# PLISKA
## STUDIA MATHEMATICA BULGARICA

# ПЛИСКА
## БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

# ON NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE OF COMPLEX DILATATIONS

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It is found a necessary and sufficient condition for the convergence of complex dilatations of quasiconformal mappings in terms of Hilbert transform.

**1. Introduction.** The convergence of quasiconformal mappings complex dilatations is studied in the present article. The history and fundamental preceding results can be found in [1-6]. The complex dilatation convergence concept induced by quasiconformal mapping locally uniform convergence has been introduced in [6]. Further this convergence will be called characteristic convergence. The given paper proves propositions on the metrizability and the sequent compactness of the characteristic space of local nature and some sufficient conditions about characteristic convergence. The main result of the article is the necessary and sufficient condition for the characteristic convergence in terms of Hilbert transform.

**2. Concept and fundamental properties of characteristic convergence.** One of analytic definitions (see [2, p. 194]) says that a quasiconformal mapping is a homeomorphic generalized solution of Beltrami equation

$$f_{\bar{z}}=\mu f_z, \tag{1}$$

where $\mu=\mu(z)$ is a measurable function that satisfies the inequality

$$|\mu(z)|\leqq q<1 \tag{2}$$

almost everywhere (a. e.). The function $\mu(z)$ is called complex dilatation of the mapping $f(z)$ or simply a characteristic.

Let $\mathfrak{M}_q$, $0<q<1$, is the class of all measurable complex functions $\mu(z)$, defined in the complex plane **C** and satisfying (2) a. e. We say that a sequence $\mu_n\in\mathfrak{M}_q$ converges to $\mu\in\mathfrak{M}_q$ in the sense of characteristics and write $\mu_n \xrightarrow{\text{ch.}} \mu$, if some sequence of quasiconformal mappings $f_n$ with characteristics $\mu_n$ is converged locally uniformly (l. u.) to some quasiconformal mapping $f$ with the characteristic $\mu$.

It is easy to show on the basis of Waierstrass theorem for analytic funcions that, if two sequences of quasiconformal mappings $f_n$ and $g_n$ with characteristics $\mu_n$

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converge l. u. to the quasiconformal mappings $f$ and $g$ with characteristics $\mu$ and $\varkappa$ correspondingly, then $\mu(z)=\varkappa(z)$ a. e. Thus:

Remark 1. The limit in the sense of characteristic convergence is unique up to modulus of equivalence.

Further two functions from $\mathfrak{M}_q$ which are equal a. e. will be called equivalent and, strictly speaking, the elements of $\mathfrak{M}_q$ will be taken the equivalence classes. Then on $\mathfrak{M}_q$ a metric generating characteristic convergence can be introduced.

Indeed let $\mathfrak{F}_q$ be the class of all quasiconformal mappings of the complex plane C on itself which preserve the points 0, 1 and $\infty$ with characteristics in $\mathfrak{M}_q$. We observe that every $\mu\in\mathfrak{M}_q$ corresponds to a unique mapping $f_\mu\in\mathfrak{F}_q$ with characteristic $\mu$ (see [7, p. 90]). Conversely each $f\in\mathfrak{F}_q$ corresponds to a unique equivalence class of his characteristics from $\mathfrak{M}_q$. According to this for any pair of $\mu_1, \mu_2\in\mathfrak{M}_q$ let

$$\rho(\mu_1, \mu_2)=\sum_{m=1}^{\infty}\frac{1}{2^m}\frac{\rho_m(\mu_1, \mu_2)}{1+\rho_m(\mu_1, \mu_2)},$$

where $\rho_m(\mu_1, \mu_2)=\max_{|z|\leq m}|f_\mu(z)-f_{\mu_2}(z)|$, $m=1, 2, \ldots$ The function $\rho$ is a metric (see [8, p. 228 and 243]) and generates on $\mathfrak{M}_q$ some convergence which is equivalent to the l. u. convergence $f_{\mu_n}\to f_\mu$. Consequently, the metric space $(\mathfrak{M}_q, \rho)$ is sequentially compact (see [8, p. 197-199] and [2, p. 76]).

