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ON SPECIAL CLASSES OF A LOCALLY DECOMPOSABLE RIEMANNIAN SPACE

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In the present paper we define subclasses of locally decomposable Riemannian space, i. e. the class of the almost recurrent and the class of the almost Ricci recurrent locally decomposable Riemannian spaces. In these cases we find that the almost recurrence covectors are decomposable. Theorem 2 (resp. Theorem 3) characterizes the class of the almost recurrent (resp. of the almost Ricci recurrent) spaces.

1. Introduction. Let M be an n-dimensional Riemannian space with a metric g and a locally product structure f and ∇ be the Riemannian connection on M. Let the indices $i, j, k, l, s, \in \{1, 2, \dots, n\}$. If the local coordinates of g and f satysfy the following conditions

$$f_i^s f_s^j = \delta_i^j, \; f_i^j \neq \pm \delta_i^j, \tag{1.1}$$

$$f_i^s f_j^k g_{sk} = g_{ij}, \tag{1.2}$$

$$\nabla_i f_k^s = 0, \tag{1.3}$$

then M is called locally decomposable Riemannian space [1]. It is known [1] that M is a locally direct product $M_1 \times M_2$ of Riemannian spaces. We suppose that $dim M_1 = m$. $0 < m < n$. Let the indices $a, b, c, d, e \in \{1, 2, \dots, m\}$ and $\alpha, \beta, \gamma, \delta, \varepsilon \in \{m+1, m+2, \dots, n\}$ In a special coordinate system we have $f_a^b = \delta_a^b$, $f_\alpha^\beta = -\delta_\alpha^\beta$ in addition g_{ab} and $g_{\alpha\beta}$ are the metric tensors of M_1 and M_2, respectively. If R_{ijk}^s is the curvature tensor of M, then R_{abc}^d and $R_{\alpha\beta\delta}^\varepsilon$ are the curvature tensors of M_1, and M_2, respectively. Similarly if, $R_{jk} = R_{ijk}^i$ is the Ricci tensor of M, then R_{ab} and $R_{\alpha\beta}$ are the Ricci tensors of M_1 and M_2, respectively. If any coordinate of the above geometric objects of M has indices as a, b, c, d, e and $\alpha, \beta, \gamma, \delta, \varepsilon$ simultaneously, then this coordinate is zero. The functions $R = R_{ij} g^{ij}$ and $\tilde{R} = R_{ij} f_s^i g^{js}$ are called scalar curvatures of M. Let $R_1 = R_{ab} g^{ab}$ and $R_2 = R_{\alpha\beta} g^{\alpha\beta}$ be the scalar curvatures of M_1 and M_2, respectively. It is clear that

$$R = R_1 + R_2; \quad \tilde{R} = R_1 - R_2. \tag{1.4}$$

Let $(x^1, x^2, \dots, x^m)$ and $(x^{m+1}, x^{m+2}, \dots, x^n)$ be the local coordinates of the points in M_1 and M_2, respectively. A smooth covector field λ_k on M in called decomposable if

$$\lambda_a = \lambda_a(x^1, x^2, \dots, x^m); \; \lambda_\alpha = \lambda_\alpha(x^{m+1}, x^{m+2}, \dots, x^n).$$

2. Almost recurrent and almost Ricci recurrent locally decomposable Riemannian spaces. Let M be locally decomposable Riemannian space and R_{ijk}^s and R_{ij} be the curvature tensor and the Ricci tensor of the metric g_{ij}, respectively. We denote $R_{ijkl} = R_{ijk}^s g_{sl}$ and $\tilde{R}_{ijkl} = R_{ijks} f_l^s$. Because of (1.1)—(1.3) we see that $\tilde{R}_{ijks}$ has the Levi Chivita's properties as R_{ijks}. We denote also $\tilde{R}_{ij} = \tilde{R}_{kijs} g^{ks}$, and have $\tilde{R}_{ij} = \tilde{R}_{ji}$.

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We call M almost recurrent space if the following condition is valid

$$\nabla_s R_{ijkl} = \lambda_s R_{ijkl} + \mu_s \widetilde{R}_{ijkl}, \tag{2.1}$$

or almost Ricci recurrent space, if

$$\nabla_s R_{jk} = \lambda_s R_{jk} + \mu_s \widetilde{R}_{jk} \tag{2.2}$$

for some covector fields λ_s, μ_s on M.

Evidently if M is almost recurrent, then M is almost Ricci recurrent space.

