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TOPOLOGICAL CHARACTERIZATIONS OF THE TYCHONOFF CUBES I^τ

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Every Tychonoff cube I^τ may be broken into a grid by a finite systems of parallel hyperplanes. Thus we can get a sequence of inscribed grids which separates points of I^τ. Conversely, it was found that, if a compact space X is spanned to its essential system $\mathscr{F}=\{F_{-s}, F_{+s}: s\in S\}$ and there exists a sequence of inscribed "grids" which separates points of X, then X is homeomorphic to I^τ, where $\tau=|S|$. Here we define a "grid" by means of a finite systems of "parallel" partitions between the sets F_{-s} and F_{+s}.

This paper sets forth to discuss some "inner" characterizations of any topological space, which is homeomorphic with some cube I^τ, where τ is a finite or infinite cardinal number. The results below are contained in the author's Ph. D. thesis "Cantor manifolds" Sofia University, 1980 (unpublished).

Preliminary Notions. Let S be a set and τ is its power. We shall say that the system $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ is a τ-system in the space X if for each $s\in S$, F_{-s} and F_{+s} are closed disjoint subsets of X. The system $\mathscr{F}$ is called essential in X, if for any choice of the partitions C_s in X between the sets F_{-s} and F_{+s} we have $\cap\{C_s: s\in S\}\neq\emptyset$.

It is well known [1, p. 67; 27) that the τ-system of all pairs $(I^\tau_{-s}, I^\tau_{+s})$ of opposite sides I^τ_{-s} and I^τ_{+s} of I^τ is essential in I^τ.

Let A and B be two closed and disjoint subsets of the normal space X. The ordered system $\lambda=(L_1, L_2, \dots, L_n)$ will be called a spectrum in X with respect to the pair (A, B) if: 1) L_ν is closed subset of X, $\nu=1, 2, \dots, n$ 2) $1\leq i<j\leq n$ implies $L_i\subset \operatorname{Int} L_j$; and 3) $A\subset \operatorname{Int} L_1$, $B\cap L_n=\emptyset$.

Evidently such spectrum λ exists in every normal space.

Further we put:

$$[L_i, L_j]_\lambda=L_j\setminus \operatorname{Int} L_i \quad \text{for} \quad 1\leq i<j\leq n,$$

$$[L_0, L_j]_\lambda=L_j \quad \text{and} \quad [L_i, L_{n+1}]_\lambda=X\setminus \operatorname{Int} L_i \quad \text{for} \quad i, j=1, 2, \dots, n.$$

Every spectrum λ with respect to (A, B) induces a closed cover $\Pi(\lambda)=\{[L_i, L_{i+1}]_\lambda: i=0, 1, 2, \dots, n\}$ of X.

Let $\mu=(M_1, M_2, \dots, M_r)$ be another spectrum in X with respect to (A, B). We shall say that μ is a refinement of λ and set $\mu<\lambda$ or $\lambda>\mu$, if $\{L_1, L_2, \dots, L_n\}\subset\{M_1, M_2, \dots, M_r\}$. In particular $n\leq r$.

Similarly we shall say that μ is a strong refinement of λ and set $\mu\ll\lambda$ or $\mu\gg\lambda$ if $\mu<\lambda$ and for each $i=0, 1, 2, \dots, n$ there exists an index $k=k(i)$ with $L_i\subset \operatorname{Int} M_k\subset M_k\subset \operatorname{Int} L_{i+1}$ (here we put $S_0=A$ and $L_{n+1}=X$).

It is clear that the relations "$<$" and "$\ll$" are transitive.

Let $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ be a τ-system in X and let put $\vec{\mathscr{F}}=\{(F_{-s}, F_{+s})\}_{s\in S}$. We shall say that a grid $\mathscr{L}$ in X with respect to $\vec{\mathscr{F}}$ is given, if for every $s\in S$ we have a spectrum $\lambda_s=(L_1^s, L_2^s, \ldots, L_{n_s}^s)$ with respect to (F_{-s}, F_{+s}). Then we put $\mathscr{L}=\{\lambda_s\}_{s\in S}$. If $\mathscr{M}=\{\mu_s\}_{s\in S}$ is another grid with respect to $\vec{\mathscr{F}}$, we shall say that $\mathscr{M}$ is a refinement (resp. a strong refinement) of $\mathscr{L}$ if $\mu_s<\lambda_s$ (resp. $\mu_s\ll\lambda_s$) for each $s\in S$. Then we put $\mathscr{M}<\mathscr{L}$ or $\mathscr{L}>\mathscr{M}$ (resp. $\mathscr{M}\ll\mathscr{L}$ or $\mathscr{L}\gg\mathscr{M}$).

