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THE NONEXISTENCE OF SOME GRIESMER ARCS IN PG(4, 5)

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Abstract. In this paper, we prove the nonexistence of arcs with parameters $(232, 48)$ and $(233, 48)$ in $\mathrm{PG}(4, 5)$. This rules out the existence of linear codes with parameters $[232, 5, 184]$ and $[233, 5, 185]$ over the field with five elements and improves two instances in the recent tables by Maruta, Shinohara and Kikui of optimal codes of dimension 5 over $\mathbb{F}_5$.

1. Introduction. The problem of finding the minimum length of a code with a prescribed dimension k and a prescribed minimum distance d over a fixed finite field $\mathbb{F}_q$ has been completely solved only for small finite fields and small dimensions. It is known that in the case of 4-dimensional codes over $\mathrm{GF}(5)$ there are eight undecided cases. All they are believed to be difficult. They are presented in the table below.

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d	$g_5(4,d)$	$n_5(4,d)$
31	41	41 – 42
32	42	42 – 43
36	47	47 – 48
37	48	48 – 49
81	103	103 –104
82	104	104 –105
161	203	203 –204
162	204	204 –205

In a recent paper Maruta et al. [9] initiated the systematic research on the optimal length problem for 5-dimensional linear codes over $\mathbb{F}_5$. The problem is still far from its final solution with more than 400 unsolved values for d. In this paper, we improve some of the results in [9] by proving the nonexistence of Griesmer codes with parameters $[232, 5, 184]_5$ and $[233, 5, 185]_5$. In order to save space, we refer to [4, 8] for the basic notions and facts from coding theory.

2. Preliminaries. Let $\mathrm{PG}(t, q)$ be the t-dimensional projective space over the finite field of order q. We denote by $\mathcal{P}$ the pointset of $\mathrm{PG}(t, q)$. A *multiset* in $\mathrm{PG}(t, q)$ is any mapping $\mathfrak{K}\colon \mathcal{P} \to \mathbb{N}_0$ from the points of $\mathrm{PG}(t, q)$ to the nonnegative integers. This mapping is extended in a natural way to the subsets of $\mathcal{P}$. Given $\mathcal{Q} \subseteq \mathcal{P}$, we call the integer $\mathfrak{K}(\mathcal{Q})$ the multiplicity of $\mathcal{Q}$. In particular, we speak of the multiplicity of a point, a line, a plane etc. The support of a multiset is the the set of all points in $\mathcal{P}$ that are of nonzero multiplicity. A multiset with $\mathfrak{K}(P) \in \{0, 1\}$ for every point $P \in \mathcal{P}$ is called a *non-weighted* or *projective* multiset. Non-weighted multisets can be viewed as sets by identifying them with their support.

A multiset in $\mathrm{PG}(t, q)$ is called an $(n, w; t, q)$-*multiarc* if

(a) $\mathfrak{K}(\mathcal{P}) = n$;

(b) $\mathfrak{K}(H) \leq w$ for any hyperplane H;

(c) there exists a hyperplane H_0 with $\mathfrak{K}(H_0) = w$.

The existence of linear a $[n, k, d]_q$ code of full length, i.e., a code with no coordinate identically zero, is equivalent to that of an $(n, n-d; k-1, q)$-arc [3]. Two linear codes with the same parameters are semilinearly isomorphic if and only if the corresponding arcs are projectively equivalent.

Given a $(n, w; k-1, q)$-arc $\mathfrak{K}$, we denote by $\gamma_i(\mathfrak{K})$ the maximal multiplicity of an i-dimensional flat in $\mathrm{PG}(k-1, q)$, i.e. $\gamma_i(\mathfrak{K}) = \max_\delta \mathfrak{K}(\delta)$, $i = 0, \ldots, k-1$, where δ runs over all i-dimensional flats in $PG(k-1, q)$. If $\mathfrak{K}$ is clear from the context we write just γ_i. In what follows, we repeatedly make use the following lemma which is proved by a straightforward counting argument.

