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UNIVALENT FUNCTIONS THAT ARE LOCAL EXTREMA OF TWO REAL FUNCTIONALS

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The class S consists of all functions $f(z)=z+c_2z^2+\dots$ that are regular and univalent in the unit disc. Let the functionals $F(c_2,\dots, c_n)$ and $\Phi(c_2,\dots, c_m)$ has nonvanishing gradient in domains containing sets of the type $\{|c_2|\leqq 2,\dots, |c_r|\leqq r\}$. A function $f_0\in S$ is found for which the functionals F and Φ attain local extremum.

Let S be the class of functions $f(z)=z+\Sigma_{k=2}^{\infty} c_k z^k$, $c_k=\zeta_{2k-1}+i\zeta_{2k}$ regular and univalent in the disc $\Delta=\{z:|z|<1\}$.

If $f,\ g\in S$ the metric $\rho(f,\ g)=\max_{|z|=1/2}|f(z)-g(z)|$ turns the class S into a compact metric space.

For a function $\varphi(z)=\Sigma_{n=0}^{\infty}\gamma_n z^n$ regular in a neighbourhood of the origin we shall use the notation $\{\varphi\}_n=\gamma_n$.

Let $F=F(c_2,\dots, c_n)$ be a real-valued function of the variables ζ_j $(3\leqq j\leqq 2n)$ defined and continuously differentiable in some domain containing the set $\{|c_2|\leqq 2,\dots, |c_n|\leqq n\}$. Moreover suppose grad $F\neq 0$ in this domain. We shall consider the function F as a functional on the class S. Let $\Phi=\Phi(c_2,\dots, c_m)$ be another functional on S of the mentioned type.

With some additional assumption concerning F and Φ we find the form of a function $f_0\in S$, which is locally extremal for both of these functionals. Earlier [1, 2, 3] (see also [4, p. 347-351]) solutions of this problem has been obtained under some restrictions on F, Φ as well as on f_0. For the couples of functionals

$$F=|c_n|,\quad \Phi=|c_m|,$$

$$F=|\{\log f'(z)\}_n|,\quad \Phi=|\{\log f'(z)\}_m|,$$

$$F=|\{\log[f(z)/z]\}_n|,\quad \Phi=|\{\log[f(z)/z]\}_m|$$

$(n\neq m$ in all the three cases) the set of functions f_0 has been described by V. V. Andreev and the author [7] for the first case (the last two are submitted for publication).

Denote $\lambda_k=2(\partial F/\partial\bar{c}_k)=(\partial F/\partial\zeta_{2k-1})+i(\partial F/\partial\zeta_{2k}),\quad k=2,\dots, n,$

$$\mu_l=2(\partial\Phi/\partial\bar{c}_l)=(\partial\Phi/\partial\zeta_{2l-1})+i(\partial F/\partial\zeta_{2l}),\quad l=2,\dots, m.$$

Let $u\in[2, n]$ and $v\in[2, m]$ be the greatest integers for which $\lambda_u\neq 0$ and $\mu_v\neq 0$ respectively. It is well known ([3, Ch. 1], [4, p. 338-343]) that if the functional

$F=F(c_2,\dots,c_n)$ posseses a local extremum at the function $w=f_0(z)=\Sigma_{k=1}^{\infty} a_k z^k$, $a_1=1$, the following differential equation is satisfied

$$\left(\frac{zw'}{w}\right)^2 Q_u(w)+P_u(z)=0, \tag{1}$$

where

$$P_u(z)=\sum_{k=2}^{u}\left[\bar{\lambda}_k\left(\sum_{j=1}^{k-1}(k-j)a_{k-j}z^{-j}\right)+(k-1)\bar{\lambda}_k a_k+\lambda_k\left(\sum_{j=1}^{k-1}(k-j)\bar{a}_{k-j}z^j\right)\right],$$

$$-Q_u(w)=\sum_{k=2}^{u}\bar{\lambda}_k q_k(w),$$

$$q_k(w)=\left\{\frac{f_0^2(z)}{f_0(z)-w}\right\}_k=-\sum_{j=1}^{k-1}\{f_0^{j+1}(z)\}_k w^{-j}.$$