Every grid $\mathscr{L}$ induces a closed cover $\square(\mathscr{L})=\Lambda\{\Pi(\lambda_s): s\in S\}$ of X. It is clear that $\mathscr{M}<\mathscr{L}$ implies that the cover $\square(\mathscr{M})$ is a refinement of the cover $\square(\mathscr{L})$.

Let us consider a sequence

$$\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n, \ldots \tag{2}$$

of grids $\mathscr{L}_n$ in X with respect to $\vec{\mathscr{F}}$ for which $\mathscr{L}_1>\mathscr{L}_2>\cdots>\mathscr{L}_n>\cdots$ We shall say that this sequence becomes tinier if for every open cover $\mathscr{U}$ of X there exists a number $k=k(\mathscr{U})$ such that the cover $\square(\mathscr{L}_k)$ refines $\mathscr{U}$. Similarly we shall say that the sequence (2) separates points of the space X if for every two distinct points p and q of X there is a number $m=m(p, q)$ and two elements P and Q of $\square(\mathscr{L}_m)$ such that $p\in P$, $q\in Q$ and $P\cap Q=\emptyset$.

Now we may formulate the main result.

Theorem. *For a compact space X the following conditions are equivalent:*

1) *X is homeomorphic to I^τ;*

2) *There exists an essential τ-system $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ in X and a sequence of grids with respect to $\vec{\mathscr{F}}$ which become tinier;*

3) *There exists an essential τ-system $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ in X and a sequence $\mathscr{L}_1>\mathscr{L}_2>\cdots>\mathscr{L}_n>\cdots$ of grids $\mathscr{L}_n$ in X with respect to $\vec{\mathscr{F}}$ with the following property: for each open cover $\mathscr{U}$ of X with $|\mathscr{U}|\leq 3$ there exists an index $k=k(\mathscr{U})$ such that $\square(\mathscr{L}_k)$ is a refinement of $\mathscr{U}$;*

4) *There exists an essential τ-system $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ in X and a sequence of grids with respect to $\vec{\mathscr{F}}$, which separates points of X:*

5) *There exists an essential τ-system $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ in X and a sequence $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n, \ldots$ of grids $\mathscr{L}_n$ with respect to $\vec{\mathscr{F}}$ which separates points of X and $\mathscr{L}_1\gg\mathscr{L}_2\gg\cdots\gg\mathscr{L}_n\gg\cdots$*

The proof of the theorem will be carried out on the scheme: (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(1).

To prove the first implication it is obviously sufficient to make use of the following well-known and elementary proposition:

Lemma 1. *For each open cover $\mathscr{U}$ of I^τ there exists a positive number δ with the property: if $A_s\subset I_s=I$ and* diam $A_s\leq\delta$ *for each $s\in S$, then the set $A=\Pi\{A_s: s\in S\}$ is contained in some element of $\mathscr{U}$ (here $\tau=|S|$).*

The implication (2)$\Rightarrow$(3) is evident. The next (3)$\Rightarrow$(4) is proved by easy verification that each sequence of grids, which satisfies the conditions of (3), also satisfies the conditions of (4).

To prove the implication (4)$\Rightarrow$(5) we employ the following simple assertion:

L e m m a 2. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ *be a sequence of spectra* λ_n *in a normal space* X *with respect to the pair* (A, B) *with* $\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots$ *Then there exists a sequence* $\mu_1, \mu_2, \ldots, \mu_n, \ldots,$ *of spectra* μ_n *in* X *with respect to* (A, B) *for which :*

1) $\mu_1 \gg \mu_2 \gg \cdots \gg \mu_n \gg \cdots$ *and*

2) $\mu_k < \lambda_k$ *for each* $k = 1, 2, \ldots$

The simple proof of this lemma will be omitted.

For the proof of (5)⇒(1) some notions will serve to the purpose.

Let $\lambda = (L_1, L_2, \ldots, L_l)$ be a spectrum in a normal space X with respect to the (A, B) and let $t_i, i = 0, 1, 2, \ldots, l+1$ be a $l+2$ points for which $-1 = t_0 < t_1 < \cdots < t_l < t_{l+1} = 1$. From the normalities of X it follows that there exists a continuous function $f: X \to I = [-1, 1]$ for which $f([L_i, L_{i+1}]_\lambda) \subset [t_i, t_{i+1}]$, $i = 0, 1, 2, \ldots, l$. Such a function will be called a function induced from λ.

Let $\mu = (M_1, M_2, \ldots, M_r)$ be another spectrum in X with respect to (A, B) with $\mu \ll \lambda$ and g be a function induced from μ. Let $-1 = u_0 < u_1 < \cdots < u_r < u_{r+1} = 1$ be the subdivision of the segment $[-1, 1]$ which defines g. We shall say that g is subordinated to f and put $g \lhd f$ or $f \rhd g$, if g satisfies the following condition: if $k_1 < k_2 < \cdots < k_m$ are all indices for which $L_i \subset M_{k_1} \subset M_{k_2} \subset \cdots \subset M_{k_m} \subset L_{i+1}$ then subdivision $t_i \leq u_{k_1} < u_{k_2} < \cdots < u_{k_m} \leq t_{i+1}$ divides the segment $[t_i, t_{i+1}]$ into equal subsegments, $i = 0, 1, 2, \ldots, l$.

