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ON A CRITERION FOR λ-SEQUENCES

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We extend some results connected with the well known theorem of Hermite — Poulain for the functions in the class L. A criterion for λ-sequences and some new results are obtained.

It will be denoted by L the class of all entire functions which are either polynomials with real coefficients and with nonreal zeros only or are limits of such polynomials in every bounded domain. Also, for the sake of convenience, by L^+ (resp. L^-) will be denoted the class of polynomials in L which take only positive (resp. negative) values on the real axis.

In [1] Iliev introduced the so-called λ-sequences with the following definition: A sequence $\lambda=\{\lambda_k\}_{k=0}^{\infty}$ of real numbers is called a λ-sequence if it maps every polynomial $P(z)=\Sigma_{k=0}^{2n} a_k z^k$ from L into a polynomial $\lambda P(z)=\Sigma_{k=0}^{2n} \lambda_k a_k z^k$ of the same class. We shall denote by λ the set of all λ-sequences.

In [2] "λ-sequences about degree $2n$" are defined as follows: A finite sequence $\lambda=\{\lambda_k\}_{k=0}^{2n}$ is called a λ-sequence about degree $2n$ if for every polynomial $P(z)$ from the class L of degree at most $2n$ the polynomial $\lambda P(z)$ belongs to the same class. It follows immediately from the properties of the λ-sequences [2] that if $\{\lambda_k\}_{k=0}^{\infty}\in\lambda$ then $\{\lambda_{2m+k}\}_{k=0}^{2n}\in\lambda$ for every natural number m. Here we shall not consider the question when a given finite λ-sequence can be extended to an infinite λ-sequence.

The set of the strictly positive sequences [3] we shall denote by s. This set is defined as follows: The sequence $\{\{\alpha_k\}_{k=0}^{\infty}$ of real numbers belongs to the class s if and only if for every polynomial $P(z)=\Sigma_{k=0}^{n} a_k z^k$ which takes positive values on the real axis the inequality $\Phi(P):=\Sigma_{k=0}^{n}\alpha_k a_k>0$ holds. A finite s-sequence can be defined analogously to the finite λ-sequence. A sequence $\{\alpha_k\}_{k=0}^{n}$ of real numbers is called a finite s-sequence about degree n if for every polynomial $P(z)=\Sigma_{k=0}^{m} a_k z^k$ of degree $m\leqq n$ such that $P(x)>0$ when $x\in\mathrm{R}$ $\Phi(p)>0$ holds.

In [2] it is proved the following

Theorem. *Let $\{\lambda_k\}_{k=0}^{\infty}$ be a sequence of real numbers and $\lambda_0>0$. Then $\{\lambda_k\}_{k=0}^{\infty}\in\lambda$ if and only if $\{\lambda_k\}_{k=0}^{\infty}\in s$.*

An analogous statement is valid also for the finite λ and s-sequences.

An essential result from the theorem just quoted is the possibility to represent the terms of a given λ-sequence as moments of a Stieltjes function. With the aid of this result we are able to prove the following theorem.

Theorem 1. *Let $\{\lambda_k\}_{k=0}^{\infty}$, $\lambda_0>0$, be a sequence of real numbers. Then $\{\lambda_k\}_{k=0}^{\infty}$ is a λ-sequence if and only if the polynomial*

(1) $$\varphi(z)=\sum_{k=0}^{2n}\frac{\lambda_k}{k!}f^{(k)}(z)$$

belongs to the class L^+ for every polynomial $f(z)$ in L^+, $2n=\deg f(z)$.

Proof. Let $\{\lambda_k\}_{k=0}^{\infty}$, $\lambda_0>0$, be a λ-sequence and let $f(z)$ be a polynomial in the class L^+, i.e. $f(z)\in L$ and for every $x\in \mathsf{R}$, $f(x)>0$. Then $\{\lambda_k\}_{k=0}^{\infty}\in s$ and there exists a monotone increasing and bounded function $g(t)$ with infinitely many points of growth such that [2]

$$\lambda_k=\int_{-\infty}^{\infty}t^k dg(t),\ k=0,\ 1,\ 2,\ldots$$

We obtain

$$\varphi(x)=\sum_{k=0}^{2n}\frac{\lambda_k}{k!}f^{(k)}(x)=\int_{-\infty}^{\infty}f(t+x)dg(t).$$

Since

$$f(z+x)=\sum_{k=0}^{2n}\frac{f^{(k)}(x)}{k!}z^k\in L^+,$$

when $x\in\mathsf{R}$ it follows that $f(t+x)>0$ for every $t\in\mathsf{R}$. Then by Lemma 2 ([3, p. 456]) in the case of an infinite interval we obtain $\varphi(x)>0$ for every $x\in\mathsf{R}$, i.e. $\varphi(z)\in L^+$.