On the other hand, if the functional $\Phi=\Phi(c_2,\dots,c_m)$ posseses a local extremum at the same function $f_0(z)$ an analogous differential equation will be satisfied. Namely

$$\left(\frac{zw'}{w}\right)^2 Q_v(w)+P_v(z)=0, \tag{2}$$

where

$$P_v(z)=\sum_{l=2}^{v}\left[\bar{\mu}_l\left(\sum_{j=1}^{l-1}(l-j)a_{l-j}z^{-j}\right)+\bar{\mu}_l(l-1)a_l+\mu_l\left(\sum_{j=1}^{l-1}(l-j)\bar{a}_{l-j}z^j\right)\right],$$

$$Q_v(w)=\sum_{l=2}^{v}\bar{\mu}_l q_l(w).$$

Theorem 1. *Let $u\geqq 2$, $v\geqq 2$ be integers,*

$$\left|\frac{\mu_{v-1}}{\mu_v}(u-3)+\frac{\lambda_{u-1}}{\lambda_u}(v-3)\right|>4|u-v| \tag{3}$$

and $f_0\in S$ satisfies simultaneously the equations (1) *and* (2). *Then*

$$f_0(z)=z(1-ze^{i\tau_1})^{-1}(1-ze^{i\tau_2})^{-1}, \tag{4}$$

where τ_1, τ_2 are real.

In the equality case in (3), *but $u\neq v$, $f_0(z)$ is also in the form* (4).

If $u=v=3$ and $(\lambda_{u-1}/\lambda_u)\neq(\mu_{v-1}/\mu_v)$ or

$$(5)\qquad \left|\frac{\lambda_{u-1}}{\lambda_u}\right| \geqq 4 \quad (\lambda_2, \ldots, \lambda_u) \neq (\mu_2, \ldots, \mu_v)$$

$f_0(z)$ *is also of the form* (4).

Proof. We shall consider first the case $u \neq v$. Let the function $f_0(z)$ satisfies (1) and (2). Following [3, p. 260-261], [4, p. 346-350] we shall show that $f_0(z)$ can be continued on $\bar{\mathbf{C}}$ as an algebraic function. Really after dividing the equation (1) by the equation (2) we obtain

$$(6)\qquad \frac{Q_u(w)}{Q_v(w)} = \frac{P_u(z)}{P_v(z)} \Leftarrow\Rightarrow w^{v-u} G(w) = z^{v-u} H(z),$$

where $H(z)$ and $G(w)$ are rational functions and $H(0)=G(0)=1$. It follows from (6) that $f_0(z)$ is an algebraic function. Hence $f_0(z)$ can be continued from Δ to $\bar{\mathbf{C}}$ except for a finite number of poles and algebraic branching points. Denote such an analytic continuation by $\mathscr{F}(z)$. The function $w=\mathscr{F}(z)$ satisfies the equations (1) and (2). It has be shown [3, p. 261], [4, p. 349] that in a neighbourhood of the origin every branch of $\mathscr{F}(z)$ has the series expansion $\mathscr{F}(z)=B_1 z+\ldots$ Extracting $(v-u)$-th root from the both sides of (6) we obtain

$$(7)\qquad \varphi(w)=e^{i\theta}\psi(z),$$

where φ and ψ are regular in a neighbourhood of the origin, $\varphi(0)=\psi(0)=0$, $\varphi'(0)=\psi'(0)=1$, $e^{i\theta(v-u)}=1$ (θ characterizes the branch we have chose). From (7) we conclude that in a neighbourhood of the origin every branch of $w=\mathscr{F}(z)$ has the series expansion

$$(8)\qquad \mathscr{F}(z)=\varphi^{-1}(e^{i\theta}\psi(z))=b_1 z+\ldots, \quad b_1=e^{i\theta}.$$

In the sequel we shall show that under the hypothesis of Theorem 1 the only possible value of b_1 is 1. Together with (7) this will give that every branch of $\mathscr{F}(z)$ coincides with $f_0(z)$ in some neughbourhood of the origin and, thus, $\mathscr{F}(z)$ is single-valued in $\bar{\mathbf{C}}$. Moreover this shows that $\mathscr{F}(z)$ is a rational function.