It is clear from the normalities of X that a subordinated function g to f exists. Also if we put $\eta = \max\{t_{i+1} - t_i : i = 0, 1, 2, \ldots, l\}$ and $\zeta = \max\{u_{i+1} - u_i : i = 0, 1, 2, \ldots, r\}$ then

(3) $$\zeta \leq \eta/2$$

and

(4) $$|f(x) - g(x)| \leq \eta \quad \text{for each} \quad x \in X.$$

Let

(5) $$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$$

be a sequence of spectra λ_n in X with respect to (A, B) for which $\lambda_1 \gg \lambda \gg \cdots \gg \lambda_n \gg \cdots$

It is easy to consctruct by induction a sequence

(6) $$h_1, h_2, \ldots, h_n, \ldots,$$

of mappings $h_n: X \to [-1, 1]$ for which:

1) h_n is induced from λ_n;

2) $h_{n+1} \lhd h_n$, $n = 1, 2, \ldots$

L e m m a 3. *The sequence* (6) *is uniformly fundamental in* X.

P r o o f. By virtue of the inequalities (3) and (4) we get $|h_{n+1}(x) - h_n(x)| \leq \eta/2^{n-1}$ for each $x \in X$, where $\eta = \max\{t_{i+1} - t_i : i = 0, 1, 2, \ldots, l\}$ and $-1 = t_0 < t_1 < \cdots < t_l < t_{l+1} = 1$ is the subdivision of the segment $[-1, 1]$ which corresponds to the first function h_1. From this inequality we easily get

$$|h_{n+p}(x) - h_n(x)| \leq \eta \left(\frac{1}{2n-1} + \frac{1}{2n} + \cdots + \frac{1}{2n+p-2}\right)$$

for each $x \in X$, $n = 1, 2, \ldots$ and $p = 1, 2, \ldots$

Now, the assertion of this lemma is clear.

Hereby we get immediately the following:

L e m m a 4. *Let us put* $h(x)=\lim_{n\to\infty}h_n(x)$. *Then the function* $h:X\to[-1,1]$ *is correctly defined, is continuous and* $h(A)\subset\{-1\}$ *and* $h(B)\subset\{1\}$.

Proof of (5)$\Rightarrow$(1). Let $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ be an essential τ-system in X and

(7)
$$\mathscr{L}_1, \mathscr{L}_2, \dots, \mathscr{L}_n, \dots$$

be a sequence of grids $\mathscr{L}_n$ in X with respect to $\vec{\mathscr{F}}$, which separates points of X and

(8)
$$\mathscr{L}_1 \gg \mathscr{L}_2 \gg \dots \gg \mathscr{L}_n \gg \dots$$

We shall construct a homeomorphism $H:X\to I^{\tau}$ with $H(\mathscr{F}_{\mp s})=I^{\tau}_{\mp s}$ for each $s\in S$ and hence $H(X\setminus\bigcup\mathscr{F})=I^{\tau}\setminus\bigcup_s I^{\tau}_{\mp s}$.

Set $\mathscr{L}_n=\{\lambda^s_n\}_{s\in S}$. Then by virtue of (8) we have $\lambda^s_1\gg\lambda^s_2\gg\dots\gg\lambda^s_n\gg\dots$ for each $s\in S$.

Let us consider a sequence

(9)
$$h^s_1, h^s_2, \dots, h^s_n, \dots$$

(as (6)) of mappings $h^s_n:X\to[-1,1]$ for which:

1) h^s_n is induced from λ^s_n;

2) $h^s_{n+1}\lhd h^s_n$, $n=1,2,\dots$

By virtue of Lemma 4 the transformations $h_s:X\to[-1,1]$ defined by means of the formula $h_s(x)=\lim_{n\to\infty}h^s_n(x)$ are mappings and $h_s(F_{-s})\subset\{-1\}$ and $h_s(F_{+s})\subset\{1\}$.

Now let us consider the diagonal mapping $H=\underset{s\in S}{\Delta}h_s:X\to I^{\tau}$. We shall prove that H is a homeomorphism of X onto I^{τ}.

First of all let us observe that $H(F_{\mp s})\subset I^{\tau}_{\pm s}$ for each $s\in S$.

To prove that H is a mapping onto I^{τ} it is sufficient to use the following elementary assertion contained in [2].

