals in R. Let $\hat{R}$ be the completion of R with respect to this topology. Clearly the multiplication $R \times Q/R \to Q/R$ has an unique continuous extension $\hat{R} \times Q/R \to Q/R$. Since $H = (Q/R)^*$, H is a topological $\hat{R}$-module, by Pontryagin's duality.

The embedding $R \to R_p$ has an unique continuous extension $e_p: \hat{R} \to R_p$, which clearly is a ring homomorphism. Thus we find the continuous ring homomorphism

(2.11) $$e = \prod_{p \in \pi} e_p: \hat{R} \longrightarrow \prod_{p \in \pi} R_p.$$

It is a well-known fact that if R is a Dedekind ring, e is a topological isomorphism. The isomorphisms (2.7) and (2.11) are closely related, since we have

(2.12) $$(\prod_{p \in \pi} h_p)(\sigma\xi) = ((e_p(\sigma))(h_p(\xi)))_{p \in \pi}$$

for each $\sigma \in \hat{R}$ and for each $\xi \in H$. We omit the proof of (2.12).

The next proposition describes the continuous homomorphisms from H to Ξ.

2.6. Proposition. *Let R be a Dedekind ring. Then for each continuous homomorphism $h: H \to \Xi$ there exist $\sigma \in \hat{R}$ and $s \in R \setminus \{0\}$ with*

$$h(\xi) = \sigma\xi/s \tag{2.13}$$

for each $\xi \in H$.

Proof. Clearly the conjugate homomorphism $h^*: Q \to Q/R$ is not a monomorphism. Now Pontryagin's duality requires that h is not an epimorphism.

Let p be a maximal ideal in R. Then $R^{\perp}_{(p)} \subset H$. We apply Proposition 2.3. to $h/R^{\perp}_{(p)}$, and find $\sigma'_p \in R_p$, $t_p \in p \setminus p^2$, $k_p = 0, 1, 2, \ldots$ with

$$h(\xi_p) = \sigma'_p \xi_p / t_p^{k_p} \tag{2.14}$$

for each $\xi_p \in R^{\perp}_{(p)}$. We may suppose, of course, that k_p in (2.14) has the smallest possible value.

Now we prove that $k_p > 0$ only for finite number of indices p. Since k_p is minimal, we have $\sigma'_p \xi_p \in R^{\perp}_{(p)} \setminus t_p R^{\perp}_{(p)}$, and so $\sigma'_p \xi_p / t_p^{k_p} \,\bar{\in}\, H$, if $k_p > 0$. If $k_p > 0$ for infinitely many p in (2.14), Theorem 2.5 would imply that $h(H)$ is dense in Ξ. Since H is compact, $h(H) = \Xi$, and this is a contradiction.

Denote by Φ the finite set of the maximal ideals p in R with $k_p > 0$. Direct checking shows that if $s = \Pi_{p \in \pi} t_p^{k_p}$, then

$$h((\xi_p)_{p \in \Phi_1}) = \frac{1}{s} (\sigma_p \xi_p)_{p \in \Phi_1} \tag{2.15}$$

for each finite subset Φ_1 of π with $\Phi_1 \supset \Phi$. Since each element $(\xi_p)_{p \in \pi}$ of H admits approximation by elements of the type $(\xi_p)_{p \in \Phi_1}$, (2.13) follows from (2.15). This completes the proof of the proposition.

2.7. Corollary. *Let R be a Dedekind ring. Then the copies of H in Ξ are the submodules of Ξ of the type*

$$\frac{1}{s} \prod_{p \in \pi} t_p^{k_p} R^{\perp}_{(p)},$$

where $s \in R \setminus \{0\}$, $t_p \in p \setminus p^2$ and $k_p = 0, 1, 2, \ldots$ $(p \in \pi)$.

3. Minimal Compact Representations of Torsion-Free Modules over Dedekind Rings. Hensel and Q-adic modules permit to describe minimal compact representations of modules over Dedekind rings. Here we consider one problem of that type.

3.A. General form of minimal compact representations. The following lemma shows that the range of a minimal compact representation of a torsion-free module over Dedekind ring has a very special form.

3.1. Lemma. *Let R be a Dedekind ring, and X be a torsion-free R-module. Then every minimal compact representation r of X has the form*

$$r: X \longrightarrow \prod_{p \in \pi} \Xi_{p^\infty}^{a_p} \times \Xi^{a}, \tag{3.1}$$

where a_p $(p \in \pi)$ and a are cardinal numbers.

Proof. Let $r\colon X\to C$ be a minimal compact representation of X. Then r is essential (see [9, p. 519]. Since X is torsion-free, it follows that C is also torsion-free. Therefore, C^* is divisible R-module. From [10] it follows that

$$C^*=\bigoplus_{p\in\pi}(R(p^\infty))^{(a_p)}\oplus Q^{(a)}.$$

Now the proposition follows from Pontryagin's duality.

The next lemma gives conditions under which an injective compact representation of the form (3.1) is minimal.

3.2. Lemma. *Let R be a Dedekind ring which is not a field, X be a torsion-free R-module, and* (3.1) *be an injective compact representation of X. Then* (3.1) *is minimal, iff for each $q\in\pi$ and for each copy $\tilde{\Xi}_{q^\infty}$ of Ξ_{q^∞} in $\Pi_{p\in\pi}\,\Xi^{a_p}_{p^\infty}\times\Xi^a$ we have*

$$r(X)\cap\tilde{\Xi}_{q^\infty}\neq\{0\}. \tag{3.2}$$

Proof. $\Pi_{p\in\pi}\,\Xi^{a_p}_{p^\infty}\times\Xi^a$ is torsion-free. Therefore, its Hensel submodules are of the type Ξ_{q^∞}. Now the lemma follows from Corollary 1.2.

If R is a field, then (3.1) reduces to

$$r\colon X\longrightarrow\Xi^a. \tag{3.3}$$

Therefore, from Lemma 3.1 it follows that (3.3) is the general form of the minimal compact representations.