Assume that $b_1 \neq 1$. Using (8) we obtain

$$[z\mathscr{F}'(z)/\mathscr{F}(z)]^2 = 1+2b_2 b_1^{-1} z + O(z^2),$$

$$[\mathscr{F}(z)]^{-j} = (b_1 z)^{-j} - j b_2 b_1^{-j-1} z^{1-j} + O(z^{2-j}).$$

Hence

$$Q_u(\mathscr{F}(z)) = -\left[\frac{\bar{\lambda}_u}{(b_1 z)^{u-1}} - \frac{\bar{\lambda}_u b_2(u-1)}{b_1^u z^{u-2}} + \frac{\bar{\lambda}_u(u-1)a_2+\bar{\lambda}_{u-1}}{b_1^{u-2} z^{u-2}} + O(z^{3-u})\right],$$

$$[z\mathscr{F}'(z)/\mathscr{F}(z)]Q_u(\mathscr{F}(z)) = -\frac{\bar{\lambda}_u}{(b_1 z)^{u-1}} + \frac{\bar{\lambda}_u b_2(u-3)}{b_1^u z^{u-2}} - \frac{\bar{\lambda}_u(u-1)a_2+\bar{\lambda}_{u-1}}{(b_1 z)^{u-2}} + O(z^{3-u}).$$

On the other hand,

$$P_u(z)=\frac{\bar{\lambda}_u}{z^{u-1}}+\frac{2\bar{\lambda}_u a_2+\bar{\lambda}_{u-1}}{z^{u-2}}+O(z^{3-u}).$$

Compairing in (1) the coefficients of the terms containing z^{2-u} and z^{1-u} respectively we obtain

$$(9)\qquad 2\bar{\lambda}_u a_2+\frac{\bar{\lambda}_u b_2(u-3)}{b_1^u}-\frac{\bar{\lambda}_u(u-1)a_2+\bar{\lambda}_{u-1}}{b_1^{u-2}}=0,$$

$$\bar{\lambda}_u-\bar{\lambda}_u b_1^{1-u}=0.$$

Hence $b_1^{1-u}=1$ and we derive

$$(10)\qquad (u-3)b_2-a_2b_1[(u-1)b_1-2]=\frac{\bar{\lambda}_{u-1}}{\bar{\lambda}_u}b_1(b_1-1).$$

Proceeding in the same way we derive from (2)

$$(11)\qquad b_1^{v-1}=1,$$

$$(12)\qquad (v-3)b_2-a_2b_1[(u-1)b_1-2]=\frac{\bar{\mu}_{u-1}}{\bar{\mu}_u}b_1(b_1-1).$$

Eliminating b_2 from (10) and (12) we find

$$a_2=\frac{1}{2(v-u)}\left[\frac{\bar{\mu}_{v-1}}{\bar{\mu}_v}(u-3)-\frac{\bar{\lambda}_{u-1}}{\bar{\lambda}_u}(v-3)\right].$$

The well known estimation $|a_2|\leqq 2$ [5] for $f_0\in S$ with extremal function the Koebe function $z(1-ze^{i\theta})^{-2}$ and (3) yield either $f_0(z)$ is the Koebe function or $b_1=1$ and hence the function $f_0(z)$ is rational. It is known [1, 6] also [4, p. 347] that if the rational function $f_0\in S$ satisfies an equation of type (1) $f_0(z)$ is of the form (4).

Let now $u=v$. From (3) and (5) it follows that with respect to w (6) is an algebraic equation of degree greater than zero. Hence $w=f_0(z)$ can be continued in $\bar{\mathbb{C}}$ except for a finite number of poles and algebraic branching points. We rewrite (6) in the form

$$(13)\qquad \frac{A_s w^s+\ldots+\bar{\lambda}_u}{\tilde{A}_r w^r+\ldots+\bar{\mu}_u}=\frac{\lambda_u z^{2(u-1)}+\ldots+\bar{\lambda}_u}{\mu_u z^{2(u-1)}+\ldots+\bar{\mu}_u},$$

where s, r are positive integers and A_s, $\tilde{A}_r\neq 0$. After subtracting $\bar{\lambda}_u/\bar{\mu}_u$ from both sides of (13) it becomes

$$(14)\qquad w^x G(w)=z^x H(z),$$

where x is some positive integer and $G(w)$, $H(z)$ are rational functions. Since the equation (14) is satisfied for $w=f_0(z)$, $G(0)=H(0)=0$. As in the first part of the proof from (1) and (14) we obtain that in a neighbourhood of the origin every branch of $\mathscr{F}(z)$ has the series expansion $\mathscr{F}(z)=b_1z+b_2z^2+\ldots.=w$, $b_1^x=1$ and the statement of the theorem will be proved if we show that $b_1=1$ for any branch of $\mathscr{F}(z)$.

