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ON THE GENERATION OF HERONIAN TRIANGLES

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Abstract. We describe several algorithms for the generation of integer Heronian triangles with diameter at most n. Two of them have running time $\mathcal{O}\left(n^{2+\varepsilon}\right)$. We enumerate all integer Heronian triangles for $n \leq 600000$ and apply the complete list on some related problems.

1. Introduction. The Greek mathematician Heron of Alexandria (c. 10 A.D. – c. 75 A.D.) was probably the first to prove a relation between the side lengths a, b, and c and the area A of a triangle,

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad \text{where } s = \frac{a+b+c}{2}\,.$$

If its area and its side lengths are rational then it is called a Heronian triangle. Triangles with integer sides and rational area were considered by the Indian

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mathematician Brahmagupta (598–668 A.D.) who gives the parametric solution

$$\begin{aligned} a &= \frac{p}{q}h(i^2+j^2) \\ b &= \frac{p}{q}i(h^2+j^2) \\ c &= \frac{p}{q}(i+h)(ih-j^2) \end{aligned}$$

for positive integers p, q, h, i, and j satisfying $ih > j^2$ and $\gcd(p,q) = \gcd(h,i,j) = 1$.

Much has been contributed [5, 6, 22, 23, 24, 28] to the enumeration of such integral triangles, but still little is known about the generation of integer Heronian triangles with diameter $n = \max(a,b,c)$. Our aim is to develop a fast algorithm for the generation of the complete set of integer Heronian triangles with diameter n.

In this context an extensive search on those triangles was made by Randall L. Rathbun [20]. He simply checked the 7818928282738 integer triangles with diameter at most 2^{17} and received 5801746 primitive, i. e. such with $\gcd(a,b,c) = 1$, integer Heronian triangles with rational area.

In the next section we introduce a new parameterization and in Section 3 we give some algorithms for the generation of integer Heronian triangles. We finish with some combinatorial problems connected to Heronian triangles.

2. A new parameterization.

The obstacle for a computational use of Brahmagupta's parametric solution is the denominator q. So we first prove a few lemmas on q.

Lemma 2.1. *We can assume that the denominator q can be written as $q = w_1w_2w_3w_4$ with pairwise coprime integers w_1, w_2, w_3, w_4 and*

$$\begin{array}{llll} (w_1,h) = w_1, & (w_1,i) = w_1, & (w_1,i^2+j^2) = 1, & (w_1,h^2+j^2) = 1, \\ (w_2,h) = w_2, & (w_2,i) = 1, & (w_2,i^2+j^2) = 1, & (w_2,h^2+j^2) = w_2, \\ (w_3,h) = 1, & (w_3,i) = w_3, & (w_3,i^2+j^2) = w_3, & (w_3,h^2+j^2) = 1, \\ (w_4,h) = 1, & (w_4,i) = 1, & (w_4,i^2+j^2) = w_4, & (w_4,h^2+j^2) = w_4, \end{array}$$

where (x,y) abbreviates $\gcd(x,y)$.

P r o o f. Suppose $q = \dfrac{q_1q_2}{\gcd(q_1,q_2)}$ with $q_1|h$ and $q_2|i^2+j^2$. Now let r be a prime divisor of $\gcd(q_1,q_2) \Longrightarrow r|h,\ r|i^2+j^2$. With $b = \dfrac{pi(h^2+j^2)}{q}$ and

$\gcd(p,q)=1$ we also have $r|i$ or $r|h^2+j^2$. In the first case we have $r|i^2+j^2 \Longrightarrow r|j^2 \Longrightarrow r|\gcd(h,i,j) \Longrightarrow r=1$. In the second case we can use $r|h^2+j^2$ and $r|h$ to conclude $r|j^2$. With this and $r|i^2+j^2$ we also get $r|i^2$ and so $r|\gcd(h,i,j)=1 \Longrightarrow r=1$. So we know $\gcd(q_1,q_2)$=1.