Suppose now that $\{\lambda_k\}_{k=0}^{\infty}$ is a sequence of real numbers, $\lambda_0>0$ such that for every polynomial $f(z)\in L^+$, $\deg f(z)=2n$ the polynomial (1) belongs to the class L^+. Since by the assumption $\varphi(z)\in L^+$, i.e. $\varphi(z)\in L$, $\varphi(x)>0$ we conclude that

$$\varphi(0)=\sum_{k=0}^{2n}\lambda_k a_k>0.$$

Then the definition of the strictly positive sequence shows that $\{\lambda_k\}_{k=0}^{\infty}$ is s-sequence for all polynomials which are strictly positive over R and consequently the sequence $\{\lambda_k\}_{k=0}^{\infty}$ is λ-sequence.

The previous theorem allows to obtain the following criterion for finite λ-sequences.

Theorem 2. *Let $\{\lambda_k\}_{k=0}^{2n}$ be a finite sequence of real numbers, $\lambda_0>0$, and let*

$$P(z)=\sum_{k=0}^{2n}\frac{\lambda_k}{k!}z^k.$$

Then the sequence $\{\lambda_k\}_{k=0}^{2n}$ is a λ-sequence about degree $2n$ if and only if the polynomial

$$P(D)f(z)=\sum_{k=0}^{2n}\frac{\lambda_k}{k!}f^{(k)}(z)$$

belongs to the class L^+ for every polynomial $f(z)$ in L^+ of degree at most $2n$.

Proof. Let $\{\lambda_k\}_{k=0}^{2n}\in\lambda$, $\lambda_0>0$. Then $\{\lambda_k\}_{k=0}^{2n}\in s$. If $f(z)$ is an arbitrary polynomial in L^+, $\deg f(z)=2m\leqq 2n$, then for every $x\in\mathsf{R}$

$$\sum_{k=0}^{2n} \frac{f^{(k)}(x)}{k!}\lambda_k>0$$

and consequently $P(D)f(z)\in L^+$.

Conversely, let $P(D)f(z)\in L^+$ for every polynomial $f(z)$ in L^+, $\deg f(z)\leqq 2n$. If $f(z)=\Sigma_{k=0}^{2m} a_k z^k$ is an arbitrary polynomial in L^+, $2m\leqq 2n$, then $P(D)f(x)>0$ for every $x\in \mathrm{R}$ and therefore $\Sigma_{k=0}^{2m} \alpha_k\lambda_k>0$. Consequently, $\{\lambda_k\}_{k=0}^{2n}\in s$ and $\{\lambda_k\}_{k=0}^{2n}\in\lambda$.

We shall consider some applications of the previous results.

Theorem 3. *Let $\{\lambda_k\}_{k=0}^{\infty}\in\lambda$, $\lambda_0>0$ and let $f(z)$ be an arbitrary polynomial in L, $\deg f(z)=2n$. Then the polynomial*

$$F(z)=\sum_{k=0}^{2n}\frac{\lambda_k}{k!}z^k f^{(k)}(z)$$

belongs to the class L.

Proof. Let $x_0\in\mathrm{R}$, $x_0\neq 0$. From one of the properties of λ-sequences it follows that $\{\lambda_k x_0^k\}_{k=0}^{\infty}\in\lambda$. Now we apply Theorem 1 and obtain that

$$P(z)=\sum_{k=0}^{2n}\frac{\lambda_k x_0^k}{k!}f^{(k)}(z)\in L,$$

i.e. $P(x)\neq 0$ for every $x\in\mathrm{R}$. In particular,

$$P(x_0)=\sum_{k=0}^{2n}\frac{\lambda_k x_0^k}{k!}f^{(k)}(x_0)=F(x_0)\neq 0.$$

But $x_0\neq 0$ is an arbitrary real number. Consequently $F(x)\neq 0$ for all real numbers $x\neq 0$ and since $F(0)=\lambda_0>0$ it follows that $F(z)\in L$.

As an immediate consequence of Theorem 3 we obtain

Theorem 4. *Let $f(z)$ be an arbitrary polynomial in L, $\deg f(z)=2n$. Then the polynomial $\Sigma_{k=0}^{2n} z^k f^{(k)}(z)$ belongs to the class L.*

Since the sequence $\{k!\}_{k=0}^{\infty}$ is λ-sequence the above theorem follows from Theorem 3 if we choose λ_k to be $k!$.

