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ПЛИСКА

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EFFICIENT SEQUENTIAL ESTIMATION IN STOCHASTIC PROCESSES

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In the present paper for a large class of processes the problem of efficient (in the sense of a quadratic function of loss under Cramer-Rao regular conditions) sequential estimation is considered. A necessary condition for Markov stopping time to be efficient is given. It is important to underline that this condition appears as a sufficient for all the processes for which the problem of efficient sequential estimation is completely solved (binominal, multinominal, Poisson processes, etc.). Some helpful results for determining the efficient plans are given, too.

The main results of this paper are announced without proofs in [5].

Let $(\Omega, \mathfrak{B}, \mathsf{P}_\theta)$ be a probability space, where $\theta \in \Theta \subset R^k$ and let $X_t: (\Omega, \mathfrak{B}, \mathsf{P}_\theta) \to (X, \mathscr{A})$ be a random vector for each $t \in T$, where $\mathbf{X} \subset R^n$, $\mathscr{A}$ is the σ-algebra of the Borel subsets and $T = [0, +\infty)$ or $T = \{0, 1, 2, \dots\}$. Let $\{S_t\}_{t \in T}$, S_t: $(\Omega, \mathfrak{B}, \mathsf{P}_\theta) \to (\mathbf{X}, \mathscr{A})$ be a sufficient statistics for the process $\{X_t\}_{t \in T}$ (see [9]). We shall suppose that the process $\{S_t\}_{t \in T}$ is continuous from the right, $T = [0, +\infty)$. Let τ be a Markov stopping time with respect to $\{\mathfrak{F}_t\}_{t>0}$, $\mathfrak{F}_t = \sigma(X_s, s \leq t)$. We shall consider only such τ for which $\mathsf{P}_\theta\{\tau > 0\} = 1$ for each $\theta \in \Theta$ and such that for each θ from some interval $I = \times_{i=1}^{k} [\alpha_i, \beta_i] \subset \Theta$, $\mathsf{P}_\theta(\tau < +\infty\} = 1$.

Definition 1. *By a sequential plan we shall mean the triplet* (τ, f, h), *where* $\int_\Omega f(S_\tau, \tau) d\mathsf{P}_\theta = h(\theta)$, $\forall \theta \in I$.

We shall consider only such h that $h \neq \text{const}$ on I. We shall suppose that for each $\theta \in I$

(1) $\mathsf{P}_\theta\{S_\tau, \tau) \in C\} = \int_C g(x, \theta) d\nu_\tau(x)$, $x \in R^{n+1}$; $C \in \mathfrak{B}_{R^{n+1}}$, where ν_τ is a σ-finite measure on $(R^{n+1}, \mathfrak{B}_{R^{n+1}})$, which does not depend on θ [7, 11].

Under the so-called Cramer-Rao regular conditions [8], we obtain the Rao-Cramer inequality for the estimator f and the estimable function h, where equality for $\theta = \theta_0 \in I$ holds iff there exist $a_i(\theta_0)$, $i = 1, 2, \dots, k$, $\sum_{i=1}^{k} [a_i(\theta_0)]^2 \neq 0$ such that

$$f - h(\theta_0) = \sum_{i=1}^{k} a_i(\theta_0) \left. \frac{\partial \log g(x, \theta)}{\partial \theta_i} \right|_{\theta=\theta_0}, \quad \theta = (\theta_1, \theta_2, \dots, \theta_k). \tag{2}$$

Definition 2. *The Markov stopping time* τ *is closed for* Θ_1, $(\Theta_1 \subset \Theta)$ *if* $\mathsf{P}_\theta\{\tau < +\infty\} = 1$ *for each* $\theta \in \Theta_1$.

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Definition 3. *The sequential plan* (τ, f, h) *is said to be efficient for* Θ_1, $(\Theta_1 \subset \Theta)$ *if* τ *is closed for* Θ_1 *and for each* $\theta \in \Theta_1$ *we have:* $E_\theta \tau < +\infty$, (1) *holds, Rao-Cramer regular conditions hold and* (2) *holds. The function* h *is called efficiently estimable for* Θ_1 *and* f *is its efficient estimator for* Θ_1.

Definition 4. *The sequential plan* (τ, f, h) *is said to be efficient if there exists some interval* I, $I \subset \Theta$, *such that* (τ, f, h) *is efficient for* I. *The function* h *is called efficiently estimable and* f *is its efficient estimator.*

Definition 5. *The Markov stopping time* τ *is said to be efficient for* Θ_1, $\Theta_1 \subset \Theta$, *if there exist* f *and* h *such that* (τ, f, h) *is efficient for* Θ_1.

Definition 6. *The Markov stopping time* τ *is said to be efficient if* τ *is efficient for some interval* I, $(I \subset \Theta)$.