Similarly we get $q=q_3q_4$ with $\gcd(q_3,q_4)=1$, $q_3|i$, and $q_4|\left(h^2+j^2\right)$.

Now we set $q_1=w_1w_2$, $q_2=w_3w_4$, $q_3=w_1w_3$, and $q_4=w_2w_4$. With $\gcd(q_1,q_2)=\gcd(q_3,q_4)=1$ we can conclude that the 4 divisibility conditions for each w_i and that the w_i are pairwise coprime. $\square$

Lemma 2.2.

$$w_4|2(i+h).$$

Proof. We consider $ai-bh=\dfrac{pih(i+h)(i-h)}{q}$ and conclude $w_4|(i-h)(i+h)$. Now we consider a prime factor r with $r|(i-h)$ and $\gcd(r,i+h)=1$. Because $r|w_4|a,b,c$ we get $r|(i^2+j^2)+(h^2+j^2)+2(ih-j^2)=(i+h)^2$, a contradiction to $\gcd(r,i+h)=1$. The proof is completed by $\gcd(i+h,i-h)|2$. $\square$

Lemma 2.3.

$$w_4\le 8n.$$

Proof. To prove the lemma we will show $w_4|8c$. From $w_4|2(i+h)$ we conclude $w_4|2(i^2+j^2)+2(h^2+j^2)-2(i+h^2)=4(j^2-ih)$ and thus $w_4|2(i+h)\dfrac{4(ih-j^2)}{w_4}=8c$. $\square$

The next step is to find a parameterization of the set of solutions which is better suited for computational purposes. Therefore we set

$$w_2=st^2$$

and

$$w_3=uv^2$$

with square-free integers s and u. Because $w_2|h^2+j^2$, $w_2|h$, $w_3|i^2+j^2$, $w_3|i$, and $\gcd(w_2,w_3)=1$ we have $stuv|j$. Thus we can set

$$\begin{aligned} h &= \alpha w_1 st^2,\\ i &= \beta w_1 uv^2,\\ j &= \gamma stuv \end{aligned}$$

with integers α, β, and γ.

With this we can give the following parameterization of the set of integer Heronian triangles.

$$\begin{aligned} a &= \frac{p\alpha u[(\beta w_1 v)^2 + (\gamma st)^2]}{w_4}, \\ b &= \frac{p\beta s[(\alpha w_1 t)^2 + (\gamma uv)^2]}{w_4}, \\ c &= \frac{p(\beta uv^2 + \alpha st^2)(\beta\alpha w_1^2 - \gamma^2 su)}{w_4}. \end{aligned}$$

3. Algorithms for the generation of integer Heronian triangles. In this section we list several algorithms to generate all integer Heronian triangles with diameter at most n. The main idea of the first algorithm is to utilize the parameterization of the previous section to run through all possible values for a, w_4 and then to determine all possible parameters p, w_1, s, t, u, v, α, β and γ. Without loss of generality we can assume that $a \geq b$ and thus $n \leq 2a - 1$. Then by Lemma 2.3 we have $w_4 \leq 8n \leq 16a$.

Algorithm 3.1 (Generation of integer Heronian triangles I)**.**

```
determine the prime factorization of all integers not greater that 16n
determine the solutions of z = x^2 + y^2 for all z ≤ 16n
for a from 1 to n
  for w4 from 1 to 16a
    loop over all quadruples (p, α, u, z) with pαuz=aw4
      loop over all pairs (x, y) with x^2 + y^2 = z
        loop over all triples (β, w1, v) with βw1v = x
          loop over all triples (γ, s, t) with γst = y
            calculate and output a, b, c
```

In order to prove the running time $\mathcal{O}\left(n^{2+\varepsilon}\right)$ of Algorithm 3.1 we rephrase two results from number theory.