3.3. Lemma. *Let R be a field, X be a linear space over R, and* (3.3) *be an injective compact representation of X. Then* (3.3) *is minimal iff for each copy $\tilde{\Xi}$ of Ξ in Ξ^a we have*

$$r(X)\cap\tilde{\Xi}\neq\{0\}. \tag{3.4}$$

Proof. Clear.

3. B. Necessary and sufficient condition. Lemmas 3.2 and 3.3 require to know more about the copies of Ξ_{q^∞} (resp. Ξ) in the right-hand side of (3.1) (resp. 3.3)). Let

$$i\colon\Xi_{q^\infty}\longrightarrow\prod_{p\in\pi}\Xi^{a_p}_{p^\infty}\times\Xi^a \tag{3.5}$$

be a continuous homomorphism. By composing i with the corresponding projections, we find continuous homomorphisms

$$i_{\alpha_p}\colon\Xi_{q^\infty}\longrightarrow\Xi_{p^\infty}\ \ (p\in\pi,\ \alpha_p\in a_p), \tag{3.6}$$

and

$$i_\alpha\colon\Xi_{q^\infty}\longrightarrow\Xi\ \ (\alpha\in a). \tag{3.7}$$

But if $p\neq q$, the unique homomorphism $R(p^\infty)\to R(q^\infty)$ is the trivial one. Therefore,

$$i_{\alpha_p}=0,\ p\neq q,\ \alpha_p\in a_p. \tag{3.8}$$

On the other hand, each homomorphism $h: R(q^\infty) \to R(q^\infty)$ has the form $h(x) = \sigma x \, (x \in R(q^\infty))$, where σ is an element of R_q. Therefore, for each $\alpha_q \in a_q$ there exists $\sigma_{\alpha_q} \in R_q$ with

$$i_{\alpha_q}(\xi) = \sigma_{\alpha_q} \xi \tag{3.9}$$

for each $\xi \in \Xi_{q^\infty}$. By Proposition 2.3., for each $\alpha \in a$ there exists $\sigma_\alpha \in Q_q$ (the q-adic field for R) with

$$i_\alpha(\xi) = \sigma_\alpha \xi \quad (\xi \in \Xi_{q^\infty}). \tag{3.10}$$

The above analysis permits to examine minimal compact representation in more detail.

3.4. Theorem. *Let R be a Dedekind ring which is not a field, and X be a torsion-free R-module. Then the minimal compact representations of X are the injective compact representations of the form* (3.1) *such that for each point*

$$(\sigma, \tau) \in R_q^{a_q} \times Q_q^{a} \setminus \{0\}, \quad q \in \pi$$

there exists $\xi \in \Xi_{q^\infty} \setminus \{0\}$ *with*

$$(0, \sigma\xi, \tau\xi) \in r(X). \tag{3.11}$$

Proof. Since the set

$$\{(0, \sigma\xi, \tau\xi) : \xi \in \Xi_{q^\infty}\}$$

is clearly a copy of Ξ_{q^∞} in $\prod_{p \in \pi} \Xi_{p^\infty}^{a_p} \times \Xi^a$, the necessity follows from Lemma 3.2.

Conversely, each copy $\tilde{\Xi}_{q^\infty}$ of Ξ_{q^∞} in $\prod_{p \in \pi} \Xi_{p^\infty}^{a_p} \times \Xi^a$ is an image of Ξ_{q^∞} by a continuous homomorphism (3.5). The coordinate homomorphisms (3.6) and (3.7) satisfy (3.8) — (3.10). Since i is a monomorphism, at least one of the conditions

$$\sigma = (\sigma_{\alpha_q})_{\alpha_q \in a_q} \neq 0 \text{ or } \tau = (\sigma_\alpha)_{\alpha \in a} \neq 0$$

holds. By assumption, there is $\xi \in \Xi_{q^\infty} \setminus \{0\}$ with (3.11). Then $i(\xi) \neq 0$, and $i(\xi) \in r(X) \cap \Xi_{q^\infty}$. Now the sufficiency follows from Lemma 3.2. This completes the proof of Theorem 3.4.

3.5. Theorem. *Let R be a field, and X be a linear space over R. Then the minimal compact representations of X are the injective compact representations of the form* (3.3) *such that for each point* $(\sigma_\alpha)_{\alpha \in a} \in Q^q \setminus \{0\}$ *there exists* $\xi \in \Xi \setminus \{0\}$ *with* $(\sigma_\alpha \xi)_{\alpha \in a} \in r(X)$.

Proof. Analogous to the proof of Theorem 3.4.

4. Minimal Compact Representation of Torsion-Free Modules in Special Cases. There are natural conditions under which minimal compact representations are of one of the forms

$$r: X \longrightarrow \prod_{p \in M} \Xi_{p^\infty}, \quad M \subset \pi \tag{4.1}$$

or

$$r: X \longrightarrow \Xi. \tag{4.2}$$

Let us note that an injective compact representation of the type (4.1) is minimal, iff for each $p\in M$ we have $r(X)\cap\Xi_{p^\infty}\neq\{0\}$ (see [9, *p.* 520] and Proposition 1.17). By Proposition 2.2, an injective representation of the form (4.2) is minimal iff for each maximal ideal p in R we have $r(X)\cap R^{\perp}_{(p)}\neq\{0\}$.

4. A. Minimal compact representations of the R-module R. By Theorem 3.4, minimal compact representations of the R-module R have the form

$$r:R\longrightarrow\Xi_{p^\infty},\quad p\in\pi. \tag{4.3}$$

Since r is a homomorphism, there is $\xi\in\Xi_{p^\infty}$ with $r(s)=s\xi$ for each $s\in R$. The density condition is fulfilled, iff $\xi\in\Xi_{p^\infty}\setminus t\,\Xi_{p^\infty}$. It is interesting to note that if the ideal p is of finite index, different ξ define equivalent compact representations. If the index of p is infinite, different ξ give equivalent compact representations iff they are colinear over R_p. By Theorem 1.18, in the last case, there are $2^{\mathrm{card}\,R/p}$ classes of equivalent minimal compact representations (4.3) of R.