Theorem 5. *Let $\{\lambda_k'\}_{k=0}^{\infty}\in\lambda$, $\{\lambda_k''\}_{k=0}^{\infty}\in\lambda$, $\lambda_0'>0$, $\lambda_0''>0$. Then $\{\Lambda_k\}_{k=0}^{\infty}\in\lambda$, where $\Lambda_k=\Sigma_{\nu=0}^{k}\lambda_\nu'\lambda_{k-\nu}''$.*

Proof. In view of Theorem 1 we need only to consider the action of $\{\Lambda_k\}_{k=0}^{\infty}$ on the polynomials in L^+. Let $f(z)$ be an arbitrary polynomial in L^+, $\deg f(z)=2n$ and let

$$\psi_1(z)=\sum_{k=0}^{2n}\frac{\lambda_k'}{k!}z^k,\quad \psi_2(z)=\sum_{k=0}^{2n}\frac{\lambda_k''}{k!}z^k,\quad \psi(z)=\sum_{k=0}^{2n}\frac{\Lambda_k}{k!},$$

$F(z)=\psi_1(z)\psi_2(z)$. Then by Theorem 1 it suffices to show that $\psi(D)f(z)\in L^+$. Consider

$$F(D)f(z)=\psi_1(D)\psi_2(D)f(z)=\psi(D)f(z).$$

Since by Theorem 1 $h(z)=\psi_2(D)f(z)\in L^+$, another application of the same theorem shows that $\psi_1(D)h(z)=\psi(D)f(z)\in L^+$. Hence by Theorem 1 the sequence $\{\Lambda_k\}_{k=0}^{\infty}$ is a λ-sequence.

Finally, we will consider another application of Theorem 1 connected with the distribution of the zeros of the so-called polynomial of Silvester

$$S(z)=1+\frac{z}{1!}+\ldots+\frac{z^n}{n!}. \tag{2}$$

In [4] it is proved that the polynomial (2) has at most one real zero. In 1970 Bojorov introduced the polynomials of the kind

$$S(f;\ z)=\sum_{k=0}^{n} f^{(k)}(z), \quad \deg f(z)=n, \tag{3}$$

as a generalization of the polynomials of Silvester. Indeed, when $f(z)=z^n/n!$ the polynomial $S(f;\ z)$ coincides with (2).

Using the idea of the proof of the mentioned statement for the polynomial (2), given in [4] one can prove statements which in general lead to the following

$$Z_{\mathbf{R}}(S(f;\ z))\leqq Z_{\mathbf{R}}(f(z)). \tag{4}$$

Here $Z_{\mathbf{R}}(f)$ denotes the numbers of real zeros of $f(z)$. It is supposed for the polynomial $f(z)$ that if it has real zeros they are simple.

It follows immediately from (4) the following

Theorem 6. *If the polynomial $f(z)$ belongs to the class L, then $S(f;\ z)$ belongs to the same class.*

It is interesting to note that the same result can be obtained independently of (4) by the application of Theorem 1. Indeed, let $f(z)\in L^+$ (resp. L^-), $\deg f(z)=2n$. Since $\{k!\}_{k=0}^{\infty}\in\lambda$ then by Theorem 1 the polynomial

$$\sum_{k=0}^{2n}\frac{k!}{k!}f^{(k)}(z)=S(f;\ z)$$

belongs to the class L^+ (resp. L^-). Consequently $S(f;\ z)\in L$.

Theorem 7. *Let $f(z)$ be a polynomial in L, $\deg f(z)=2n$ and let $\{\lambda_k\}_{k=0}^{\infty}\in\lambda$, $\lambda_0>0$. Then the polynomial*

$$\sum_{k=0}^{2n}\lambda_k f^{(k)}(z)$$

also belongs to the class L.

In order to prove this theorem it is sufficient to note that from $\{\lambda_k\}_{k=0}^{\infty} \in \lambda$, $\{k!\}_{k=0}^{\infty} \in \lambda$ it follows $\{k!\, \lambda_k\}_{k=0}^{\infty} \in \lambda$. The application of Theorem 1 shows that the polynomial

$$\sum_{k=0}^{2n} \frac{k!\, \lambda_k}{k!} f^{(k)}(z) = \sum_{k=0}^{2n} \lambda_k f^{(k)}(z)$$

belongs to L.

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