Theorem 3.2 (Theorem 317 [13])**.** *For* $\varepsilon > 0$ *and* $n > n_0(\varepsilon)$

$$\tau(n) < 2^{(1+\varepsilon)\frac{\log n}{\log\log n}}$$

where $\tau(n)$ denotes the number of divisors of n.

So for each $\varepsilon > 0$, $f \leq 16n^2$ there are only $\mathcal{O}(n^\varepsilon)$ quadruples (f_1, f_2, f_3, f_4) with $f_1 f_2 f_3 f_4 = f$.

Lemma 3.3. *The equation $z = x^2 + y^2$ has at most $\mathcal{O}(z^\varepsilon)$ solutions in positive integers x, y for each $\varepsilon > 0$.*

Proof. If we denote the number of solutions of $z = x^2 + y^2$ in pairs (x, y) of integers by $r_2(z)$ then we have [12, 27]

$$r_2(z) = 4 \cdot \sum_{d|z} \sin\left(\frac{1}{2}\pi d\right) \in \mathcal{O}(z^\varepsilon). \qquad \square$$

Thus for each $\varepsilon > 0$ and each $z \leq 16n^2$ there are only $\mathcal{O}(n^\varepsilon)$ integer solutions of $z = x^2 + y^2$. Consequently there exists an implementation of Algorithm 3.1 with running time $\mathcal{O}\left(n^{2+\varepsilon}\right)$. Furthermore we can conclude that there are $\mathcal{O}\left(n^{1+\varepsilon}\right)$ integer Heronian triangles with diameter n. Maybe a faster algorithm can be designed by using refined number theoretic conditions on w_4. Unfortunately we were not able to find estimations on the number of integer Heronian triangles in the literature. Therefore we are unable to give a lower bound for the complexity of generating integer Heronian triangles.

In order to derive a second algorithm for the determination of integer Heronian triangles we utilize the Heron formula for the area of a triangle $\Delta = (a, b, c)$ and consider

$$16A^2 = (p - c)(p + c)(c - q)(c + q)$$

with $p = a + b$ and $q = a - b$.

The idea is to run trough all possible values for $4A$ and then determine a, b and c by factorising $16A^2$.

Algorithm 3.4 (Generation of integer Heronian triangles II).
`loop over` *all m and the prime factorization of m^2 with $1 \leq m \leq \sqrt{3}n^2$*
  `loop over` *all $p - c$, $p + c$, $c - q$, $c + q$ with $m^2 = (p - c)(p + c)(c - q)(c + q)$*
    determine a, b, and c
      `if` *a, b, and c are positive integers satisfying the triangle conditions* `then` *output a, b, and c*

Since $16A^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c) \leq 3n^4$ we have $m = 4A \leq \sqrt{3}n^2$. For the factorization of m we may use an arbitrary algorithm with running time $\mathcal{O}(m^\varepsilon)$ [21]. If we are allowed to use $\Omega(n^2)$ space a less sophisticated possibility would be to use the Sieve of Eratosthenes on the numbers 1 to $\sqrt{3}n^2$. Thus Algorithm 3.4 can be implemented with running time $\mathcal{O}\left(n^{2+\varepsilon}\right)$.

For completeness we would also like to give the pseudocode of the algorithm mentioned in the introduction.

Algorithm 3.5 (Generation of integer Heronian triangles III)**.**

```
for a from 1 to n
  for b from ⌈(a+1)/2⌉ to a
    for c from a+1-b to b
      if (a+b+c)(a+b-c)(a-b+c)(-a+b+c) is the square of an integer
      then output a, b, and c
```

The running time of Algorithm 3.5 is $\mathcal{O}\left(n^3\right)$. It has the advantage of producing only one representative of each equivalence class of integer Heronian triangles in a canonical ordering. Due to the overhead of Algorithm 3.1 and Algorithm 3.4 the trivial Algorithm 3.5 is faster for *small* values of n.