Let (τ, f, h) be an efficient sequential plan. According to (2) we usually obtain for some natural m and some vector B $(B = \{b_i\}_{i=1}^{m+2}$, $\Sigma_{i=1}^{m+1} b_i^2 \neq 0$, $b_{m+2} > 0)$ that

$$\sum_{i=1}^{m} b_i V_i(S_\tau, \tau) + b_{m+1}\tau = b_{m+2} \tag{3}$$

almost surely with respect to P_θ for each $\theta \in I$, where $V_i : (R^{n+1}, \mathfrak{B}_{R^{n+1}}) \to (R, \mathfrak{B}_R)$

We shall suppose that m and $\{V_i\}_{i=1}^m$ are fixed (i.e. they do not depend on sequential plans and intervals I). For each B described above, we define the set

$$K(B) = \{\bar{x} \in R^{m+1} : \bar{x} = \{x_i\}_{i \leq m+1}, \ \sum_{i=1}^{m+1} b_i x_i = b_{m+2}\}.$$

Moreover, for each B we suppose that the following assumptions are satisfied

(i) $V_i(S_0, 0) \equiv 0$, $i = 1, 2, \ldots, m$,

(ii) $\{Y(t)\}_{t \in T}$ is a continuous from the right stochastic process (when $T = [0, +\infty)$), where $Y(t) = \Sigma_{i=1}^m b_i V_i(S_t, t)$.

Let $\{\tau_s : s = 0, 1, 2, \ldots\}$ be a set of Markov stopping times with respect to $\{\mathfrak{F}_t\}_{t>0}$, such that $\tau_0 \equiv 0$, $\tau_{s+1} \geq \tau_s$, $s = 0, 1, 2, \ldots$ and $\{\tau_{s+1} - \tau_s\}_{s \geq 0}$ is a sequence of independent identically distributed random variables (i.i.d.r.v.) and $0 < E_\theta(\tau_{s+1} - \tau_s) < +\infty$ for each $\theta \in \Theta$ and:

(iii) $\{Z_{s+1} - Z_s\}_{s \geq 0}$ and $\{\tilde{Z}_{s+1} - \tilde{Z}_s\}_{s \geq 0}$ are sequences of i.i.d.r.v., with a finite second moment, where $Z_s = Y(\tau_s) + b_{m+1}\tau_{s-1}$, $\tau_{-1} \equiv 0$, $\tilde{Z}_s = Y(\tau_s) + b_{m+1}\tau_{s+1}$;

(iv) For each s_0 the σ-algebras

$\sigma\{(Z_{s+1} - Z_s) : s = s_0, s_0 + 1, \ldots\}$ and $\sigma\{\mathfrak{F}_{\tau_{s_0 - 1}} \cup \sigma(Z_{s_0})\}$ are independent and respectively the σ-algebras $\sigma\{(\tilde{Z}_{s+1} - \tilde{Z}_s) : s = s_0, s_0 + 1, \ldots\}$ and $\sigma\{\mathfrak{F}_{\tau_{s_0}} \cup \sigma(\tilde{Z}_{s_0})\}$ are independent too;

(v) There exists $d \geq 0$ such that for each $s \geq 0$ and for each $t < \tau_{s+1} - \tau_s$: $|Y(\tau_s + t) - Y(\tau_s)| \leq d$ almost surely (P_θ, for each $\theta \in \Theta$), where the set $\{\tau_s : s = 0, 1, 2, \ldots\}$ may be such that $d < \varepsilon$ for previously given $\varepsilon > 0$.

(vi) For each efficient Markov stopping time τ and for each $s = 1, 2, \ldots$ and $i = 1, 2, \ldots, m$

$$E_\theta V_i(S_\tau, \tau) = \varphi_i(\theta) E_\theta \tau, \qquad \theta \in I,$$
$$E_\theta V_i(S_{\tau_s}, \tau_s) = \varphi_i(\theta) E_\theta \tau_s, \qquad \theta \in \Theta,$$

where $\varphi_i(\cdot)$ is a continuous function for each i.

If $b_{m+1}+\Sigma_{i=1}^{m}b_i\varphi_i(\theta_0)>0$ for some θ_0, then from (vi) we have $\mathsf{E}_{\theta_0}(Z_{s+1}-Z_s)>0$ and respectively $\mathsf{E}_{\theta_0}(\tilde{Z}_{s+1}-\tilde{Z}_s)>0$. So that, for each $s_0\geqq 0$, and for each $v_2>0$, there exists a natural l, such that:

$$(*)\qquad \mathsf{P}_{\theta_0}\{Z_{s_0+l}-Z_{s_0}>v_2,\quad Z_s-Z_{s_0}>0,\quad s=s_0+1,\ldots,s_0+l-1\}>0,$$

and respectively

$$(**)\qquad \mathsf{P}_{\theta_0}\{\tilde{Z}_{s_0+l}-\tilde{Z}_{s_0}>v_2,\quad \tilde{Z}_s-\tilde{Z}_{s_0}>0,\quad s=s_0+1,\ldots,s_0+l-1\}>0.$$