Lemma 1. *Let $\mathfrak{K}$ be an $(n, n-d; k-1, q)$-arc, and let Π be an $(s-1)$-dimensional flat in $PG(k-1, q)$, $2 \le s < k$, with $\mathfrak{K}(\Pi) = w$. Then, for any $(s-2)$-dimensional flat Δ contained in Π, we have*

$$\mathfrak{K}(\Delta) \le \gamma_{s-1}(\mathfrak{K}) - \frac{n-w}{q^{k-s} + \ldots + q}.$$

Let $\mathfrak{K}$ be an $(n, n-d; k-1, q)$ arc and denote by a_i the number of hyperplanes Δ in $\mathrm{PG}(k-1, q)$ with $\mathfrak{K}(\Delta) = i$, $i = 0, 1, \ldots$, and let λ_j be the number of points P from $\mathcal{P}$ with $\mathfrak{K}(P) = j$. The sequence $(a_0, a_1, \ldots)$ is called *the spectrum* of $\mathfrak{K}$. Simple counting arguments yield the following identities, which are equivalent to the first three MacWilliams identities for linear codes [6]:

$$\sum_{i=0}^{n-d} a_i = \frac{q^k - 1}{q - 1}, \tag{1}$$

$$\sum_{i=1}^{n-d} i a_i = n \cdot \frac{q^{k-1} - 1}{q - 1}, \tag{2}$$

$$\sum_{i=2}^{n-d} \binom{i}{2} a_i = \binom{n}{2} \frac{q^{k-2} - 1}{q - 1} + q^{k-2} \cdot \sum_{i=2}^{\gamma_0} \binom{i}{2} \lambda_i. \tag{3}$$

Let $\mathfrak{K}$ be an $(n, w; t, q)$-multiarc. Fix an u-dimensional flat U in $\mathrm{PG}(t, q)$. Let further V be a v-dimensional flat in $\mathrm{PG}(t, q)$ with $u+v = t-1$ and $U \cap V = \emptyset$. Define the projection $\varphi = \varphi_{U,V}$ from U onto V by

$$\varphi_{U,V}\colon \left\{ \begin{array}{lcl} \mathcal{P} \setminus U & \to & V \\ P & \to & V \cap \langle U, P \rangle \end{array} \right. , \tag{4}$$

where $\mathcal{P}$ is the pointset of $\mathrm{PG}(t, q)$. Note that $\varphi_{U,V}$ maps $(u+s)$-dimensional flats containing U into $(s-1)$-dimensional flats contained in V. The *induced multiarc* $\mathfrak{K}^\varphi$ is defined on the points of V by

$$\mathfrak{K}^\varphi\colon \left\{ \begin{array}{lcl} \mathcal{P}(V) & \to & \mathbb{N}_0 \\ P & \to & \sum_{Q\colon \varphi(Q)=P} \mathfrak{K}(Q) \end{array} \right. .$$

If S is a t'-dimensional flat in V then $\mathfrak{K}^{\varphi}(S) = \mathfrak{K}(\langle S, U\rangle) - \mathfrak{K}(U)$. Here $\langle S, U\rangle$ denotes the projective subspace of $\mathrm{PG}(t, q)$ generated by S and δ. Clearly, $\mathfrak{K}^{\varphi}$ is an $(n - \mathfrak{K}(U), w - \mathfrak{K}(U))$-multiarc in $V \cong \mathrm{PG}(v, q)$. When U and the arc $\mathfrak{K}$ remain fixed, we get equivalent induced multiarcs for all choices of V. Similarly, if $\mathfrak{K}$ is a (n, w)-minihyper then $\mathfrak{K}^{\varphi}$ is an $(n - \mathfrak{K}(U), w - \mathfrak{K}(U))$-minihyper in V.

In this paper, we shall constantly exploit the fact that the existence of linear codes with parameters $[n, k, d]_q$ is equivalent to the existence of $(n, n-d)$-arcs in $\mathrm{PG}(k-1, q)$. In fact, we prove our results for arcs and formulate as corollaries the corresponding results for codes.

3. The $(48, 11)$-arcs in PG$(3, 5)$. The problem of the existence of $(48, 11)$-arcs in $\mathrm{PG}(3, 5)$ is still unresolved and seems to be very difficult. Nevertheless, it is possible to prove some restrictions on the spectrum of such arcs that turns out to be useful for the investigation of code parameters in higher dimensions. In this section, we prove the nonexistence of an empty plane in $(48, 11; 3, 5)$-arcs.

First, let us recall the spectra of the two $(11, 3)$-arcs in $\mathrm{PG}(2, 5)$.