In the case in question the equations (10) and (12) become

$$(15)\qquad (u-3)b_2-a_2b_1[(u-1)b_1-2]=\frac{\bar{\lambda}_{u-1}}{\bar{\lambda}_u}(b_1-1)b_1$$

and

$$(16)\qquad (u-3)b_2-a_2b_1[(u-1)b_1-2]=\frac{\bar{\mu}_{u-1}}{\bar{\mu}_u}(b_1-1)b_1.$$

If $b_1\neq 1$ (15) and (16) yield $(\lambda_{u-1}/\lambda_u)=(\mu_{u-1}/\mu_u)$ which contradicts (3). Let $u=v=3$ and $b_1\neq 1$. From (15) and (16) we obtain

$$-2a_2=\frac{\bar{\lambda}_{u-1}}{\bar{\lambda}_u}=\frac{\bar{\mu}_{u-1}}{\bar{\mu}_u}.$$

Hence, having in mind (5), we obtain $|a_2|\geqq 2$, which is possible only for the Koebe function.

So either $f_0(z)$ is the Koebe function or $b_1=1$ and then $f_0(z)$ is of the form (4). This completes the proof of Theorem 1.

Corollary 1. *If the functionals F, Φ posses a local extremum at $f_0\in S$ and satisfy the condition of Theorem 1, then $f_0(z)$ is of the form* (4).

It is known [1, 4] that the statement of Theorem 1 is valid provided the restrictions (3) and (5) are replaced by the restriction $(u-1)$ and $(v-1)$ to be relatively prime.

Theorem 2. Let

a) $\psi_1(\xi)=\Sigma_{l=0}^{\infty}A_l(\xi-1)^l$ *and* $\psi_2(\xi)=\Sigma_{l=0}^{\infty}B_l(\xi-1)^l$ *be two functions regular in a neighbourhood of the point* $\xi=1$ *with* $A_1B_1\neq 0$,

b) $n\geqq 2$, $m\geqq 2$, $n\neq m$ *be integers,*

c) α, $\beta\in\mathbf{C}$, $\alpha\neq n/(n-1)$, $\beta\neq m/(m-1)$.

Denote

$$\sigma_n(j)=\begin{cases}1, & j=(n+1)/2,\\ 2, & j\neq(n+1)/2,\end{cases}\qquad \sigma_m(j)=\begin{cases}1, & j=(m+1)/2,\\ 2, & j=(m+1)/2.\end{cases}$$

Suppose

$$(17)\qquad \frac{A_2\sigma_n(j)[\alpha+(1-\alpha)(n-j+1)][\alpha+(1-\alpha)j]}{A_1[\alpha+(1-\alpha)n]}(m-2j+1)$$
$$-\frac{B_2\sigma_m(j)[\beta+(1-\beta)(m-j+1)](\beta+(1-\beta)j]}{B_1[\beta+(1-\beta)m]}(n-2j+1)\neq(n-m)j$$

for all integers j, $2 \leqq j \leqq \min(n, m)$. *If the functionals*

$$F = F(c_2, \dots, c_n) = \mathrm{Re}[\{\psi_1(\alpha \frac{f(z)}{z} + (1-\alpha) f'(z))\}_{n-1}]$$

and

$$\Phi = \Phi(c_2, \dots, c_m) = \mathrm{Re}[\{\psi_2(\beta \frac{f(z)}{z} + (1-\beta) f'(z))\}_{m-1}]$$

posses local extremum at $f_0 \in S$, *then*

$$f_0(z) = \frac{z}{\sqrt[d_0]{(1-e^{i\tau_1} z^{d_0})(1-e^{i\tau_2} z^{d_0})}}, \quad \sqrt[d_0]{1} = 1, \tag{18}$$

where τ_1, τ_2 *are real,* d_0 *is a common divisor of* $(n-1)$ *and* $(m-1)$.