Definition 7. *We say that $K(B)$ can be passed when $b_{m+1}\geqq 0$, or respectively when $b_{m+1}\leqq 0$ if there exists s_0 such that the event "$Z_{s_0}>b_{m+1}+d$* and for each $t\leqq\tau_{s_0-1}$ *the equality $b_{m+1}t+Y(t)=b_{m+2}$ is not true", or respectively "$\tilde{Z}_{s_0}>b_{m+2}+d$ and for each $t\leqq\tau_{s_0}$ the equality $b_{m+1}t+Y(t)=b_{m+2}$ is not true", has a positive P_θ — probability for each $\theta\in\Theta$.*

Denote by $\mathscr{K}$ the family of all $K(B)$, which cannot be passed.

Definition 8. *Every Markov stopping time τ for which* (3) *holds will be called similar to the moment of the first attaining of $K(B)$ (denote it by $\tau_{K(B)}$) by the process $(\{V_i(S_t,t)\}_{i=1}^{m},t)_{t\in T}$.*

Theorem. *Under the assumptions* (i) — (vi) *made above, the Markov stopping time τ to be efficient, it is necessary that there exists $K(B)\in\mathscr{K}$, such that τ is similar to $\tau_{K(B)}$. The following set $\Theta_1=\{\theta\in\Theta:\Sigma_{i=1}^{m}b_i\varphi_i(\theta)+b_{m+1}>0\}$ is the widest set for which $\tau_{K(B)}$ or similar to it stopping times may be efficient. If $\tau_{K(B)}$ is efficient, then it is the only similar to $\tau_{K(B)}$ which is efficient. $\tau_{K(B)}$ is closed for Θ_1, $K(B)\in\mathscr{K}$.*

Proof. Let τ be efficient. From (3) and (vi) we have that $\mathsf{E}_\theta\tau=b_{m+2}/(\Sigma_{i=1}^{m}b_i\varphi_i(\theta)+b_{m+1})$, $\theta\in I$. Thus the set Θ_1 is the widest for which $\tau_{K(B)}$ or similar to it stopping times may be efficient.

Let $\tau_{K(B)}$ and a similar to it stopping time τ be efficient. Then we have $\mathsf{E}_\theta\tau=\mathsf{E}_\theta\tau_{K(B)}$ for $\theta\in I$ and of course $\tau_{K(B)}\leqq\tau$. Thus $\tau=\tau_{K(B)}$ almost surely for each $\theta\in I$.

Suppose now that the set $K(B)$, connected with the efficient stopping time τ (the connection between $K(B)$ and τ is expressed by (3)) does not belong to $\mathscr{K}$. Of course the interval I, for which τ is efficient must belong to Θ_1. Let $\theta_0\in I$. Consider the case when $b_{m+1}\geqq 0$. According to Definition 7 there exists s_0 such that:

$$p(s_0)=\mathsf{P}_{\theta_0}\{Z_{s_0}>b_{m+2}+d,\quad b_{m+1}t+Y(t)\neq b_{m+2},\quad t\leqq\tau_{s_0-1}\}>0.$$

Taking into account (i) — (v) and (*) one can easily see that for each $v_2>0$ there exists l, such that for each s, $s>s_0+l$:

$$\mathsf{P}_{\theta_0}\{\tau>\tau_s\}\geqq p(s_0)\mathsf{P}_{\theta_0}\{Z_{s_0+l}-Z_{s_0}>v_2,\quad Z_{s_1}-Z_{s_0}>0,\quad s_1=s_0+1,\ldots,s_0+l-1\}$$

$$\times\mathsf{P}_{\theta_0}\{Z_{s_0+l+1}-Z_{s_0+l}>-v_2,\ldots,Z_s-Z_{s_0+l}>-v_2\}.$$

From (vi) we have

$$\mathsf{E}_\theta\{Z_{s+1}-Z_s)=(\Sigma_{i=1}^{m}b_i\varphi_i(\theta)+b_{m+1})\mathsf{E}_\theta(\tau_{s+1}-\tau_s),\quad s\geqq 0.$$

Thus $c=\mathsf{E}_{\theta_0}(Z_{s+1}-Z_s)>0$ for each $s\geqq 0$. But

$$\mathsf{P}_{\theta_0}\{Z_{s_0+l+1}-Z_{s_0+l}>-v_2, \ldots, \quad Z_s-Z_{s_0+l}>-v_2\}$$
$$\geqq \mathsf{P}_{\theta_0}\{[Z_{s_0+l+1}-Z_{s_0+l}-\mathsf{E}_{\theta_0}(Z_{s_0+l+1}-Z_{s_0+l})]^2<[c+v_2]^2, \ldots$$
$$[Z_s-Z_{s_0+l}-\mathsf{E}_{\theta_0}(Z_s-Z_{s_0+l})]^2<[c(s-s_0-l)+v_2]^2\}.$$

