PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

CONTROLLED PROCESS WITH RANDOM BREAKDOWNS AND REPEAT ACTIONS

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Some models minimizing the duration of the unreliable process by the help of statement copies are considered.

1. Introduction. Let during the operating time of a process (for example by computing, by constructing) in a random way some undesirable events (breakdowns, catastrophes) arise, which leads to process interruption. If one wants to have a successful finish, the process must be repeated from the origin (Fig. 1). It is profitable to introduce a strategy for making copies and remember the process states in some chosen moments. If a catastrophe appear the process continues from the last copied state. In the present paper a strategy for process control is proposed. This strategy minimizes a functional of the pure process operating time.

2. The basic model. Let X be the pure duration of the process without breakdowns, $\{\xi_n\}$ a random flow of discrete events (catastrophes) and $\tau(X)$ the full duration of the process up to finish. The process starts at $t=0$. If the process restarts after every catastrophe and k is the first number k for which $\xi_k \geq X$, then $\tau(X)=\Sigma_{i=1}^{k-1}\xi_i+X$. Evidently, $\tau(X)\geq X$.

3. The copies control. Let us consider a sequence $\{a_k\}$ of time intervals between copies and the sequences $\{\theta_k\}$ of copiring time durations. After every time interval a_i one makes a copy of the process state in a time θ_i. The breakdowns are possible and during the copies. The process restarts from the last copiring state (Fig. 2). In that case the full process duration is denoted by $\tau(X, \{a_k\}, \{\theta_k\})$ (controlled time duration). Sometimes the short notation $\tau(X)$ will be used (e. g. in the expectations).

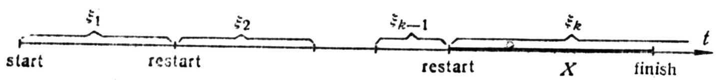

Fig. 1

Fig. 2

4. Some simplifying suppositions. Under general assumptions no one can define any problem. We suppose, that X is a random variable (r.v.) with a given distribution function (d.f.) $A(x)=\mathsf{P}\{X<x\}$; $\{\xi_n\}$ is a Poisson process with an intensity $\gamma>0$; $\{a_k\}$ is a deterministic process, $\{\theta_k\}$ forms a renewable one, and $\mathsf{P}\{\theta_k<x\}=F(x)$. Then $\tau(X, \{a_k\}, \{\theta_k\})=\tau(X, \{a_k\}, \lambda, \theta)$ is a r.v.

5. The optimal control problem. Under the above assumptions the question is: how to choose $\{a_k^*\}$ so that

$$\inf_{\{a_k\}} \mathsf{E}\,\tau(X, \{a_k\}, \theta)=\mathsf{E}\,\tau(X, a_k^*\}, \theta).$$

The sequence $\{a_k^*\}$ will be called *optimal copy schedule.*

6. Some exact results. Further we suppose the assumptions 5 are fulfilled. We introduce the functions

$$\tau^*(s)=\mathsf{E}\,e^{-s\tau(X)}, \quad \tau^{**}(s)=\mathsf{E}\exp[-s\tau(X, \{a_k\}, \theta)],$$

which are the Lapace-Stilties (L.S.) transformations of the probable distributions of r.v. $\tau(X)$ and $\tau(X, \{a_k\}, \theta)$ accordingly.

6.1. The case $X=$const.

Lemma 1. *If* $\mathsf{P}\{X=x\}=1$, *then*

$$\tau^*(s)=\frac{s+\gamma}{\gamma+s\exp[(s+\gamma)x]}.$$

The proof is a simple consequence of the next recurrent relation:

$$\tau^*(s)=e^{-(s+\gamma)x}+\frac{\gamma}{s+\gamma}[1-e^{-(s+\gamma)x}]\tau^*(s). \tag{1}$$

One can derive (1), using the probabilistic interpretation of L.S. transformations [5].