Proof. For the functionals in the statement we have

$$\bar{\lambda}_n = A_1[\alpha + (1-\alpha)n] \neq 0, \quad \bar{\mu}_m = B_1[\beta + (1-\beta)m] \neq 0.$$

Of course the function $w = f_0(z) = z + \Sigma_{j=2}^{\infty} a_j z^j$ we seek satisfies the equations (1) and (2). Hence, as in the proof of Theorem 1, we arrive to the conclusion that $f_0(z)$ is an algebraic function. In a neighbourhood of the origin any branch of its analytic continuation $\mathscr{F}(z)$ has the series expansion

$$w = \mathscr{F}(z) = \sum_{j=1}^{\infty} b_j z^j, \quad b_1^{n-m} = 1, \tag{19}$$

where the different coefficients b_1 correspond to different brances of $\mathscr{F}(z)$. The function $\mathscr{F}(z)$ satisfies the equations (1) and (2) as well. If in (19) $b_1 = 1$ for every branch of $\mathscr{F}(z)$, then (see the proof of Theorem 1) $\mathscr{F}(z)$ is a rational function and as earlier we conclude that $f_0(z)$ is of the form (18).

Assume that for some branch of $\mathscr{F}(z)$ we have $b_1 \neq 1$ in (19) and let d be the smallest positive integer for which $b_1^d = 1$. Using induction we shall prove that a_j, b_j, λ_{n-j+1}, μ_{m-j+1} can differ from zero only if $j = dk+1$ (for some integer $k \geqq 0$).

1) Let $j = 2$. It is easily calculated that

$$\bar{\lambda}_{n-1} = A_2 a_2 \sigma_n(2)[\alpha + (1-\alpha)2][\alpha + (n-1)(1-\alpha)],$$

$$\bar{\mu}_{m-1} = B_2 a_2 \sigma_m(2)[\beta + (1-\beta)2][\beta + (m-1)(1-\beta)].$$

For the functionals in the statement the equalities (9)-(12) hold if we put $u = n$, $v = m$. From (9) and (11) we conclude that d is a common divisor of $(n-1)$ and $(m-1)$. Elimination of b_2 from (10) and (12) gives

$$2a_2(n-m)=\frac{\bar{\lambda}_{n-1}}{\bar{\lambda}_n}(m-3)-\frac{\bar{\mu}_{m-1}}{\bar{\mu}_m}(n-3)$$

$$=a_2\left[\frac{A_2\sigma_n(2)[\alpha+(1-\alpha)2][\alpha+(1-\alpha)(n-1)]}{A_1[\alpha+(1-\alpha)n]}(m-3)\right.$$

$$\left.-\frac{B_2\sigma_m(2)[\beta+(1-\beta)2][\beta+(1-\beta)(m-1)]}{B_1[\beta+(1-\beta)m]}(n-3)\right],$$

which contradicts (17) provided $a_2\neq 0$. Hence $a_2=0$, $\lambda_{n-1}=\mu_{m-1}=0$. From (10) we obtain $b_2=0$. This completes the proof of our assertion when $j=2$.

2) Suppose our assertion has been proved for all positive integers less than $kd+l+1$ for some integers k and l ($k\geqq 0$, $0<l<d$). We first consider the case $kd+l+1<\min(n, m)$. From the inductive assumption it follows

$$\bar{\lambda}_{n-kd-l}=A_2a_{kd+l+1}\sigma_n(kd+l+1)[\alpha+(1-\alpha)(kd+l+1)][\alpha+(1-\alpha)(n-kd-l)]$$

so that for positive integers $j<kd+l$ the quantity λ_{n-j} can be different from zero only if $j=rd$ (for some integer $r\geqq 0$). In the following $R_j(\xi)$, $R_j(0)=0$ will stand for a polynomial of ξ of degree at most j, and $h_j(\xi)=\sum_{\nu=j}^{\infty}\gamma_\nu\xi^\nu$ for a regular function. We have

$$[\mathscr{F}(z)]^{-j}=(b_1z)^{-j}[1+R_k(z^d)-jb_{kd-l-1}b_1^{-1}z^{kd+l}+h_{kd+l+1}(z)],$$