From the definition of characteristic convergence, Remark 1 and the general properties of sequentially compact spaces (see [8, p. 203]) we obtain that the convergence $\mu_n\xrightarrow{\text{ch.}}\mu$ is equivalent to the convergence $\rho(\mu_n, \mu)\to 0$. Thus:

Proposition 1. *The characteristic convergence generates on $\mathfrak{M}_q$ some topology which is metrizable and sequentially compact.*

Hence we obtain (see [8, p. 203]):

Lemma 1. *If all subsequences $\{\mu_n\}$, $\mu_n\in\mathfrak{M}_q$, $n=1, 2,\ldots$, converging in the sense of characteristics have the same limit $\mu\in\mathfrak{M}_q$, then $\mu_n\xrightarrow{\text{ch.}}\mu$.*

We observe also a local nature of the characteristic convergence:

Proposition 2. *Let $\mu$ and $\mu_n\in\mathfrak{M}_q$, $n=1, 2,\ldots$ Then $\mu_n\xrightarrow{\text{ch.}}\mu$ if and only if $\mu_n\xrightarrow[K]{\text{ch.}}\mu$ on every compact set $K$ from* C.

Here $\mu_n\xrightarrow[K]{\text{ch.}}\mu$ means that $\mu_n\chi_K\xrightarrow{\text{ch.}}\mu\chi_K$, where $\chi_K(z)$ is the characteristic function of the set $K$.

Proof. 1) Let $\mu_n\xrightarrow{\text{ch.}}\mu$ and $\tilde\mu=\mu\chi_K$, $\tilde\mu_n=\mu_n\chi_K$, $n=1, 2,\ldots$ According to Lemma 1 for proving that $\tilde\mu_n\xrightarrow{\text{ch.}}\tilde\mu$ we consider any subsequence $\tilde\mu_{n_m}$ converging in the sense of characteristics to some $\varkappa$ and will show that $\varkappa=\tilde\mu$ a. e.

By the definition of characteristic convergence we have that there are some sequences of quasiconformal mappings $f_m$ and $g_m$ with characteristics $\mu_{n_m}$ and $\tilde\mu_{n_m}$

which converge l. u. to some quasiconformal mappings $f$ and $g$ with characteristics $\mu$ and $\varkappa$ correspondingly. On the one hand, by Strebel's theorem [5] $\varkappa(z)=0$ a. e. on $\mathbf{C}\setminus K$. On the other hand, the characteristics of the sequence $f_m \circ g_m^{-1}$ vanish on the set $K$ and we obtain that $\varkappa(z)=\mu(z)$ a. e. on $K$ again by Strebel's theorem. Thus $\varkappa(z)=\tilde{\mu}(z)$ a. e.

2) Let $\mu_n \xrightarrow[K]{\text{ch.}} \mu$ for any compact set $K$ from $\mathbf{C}$. Then, in particular, this is true for all disks $K_l=\{z\in\mathbf{C},\ |z|\leqq l\}$, $l=1, 2,\dots$, which form a countable exhaustion of the plane $\mathbf{C}$.

In order to use Lemma 1 we consider any converging subsequence $\mu_{n_m} \xrightarrow{\text{ch.}} \varkappa$. By the first part of the proof $\mu_{n_m} \xrightarrow[K_l]{\text{ch.}} \varkappa$, $l=1, 2,\dots$ Consequently, $\varkappa(z)=\mu(z)$ a. e. on $K_l$, $l=1, 2,\dots$, what is established by analogy to the preceding. Finally, because of countablity of the exhaustion $\{K_l\}$ we have that $\varkappa(z)=\mu(z)$ a. e. and theorem is proved.

By analogy, on the basis of the Lemma 1, Bers theorem ([2, p. 197]) and the properties of measure convergence ([10, p. 158]) it can be proved the following:

Proposition 3. *Let $\mu$ and $\mu_n\in\mathfrak{M}_q$, $n=1, 2,\dots$ Then in order to have the convergence $\mu_n \xrightarrow{\text{ch.}} \mu$ it is sufficient $\{\mu_n\}$ to satisfy any of the following conditions*:

1. $\mu_n\to\mu$ *a. e.*;
2. $\mu_n\to\mu$ *locally by measure*;
3. $\mu_n\to\mu$ *locally in* $\mathscr{L}^p$, $1\leqq p\leqq\infty$.

*However no enumerated conditions are necessary for the convergence* $\mu_n \xrightarrow{\text{ch.}} \mu$.