Spectra of the $(11, 3)$-arcs in $\mathrm{PG}(2, 5)$

type	a_0	a_1	a_2	a_3
A1	4	4	7	16
A2	5	1	10	15

Lemma 2. *Let $\mathfrak{K}$ be a $(48, 11)$-arc in* $\mathrm{PG}(3, 5)$. *Then there* $a_0 = 0$.

P r o o f. Let $\mathfrak{K}$ have spectrum (a_i), $i = 0, 1, \ldots, 11$. From identities (1–3), we obtain

$$\sum_{i=0}^{9} \binom{11-i}{2} a_i = 468. \tag{5}$$

Clearly, $a_2 = a_7 = 0$ since there exist no $(2, 1)$- and no $(7, 2)$-arcs in $\mathrm{PG}(2, 5)$.

Now assume that $a_0 \geq 1$. Then $a_0 = 1$ and $a_3 = 0$. Let us count the number of 4-, 5- and 6-planes through the lines of an $(11, 3)$-arc in $\mathrm{PG}(2, 5)$. Note that a 4-, 5- or 6-plane does not meet an 11-plane in a 3-line; through a 2-line in an 11-plane there is at most one i-plane, $i = 4, 5, 6$, through an 1-line there are at most two such planes and through a 0-line at most three such planes.

We must take into account that one of the 0-lines in the 11-plane is incident with the 0-plane and hence with at most one 4-, 5- or 6-plane. Now using the spectra of the $(11, 3)$-arcs, we get that $a_4 + a_5 + a_6 \leq 25$.

Suppose $a_4 \neq 0$ and π_0 is an arbitrary fixed 4-plane. Each 4-plane contains thirteen 0-lines, twelve 1-lines and six 2-lines. We fix a 2-line l in π_0 and denote by π_i, $i = 1, \dots, 5$ the other planes through l. Consider the projection $\varphi = \varphi_{P,\pi}$ of $PG(3, 5)$ from a 1-point P on l, onto a plane π not incident with P. Let $l_i = \varphi(\pi_i)$, $i = 0, \dots, 5$. Then the type of l_0 is $(1, 0, 0, 0, 1, 1)$, the type of l_1 is $(1, 0, 2, 2, 2, 2)$ or $(1, 1, 1, 2, 2, 2)$ and the type of the remaining lines $l_2, \dots, l_5$ is $(1, 1, 2, 2, 2, 2)$.

Case 1. Let the type of l_1 be $(1, 0, 2, 2, 2, 2)$. Then the set

$$T = \{X \mid X \in l_0, \mu(X) = 1\} \bigcup \{Y \mid \mu(Y) = 2\}$$

is a (23, 5)-arc. Otherwise there exists a line s of type $(1, 2, 2, 2, 2, 2)$, which leads to the existence of a plane $\alpha = \alpha(P, s)$ with $\mathfrak{K}(\alpha) = 12$, a contradiction. The fact that the smallest nontrivial blocking set in PG$(2, 5)$ is of size 9 implies the existence of a unique line m of type $(0, 0, 1, 1, 1, 1)$. All lines different from l_0 through any of the two points to $T \setminus m$ are of type $(0, 0, 2, 2, 2, 2)$ and four of type $(0, 1, 2, 2, 2, 2)$. Hence, there are exactly four 1-lines in π_0 such that through each of them there is a 5-plane (all other planes are 11-planes). The planes through each of the remaining 1-lines in π_0 are one 9-plane and four 10-planes.

Counting the maximal possible contributions of the planes through the lines of π_0 to the left-hand side of (5) and taking into account that $a_4 + a_5 + a_6 \leq 25$, we get

$$1.21 + 1.55 + 8.25 + 4.24 + 4.15 + 8.1 = 440 \geq 468,$$

a contradiction.

Case 2. Let the type of l_1 be $(1, 1, 1, 2, 2, 2)$. It is easy to check that if a 1-line through P in π_0 is incident with a 6-plane then each of the other 1-lines through P in π_0 is incident with one 9-plane and four 10-planes. If none of 1-lines through P in π_0 lie in a 6-plane then each of of them is incident with at most two 8-planes.

Counting once again the maximal possible contributions of the planes through the lines of π_0 to the left side of (5), recalling that $a_4 + a_5 + a_6 \leq 25$:

$$1.21 + 1.55 + 12.25 + 12.6 = 448 \geq 468,$$

a contradiction. Hence $a_4 = 0$.