$$[z\mathscr{F}'(z)/\mathscr{F}(z)]^2=1+R_k(z^d)+2(kd+l)b_{kd+l+1}b_1^{-1}z^{kd+l}+h_{kd+l+1}(z),$$

$$Q_n(w)=\sum_{r=0}^{k}\bar{\lambda}_{n-rd}q_{n-rd}(w)+\bar{\lambda}_{n-kd-l}q_{n-kd-l}(w)+\sum_{\nu=2}^{n-kd-l-1}\lambda_\nu q_\nu(w),$$

where

$$q_{n-rd}(w)=-\sum_{j=1}^{n-rd-1}\{f_0^{j+1}(z)\}_{n-rd}w^{-j}.$$

Further

$$\{f_0^{j+1}(z)\}_{n-rd}=\{[1+R_k(z^d)+a_{kd+l+1}z^{kd+l}+h_{kd+l+1}(z)]^{j+1}\}_{n-rd-j-1}$$

can be $\neq 0$ only for $j=n-rd-1, n-rd-d-1, \ldots, n-(r+k)d-1, n-(r+k)d-1-l$, as well as for $j<n-(r+k)d-1-l$. Moreover

$$\{f_0^{n-kd-l}(z)\}_n=(n-kd-l)a_{kd+l+1}.$$

Consequently

$$Q_n(w)=-\frac{\bar{\lambda}_n}{w^{n-1}}\left[1+R_k(w^d)+\left(\frac{\bar{\lambda}_{n-kd-l}}{\bar{\lambda}_n}+(n-kd-l)a_{kd+l+1}\right)w^{kd+l}+h_{kd+l+1}(w)\right].$$

Thus

$$Q_n(\mathscr{F}(z))=\frac{\bar{\lambda}_n}{z^{n-1}}\Bigg[1+R_k(z^d)-(n-1)\frac{b_{kd+l+1}}{b_1}z^{kd+l}$$
$$+\left(\frac{\bar{\lambda}_{n-kd-l}}{\bar{\lambda}_n}+(n-kd-l)a_{kd+l+1}\right)b_1^l z^{kd+l}+h_{kd+l+1}(z)\Bigg].$$

So we obtain

$$[z\mathscr{F}'(z)/\mathscr{F}(z)]^2Q_n(\mathscr{F}(z))=-\frac{\bar{\lambda}_n}{z^{n-1}}[1+R_k(z^d)+z^{kd+l}\Bigg(\frac{\bar{\lambda}_{n-kd-l}}{\bar{\lambda}_n}b_1^l$$
$$+b_1^l(n-kd-l)a_{kd+l+1}+\frac{b_{kd+l+1}}{b_1}(2(kd+l)-n+1)\Bigg)+h_{kd+l+1}(z)].$$

From the inductive assumption it follows that

$$P_n(z)=z^{1-n}[\bar{\lambda}_n+R_k(z^d)+z^{kd+l}(\bar{\lambda}_n(kd+l+1)a_{kd+l+1}+\bar{\lambda}_{n-kd-l})+h_{kd+l+1}(z)].$$

Comparing in (1) the coefficients preceeding $z^{kd+l+1-n}$ we obtain

$$(n-1-2(kd+l))b_{kd+l+1}-b_1(b_1^l(n-kd-l)-(kd+l+1))a_{kd+l+1}$$
$$=\frac{\bar{\lambda}_{n-kd-l}}{\bar{\lambda}_n}(b_1^l-1)b_1. \tag{20}$$

Analogous reasoning for the functional Φ gives

$$(m-1-2(kd+l))b_{kd+l+1}-b_1(b_1^l(m-kd-l)-(kd+l+1))a_{kd+l+1}$$
$$=\frac{\bar{\mu}_{m-kd-l}}{\bar{\mu}_n}(b_1^l-1)b_1. \tag{21}$$

Here

$$\bar{\mu}_{m-kd-l}=B_2a_{kd+l+1}\sigma_m(kd+l+1)[\beta+(1-\beta)(kd+l+1)][\beta+(1-\beta)(m-kd-l)].$$