L e m m a 5. *If* $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ *is an essential* τ*-system in a compact space* X *and* $\varphi:X\to I^{\tau}$ *is a mapping with* $\varphi(F_{\mp s})\subset I^{\tau}_{\mp s}$ *for each* $s\in S$, *then* $\varphi(X)=I^{\tau}$.

Thus in order to complete our proof it suffices to verify that H is one-to-one.

Let p and q be two different points of X. Since the sequence (7) separates points of X there exists an index k and two elements P and Q of the cover $\square(\mathscr{L}_k)$ for which $p\in P$, $q\in Q$ and $P\cap Q=\emptyset$. Setting $\lambda^s_k=(L^k_{s,1}, L^k_{s,2}, \dots, L^k_{s,m(s,k)})$ we get

$$P=\bigcap_{s\in S}[L^k_{s,i_s}, L^k_{s,i_s+1}]_{\lambda^s_k} \quad\text{and}\quad Q=\bigcap_{s\in S}[L^k_{s,j_s}, L^k_{s,j_s+1}]_{\lambda^s_k},$$

where $0\leq i_s, j_s\leq m(s,k)$.

Now we shall observe that $|i_s-j_s|\geq 2$ at least for one index $s\in S$. Really, assuming the contrary we get $|i_s-j_s|\leq 1$ for each $s\in S$.

Then putting

$$C_s=\begin{cases}\operatorname{Fr} L^k_{s,i_s+1}, & \text{if } i_s=j_s \text{ and } i_s=j_s<m(s,k);\\ \operatorname{Fr} L^k_{s,i_s}, & \text{if } i_s=j_s \text{ and } i_s=j_s=m(s,k);\\ \operatorname{Fr} L^k_{s,j_s}, & \text{if } j_s=i_s+1;\\ \operatorname{Fr} L^k_{s,i_s}, & \text{if } i_s=j_s+1,\end{cases}$$

we define partitions C_s in X between F_{-s} and F_{+s}. Since $\mathscr{F}$ is essential in X we get $C=\cap\{C_s : s\in S\}\neq\emptyset$. On the other hand it is clear that $C_s\subset[L^k_{s,i_s}, L^k_{s,i_s+1}]_{\lambda^s_k}$ and $C_s\subset[L^k_{s,j_s}, L^k_{s,j_s+1}]_{\lambda^s_k}$ (since $|i_s-j_s|\leq 1$) and hence $C\subset P\cap Q$, which contradicts $P\cap Q=\emptyset$. Thus we proved $|i_{s_0}-j_{s_0}|\geq 2$ at least for one $s_0\in S$. Now we have two cases: $i_{s_0}-j_{s_0}\geq 2$ or $j_{s_0}-i_{s_0}\geq 2$. By means of symmetry we can get the first case: $i_{s_0}-j_{s_0}\geq 2$ i. e. $j_{s_0}+2\leq i_{s_0}$. Therefore, since $p\in[L^k_{s_0,i_{s_0}}, L^k_{s_0,i_{s_0}+1}]_{\lambda^{s_0}_k}$ and $q\in[L^k_{s_0,j_{s_0}}, L^k_{s_0,j_{s_0}+1}]_{\lambda^{s_0}_k}$ from the properties 1) and 2) of the sequence (9) it follows that $h_{s_0}(q)<h_{s_0}(p)$ and hence $H(p)\neq H(q)$. Thus the Theorem is proved.

To end up with we shall complete with the following remark.

For a compact space X we may weaken the condition 2) of the Theorem as follows:

($*_\tau$) There exists an essential τ-system $\mathscr{F}=\{F_{-s}, F_{+s}\}_{s\in S}$ in X with the property: for each finite open cover $\mathscr{U}$ of X there exists a grid $\mathscr{L}$ in X with respect to $\vec{\mathscr{F}}$ such that the cover $\square(\mathscr{L})$ refines the cover $\mathscr{U}$.

Then it is natural to ask whether the condition ($*_\tau$) implies that X is homeomorphic to I^τ. The answer is "no". To get convinced in this, let us consider the space "Extended Long Line" (for example [3, p. 71]). This space X is constructed from the ordinal space $[0, \omega_1]$ (where ω_1 is the first uncountable ordinal) by placing between each ordinal α and its successor $\alpha+1$ a copy of the interval $(0, 1)$. X is then linearly ordered, and we give it the order topology. This space is evidently compact, connected and locally connected. By putting $F_{-1}=\{0\}$ and $F_{+1}=\{\omega_1\}$ we get an essential 1-system $\mathscr{F}=\{F_{-1}, F_{+1}\}$ in X. There are no difficulties to verify that X satisfies the condition ($*_1$). On the other hand it is well-known [3] that X is not homeomorphic to I. By multiplication of X on I, I^2, etc. we can get other examples of higher dimension. However, let us note that the condition ($*_\tau$) implies $\dim X=\tau$, when $\tau<\infty$.

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