If the ring R is a field, by Theorem 3.5 the minimal compact representations of R are of the type

$$r:R\longrightarrow\Xi. \tag{4.4}$$

Since the unique closed submodules of Ξ are 0 and Ξ, each injective representation (4.4) is minimal. Clearly for each r there exists $\xi\in\Xi\setminus\{0\}$, with $r(s)=s\xi$ for each $s\in R$. The density condition is always fulfilled.

4. B. Minimal compact representations of the R-module Q. The problem is already considered, if R is a field. Let the Dedekind ring R be not a field. It follows from Theorem 3.4 that the minimal compact representations of Q have the form

$$r:Q\longrightarrow\Xi. \tag{4.5}$$

Since H is a closed submodule of Ξ, and $H=\prod_{p\in\pi}\Xi_{p^\infty}$, by Theorem 2.4 it follows that if R has at least two maximal ideals, Q has no minimal compact representations. If the ring R is local, an injective compact representation (4.5) is minimal, iff $r(Q)\cap H\neq\{0\}$. By Theorem 2.5, the last condition guaranties also the density of $r(Q)$ in Ξ.

4. C. The case $\mathrm{card}\,X<2^{\aleph_0}$. This is another condition under which the conclusion of Lemma 3.1 is simplified.

4.1. Proposition. *Let R be a Dedekind ring which is not a field, and X be a torsion-free R-module with* $\mathrm{card}\,X<2^{\aleph_0}$. *Then the minimal compact representations of X have the form* (4.1) *with* $\mathrm{card}\,M\leq\mathrm{rank}\,X$, *or* (4.2).

Proof. Let p be a maximal ideal in R. Then

$$\mathrm{card}\,R_p\geq 2^{\aleph_0}. \tag{4.6}$$

Indeed, let $t\in p\setminus p^2$. Then the series $\Sigma_{\nu=1}^{\infty}s_\nu t^\nu$ is clearly convergent. The equality $\Sigma_{\nu=0}^{\infty}\varepsilon_\nu t^\nu=\Sigma_{\nu=0}^{\infty}\varepsilon'_\nu t^\nu$, where $\varepsilon_\nu,\ \varepsilon'_\nu\in\{0,1\}$, is fulfilled, iff $\varepsilon_\nu=\varepsilon'_\nu$ for each $\nu=0,1,2,\dots$ Thus (4.6) is proved.

By Lemma 3.1, every minimal compact representation of X has the form

$$r: X \longrightarrow \prod_{p \in \pi} \Xi_{p^\infty}^{a_p} \times \Xi^a. \tag{4.7}$$

Assume that there exists a minimal compact representation of X with a form different from (4.1) and (4.2). Since Ξ contains a copy of each Ξ_{p^∞} $(p \in \pi)$, the right-hand side ot (4.7) contains a copy of $\Xi_{p^\infty}^2$ for a $p \in \pi$. Let $\sigma \in R_p$. Then the set

$$M_\sigma = \{(\xi, \sigma\xi) : \xi \in \Xi_{p^\infty}\} \tag{4.8}$$

is a non-zero closed submodule of $\Xi_{p^\infty}^2$. By (4.6), the modules (4.8) are at least $2^{\aleph_0}$. Moreover, $M_\sigma \cap M_{\sigma'} = \{0\}$, if $\sigma \neq \sigma'$. Since r has to be essential, then card $X \geq 2^{\aleph_0}$, in contradiction with card $X < 2^{\aleph_0}$. This completes the proof of the proposition.

4. D. Minimal compact representations of finitely generated torsion-free modules. It turns out that in general minimal compact representations of a finitely generated torsion-free module over Dedekind ring also have the form (4.1) or (4.2). The following Lemma is basic for the proof.

4.2. Lemma. *Let $\mathscr{K}$ be a field, and, L be an extension of $\mathscr{K}$. Suppose there are finitely many elements $\xi_1, \xi_2, \ldots, \xi_n$ of L such that for each $\xi \in L$ there exist $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ from $\mathscr{K}$ with*

$$\xi = \sum_{\nu=1}^{n} a_\nu \xi_\nu \Big/ \sum_{\nu=1}^{n} b_\nu \xi_\nu. \tag{4.9}$$

Then L is a finite algebraic extension of $\mathscr{K}$.

Proof. Assume the contrary. Then there exists a non-void transcendental basis $\{x_1, x_2, \ldots, x_m\} \subset \{\xi_1, \xi_2, \ldots, \xi_n\}$. Then the subfield of L generated by $x_1, x_2, \ldots, x_m$ coincides with the field $\mathscr{K}(x_1, x_2, \ldots, x_m)$ of the rational functions of m variables and coefficients from $\mathscr{K}$. At the same time L is a finite algebraic extension of $\mathscr{K}(x_1, x_2, \ldots, x_m)$.

If $L = \mathscr{K}(x_1, x_2, \ldots, x_m)$, the right-hand side of (4.9) is a rational function of $x_1, x_2, \ldots, x_m$ with bounded degree such that the upper bound does not depend on a_ν and b_ν. Therefore, (4.9) can not be fulfilled for each $\xi \in L$. Thus the case $L = \mathscr{K}(x_1, x_2, \ldots, x_m)$ is impossible.