Obviously $cv_3+v_2\geqq(\frac{4}{3}cv_3)^{3/4}(4v_2)^{1/4}$, $v_3>0$. It allows us to write

(4) $$\mathsf{P}_{\theta_0}\{Z_{s_0+l+1}-Z_{s_0+l}>-v_2, \ldots, \quad Z_s-Z_{s_0+l}>-v_2\}$$
$$\geqq \mathsf{P}_{\theta_0}\{[Z_{s_0+l+1}-Z_{s_0+l}-\mathsf{E}_{\theta_0}(Z_{s_0+l+1}-Z_{s_0+l})]^2<(4v_2)^{1/2}(\frac{4}{3}c)^{3/2},$$
$$\ldots, (Z_s-Z_{s_0+l}-\mathsf{E}_{\theta_0}(Z_s-Z_{s_0+l})]^2<(4v_2)^{1/2}[\frac{4}{3}c(s-s_0-l)]^{3/2}\}.$$

Let $A=\mathsf{D}_{\theta_0}(Z_2-Z_1)\sum_{n=1}^{\infty}(4v_2)^{-1/2}(\frac{4}{3}cn)^{-3/2}$. From Hajek-Renyi-Chow inequality (see [10]), we obtain that the right side in (4) is not smaller than $1-A$ But we can take v_2 such that $A<1$. So that for sufficiently large s we have $\mathsf{P}_{\theta_0}\{\tau>\tau_s\}\geqq v_4>0$, where v_4 does not depend on s. From the obvious fact (which follows from the law of large numbers)

$$\mathsf{P}_{\theta_0}\{\tau_s>s[\mathsf{E}_{\theta_0}(\tau_2-\tau_1)-\varepsilon]\}\underset{s\to\infty}{\longrightarrow}1, \quad \varepsilon>0$$

we get that there exists t_0 such that for each $t>t_0$

$$\mathsf{P}_{\theta_0}\{\tau>t\}\geqq v_5>0,$$

where v_5 does not depend on t, which yields $\mathsf{P}_{\theta_0}\{\tau=+\infty\}>0$.

In a similar way using $\{\tilde{Z}_s\}_{s\geqq 0}$ the case when $b_{m+1}\leqq 0$ can be considered So that a Markov stopping time τ, which is connected with $K(B)\notin\mathscr{K}$, cannot be efficient.

Consider now $\tau_{K(B)}$, where $K(B)\in\mathscr{K}$ and $b_{m+1}\geqq 0$. Let $\theta_0\in\Theta_1$. From the law of iterated logarithm we have

(5) $$\mathsf{P}_{\theta_0}\{\liminf_{s}\frac{Z_s-s\mathsf{E}_{\theta_0}(Z_2-Z_1)}{\sqrt{D_{\theta_0}(Z_2-Z_1)}\sqrt{2s\ln\ln s}}=-1\}=1.$$

Let $\varepsilon>0$. From (5) we obtain that

$$\mathsf{P}_{\theta_0}\{\exists_{s_0}\forall s\geqq s_0: Z_s>s\mathsf{E}_{\theta_0}(Z_2-Z_1)-(1+\varepsilon)\sqrt{\mathsf{D}_{\theta_0}(Z_2-Z_1)2s\ln\ln s}\}=1.$$

From the facts that $\sqrt{2s\ln\ln s}/s\underset{s\to\infty}{\longrightarrow}0$ and $\mathsf{E}_{\theta_0}(Z_2-Z_1)>0$ it follows

(6) $$\mathsf{P}_{\theta_0}\{\exists s_1\forall s\geqq s_1: Z_s>b_{m+2}+d\}=1.$$

But it is easy to see, taking into account Definition 7, that (6) suffices to get that $\tau_{K(B)}$ is closed for $\theta_0\in\Theta_1$. The case $b_{m+1}\leqq 0$, can be considered in the same way, using $\{\tilde{Z}_s\}_{s\geqq 0}$.

This completes the proof of the Theorem.

Examples. It is easy to see that the assumptions (i) — (vi) are fulfilled for the following processes, if we take the corresponding set $\{\tau_s : s=0, 1, 2\ldots\}$ of Markov stopping times, as follows:

1) for the multinomial process [1, 2] $\tau_s = s$;

2) for the Poisson process [3, 4, 6] τ_s = the moment of the s-th jump of the process;

3) for the jumping processes belonging to the exponential class of processes [7] τ_s = the moment of the s-th jump of the process.

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