For a practical implementation we describe some useful tricks to enhance Algorithm 3.5 a bit. We observe that if $(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$ is a square then it must also be a square if we calculate in the ring $\mathbb{Z}_m$ for all $m \in \mathbb{N}$. In our implementation we have used the set of divisors of 420 for m. In a precalculation we have determined all possible triples in $\mathbb{Z}_{420}^3$. Hereby the number of candidates is reduced by a factor of $\frac{14744724}{74088000} \approx 0.199$. Additionally we determine the square-free parts of the integers at most $3n$ in a precalculation. Instead of determining the square root of a big integer we determine square-free parts of integers. If $\operatorname{sfp}(f)$ gives the square-free part of f then we have

$$\operatorname{sfp}(f_1 \cdot f_2) = \frac{\operatorname{sfp}(f_1) \cdot \operatorname{sfp}(f_2)}{\gcd\left(\operatorname{sfp}(f_1), \operatorname{sfp}(f_2)\right)^2}.$$

Thus we can avoid high precision arithmetic by using a gcd-algorithm. Without it we would have to deal with very large numbers – since we compute up to $n = 600000$. A complete list of the integer Heronian triangles of diameter at most 600000 can be obtained upon request by the author. In the following sections we will use this list to attack several combinatorial problems.

4. Maximal integral triangles. A result due to Almering [1] is the following. Given any rational triangle $\Delta = (a, b, c) \in \mathbb{Q}^3$ in the plane, i. e., a triangle with rational side lengths, the set of all points x with rational distances to the three corners of Δ is dense in the plane. Later Berry [2] generalized this result to triangles whose side lengths are rational when squared and with one rational side length. If we proceed to integral side lengths and integral coordinates the situation is a bit different. In [15] the authors search for inclusion-maximal integral triangles over $\mathbb{Z}^2$ and answer the existence question from [9] positively. They exist but appear to be somewhat rare. There are only seven inclusion-maximal integral triangles with diameter at most $5,000$.

Here we have used the same algorithm as in [15] to determine inclusion-maximal integral triangles over $\mathbb{Z}^2$ with diameter at most $15,000$. Up to symmetry the complete list is given by:
(2066,1803,505), (2549,2307,1492), (3796,2787,2165), (4083,2425,1706), (4426,2807,1745), (4801,2593,2210), (4920,4177,985), (5044,4443,2045), (5045,4803,244), (5186,5163,745), (5905,5763,1586), (5956,4685,2427), (6120,5953,409), (6252,3725,3253), (6553,5954,3099), (6577,5091,1586), (6630,5077,1621), (6787,5417,1546), (6855,6731,130), (6890,6001,1033), (6970,4689,4217), (6987,5834,1585), (7481,6833,5850), (7574,4381,3207), (7717,6375,1396), (7732,7215,541), (7734,6895,4537), (7793,4428,3385), (7837,6725,1308), (7913,6184,1745), (7985,7689,298), (8045,7131,1252), (8187,6989,1252), (8237,7899,4036), (8249,7772,879), (8286,5189,3865), (8375,6438,1949), (8425,4706,3723), (8644,7995,1033), (8961,8633,740), (9683,8749,4632), (9745,5043,4706), (9771,7373,5044), (9840,8473,2089), (9939,6388,3845), (9953,6108,4825), (10069,9048,6421), (10081,8705,1378), (10088,8886,4090), (10090,9606,488), (10100,5397,5389), (10114,5731,4405), (10372,7739,2775), (10394,8499,1993), (10441,6122,5763), (10595,10283,340), (10600,6737,3881), (10605,8957,1754), (10615,10119,562), (10708,9855,1069), (10804,8691,7013), (10825,8259,3242), (10875,9805,1076), (10993,8164,3315), (11133,10250,6173), (11199,10444,757), (11283,8788,4229), (11332,9147,6029), (11434,6159,5305), (11441,7577,3880), (11559,6145,5416), (11765,10892,877), (11787,9341,3172), (12053,8979,3076), (12676,9987,3845), (12745,12603,1586), (12757,11544,1237), (12810,12077,2669), (12818,11681,1601), (12946,9523,3425), (12953,8361,4930), (12965,12605,5406), (13012,11405,2091), (13061,9745,8934), (13100,12875,1011), (13106,11908,6198), (13115,11492,1709), (13130,12097,2329), (13309,12916,8585), (13350,7901,5645), (13369,12867,698), (13385,11931,1618), (13445,9750,3701), (13466,8665,4803), (13683,8042,6841), (13700,11115,2621), (13710,13462,260), (13740,8053,5951), (13780,12002,2066), (13876,10657,3315), (13940,9378,8434), (13940,13647,12775), (13951,11785,9608), (13971,10804,8933), (14065,10984,3831),