So it only remains to consider the case $L \neq \mathscr{K}(x_1, x_2, \ldots, x_m)$. Then there are algebraic over $\mathscr{K}(x_1, x_2, \ldots, x_m)$ elements $y_1, y_2, \ldots, y_t$ of L and rational functions $R_\nu = R_\nu(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_t)$ with coefficients from $\mathscr{K}$ such that for each $\xi \in L$ there exist elements $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ of $\mathscr{K}$ with

$$\xi = \sum_{n=1}^{n} a_\nu R_\nu \Big/ \sum_{\nu=1}^{n} b_\nu R_\nu. \tag{4.10}$$

We shall use induction with respect to t. The case $t=0$ is already considered. Let $t=1, 2, \ldots,$ and we have obtained a contradiction for $t-1$. Let $l=1, 2, \ldots$ be the degree of y_t over the field

$$L' = \mathscr{K}(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_{t-1}).$$

Then

$$R_\nu = \sum_{\lambda=0}^{l-1} R_{\nu\lambda}\, y_t^\lambda, \qquad \nu = 1, 2, \ldots, n, \tag{4.11}$$

where

$$R_{\nu\lambda} = R_{\nu\lambda}(x_1, x_2, \ldots, x_m;\ y_1, y_2, \ldots, y_{t-1}) \tag{4.12}$$

are rational functions with coefficients from $\mathscr{K}$. From (4,10) and (4.11) it follows

$$\sum_{\lambda=0}^{l-1} [\xi \sum_{\nu=1}^{n} b_\nu R_{\nu\lambda} - \sum_{\nu=1}^{n} a_\nu R_{\nu\lambda}]\, y_t^\lambda = 0. \tag{4.13}$$

If $\xi \in L'$, (4.13) is an equation with coefficients from L'. Since the degree of y_t is l, the coefficients of (4.13) have to be zero, i. e.

$$\xi \sum_{\nu=1}^{n} b_\nu R_{\nu\lambda} - \sum_{\nu=1}^{n} a_\nu R_{\nu\lambda} = 0, \qquad \lambda = 0, 1, \ldots, l-1. \tag{4.14}$$

but

$$\sum_{\lambda=0}^{l-1} \sum_{\nu=1}^{n} b_\nu R_{\nu\lambda} = \sum_{\nu=1}^{n} b_\nu R_\nu \neq 0,$$

by (4.10). Hence there exists $\lambda = 0, 1, 2, \ldots, l-1$ with $\Sigma_{\nu=1}^{n} b_\nu R_{\nu\lambda} \neq 0$. Now (4.14) implies

$$\xi = \sum_{\nu=1}^{n} a_\nu R_{\nu\lambda} \Big/ \sum_{\nu=1}^{n} b_\nu R_{\nu\lambda} = \sum_{\nu=1}^{n} \sum_{\lambda=1}^{l-1} a_{\nu\lambda} R_{\nu\lambda} \Big/ \sum_{\nu=1}^{n} \sum_{\lambda=0}^{l-1} b_{\nu\lambda} R_{\nu\lambda}, \tag{4.15}$$

where $a_{\nu\lambda}$ and $b_{\nu\lambda}$ are suitable elements of $\mathscr{K}$. Thus we found for L' a representation (4.15). But by (4.12), the number of variables y in (4.15) are $t-1$, in contradiction to the inductive hypothesis. This completes the proof of the lemma.

4.3. Proposition. *Let R be a Dedekind domain which is not a field, and for each $p \in \pi$ the rank of the R-module R_p be infinite. Suppose that X is a finitely generated torsion-free R-module. Then the minimal compact representations of X have the form* (4.1) *with* card $M \leq$ rank X, *or* (4.2).

Proof. Assume the contrary. As has been remarked before in the proof of Proposition 4.1, there exists $p \in \pi$ such that $r(X)$ contains non-zero elements of each of the modules (4.8). Since the rank of X is finite, there are elements $\sigma_1, \sigma_2, \ldots, \sigma_n$ of R_p and elements $\xi_1, \xi_2, \ldots, \xi_n$ of $\Xi_{p\infty}$ such that for each $\sigma \in R_p$ there exist $\xi_\sigma \in \Xi_{p\infty} \setminus \{0\}$, $a_\sigma \in R \setminus \{0\}$, and $a_{\nu\sigma} \in R\, (\nu = 1, 2, \ldots, n)$ with

$$a_\sigma \xi_\sigma = \sum_{\nu=1}^{n} a_{\nu\sigma} \xi_\nu, \tag{4.16}$$

$$\sigma a_\sigma \xi_\sigma = \sum_{\nu=1}^{n} \sigma_\nu a_{\nu\sigma} \xi_\nu.$$

Let

$$\xi_1, \xi_2, \ldots, \xi_m, \qquad m \leq n, \tag{4.17}$$

be a maximal subsystem of $\{\xi_\nu\}_{\nu=1}^n$, independent over R_p. Then there exist $\sigma_0 \in R_p \setminus \{0\}$ and $\lambda_{\mu\nu} \in R_p$ with

(4.18) $$\sigma_0 \xi_\nu = \sum_{\mu=1}^{m} \lambda_{\mu\nu} \xi_\mu, \qquad \nu = m+1, \ldots, n.$$

From (4.15), (4.17) and (4.18) it follows

(4.19) $$\sigma\sigma_0 a_\sigma \xi_\sigma = \sum_{\mu=1}^{m} \sigma\sigma_0 a_{\mu\sigma} \xi_\mu + \sum_{\nu=m+1}^{n} \sigma a_{\nu\sigma} \sum_{\mu=1}^{m} \lambda_{\mu\nu} \xi_\mu,$$

and

(4.20) $$\sigma\sigma_0 a_\sigma \xi_\sigma = \sum_{\mu=1}^{m} \sigma_\mu \sigma_0 a_{\mu\sigma} \xi_\mu + \sum_{\nu=m+1}^{n} \sigma_\nu a_{\nu\sigma} \sum_{\mu=1}^{m} \lambda_{\mu\nu} \xi_\mu.$$

Since $\sigma\sigma_0 a_\sigma \xi_\sigma \neq 0$, there is μ such that the coefficients of ξ_μ in (4.19) and (4.20) are different than zero. If, for example, $\mu = 1$, it follows from (4.19) and (4.20)

$$\sigma(\sigma_0 a_{1\sigma} + \sum_{\nu=m+1}^{n} \lambda_{1\nu} a_{\nu\sigma}) = \sigma_1 \sigma_0 a_{1\sigma} + \sum_{\nu=m+1}^{n} \lambda_{1\nu} \sigma_\nu a_{\nu\sigma}.$$

Hence

$$\sigma = \left[a_{1\sigma} + \sum_{\nu=m+1}^{n} a_{\nu\sigma} \frac{\lambda_{1\nu}}{\sigma_0} \right] \Big/ \left[\sigma_1 a_{1\sigma} + \sum_{\nu=m+1}^{n} a_{\nu\sigma} \frac{\lambda_{1\nu}\sigma_\nu}{\sigma_0} \right].$$

Since $\lambda_{1\nu}$ and σ_ν do not depend on σ, and $a_{\nu\sigma} \in k$, Lemma 4.2 implies that the rank of R_p over R is finite, in contradiction to the condition of the proposition. This completes the proof.