Now that we know that 4-planes are impossible, we count once again the maximal possible contribution of the planes through the lines of the empty plane to the left side of (5), which gives

$$55 + 25.15 + 6.6 = 466 \geq 468,$$

a contradiction. Hence $a_0 = 0$. $\square$

4. The nonexistence of (232, 48)- and (233, 48)-arcs in PG(3, 5). In this section, we prove the nonexistence of $(232, 48)$- and $(233, 48)$-arcs in $\mathrm{PG}(3, 5)$, and equivalently, the nonexistence of Griesmer $[232, 5, 184]_5$- and $[233, 5, 185]_5$-codes. The proof relies on two classical results by Ward [10] and Hill and Lizak [5] from coding theory that are given here in their geometric form [7].

Theorem 3. (H.N. Ward) *Let $\mathfrak{K}$ be a Griesmer (n, w)-arc in $\mathrm{PG}(k-1, p)$, with $n - w \equiv 0 \bmod p^e$, $e \geq 1$. Then $\mathfrak{K}(H) \equiv n \bmod p^e$ for every hyperplane H.*

Theorem 4. (R. Hill, P. Lizak) *Let $\mathfrak{K}$ be an $(n, w; k-1, q)$-arc with $\gcd(n-w, q) = 1$. Assume that the multiplicities of all hyperplanes are congruent to n or w modulo q. Then $\mathfrak{K}$ can be extended to an $(n + 1, w)$-arc.*

Lemma 5. *Let $\mathfrak{K}$ be a $(233, 48)$-arc in $\mathrm{PG}(4, 5)$. Then $a_i = 0$ for all $i \neq 23, 38, 48$.*

P r o o f. By Theorem 3, all multiplicities w of hyperplanes are congruent to 3 modulo 5. By Lemma 1, $w \neq 3, 8, 13, 18, 23, 28, 33, 43$, hence the result. $\square$

Theorem 6. *There is no $(233, 48)$-arc in $\mathrm{PG}(4, 5)$.*

P r o o f. Assume otherwise. A 48-hyperplane in $\mathrm{PG}(4, 5)$ cannot have a 10-plane (since all hyperplanes through the 10-planes are 48-hyperplanes). Similarly, a 48-hyperplane cannot have an 8-plane.

Now consider a 48-hyperplane H and a fixed 3-line l in it. All the planes in H through l are 9- or 11-planes. Denote the number of 11-planes in H through l by t. Then $3 + 8t + 6(6 - t) = 48$, which is impossible since t is an integer. $\square$

Now we turn to the proof of the nonexistence of $(232, 48)$-arcs in $\mathrm{PG}(4, 5)$.

Lemma 7. *Let $\mathfrak{K}$ be a $(232, 48)$-arc in $\mathrm{PG}(4, 5)$. Then $a_i = 0$ for all $i \neq 12, 22, \ldots, 26,\ 32, 37, 38, 39, 42, 47, 48$.*

P r o o f. This result follows by Lemma 1 and the fact that a $(48, 11)$-arc does not have a 0-plane (Lemma 2). $\square$

Let $\mathfrak{K}$ be a $(232, 48)$-arc in $\mathrm{PG}(4, 5)$ with spectrum $(a_i)_{i\geq 0}$. By (1)–(3), one gets

$$\sum_{i=0}^{46} \binom{48-i}{2} a_i = 10620. \tag{6}$$

Theorem 8. *There is no* $(232, 48)$*-arc in* $\mathrm{PG}(4, 5)$.

Proof. The idea of this proof is to rule out the existence of solids of multiplicity $12, 22, \ldots, 26, 39$. Then a $(232, 48)$-arc has to be extendable by Hill and Lizak's extendability lemma [5], a contradiction to Theorem 6.

Assume there is a 12-hyperplane in $\mathrm{PG}(4, 5)$. The restriction to this hyperplane is a $(12, 4)$-arc, which is associated with the unique near-MDS code with parameters $[12, 4, 8]_5$ [2]. The spectrum of the $(12, 4)$-arc is

$$\tilde{a}_0 = 8, \tilde{a}_1 = 48, \tilde{a}_2 = 24, \tilde{a}_4 8 = 48, \tilde{a}_4 = 28, \tilde{a}_i = 0, i \neq 0, \ldots, 4.$$

Counting the contribution to the left-hand side of (6) given by the hyperplanes through the planes of the 12-hyperplane, we get the following:

- the maximal contribution of the hyperplanes through a 0-plane is 120;
- the maximal contribution of the hyperplanes through a 1-plane is 51;
- the maximal contribution of the hyperplanes through a 2-plane is 45;
- the maximal contribution of the hyperplanes through a 3-plane is 0;
- the maximal contribution of the hyperplanes through a 4-plane is 0.