(14065,12531,5378), (14126,12135,4357), (14172,12725,1933), (14185,9879,8194), (14282,8665,5619), (14331,7517,6964), (14356,14019,2837), (14379,10685,3748), (14545,13274,2427), (14615,11332,3441), (14625,13060,1799), (14633,12329,2320), (14637,11170,6641), (14677,11436,6145), (14690,11353,5641), (14700,11861,2845), (14705,14351,8482), (14775,10673,5284), (14785,12219,3242), (14819,13810,1695), and (14962,13666,11700).

Thus with 126 examples the situation changes a bit. There do exist lots of inclusion-maximal integral triangles over $\mathbb{Z}^2$. Some triangles from this list may be derived from others since $\left(\frac{a}{g},\frac{b}{g},\frac{c}{g}\right)$ is an inclusion-maximal integral triangle over $\mathbb{Z}^2$ for $g=\gcd(a,b,c)$ if (a,b,c) is an inclusion-maximal integral triangle over $\mathbb{Z}^2$. Here the limiting factor is the algorithm from [15] and not the generation of integral Heronian triangles. We note that there are also inclusion-maximal integral tetrahedrons over $\mathbb{Z}^3$ [15].

5. $\mathbf{n_2}$-cluster. A n_2 cluster is a set of n lattice points in $\mathbb{Z}^2$ where all pairwise distances are integral, no three points are on a line, and no four points are on a circle [19]. The existence of a 7_2-cluster is an unsolved problem of [11, Problem D20] and [19]. Since a 7_2-cluster is composed of Heronian triangles and a special case of a plane integral point set, we can use the exhaustive generation algorithms described in [17, 18] to search for 7_2-clusters. The point set with coordinates

$$\Big\{(0,0),(375360,0),(55860,106855),(187680,7990),\\ (187680,82688),(142800,190400),(232560,190400)\Big\}$$

and distance matrix

$$\begin{pmatrix} 0 & 375360 & 120575 & 187850 & 205088 & 238000 & 300560\\ 375360 & 0 & 336895 & 187850 & 205088 & 300560 & 238000\\ 120575 & 336895 & 0 & 164775 & 134017 & 120575 & 195455\\ 187850 & 187850 & 164775 & 0 & 74698 & 187850 & 187850\\ 205088 & 205088 & 134017 & 74698 & 0 & 116688 & 116688\\ 238000 & 300560 & 120575 & 187850 & 116688 & 0 & 89760\\ 300560 & 238000 & 195455 & 187850 & 116688 & 89760 & 0 \end{pmatrix}$$

is an integral point set over $\mathbb{Z}^2$, see Figure 1. Unfortunately the points 1, 2, 6 and 7 are on a circle. But, no three points are on a line and no other quadruple is situated on a circle. If we add $(319500, 106855)$ as an eighth point we receive an integral point set $\mathcal{P}$ over $\mathbb{Z}^2$ where no three points are situated on a line. There are exactly three quadruples of points which are situated on a circle: $\{1,2,3,8\}$, $\{1,2,6,7\}$, and $\{3,6,7,8\}$.