The following two propositions due to V. Tchoukanov (unpublished) elucidate the range of applications of Proposition 4.3.

4.4. Proposition. *Let R be a Dedekind domain, and p be a maximal ideal in R such that the rank of R_p over R is one. Then $R_p = R$.*

4.5. Proposition. *Let R be a Dedekind ring with characteristic zero, and p be a maximal ideal in R such that the rank of R_p over R is finite. Then $R_p = R$.*

5. A Topological Characterization of Maximal Hensel Modules over a Dedekind Ring. Here we generalize a part of [9]. Namely we shall describe those compact modules C over a given Dedekind ring, which induce minimal topology on each submodule $X \subset C$. It turns out that in general the torsion-free modules with that property coincide with the maximal Hensel modules.

5. A. The statement of the main result. Let R be a Dedekind ring. We call a compact R-module C essential when C induces minimal precompact topology on each submodule $X \subset C$. Equivalently, C is essential if for each submodule $X \subset C$ the embedding $X \subset \bar{X}$ is a minimal compact representation of X. Also C is essential, iff for each submodule $X \subset C$ every non-zero closed submodule of $\bar{X}$ contains non-zero elements of X (see [9, p. 519]. The following theorem gives a concrete description of the essential modules.

5.1. Theorem. *Let R be a Dedekind ring.*

Then the periodic essential R-modules are the products of the form

$$(5.1) \qquad C = \Xi_{p_1^{k_1}} \times \Xi_{p_2^{k_2}} \times \cdots \times \Xi_{p_l^{k_l}},$$

where l and k_λ are positive integers, p_λ are non-zero maximal ideals in R, and certain ideals $p_{\lambda_1}, p_{\lambda_2}, \ldots, p_{\lambda_m}$ may coincide iff they have finite index.

If the ring R is non-compact with respect to any of the p-adic topologies, the non-periodic essential R-modules are

$$(5.2) \qquad C = \Xi_{p^\infty}, \quad p \in \pi.$$

If the ring R is compact with respect to certain p-adic topology, then R has unique maximal ideal p, and then the non-periodic essential R-modules are the products of the form

$$(5.3) \qquad C = \Xi_{p^\infty}^m \times \Xi_{p^{k_1}} \times \cdots \times \Xi_{p^{k_l}},$$

where m, k_1, $k_2, \ldots, k_l$ are positive integers, and l is a non-negative integer.

The proof of that theorem is long, and is broken into parts in accordance with the titles of the subsections.

5. B. The modules (5.1), (5.2) and (5.3) are essential. Consider first the modules (5.1). Let $p_1, p_2, \ldots, p_l$ be different maximal ideals in R. Since each submodule X of C is periodic, we have

$$X = \prod_{\lambda=1}^{l} X_\lambda,$$

where $X_\lambda = X \cap \Xi_{p_\lambda^{k_\lambda}}$. Then

$$\overline{X} = \prod_{\lambda=1}^{l} \overline{X}_\lambda.$$

Being a closed submodule of $\Xi_{p_\lambda^{k_\lambda}}$, the module $\overline{X}_\lambda$ has the form $\Xi_{p_\lambda^{m_\lambda}}$. Therefore, without loss of generality, we may assume that $\overline{X}_\lambda = \Xi_{p_\lambda^{k_\lambda}}$. Let M be a closed non-zero submodule of C. Since $M = \prod_{\lambda=1}^{l} M_\lambda$, there is λ with $M_\lambda \neq \{0\}$. Hence $M_\lambda = s\,\Xi_{p_\lambda^{k_\lambda}}$, where $s \bar{\in} p_\lambda^{k_\lambda}$. Now it is clear that

$$X \cap M \supset X_\lambda \cap M_\lambda \neq \{0\}$$

and so the embedding $X \subset \overline{X}$ is essential.

If certain ideals p_λ coincide, they will have by assumption a finite index, and so the modules $\Xi_{p_\lambda^{k_\lambda}}$ shall be finite. Now it is clear that grouping together the multiples with equal indices p, we may establish that (5.1) are essential in the same way.

Since each non-zero submodule of Ξ_{p^∞} is of the same type, the statement about (5.2) will be proved if we show that for each dense submodule X of

$\Xi_{p\infty}$ the embedding $X\subset\Xi_{p\infty}$ is essential, and that is clear because the closed non-zero submodules of $\Xi_{p\infty}$ have the form $s\,\Xi_{p\infty}$ $(s\in R\setminus\{0\})$.

At the end we consider the modules (5.3). Then $R=R_p$, and since R_p is compact, then $\Xi_{p\infty}=R_p$, by Corollary 1.12. Now it is clear that the closed submodules of (5.3) are also R_p-modules. Moreover, as R_p-modules they are finitely generated. Since R_p is a principal ideal domain, the closed R-submodules of (5.3) are of the same form. Hence the statement will be proved if we show that for each dense submodule X of (5.3) the embedding $X\subset C$ is essential. Of course, X is finitely generated, and since R_p is compact, X is also compact. Therefore, $X=C$, and the modules (5.3) are essential.