Lemma 2. *If* $\mathsf{P}\{X=x\}=1$, *then*

$$\tau^{**}(s)=\frac{s+\gamma}{\gamma+s.\exp[(s+\gamma)(x-\sum_{k=1}^{n_x} a_k)}\prod_{k=1}^{n_x}\frac{(s+\gamma)\varphi(s+\gamma)}{\gamma\varphi(s+\gamma)+s.\exp[a_k(s+\gamma)]}, \tag{2}$$

where $\varphi(s)=\mathsf{E}\{\exp(-s\theta_k)\}$, $n_x=\max[n;\ a_1+a_2+\cdots+a_n\leq x]$.

Proof. First of all, we mention, that a copy interval a_k and the next copy time θ_k form an interval of length τ_k in the full process duration $\tau(X, \{a_k\}, \theta)$. There are n_x such of intervals and the last one doesn't need a copy time; its duration is equal to $\tau(x-\Sigma_{k=1}^{n_x}a_k)$. The "lack of memory" property of the exponential distribution shows, that the r.v. $\{\tau_k\}$ are independent, and

$$\tau(x, \{a_k\}, \theta)=\sum_{k=1}^{n_x}\tau_k+\tau(x-\sum_{k=1}^{n_x}a_k). \tag{3}$$

Further, we introduce $\tau_k^*(s)=\mathsf{E}\,e^{-s\tau_k}$. Using the "lack of memory" property, we get again

$$\tau_k^*(s)=e^{-(s+\gamma)a_k}\varphi(s+\gamma)+\frac{\gamma}{s+\gamma}[1-e^{-(s+\gamma)a_k}\varphi(s+\gamma)]\tau_k^*(s).$$

It gives the relation

$$\tau_k^*(s)=\frac{(s+\gamma)\varphi(s+\gamma)}{\gamma\varphi(s+\gamma)+se^{(s+\gamma)a_k}}. \tag{4}$$

Now (2) follows from Lemma 1, (3) and (4).

Lemma 3. *The expected full time duration of the copy controlled process in the case* $P\{X=x\}=1$ is

$$E\tau(x,\{a_k\},\theta)=\frac{1}{\gamma}\sum_{k=1}^{n_x}\left[\frac{e^{\gamma a_k}}{\varphi(\gamma)}-1\right]+\frac{1}{\gamma}\left[e^{\gamma(x-\sum_{k=1}^{n_x}a_k)}-1\right].$$

The proof follows from Lemma 2 and the well-known relation

$$E\xi=-\frac{d}{dx}(E\,e^{-s\xi})\,|_{s=0}.$$

Theorem 1. *If* $P\{X=x\}=1$ *then the optimal copy schedule satisfies the condition* $a_k^*=a^*=\text{const}$.

Proof. Let us denote $b_0=0$, $b_k=a_1+a_2+\cdots+a_k$ and $n_x=\max(k, b_k\leq x)$. Then $b_k+k\theta$ is the moment of k-th copy from the origin, if there is no catastrophes. The result of Lemma 3 can be rewritten in the form

$$E\,\tau(x,\{b_k\},\theta)=\frac{1}{\gamma}\sum_{k=1}^{n_x}\left[\frac{e^{\gamma(b_k-b_{k-1})}}{\varphi(\gamma)}-1\right]+\frac{1}{\gamma}\left[e^{\gamma(x-b_{n_x})}-1\right]. \tag{5}$$

The optimal copy schedule $\{a_k^*\}$ gives the sequence $\{b_k^*\}$, which is a solution of the system of equations

$$\frac{\partial\,E\,\tau(x,\{a_k\},\theta)}{\partial b_k}=0,\quad k=1,2,\ldots. \tag{6}$$

It is easy to see, that in the case (5) the system (6) becomes the form $\frac{1}{\varphi(\gamma)}e^{\gamma(b_k-b_{k-1})}-\frac{1}{\varphi(\gamma)}e^{\gamma(b_{k+1}-b_k)}=0$, $k=1,2,\ldots$, which proves the theorem. For an abbreviation we introduce the notations $k_x=k(a,x)=[x/a]$ the integer part of x/a;

$$I_k(a)=\begin{cases}1 & \text{if } a\in[x/(k+1),\ x/k],\\ 0 & \text{otherwise};\end{cases}$$

$$T_k(x,a)=\frac{k}{\gamma}\left[\frac{e^{\gamma a}}{\varphi(\gamma)}-1\right]+\frac{1}{\gamma}\left[e^{\gamma(x-k_xa)}-1\right].$$

The result of Lemma 3 in the case $a_k=a$ for $k=1,2,\ldots$ can be written as follows.