Lemma 1. *Let $M = M_1 \times M_2$ be an almost recurrent locally decomposable Riemannian space. If the scalar curvatures R_1 and R_2 of M_1 and M_2 respectively do not vanish, then $\mu_k = f_k^i \lambda_i$, where λ_k and μ_k are the almost recurrence covectors.*

Proof. Since M is an almost recurrent space we have (2.1) and (2.2). From (2.2) we get

$$\partial_k R = \lambda_k R + \mu_k \widetilde{R}, \tag{2.3}$$

$$\partial_k \widetilde{R} = \lambda_k \widetilde{R} + \mu_k R, \tag{2.4}$$

where R and $\widetilde{R}$ are the scalar curvatures of M. We denote $\widetilde{\lambda_k} = f_k^i \lambda_i$ and $\widetilde{\mu_k} = f_k^i \mu_i$. Multiplying (2.3) with f_i^k and using the formula $f_i^k \partial_k R = \partial_k \widetilde{R}$ [1], we find

$$\partial_k \widetilde{R} = \widetilde{\lambda}_k R + \widetilde{\mu}_k \widetilde{R}. \tag{2.5}$$

Comparing (2.4) and (2.5), we obtain

$$R(\widetilde{\lambda}_k - \mu_k) + \widetilde{R}(\widetilde{\mu}_k - \lambda_k) = 0. \tag{2.6}$$

Similarly we can get

$$\widetilde{R}(\widetilde{\lambda}_k - \mu_k) + R(\widetilde{\mu}_k - \lambda_k) = 0. \tag{2.7}$$

Dew to $R_1 \neq 0$, $R_2 \neq 0$ and (1.4) we have $|R| \neq |\widetilde{R}|$. Then (2.6) and (2.7) imply $\mu_k = \widetilde{\lambda}_k$ Thus the lemma is proved.

According to Lemma 1, conditions (2.1)—(2.4) become as follows

$$\nabla_s R_{ijkl} = \lambda_s R_{ijkl} + \widetilde{\lambda}_s \widetilde{R}_{ijkl}, \tag{2.8}$$

$$\nabla_s R_{jk} = \lambda_s R_{jk} + \widetilde{\lambda}_s \widetilde{R}_{jk}, \tag{2.9}$$

$$\partial_s R = \lambda_s R + \widetilde{\lambda}_s \widetilde{R}, \tag{2.10}$$

$$\partial_s \widetilde{R} = \lambda_s \widetilde{R} + \widetilde{\lambda}_s R. \tag{2.11}$$

Theorem 1. *Let $M = M_1 \times M_2$ be an almost recurrent locally, decomposable Riemannian space. If the scalar curvatures of M_1 and M_2 don't vanish, then the almost recurrence covectors are decomposable.*

Proof. Let M be an almost recurrent locally decomposable Riemannian space. Then we have (2.8)–(2.11). Multiplying (2.10) with R and (2.11) with $\widetilde{R}$ and subtracting the obtained results, we get

$$\lambda_k = \partial_k \ln \sqrt{|R^2 - \widetilde{R}^2|}\,. \tag{2.12}$$

From (2.12) using (1.4), we find

(2.13) $$\lambda_k = \partial_k \ln \sqrt{|R_1|} + \partial_k \ln \sqrt{|R_2|},$$

Similarly we get

(2.14) $$\tilde{\lambda}_k = \partial_k \ln \sqrt{|R_1|} - \partial_k \ln \sqrt{|R_2|}.$$

Now from (2.13) and (2.14) we find respectively

(2.15) $$\lambda_a = \tilde{\lambda}_a = \partial_a \ln \sqrt{|R_1|},$$

(2.16) $$\lambda_\alpha = -\tilde{\lambda}_\alpha = \partial_\alpha \ln \sqrt{|R_2|}.$$

So the theorem is proved.

Theorem 2. *Let* $M = M_1 \times M_2$ *be a locally decomposable Riemannian space and* $R_1 \neq 0$, $R_2 \neq 0$, *where* R_1, R_2 *are the scalar curvatures of* M_1, M_2, *respectively. Then* M *is an almost recurrent space if and only if, when* M_1 *and* M_2 *are recurrent spaces.*

Proof. If M is an almost recurrent, then we have (2.8), (2.15) and (2.16). Because of (2.15) and (2.16) the condition (2.8) gives us

(2.17) $$\nabla_a R_{bcde} = 2\lambda_a R_{bcde},$$

(2.18) $$\nabla_\alpha R_{\beta\gamma\delta\varepsilon} = 2\lambda_\alpha R_{\beta\gamma\delta\varepsilon}.$$

In the same way we see that the other relations from (2.8), where there are indices as a, b, c, d, e and indices as $\alpha, \beta, \gamma, \delta, \varepsilon$ simultaneously are satisfied identically. Thus condition (2.17) (resp. 2.18) means that M_1 (resp. M_2) is a recurrent space. Conversely let M_1 and M_2 be recurrent spaces. Supposing that M_1 is subjected to condition (2.17) and M_2 is subjected to condition (2.18), putting $\lambda_i = (\lambda_a, \lambda_\alpha)$, we can state that for $M = M_1 \times M_2$ condition (2.8) is satisfied. So the theorem is proved.

In the same way we can prove the following theorem.

Theorem 3. *Let* $M = M_1 \times M_2$ *be a locally decomposable Riemannian space and* $R_1 \neq 0$, $R_2 \neq 0$ *where* R_1, R_2 *are the scalar curvatures of* M_1, M_2, *respectively. Then* M *is an almost Ricci recurrent space if and only if when* M_1 *and* M_2 *are Ricci recurrent spaces.*

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