5. C. Lemmas. This section contains auxiliary statements used in the proof of the uniqueneses in Theorem 5.1.

5.2. Lemma. *Let R be a Dedekind ring, C be essential R-module, and X_1, X_2 be submodules of C such that $X_1\cap X_2=\{0\}$ and X_1 is closed. Then $X_1\cap\bar{X}_2=\{0\}$, and the natural mapping $h\colon X_1\times X_2\to X_1+X_2$ is a homeomorphism.*

Proof. See [9, p. 534].

5.3. Lemma. *Let R be a Dedekind ring, and p, q be different maximal ideal in R. If C is an essential R-module, and $k=1,2,\ldots,\infty$, then C cannot contain copies of $\Xi_{p\infty}$ and Ξ_{q^k} simultaneously.*

Proof. Assume that $\Xi_{p\infty}\subset C$ and $\Xi_{q^k}\subset C$. Since $\Xi_{p\infty}$ and Ξ_{q^k} have not common non-zero submodules, by compactness we have

$$\Xi_{p\infty}\times\Xi_{q^k}\subset C.$$

Let $\xi\in\Xi_{p\infty}$, and $\eta\in\Xi_{q^k}$ generate dense submodules of $\Xi_{p\infty}$ and Ξ_{q^k}. Consider the submodule

$$E=\{(s\xi,\ s\eta): s\in R\}$$

of $\Xi_{q\infty}\times\Xi_{q^k}$.

First we prove that $(\xi,0)\in\bar{E}$. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of elements of R convergent to 1 in p-adic topology, and to 0 — in q-adic. Then

$$\lim_{n\to\infty}(s_n\xi,\ s_n\eta)=(\xi,\ 0).$$

Therefore, $(\xi,0)\in\bar{E}$, and hence $\Xi_{p\infty}\subset\bar{E}$ Similarly, $\Xi_{q^k}\subset\bar{E}$, and so E is dense in $\Xi_{p\infty}\times\Xi_{q^k}$. On the other hand, $E\cap\Xi_{q^k}=\{0\}$, and it turns out that C is not essential. This completes the proof of the lemma.

5.4. Lemma. *Let R be a Dedekind ring, C be a compact R-module, and $\{\Xi_n\}_{n=1}^{\infty}$ be an infinite sequence of Hensel submodules of C. If*

$$(\Xi_1+\Xi_2+\cdots+\Xi_n)\cap\Xi_{n+1}=\{0\} \tag{5.4}$$

for each $n=1,2,\ldots$, then the module C is not essential.

Proof. The direct sum

$$E=\Xi_1\oplus\Xi_2\oplus\cdots\oplus\Xi_n\oplus\cdots \tag{5.5}$$

is clearly isomorphic with a submodule of C. Assume that C is essential. First we prove that the natural embedding

$$\varphi : E \longrightarrow \prod_{n=1}^{\infty} \Xi_n \tag{5.6}$$

is continuous. Let

$$\pi_n : \prod_{n=1}^{\infty} \Xi_n \longrightarrow \Xi_n, \; n = 1, 2, \ldots,$$

be the n-th projection. By Lemma 5.2 with $X_1 = \Xi_n$, $X_2 = \Xi_1 \oplus \cdots \oplus \Xi_{n-1} \oplus \Xi_{n+1} \oplus \cdots$ the mappings $\pi_n \varphi$ are continuous, and hence the same is true for φ.

Since $\prod_{n=1}^{\infty} \Xi_n$ is compact, φ has a continuous extension

$$\bar{\varphi} : \bar{E} \longrightarrow \prod_{n=1}^{\infty} \Xi_n.$$

Clearly $E \cap \operatorname{Ker} \bar{\varphi} = \{0\}$, and since the embedding $E \subset \bar{E}$ is essential, then $\operatorname{Ker} \bar{\varphi} = \{0\}$. Therefore, $\bar{\varphi}$ is a homeomorphism. Hence without loss of generality we may assume that $\Pi_{n=1}^{\infty} \Xi_n$ is a submodule of C, and φ is the natural embedding.

First we consider the case when infinitely many Ξ_n coincide. We may assume that this is true for all Ξ_n. Let Δ be the diagonal of $\Pi_{n=1}^{\infty} \Xi_n$. It is clear that $E \cap \Delta = \{0\}$, since all but a finite number of the coordinates of the elements of E are zero, by (5.5). This contradiction proves the statement, if infinitely many Ξ_n coincide.

Let now all Ξ_n be different. Then $\Pi_{n=1}^{\infty} \Xi_n$ has non-torsion element ξ. Then the module $R\xi$ is isomorphic with R, and the induced topology is minimal and precompact. Now by 4.A, there exists a maximal ideal p in R with $\overline{R\xi} = \Xi_{p^\infty}$. Therefore, $\Pi_{n=1}^{\infty} \Xi_n$ contains the Hensel module Ξ_{p^∞}. If among the ideals associated with Ξ_n there exists one different than p, we obtain a contradiction with Lemma 5.3. Hence we have only to consider the case $\Xi_n = \Xi_{p^{k_n}}$. Let

$$\psi_n : \Xi_{p^\infty} \longrightarrow \Xi_n, \quad n = 1, 2, \ldots,$$

be the corresponding natural homomorphisms, and

$$\psi : \Xi_{p^\infty} \longrightarrow \prod_{n=1}^{\infty} \Xi_n$$

be their product. Now $E \cap \psi(\Xi_{p^\infty}) = \{0\}$, and this is a contradiction. This completes the proof of the lemma.