**3. Normal solutions of Beltrami equation and characteristic convergence.** Let $\mathfrak{M}_q(K)$ be a subclass of $\mathfrak{M}_q$ consisting of all functions with support in the compact set $K$. If $\mu\in\mathfrak{M}_q(K)$, then the normal solution of Beltrami equation (1) is his generalized solution such that $f(0)=0$ and $f_z-1\in\mathscr{L}^p=\mathscr{L}^p(\mathbf{C})$, $p>2$ (see [7, p. 86]). The class of normal solutions of Beltrami equation with characteristics $\mu\in\mathfrak{M}_q(K)$ will be denoted by $\mathfrak{N}_q(K)$. It is known that the normal solutions of Beltrami equation are quasiconformal homeomorphism ([7, p. 89]).

Lemma 2. *The class $\mathfrak{N}_q(K)$ is sequentially compact with respect to the locally uniform convergence.*

Proof. Since the normal solutions of Beltrami equation with characteristics from $\mathfrak{M}_q(K)$ are some $Q$-quasiconformal homeomorphisms of $\bar{\mathbf{C}}$ on itself, $Q=(1+q)/(1-q)$, that satisfy the conditions $f(0)=0$, $f(\infty)=\infty$ (see [7, p. 89]), they form a normal family ([2, p. 76]). Consequently, every sequence $f_n\in\mathfrak{N}_q(K)$ contains some subsequence $f_{n_m}$ which converges l. u. with respect to the spherical metric. Moreover a limit function $f$ is a $Q$-quasiconformal mapping of $\bar{\mathbf{C}}$ onto itself or $f(z)\equiv 0$, $z\in\mathbf{C}$, and $f(\infty)=\infty$ or $f(z)\equiv\infty$, $z\in\bar{\mathbf{C}}\setminus\{0\}$, and $f(0)=0$ (see [2, p. 76-77]). We show that these two cases should be excluded.

1) In the first case $f_{n_m}\to 0$ l. u. in $\mathbf{C}$. By Green's formula

$$\iint_R (f_{n_m})_z\, dxdy \to 0$$

for any rectangle $R$. However this contradicts with the inequality (see [7, p. 86]):

$$\|f_z - 1\|_p \leqq \frac{q\,C_p}{1 - q\,C_p}(S_K)^{1/p} = C = C(q,\ p,\ K),$$

where $S_K$ is the area of $K$, $C_p$ is the $\mathscr{L}^p$ – norm of Hilbert transformation, $p>2$ such that $q\,C_p<1$ (see [7, p. 83]). Indeed for any rectangle $R$ we have:

$$\|(f_{n_m})_z - 1\|_p \geqq \|(f_{n_m})_z - 1\|_{\mathscr{L}^p(R)}$$

$$\geqq (S_R)^{-(p-1)/p}\|(f_{n_m})_z - 1\|_{\mathscr{L}^1(R)}$$

$$\geqq (S_R)^{-(p-1)/p}\left|\iint_R [(f_{n_m})_z - 1]\, dxdy\right|$$

$$\geqq (S_R)^{-(p-1)/p}\left\|\iint_R (f_{n_m})_z dxdy\right| - S_R| \to (S_R)^{1/p}.$$

Consequently, because of the arbitrariness of $R$ the norm $\|(f_{n_m})_z - 1\|_p$ converges to $\infty$ what gives the desied contradiction and the considered case should be excluded.

2) In the second case $f_{n_m} \to \infty$ l. u. with respect to the spherical metric in $\overline{\mathbb{C}}\setminus\{0\}$. According to the homeomorphism of the mappings $f_{n_m}$ and to the equality $f_{n_m}(0)=0$ we obtain that the area $S(f_{n_m})(\Delta_r))$, where $\Delta_r = \{z : |z| < r\}$, converges to $\infty$ for any $r>0$. However this contradicts to the following inequality for the class $\mathfrak{N}_q(K)$:

$$[S(f(\Delta_r))]^{1/2} = \left(\iint_{|z|\leqq r} (1 - |\mu(z)|^2)|f_z|^2\, dxdy\right)^{1/2}$$

$$\leqq \|f_z\|_{\mathscr{L}^2(\Delta_r)} \leqq \|f_z - 1\|_{\mathscr{L}^2(\Delta_r)} + \pi^{1/2} r$$

$$\leqq \|f_z - 1\|_{\mathscr{L}^2(\mathbb{C})} + \pi^{1/2} r \leqq \frac{q}{1-q}(S_K)^{1/2} + \pi^{1/2} r$$

(see [7, p. 86]). Thus also the second case is excluded.