Elimination of b_{kd+l+1} from (20) and (21) gives

$$(b_1^l-1)a_{kd+l+1}(kd+l+1)(n-m) \tag{22}$$
$$=(b_1^l-1)\left[\frac{\bar{\lambda}_{n-kd-l}}{\bar{\lambda}_n}(m-1-2(kd+l))-\frac{\bar{\mu}_{m-kd-l}}{\bar{\mu}_m}(n-1-2(kd+l))\right].$$

Since $b_1^l\neq 1$ then if $a_{kd+l+1}\neq 0$ it follows from (22) that

$$(kd+l+1)(n-m)=\frac{A_2\sigma_n(kd+l+1)[\alpha+(1-\alpha)(kd+l+1)][\alpha+(1-\alpha)(n-kd-l)]}{A_1[\alpha+(1-\alpha)n]}$$

$$\times(m-1-2(kd+l))-\frac{B_2\sigma_m(kd+l+1)[\beta+(1-\beta)(kd+l+1)][\beta+(1-\beta)(m-kd-l)]}{B_1[\beta+(1-\beta)m]}$$

$$\times(n-1-2(kd+l)),$$

which contradicts (17). Hence $a_{kd+l+1}=0$, $\lambda_{n-kd-l}=\mu_{m-kd-l}=0$. From (20) we obtain $b_{kd+l+1}=0$. We may assume that $\min(n, m)=n$ which means we have proved our assertion for $j\in[2, n]$. It has been shown that $Q_n(w)$ and $P_n(z)$ contain only these powers of the variables which are multiples of d.

Now let $kd+l+1>\min(m, n)=n$ ($l\neq 0$ yields $kd+l+1\neq m, n$). Denote $n_0-1=(n-1)/d$, $k\geqq n_0-1$. Then

$$Q_n(w)=-\frac{\bar{\lambda}_{n-1}}{w^{n-1}}[1+R_{n-2}(w^d)],$$

$$Q_n(\mathscr{F}(z))-\frac{\bar{\lambda}_n}{b_1^{n-1}z^{n-1}}[1+R_k(z^d)-(n-1)\frac{b_{kd+l+1}}{b_1}z^{kd+l}+h_{kd+l+1}(z)],$$

$$[z\mathscr{F}'(z)/\mathscr{F}(z)]^2Q_n(\mathscr{F}(z))=-\frac{\bar{\lambda}_n}{z^{n-1}}[1+R_k(z^d)+\frac{b_{kd+l+1}}{b_1}(2kd+2l-n+1)z^{kd+l}$$

$$+h_{kd+l+1}(z)].$$

On the other hand, $P_n(z)$ does not contain $z^{kd+l+1+n}$. Thus comparing the coefficients in (1) we obtain $b_{kd+l+1}=0$ for every branch of $\mathscr{F}(z)$. In particular $a_{kd+l+1}=0$. Hence for $2\leqq j\leqq m$ our assertion concerning μ_j is true.

This completes the proof.

So

$$\text{(A)}\qquad f_0(z)=z+\sum_{k=1}^{\infty}a_{kd+1}z^{kd+1}$$

and $P_n(z)$, $P_m(z)$, $Q_n(w)$, $Q_m(w)$ contain only these powers of the variables which are multiples of d.

Let d_ν, $1\leqq\nu\leqq s$, be different positive integers with the property: for every d_ν there exists a branch $\mathscr{F}(z)=b_1z+\dots$ such that $b_1^{d_\nu}=1$, $b_1^l\neq 1$ for all integers $l\in[1, d_\nu)$. The number d earlier introduced is among the number d_ν. Let d_0 be the smallest positive integer that is multiple of $d_1,\dots, d_s$. Since (A) holds for each $d_1,\dots, d_s$

$$f_0(z)=z+\sum_{k=1}^{\infty}a_{kd_0+1}z^{kd_0+1}$$

and $P_n(z)$, $P_m(z)$, $Q_n(w)$, $Q_m(w)$ include only such powers of the variables which are multiples of d_0. Moreover $b_1^{d_0}=1$ for any branch $\mathscr{F}(z)=b_1z+\dots$, i.e. in a neighbourhood of the origin

$$w^{d_0}=\mathscr{F}^{d_0}(z)=z^{d_0}+\dots \tag{23}$$

This allows to make in (1) and (2) the substitution $w_1=w^{d_0}$, $z_1=z^{d_0}$. Denote $f_1(z_1)=f_1(z^{d_0})=f_0^{d_0}(z)$, $f_1\in S$ and with $n_1-1=(n-1)/d_0$

$$\tilde{P}_{n_1}(z_1)=\tilde{P}_{n_1}(z^{d_0})=P_n(z),\ \tilde{Q}_{n_1}(w_1)=\tilde{Q}_{n_1}(w^{d_0})=Q_n(w).$$