5.5. Lemma. *Let R be a Dedekind ring, and C be an essential R-module. Then the number of the maximal ideals in R such that C contains Hensel modules associated with p is finite.*

Proof. Assume the contrary. Then there exist a sequence $\{p_n\}_{n=1}^{\infty}$ of different maximal ideals in R and a sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers with

$$\Xi_{p_n^{k_n}} \subset C.$$

It is not difficult to prove by induction that

$$\Xi_{p_1^{k_1}} + \Xi_{p_2^{k_2}} + \cdots + \Xi_{p_n^{k_n}} = \Xi_{p_1^{k_1}} \times \Xi_{p_2^{k_2}} \times \cdots \times \Xi_{p_n^{k_n}}.$$

Now Lemma 5.4 leads to contradiction. This completes the proof of the lemma

5. D. The uniqueness of the periodic essential modules. Let R again be a Dedekind ring. We prove that every periodic essential R-module C has the form (5.1). Since C has periodic elements, R is not a field. Therefore, the maximal ideals of R are different than zero. By Lemma 5.5, there is only a finite nimber of maximal ideals $p_1, p_2, \ldots, p_n$, such that C contains associated modules of Hensel. Since C is periodic, $C = \bigoplus_{p \in \pi} C_p$, where C_p are the corresponding p-components. It is not difficult to see that if $C_p \neq \{0\}$, then $p = p_\nu$ $(\nu = 1, 2, \ldots, n)$. Hence for each $x \in C$ there exists $m = 1, 2, \ldots$ with $(p_1 p_2 \ldots p_n)^m x = 0$. Therefore,

$$C = \bigcup_{m=1}^{\infty} C_m, \tag{5.7}$$

where $C_m = \{x \in C : (p_1 p_2 \ldots p_n)^m x = 0\}$. By Bair's category theorem, there exists m such that C_m is open in C. Therefore, the group C/C_m is compact and discrete, and thus finite. Now (5.7) implies that $(p_1 p_2 \ldots p_n)^m C = 0$, and hence $C_p = \{x \in C : p^m x = 0\}$. In this way we have proved that the modules C_p are closed. Therefore, $C = \Pi_{\nu=1}^{n} C_{p_\nu}$.

Now we prove that each element ξ of C_{p_ν} belongs to a Hensel submodule. $\Xi_{p_\nu^k} \subset C_{p_\nu}$. Let (ξ) be the R-module generated by ξ. Then $(\xi) = R/p_\nu^k = R(p_\nu^k)$

Let η be element of $R(p_\nu^k)$ with $p\eta = 0$, and $\chi_0 : R(p_\nu^k) \to \mathrm{T}$ be a continuous character with $\chi_0(\eta) \neq 0$. Then the R-module $R\chi_0$ separates the points of $R(p_\nu^k)$ Since the topology of $R(p_\nu^k)$ is minimal, $R\chi_0$ is the module of all continuous characters of $R(p_\nu^k)$(see [9,p. 519]). On the other hand, $R\chi_0 = R(p_\nu^k)$. Hence $\overline{(\xi)} = (R(p_\nu^k))^* = \Xi_{p_\nu^k}$.

First we consider the case when the ideal p_ν has an infinite index. Denote by $\Xi_{p_\nu^k}$ the Hensel submodule of C_{p_ν} with maximal k. We prove that $C_{p_\nu} = \Xi_{p_\nu^k}$. Let $\Xi_{p_\nu^l}$ be a Hensel submodule of C_{p_ν}. By induction we shall show that $\Xi_{p_\nu^l} \subset \Xi_{p_\nu^k}$. Let $l = 1$. Since Ξ_{p_ν} has not proper closed submodules, from $\Xi_{p_\nu} \not\subset \Xi_{p_\nu^k}$ it follows $\Xi_{p_\nu} \cap \Xi_{p_\nu^k} = \{0\}$. Therefore, $\Xi_{p_\nu} \times \Xi_{p_\nu^k} \subset C$, and so $\Xi^2_{p_\nu} \subset C$. Since the index of p_ν is infinite, then

$$\operatorname{card} \Xi_{p_\nu} = \operatorname{card} (R/p_\nu)^* = 2^{\operatorname{card} R/p_\nu}.$$

Therefore, Ξ_{p_ν} has at least two elements ξ_1 and ξ_2, independent over R/p_ν. Consider the R-submodule $M = \{(s_1\xi_1, s_2\xi_2) : s_1, s_2 \in R/p_\nu\}$ of $\Xi^2_{p_\nu}$. Clearly M is dense in $\Xi^2_{p_\nu}$. Hence the embedding $M \subset \Xi^2_{p_\nu}$ is essential. But from the independence of ξ_1 and ξ_2 it follows that M does not contain elements of the diagonal of $\Xi^2_{p_\nu}$ different than zero, which is a contradiction. Therefore, $\Xi_{p_\nu} \subset \Xi_{p^k_\nu}$ and the statement is proved for $l=1$. Now it follows that if $\zeta \in C_{p_\nu}$ and $p_\nu \zeta = 0$, then $\zeta \in \Xi_{p^k_\nu}$.

Suppose now that the assertion is true for $l-1$, and consider a submodule $\Xi_{p^l_\nu}$ of C_{p_ν}. Then $t\Xi_{p^l_\nu} = \Xi_{p^{l-1}_\nu} \subset \Xi_{p^k_\nu}$ $(t \in p_\nu \setminus p^2_\nu)$. If $\xi \in \Xi_{p^l_\nu}$, then $t\xi \in \Xi_{p^k_\nu}$. Since k is maximal, $l \leq k$. On the other hand, $t^{l-1}(t\xi)=0$. Hence there exists $\eta \in \Xi_{p^k_\nu}$ with $t\eta = t\xi$. Therefore, $t(\eta-\xi)=0$, and $\eta-\xi \in \Xi_{p^k_\nu}$. Thus we see that $\xi \in \Xi_{p^k_\nu}$, and the statement is proved if the index of p_ν is infinite.

Let now the index of p_ν be finite. Then $\Xi_{p^k_\nu} = R(p^k_\nu)$. Let $M_k = \{x \in C_{p_\nu} : p^k_\nu x = 0\}$ $(k=1, 2, \ldots)$. Clearly M_1 is a linear space over R/p_ν. The dimension of M_1 is finite, by Lemma 5.4. Therefore, M_1 is a finite set. From the exactness of the sequence $0 \longrightarrow M_1 \longrightarrow M_k \xrightarrow{t} M_{k-1}$ by induction it follows that M_k is finite for each positive integer k. But there exists k with $C_{p_\nu} = M_k$. Therefore, C_{p_ν} is a finite module. Since C_{p_ν} is a module over the ring R_{p_ν} which is a principal ideal domain,

$$C_{p_\nu} = \prod_{\lambda=1}^{l} R_{p_\nu} / p_\nu^{k_\lambda} = \prod_{\lambda=1}^{l} \Xi_{p_\nu^{k_\lambda}}.$$

Thus the uniqueness of the periodic essential modules is established.