Lemma 4. *For every fixed* $a>0$ *we have*

$$E\,\tau(x,\{a_k\},\theta)=T(x)=\sum_{k=0}^{\infty}I_k(a)T_k(x,a).$$

The graphs of $T_k(x,a)$ and $T(x,a)$ are given on Fig. 3.

The next properties one can derive, using the arguments of [2].

Lemma 5. *The absolute minimum of* $T_k(x, a)$ *is*

$$t_k = e^{\gamma[x+\ln\varphi(\gamma)]/(k+1)} \frac{1}{\gamma}\left[\frac{k}{\varphi(\gamma)}+1\right] - \frac{k+1}{\gamma}$$

and it gets in the point $\widehat{a}_0 = x/(k+1) + \ln\varphi(\gamma)/\gamma(k+1) < x/(k+1)$.

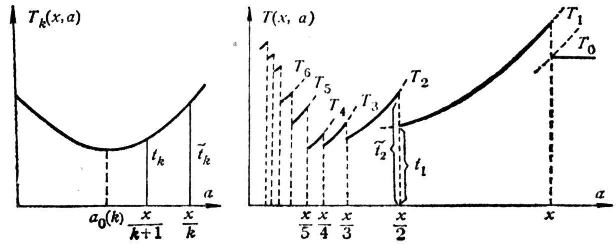

Fig. 3

Lemma 6. *If* $t_{k-1} = \lim_{a\downarrow x/k} T(x)$, $\tilde{t}_k = \lim_{t\uparrow x/k} T(x)$, *then*

$$t_{k-1} = \frac{k-1}{\gamma}\left(\frac{e^{\gamma x/k}}{\varphi(\gamma)} - 1\right) + \frac{1}{\gamma}\left(e^{\gamma x/k} - 1\right),\quad \tilde{t}_k = \frac{k}{\gamma}\left(\frac{e^{\gamma x/k}}{\varphi(\gamma)} - 1\right)$$

and always $t_{k-1} < \tilde{t}_k$.

Lemma 7. *For any fixed* x *and* γ

$$T_0(x) = \inf_a T(x) = \inf_k t_k = t_{k_0},$$

where k_0 *is finite. The optimal copy interval is* $a^* = x/(k_0+1)$.

Theorem 2. *Under the assumptions of Theorem 1 for all* X *in the interval* $[x_k, x_{k+1}]$ *is fulfilled* $a^* = X/(k+1)$, *where* $x_0 = 1$ *and the sequence* $\{x_k\}$ *is determined for all* k *by the equation*

$$\left[1+\frac{k+1}{\varphi(\gamma)}\right]e^{\gamma x_k/(k+2)} - \left[1+\frac{k}{\varphi(\gamma)}\right]e^{\gamma x_k/(k+1)} = 1. \tag{7}$$

Proof. Obviously, Lemma 7 gives one and the same number of copies k_0 for different x, because $k_0 = k_0(x)$ is a stepwise function of x. It follows that for any k there is a x_k, for which $k_0 = k$ or $k_0 = k+1$ and

$$T_0(x_k) = t_k = t_{k+1}. \tag{8}$$

Here t_k and $T_0(x)$ are as in Lemma 6 and Lemma 7. The equation (7) follows from (8) in view of the expressions for t_k, given in Lemma 6.