Hence, by (6),

$$10620 = \sum_{i=0}^{46} \binom{48-i}{2} a_i \leq 8 \cdot 120 + 48 \cdot 51 + 24 \cdot 45 + \binom{36}{2} = 5118,$$

a contradiction.

The other multiplicities are ruled out in a similar way. It has to be noted that the restriction of $\mathfrak{K}$ to a hyperplane with multiplicity 22, 23, 24, 25, or 26 is a cap. A 26-cap in $\mathrm{PG}(3, 5)$ consists of the points of an elliptic quadric and is known to be unique. The size of the second largest irreducible cap is known to be 20 [1]. Hence all caps of sizes $21, \ldots, 25$ are obtained from the elliptic quadric by deleting the appropriate number of points. The spectra of such caps are given below.

	a_6	a_5	a_4	a_3	a_2	a_1	a_0
$(26,6)$	130	0	0	0	0	26	0
$(25,6)$	100	30	0	0	0	25	1
$(24,6)$	76	48	6	0	0	24	2
$(23,6)$	57	57	15	1	0	23	3
$(22,6)$	43	56	30	0	1	22	4
$(22,6)$	42	60	24	4	0	24	4

The spectrum of a $(39,9)$-arc in $\mathrm{PG}(3,5)$ is also known to be unique, it is $\tilde{a}_9 = 117$, $\tilde{a}_4 = 39$, $\tilde{a}_i = 0$, for $i \neq 4, 9$.

Now we demonstrate how to rule out the existence of hyperplanes of multiplicity 22. Assume there is a 22-hyperplane Δ_0 and for an arbitrarily fixed line plane π in Δ_0 denote by Δ_i, $i = 1, \ldots, 5$, the other five hyperplanes through π. We have that

- the maximal contribution of the hyperplanes through a 0-plane is 241 and is obtained for $(\Delta_1, \ldots, \Delta_5) = (47, 47, 47, 43, 25)$ (note that a 48-hyperplane does not have a 0-plane);

- the maximal contribution of the hyperplanes through a 1-plane is 300 and is obtained for $(\Delta_1, \ldots, \Delta_5) = (48, 48, 48, 48, 23)$;

- the maximal contribution of the hyperplanes through a 2-plane is 135 and is obtained for $(\Delta_1, \ldots, \Delta_5) = (48, 48, 48, 42, 34)$;

- the maximal contribution of the hyperplanes through a 3-plane is 55 and is obtained for $(\Delta_1, \ldots, \Delta_5) = (47, 47, 47, 47, 37)$;

- the maximal contribution of the hyperplanes through a 4-plane is 45 and is obtained for $(\Delta_1, \ldots, \Delta_5) = (48, 48, 48, 48, 38)$;

- the maximal contribution of the hyperplanes through a 5-plane is 10. and is obtained for $(\Delta_1, \ldots, \Delta_5) = (48, 48, 48, 48, 43)$;

- the maximal contribution of the hyperplanes through a 6-plane is 0.

Hence, by (6), we obtain the following inequalities for the two possible spectra of $(22, 6; 3, 5)$-arcs:

$$10620 = \sum_{i=0}^{46} \binom{48-i}{2} a_i \leq$$
$$4 \cdot 241 + 22 \cdot 300 + 1 \cdot 135 + 0 \cdot 55 + 30 \cdot 45 + 56 \cdot 10 + 43 \cdot 0 + \binom{26}{2} = 9934,$$

and

$$10620 = \sum_{i=0}^{46} \binom{48-i}{2} a_i \leq$$
$$4 \cdot 241 + 24 \cdot 300 + 0 \cdot 135 + 4 \cdot 55 + 24 \cdot 45 + 60 \cdot 10 + 42 \cdot 0 + \binom{26}{2} = 10389,$$

a contradiction in both cases.

Hyperplanes of multiplicity $23, \ldots, 26, 39$ are ruled out in a similar way. This completes the proof. $\square$

Corollary 9. *Linear codes with parameters* $[232, 5, 184]_5$ *and* $[233, 5, 185]_5$ *do no exist.*

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