5. E. The uniqueness of the non-periodic essential modules when p-adic topologies are non-compact. Let R be a Dedekind ring. We prove that, if R is non-compact with respect to p-adic topologies, then every non-torsion essential R-module C has the form (5.2).

Let ξ be a non-torsion element of C. As has been remarked in the proof of Lemma 5.4, $\overline{R\xi} = \Xi_{p^\infty} \subset C$. Now from Lemma 5.3 if follows that each Hensel submodule of C is associated with p.

We now prove that Ξ_{p^∞} has at least two elements ξ_1 and ξ_2, independent over R. If R_p is non-compact, this follows from Theorem 1.18. Let R_p be compact. Then $R \neq R_p$. Now Proposition 4.4 implies that the rank of R_p over R is at least two. On the other hand, $\Xi_{p^\infty} = R_p$, by Corollary 1.12. Therefore, the rank of Ξ_{p^∞} over R is at least two.

Without loss of generality we may suppose that ξ_1 does not belong to the maximal submodule of Ξ_{p^∞}. We are ready to prove that C is torsion-free. Assume that a is a periodic element of C, and consider the submodule $X = \{s_1\xi_1 + s_2\xi_2 + s_2 a : s_1, s_2 \in R\}$ of C. Since ξ_1 and ξ_2 are independent, X is torsion-free. On the other hand, the embedding $X \subset \overline{X}$ is essential. Therefore, $\overline{X}$ is torsion-free. Let $s_2 = 1$. Since $R\xi_1$ is a dense submodule of Ξ_{p^∞}, there exists a sequence of values of s such that $s\xi_1$ converges to $-\xi_2$. Therefore, $a \in \overline{X}$ which is a contradiction.

Assume that $\Xi^2_{p^\infty} \subset C$. Consider the submodule $X = \{(s_1\xi_1, s_2\xi_2) : s_1, s_2 \in R\}$ of $\Xi^2_{p^\infty}$. Clearly X is dense in $\Xi^2_{p^\infty}$ and so the embedding $X \subset \Xi^2_{p^\infty}$ has to be essential. But X has not non-zero common points with the diagonal of $\Xi^2_{p\infty}$ which is a contradiction. Therefore, the copies of $\Xi_{p\infty}$ in C have non-trivial intersections. Now Lemma 5.6 implies that the unique Hensel modules in C have the form $\Xi_{p\infty}$. It is not difficult to deduce from here that if Ξ' and Ξ'' are Hensel modules in C, then $\Xi'\subset\Xi''$ or $\Xi''\subset\Xi'$.

We are ready to prove that among the copies of $\Xi_{p\infty}$ in C there is a maximal one. Assume the contrary. From the elementary properties of $\Xi_{p\infty}$ it follows $Q\subset C$. If R has at least two maximal ideals,this contradicts 4. B. Let now R be local, and p be the unique maximal ideal of R. Then $\overline{Q}\supset\Xi$. But there are submodules of Ξ such that the induced topology is not minimal. To see that, let $\xi\neq 0$ be element of Ξ with $s\xi\in H$ for each $s\in R\setminus\{0\}$. By Theorem 2.5, $Q\xi$ is a dense submodule of Ξ, and the embedding $Q\xi\subset\Xi$ is not essential, since $Q\xi\cap H=\{0\}$. The contradiction shows that there is a maximal copy of $\Xi_{p\infty}$ in C.

Let $c\in C$. Then $\overline{Rc}\Rightarrow\Xi_{p\infty}\subset C$, and so c belongs to the maximal copy of $\Xi_{p\infty}$ in C. Therefore $C=\Xi_{p\infty}$, and the proof is completed.

5. F. The uniqueness of the non-periodic essential modules when a p-adic topology is compact. Let R be a Dedekind ring, and p be such a maximal ideal in R that the p-adic topology in R is compact. Then $R=R_p$ and there are not maximal ideals in R different than p. By Corollary 1.12, $R_{p\infty}=R_p$.

Now Lemma 5.4 implies that the rank of C over R is finite. Let rank $C=m$, and $\Xi^m_{p\infty}$ be a copy of R^m_p in C. For arbitrary $l=1, 2, \dots,$ let $M_l=\{\xi\in C: p^l\xi\subset\Xi^m_{p\infty}\}$. Clearly

$$C=\bigcup_{l=1}^{\infty} M_l,$$

and the Bair theorem implies that there exists $l=1, 2, \dots$ with

$$p^lC\subset\Xi^m_{p\infty}. \tag{5.8}$$

Let T be the periodic submodule of C. Clearly $T=T_p$. If $Z_k=\{\xi\in T: p^k\xi=0\}$, it follows from Lemma 5.4 that Z_1 is a finite-dimensional linear space over the field R_p/pR_p. Since the index of p is finite, Z_1 is also finite. By induction, each of the sets Z_k is finite. On the other hand, it follows from (5.8) that $Z_l=T$, and hence T is finite.

Since p^lC is a submodule of the free R_p-module $\Xi^m_{p\infty}$, and R_p is a principal ideal domain, the module p^lC is free. Let $t^l\xi_1, t^l\xi_2, \dots, t^l\xi_n$ be a basis of p^lC. Clearly C is generated by $\xi_1, \xi_2, \dots, \xi_n$ and the elements of T. Hence C is a finitely-generated R_p-module. Therefore, C has the form (5.3), and the desirable uniqueness is established.

In this way the proof of Theorem 5.1 is also completed.

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