Corollary. *In the case* $\mathsf{P}\{\theta = \text{const}\} = 1$ *the tresholds* x_k *are solutions of the equations*

$$e^{\gamma x_k/(k+2)}\left[1+(k+1)e^{\gamma\theta}\right] - e^{\gamma x_k/(k+1)}\left[1+ke^{\gamma\theta}\right] = 1.$$

Denoting $z = \gamma\theta$, $x_k = \theta y_k$, the last equation get over

$$e^{zy_k/(k+2)}\left[1+(k+1)e^{z}\right] - e^{zy_k/(k+1)}\left[1+ke^{z}\right] = 1. \tag{9}$$

Table 1

k	Tresholds y_k, $\theta=1$				
	$z^{-1}=30$	$z^{-1}=60$	$z^{-1}=90$	$z^{-1}=120$	$z^{-1}=120$
0	11.03	15.54	19.02	21.95	24.53
1	28.37	26.21	32.23	37.31	41.79
2	25.56	36.66	45.18	52.37	58.70
3	32.71	47.05	58.04	67.33	75.49
4	39.84	57.39	70.87	82.24	92.25
5	46.96	67.72	83.68	97.14	108.92
6	54.08	78.06	96.47	112.01	125.69
7	61.18	88.37	109.24	126.89	142.37
8	68.29	98.69	122.04	141.76	159.09
9	75.39	109.00	134.83	156.52	75.71 1
10	82.49	119.31	147.54	171.47	192.33
11	89.57	129.60	160.35	186.30	209.05
12	96.70	139.92	173.01	201.08	225.75
13	103.78	150.27	185.92	215.00	242.41
14	110.85	160.46	198.60	230.86	258.57
15	117.98	170.88	211.37	245.56	275.69
16	125.07	181.03	224.15	260.41	292.49
17	132.17	191.47	236.91	274.91	308.68
18	139.25	201.74	249.71	290.40	325.52
19	146.36	212.06	262.44	304.53	342.65

If $z=\gamma\theta=\text{const}$, then the tresholds satisfy the equality $x_k=\theta y_k$, where y_k are the solutions of (9). Hence, to have the tresholds x_k for any γ and θ it is enough to have the tables of the tresholds y_k, for $\theta=1$ and for different values of z. The Table 1 gives us an example for the solutions of (9).

6.2. The case when X is r.v. and $\mathsf{P}\{X<x\}=A(x)$. Let γ and the copy interval a be given, and $\mathsf{E}\,e^{-s\theta}=\varphi(s)$. Denote $\tau(a)=\mathsf{E}\,\tau(X,\{a\},\theta)$.

Theorem 3. *We have*

$$\tau(a)=\frac{1}{\gamma}\sum_{k=0}^{\infty} e^{-\gamma ka}\int_{ka}^{(k+1)a} e^{\gamma x}dA(x)+\frac{1}{\gamma}\left[\frac{e^{\gamma a}}{\varphi(\gamma)}-1\right]\sum_{k=0}^{\infty} k\,[A((k+1)a)-A(ka)]-\frac{1}{\gamma}\cdot$$

Proof. It follows from Lemma 4 and the equation

$$\tau(a)=\int_0^\infty \mathsf{E}\,\tau(x,\{a\}\ \theta)dA(x).$$

6.3. The case $\mathsf{P}\{X>x\}=e^{-\lambda x}$, $\lambda>0$. In this particular case the following statement holds.

Theorem 4. *If*

(i) $\lambda\neq\gamma$

then $\tau(a)=\frac{1}{\gamma}\left\{\frac{\lambda}{\gamma-\lambda}[e^{(\gamma-\lambda)a}-1]+\frac{e^{(\gamma-\lambda)a}}{\varphi(\gamma)}-1\right\}/(1-e^{-\lambda a})$ *and if*

(ii) $\lambda=\gamma$

then $\tau(a)=\frac{1}{\gamma}\left[a\gamma+\frac{1}{\varphi(\gamma)}-1\right]/[1-e^{-\gamma a}]$.

Proof. The proof follows from Theorem 3. One must calculate the integrals and the obtained sums.