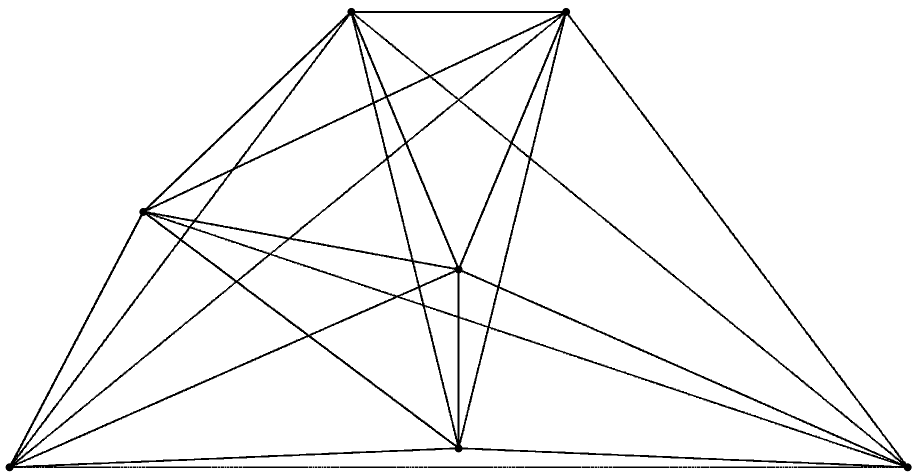

Figure 1: Almost a 7_2-cluster.

We would like to remark that the automorphism group of an n_2-cluster for $n \geq 6$ must be trivial. In [17] the possible automorphism groups of planar integral point sets were determined to be isomorphic to id, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$ or S_3. It was also shown that an automorphism of order 3 is only possible for characteristic 3. Since n_2-cluster have characteristic 1 such an automorphism cannot exist. If we would have an automorphism of order 2 then for $n \geq 6$ either three points are collinear or four points are situated on a circle.

Using our list of Heronian triangles we have performed an exhaustive search for 7_2-clusters up to diameter 600000, unfortunately without success. If we relax the condition of integral coordinates, then examples do exist, see [16].

6. Perfect pyramids. In [3] the author considers tetrahedra with integral side lengths, integral face areas, and integral volume, see also [20]. The smallest such example has side lengths $(a, b, c, d, e, f) = (117, 84, 51, 52, 53, 80)$ using the notation from Figure 2. In the plane a triangle with integral edge lengths and rational area is forced to have an integer area. The situation changes slightly in three-dimensional space. Here it is possible that the edge lengths of a tetrahedron are integer and that the volume is genuinely rational. If the edge lengths are integral and the face areas and the volume are rational, then all values must be integer, see [8]. In [3] it was also shown that a perfect pyramid with at most two different edge lengths cannot exist. For three different edge lengths a parameter solution of an infinite family is given.

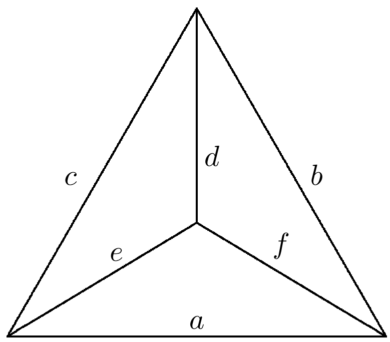

Figure 2: The six edges of a tetrahedron (a, b, c, d, e, f).

The authors of [3, 8] consider all possibilities of coincidences of edge lengths. Up to symmetry we have the following configurations:

1-parameter	(i) $a = b = c = d = e = f$,	
2-parameter	(i) $a = b = c = d = e, f$,	(ii) $a = b = c = d, e = f$,
	(iii) $a = c = d = f, b, e$,	(iv) $a = b = c, d = e = f$,
	(v) $a = d = f, b = c = e$,	
3-parameter	(i) $a = b = c = d, e, f$,	(ii) $a = c = d = f, b, e$,
	(iii) $a = b = c, d = e, f$,	(iv) $a = d = f, b = c, e$,
	(v) $a = d = f, b = e, c$,	(vi) $a = d, b = e, c = f$,
	(vii) $a = e, b = f, c = d$,	(viii) $a = b, c, d = e = f$,
	(ix) $a = d, b = f, c = e$,	
4-parameter	(i) $a = b = c, d, e, f$,	(ii) $a = b = d, c, e, f$,
	(iii) $a = b = f, c, d, e$,	(iv) $a = d, b = e, c, f$,
	(v) $a = d, b = f, c, e$,	(vi) $a = b, d = f, c, e$,
	(vii) $a = b, d = e, c, f$,	
5-parameter	(i) $a = d, b, c, e, f$,	(ii) $a = b, c, d, e, f$,
6-parameter	(i) a, b, c, d, e, f.	

In the cases 1(i), 2(i-v), 3(i), 3(iii), 3(iv), 3(vii), and 4(i) there is no solution. There are parameter solutions of infinite families known for the cases 3(ii), 3(v), 3(vi), 3(viii), and 3(ix). Unfortunately in case 3(viii) all known solutions are degenerate, meaning that the corresponding pyramid has volume 0. It is a conjecture of [8] that no non-degenerate solution exists in this case. For the cases 4(iv), 4(vii), 5(ii), and 6(i) sporadic solutions are known. Cases 4(ii), 4(iii), 4(v), 4(vi), and 5(i) remain open problems.

In order to answer some of these questions we have performed an exhaustive search on perfect pyramids up to diameter 600000 using the following algorithm: Let n be the maximum diameter, $\kappa \in \mathbb{N}$, and $\varphi : \{1,\dots,n\}\times\{1,\dots,n\} \to \{0,\kappa-1\}$ a mapping. For $0 \le i \le \kappa-1$ let $\mathcal{L}_i$ contain integers $1 \le c \le n$ such that there exits integers $a, b \le n$ fulfilling $\phi(a,b) = i$ and where (a,b,c) is a Heronian triangle.

Algorithm 6.1 (Generation of perfect pyramids)**.**

```
for d from 1 to n
  loop over all (a_1, b_1), (a_2, b_2) where (d, a_1, b_1) and (d, a_2, b_2) are Heronian triangles
    lb = max{ |a_1 - a_2|, |b_1 - b_2| } + 1
    ub = min{ a_1 + a_2 - 1, b_1 + b_2 - 1, d }
    if ub - lb + 1 = min{ ub - lb + 1, |L_φ(a_1,a_2)|, |L_φ(b_1,b_2)| } then
      for x from lb to ub
        if P = (d, a_1, a_2, b_1, b_2, x) is a perfect pyramid then output P
    if |L_φ(a_1,a_2)| < min{ ub - lb + 1, |L_φ(b_1,b_2)| } then
      for j from 1 to |L_φ(a_1,a_2)|
        x = L_φ(a_1,a_2)(j)
        if P = (d, a_1, a_2, b_1, b_2, x) is a perfect pyramid then output P
    if |L_φ(b_1,b_2)| < min{ ub - lb + 1, |L_φ(a_1,a_2)| } then
      for j from 1 to |L_φ(b_1,b_2)|
        x = L_φ(b_1,b_2)(j)
        if P = (d, a_1, a_2, b_1, b_2, x) is a perfect pyramid then output P
```

The efficiency of Algorithm 6.1 depends on a suitable choice of κ and φ in order to keep the lists $\mathcal{L}_i$ small. From a theoretical point of view for given integers a and b there do exist at most $4 \cdot \tau(ab)^2$ different values c such that (a,b,c) is a Heronian triangle [14]. Here $\tau(m)$ denotes the number of divisors of

m and we have $\tau(m) \in \mathcal{O}(m^{\varepsilon})$ for all $\varepsilon > 0$.

Unfortunately we have found no examples for one of the open cases. Thus possible examples have a diameter greater than 600000.