Analogously for $m_1-1=(m-1)/d_0$

$$\tilde{P}_{m_1}(z_1)=P_m(z),\quad \tilde{Q}_{m_1}(w_1)=Q_m(w).$$

Then (1) and (2) become

$$\left(\frac{z_1}{w_1}\frac{dw_1}{dz_1}\right)^2\tilde{Q}_{n_1}(w_1)+\tilde{P}_{n_1}(z_1)=0 \tag{24}$$

and

$$\left(\frac{z_1}{w_1}\frac{dw_1}{dz_1}\right)^2\tilde{Q}_{m_1}(w_1)+\tilde{P}_{m_1}(z_1)=0. \tag{25}$$

On the other hand (6) becomes

$$w_1^{m_1-n_1}G_1(w_1)=z_1^{m_1-n_1}H_1(z_1),$$

where $G_1(w_1)$ and $H_1(z_1)$ are rational functions, $G_1(0)=H_1(0)=1$. We mark that $f_1(z_1)$ satisfies the equations (24) and (25). This yields that $w_1=w_1(z_1)$ is an algebraic function. As earlier we obtain that in a neighbourhood of the origin any branch of $w_1(z_1)$ has the expansion $w_1(z_1)=B_1z_1+\dots$ From (23) it follows that $B_1=1$. It means (see the proof of Theorem 1) that $w_1(z_1)$ is a rational function, i.e. $f_1(z_1)$ is rational. But a rational function in S that satisfies an equation of the form (24) has the form (4) [1], [6], [4, p. 347], i.e.

$$f_1(z_1)=\frac{z_1}{(1-z_1e^{i\tau_1})(1-z_1e^{i\tau_2})}.$$

Consequently $f_0(z)$ has the form (18).

Theorem 2 is proved.

For $\alpha=\beta=1$ and $\psi_1(\xi)=\psi_2(\xi)$ we derive from Theorem 2.

Corollary 2. *Let the function $\psi(\xi)=\Sigma_{l=0}^{\infty}A_l(\xi-1)^l$, $A_1=0$ be regular in a neighbourhood of the point $\xi=1$ and $n\geqq 2$, $m\geqq 2$, $n\neq m$, be integers. Let the integer $j\neq(n+1)/2$ and $j\neq(m+1)/2$. Suppose that $-A_2/A_1\neq j/2$, $2\leqq j\leqq \min(n, m)$ and $-A_2/A_1\neq(n+1)/2$, $(m+1)/2$ if $(n+1)$, $(m+1)$ is even respectively.*

If the functionals

$$\operatorname{Re}\left[\left\{\psi\left(\frac{f(z)}{z}\right)\right\}_n\right], \quad \operatorname{Re}\left[\left\{\psi\left(\frac{f(z)}{z}\right)\right\}_m\right]$$

posses local extremum at $f_0 \in S$, *then the function* $f_0(z)$ *is of the form* (18).

Corollary 3. *By the initial assumption of Corollary 2 suppose*

$$2\frac{A_2}{A_1}\left[(j-1)\frac{n+m-1-2j}{nm}-1\right]\neq 1$$

and $-A_2/A_1 \neq 2n/(n+1)$, $2m/(m+1)$ if $(n+1)$, $(m+1)$ *is even respectively.*

If the functionals $\operatorname{Re}[\{\psi(f'(z))\}_{n-1}]$ *and* $\operatorname{Re}[\{\psi(f'(z))\}_{m-1}]$ *posses local extremum at* $f_0 \in S$, *then the function* $f_0(z)$ *is of the form* (18).

Chosing $\psi(\xi)=\xi$ and $\psi(\xi)=\log\xi$ the results obtained by V. V. Andreev and the author appear.

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