Theorem 5. *The optimal copy interval a^* is the unique solution of the equation*

$$[e^{-\gamma a}-e^{-\lambda a}]/(\gamma-\lambda)+(1-e^{-\lambda a})/[\lambda\varphi(\gamma))]=[\varphi(\gamma)-1]/[\gamma\varphi(\gamma)] \tag{10}$$

in the case $\lambda\neq\gamma$, and of the equation

$$(\gamma a-1)e^{-\gamma a}=\varphi(\gamma) \tag{11}$$

in the case $\lambda=\gamma$.

If $\lambda\neq\gamma$, a^* exists if and only if

$$\lambda<\gamma/[1-\varphi(\gamma)]. \tag{12}$$

Proof. The optimal copy interval a^* satisfies $\tau(a^*)=\inf_a \tau(a)$, and it is a solution of the equation

$$\frac{\partial}{\partial a}\tau(a)=0. \tag{13}$$

The exact analysis of (13) for the functions $\tau(a)$ from Theorem 4 goes to (10) or (11). When the inequality (12) is not valid, the optimal copy interval is $a^*=\infty$.

7. Applications. For given γ and θ it is not difficult to calculate $a^*=a^*(\lambda)$ for the suitable values of λ; a^* depends on γ and θ, which also can be taken into account. In the practice usually γ, θ are given for the process realization, but the pure time duration X can change (depending on λ). To make optimal copies one needs the tables of $a^*(\lambda)$.

8. Generalizations

a. To introduce a cost function $\alpha(x)$, instead of time duration $\tau(x)$.

If the process duration is x and there are not breakdowns, it costs $c(x)$; if the copy duration is θ, it costs $b(\theta)$; if the repeated time is y, it costs $h(y)$.

Under the assumptions: the breakdowns form a Poisson process with a parameter $\gamma>0$, the copy duration θ is constant, the breakdowns may occur during a copy and the restart begins from the last successive copy, we conclude, that the copies must be made at equidistant moments of length a.

Theorem 6. *For a fixed time duration x of the process X (without breakdowns) and given a, the expected value of the cost function is*

$$\begin{aligned}A_k(x, a)&=\mathsf{E}\,\alpha(x, a, \theta, \gamma)\\
&=e^{\gamma(a+\theta)}c(x)e^{-\gamma(x+k\theta)}+(e^{\gamma(a+\theta)}-1)\sum_{\nu=1}^{k-1} c(x-\nu a)e^{-\gamma(x-\nu a+(k-\nu)a)}\\
&\quad+c(x-ka)+\int_0^{x-ka}(c(u)+h(u))\gamma e^{-\gamma u}du\\
&\quad+ke^{\gamma(a+\theta)}\Big(\int_0^a h(u)\gamma e^{-\gamma u}du+e^{-\gamma a}(1-e^{-\gamma\theta})h(a)+e^{-\gamma\theta}b(\theta)\\
&\quad+\int_0^\theta b(u)\gamma e^{-\gamma u}du\Big)+\sum_{\nu=1}^{k}((k-\nu+1)e^{\gamma(a+\theta)}-(k-\nu))e^{-\gamma(\nu-1)\theta}\\
&\quad\times\Big(\int_{(\nu-1)a}^{\nu a} c(u)\gamma e^{-\gamma u}du+e^{-\gamma\nu a}(1-e^{-\gamma\theta})c(\nu a)\Big).\end{aligned}$$

The general form of the expected cost function is

$$A(x, a) = \sum_{k=0}^{\infty} I_k(a) \,.\, A_k(x, a),$$

where $I_k(a)$ was introduced above.

Further one can use the same techniques to find the optimal value a_0.

b. To use some more general forms of d.f. $A(x)$ and d.f. of $\{\xi_n\}$.

If $B(x) = \mathrm{P}\{\xi_n < x\}$ is not the exponential distribution, then the copies will be not equidistant. We have an optimal scheduling problem, which can be solved if $B(x)$ has a monotone failure rate.

c. To introduce a process $\{X_n\}$ instead of one fixed random duration X.

d. To consider the repair time durations caused from catastrophes, which must be made before restarts.

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