In [7] the authors have considered rational tetrahedra with edges in arithmetic progression. They proved that tetrahedra with integral edge lengths, rational face areas and rational volume do not exist. If only one face area is forced to be rational then there exists the example $(a, b, c, d, e, f) = (10, 8, 6, 7, 11, 9)$, which is conjectured to be unique up to scaling. We have verified this conjecture up to diameter 600000.

surface area	**volume**	**a**	**b**	**c**	**d**	**e**	**f**
6384	8064	160	153	25	39	56	120
64584	170016	595	208	429	116	325	276
64584	200928	595	116	507	208	325	276
69058080	14985432000	11660	5512	10812	1887	9945	5215
69058080	15020132400	11687	5215	10812	1887	9945	5512
69058080	16198182000	11687	1887	11660	5215	9945	5512

Table 1: Sets of primitive perfect pyramids with equal surface area.

Now we consider sets of primitive perfect pyramids (here the greatest common divisor of the six edge length must be equal to one) which have equal surface area, see [20]. We have performed an exhaustive search on the perfect pyramids up to diameter 600000 and list the minimal sets, with respect to the surface area, in Table 1. We would like to remark that six triples with equal surface area were found. Clearly the same question arises for equal volume instead of equal surface area. In [20] the smallest examples for up to three pyramids with equal volume are given. Unfortunately we have found no further examples consisting of three or more primitive perfect pyramids up to diameter 600000. Additionally we have searched, without success, for a pair of perfect pyramids with equal surface area and equal volume.

Clearly the concept of perfect pyramids can be generalized to higher dimensions. The m-dimensional volume $V_m(\mathcal{S})$ of a simplex $\mathcal{S} = (v_0, \ldots, v_m)$ can be expressed by a determinant [25]. Therefore let $D := \left(\|v_i - v_j\|_2^2 \right)_{0 \le i,j \le m}$. With this we have $V_m(\mathcal{S})^2 = \frac{(-1)^{m+1}}{2^m (m!)^2} \cdot \det B$, where B arises from D by bordering D with a top row $(0,1,\ldots,1)$ and a left column $(0,1,\ldots,1)^T$. We call an m-dimensional simplex $\mathcal{S}$ with integer edge lengths a perfect pyramid if $V_r\Big(\{i_0,\ldots,i_r\}\Big)\in\mathbb{Q}_{>0}$ for all $2\le r\le m$ and all $\{i_0,\ldots,i_r\}\subseteq\mathcal{S}$. Up to diameter 600000 we have only found some degenerate perfect pyramids where $V_4(\mathcal{S})=0$.

7. Heronian triangles and sets of Heronian triangles with special properties. In [26] it is shown that for every positive integer N there exists an infinite family parameterized by $s\in\mathbb{Z}_{>0}$, of N-tuples of pairwise non-similar Heron triangles, all N with the same area $A(s)$ and the same perimeter $p(s)$, such that for any two different s and s' the corresponding ratios $A(s)/p(s)^2$ and $A(s')/p(s')^2$ are different. Randall Rathbun found the smallest N-tuples for $N\le 9$. In Table 2 we give the perimeter and the area of the smallest N-tuples for $N\le 10$. The smallest 11-tuple has a perimeter greater than 1200000.

N	perimeter	area	N	perimeter	area
1	12	24	6	2340	786240
2	70	840	7	11700	19656000
3	98	1680	8	84630	365601600
4	448	26880	9	142912	2117955840
5	1170	196560	10	441784	30324053760

Table 2: Minimum perimeter of N-tuples of Heronian triangles with equal perimeter and equal area.

Some authors consider Heronian triangles with rational medians, see e. g. [4, 5]. Infinite families and some sporadic examples of Heronian triangles with two rational medians are known. Whether there exists a Heronian triangle with three rational medians is an open question. For this problem we can only state that the lists in [4, 5] are complete up to diameter 600000.

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