3) It remains to consider the case in which $f_{n_m} \to f$ l. u., where $f$ is some $Q$-quasiconformal mapping. In this case $[(f_{n_m})_z - 1] \to (f_z - 1)$ weakly in $\mathscr{L}^p$ (see [2, p. 196], [7, p. 86] and [11, p. 44]). According to the inequality from the first part of the proof and to the weakly compactness of the balls in $\mathscr{L}^p$ we obtain:

$$\|f_z - 1\|_p \leqq C(q, p, K).$$

That the characteristic $\mu$ of the mapping $f$ vanishes a.e. on $\mathbf{C}\setminus K$ and that $|\mu(z)| \leqq q$ a.e. on $K$ follows from Strebel's theorem. Thus $f \in \mathfrak{N}_q(K)$ and the lemma is proved.

On the basis of Lemmas 1 and 2 (see also [8, p. 203]) analogously to Proposition 2 it can be proved following:

Lemma 3. *Let $\mu$, $\mu_n \in \mathfrak{M}_q(K)$, $n = 1, 2, \ldots$, and $f$, $f_n \in \mathfrak{N}_q(K)$, $n = 1, 2, \ldots$, are the corresponding normal solutions of Beltrami equation. Then $\mu_n \xrightarrow{\text{ch.}} \mu$ if and only if $f_n \to f$ l.u.*

**4. A criterion for characteristic convergence.** Let $N$ be a nonlinear operator defined on the class $\mathfrak{M}_q(K)$ by the relation

$$N(\mu) = T(\mu) + T(\mu T(\mu)) + \ldots,$$

where the convergence is in the sense of $\mathscr{L}^2$ ([2, p. 224]) and $T$ is the Hilbert operator

$$T(\mu) = \frac{1}{2\pi i} \iint_{\mathbf{C}} \frac{\mu(z)}{(z-\zeta)^2} dz d\bar{z}.$$

According to the local nature of the characteristic convergence it is sufficient to study the question on the class $\mathfrak{M}_q(K)$ only.

Theorem. *Let $\mu$ and $\mu_n \in \mathfrak{M}_q(K)$, $n = 1, 2, \ldots$ Then $\mu_n \xrightarrow{\text{ch.}} \mu$ if and only if $N(\mu_n) \to N(\mu)$ weakly in $\mathscr{L}^2$.*

Proof. 1) Let $\mu_n \xrightarrow{\text{ch.}} \mu$. Then by Lemma 3 $f_n \to f$ l.u., where $f$ and $f_n \in \mathfrak{N}_q(K)$ are the corresponding normal solutions of Beltrami equation. Consequently (see [2, p. 196]) for any rectangle $R$:

$$\iint_R (f_n)_z \, dx dy \to \iint_R f_z dx dy.$$

On the other hand, $(f_n)_z = 1 + N(\mu_n)$ and $f_z = 1 + N(\mu)$ for normal solutions (see [7, p. 85-86]). Thus for any rectangle $R$:

$$\iint_R N(\mu_n) dx dy \to \iint_R N(\mu) dx dy.$$

According to the isometric relation for Hilbert operator ([7, p. 81]) the operator $N$ is bounded in $\mathscr{L}^2$ and hence ([11, p. 44]) $N(\mu_n) \to N(\mu)$ weakly in $\mathscr{L}^2$.

2) Let $N(\mu_n) \to N(\mu)$ weakly in $\mathscr{L}^2$. To establish that this condition is sufficient for $\mu_n \xrightarrow{\text{ch.}} \mu$ according to Lemma 1 we consider any converging subsequence

$\mu_{n_m} \xrightarrow{\text{ch.}} \mu_0$ and will show that $\mu_0(z)=\mu(z)$ a.e. Indeed from the first part we obtain that $N(\mu_{n_m})\to N(\mu_0)$ weakly in $\mathscr{L}^2$ and according to the uniqueness of the weak limit we have $N(\mu_0)=N(\mu)$. Let $\varphi=N(\mu_0)=N(\mu)$. Applying the operator $P_\mu(\psi) =\psi-T(\mu\psi)$ to the equality $1+\varphi=1+N(\mu)$ we obtain the equality $\varphi=T(\mu(1+\varphi))$. Analogously applying the operator $P_{\mu_0}(\psi)=\psi-T(\mu_0\psi)$ to the equality $1+\varphi=1+N(\mu_0)$ we obtain $\varphi=T(\mu_0(1+\varphi))$. Consequently, $T((\mu-\mu_0)\times(1+\varphi))=0$. According to the isometric relation for Hilbert operator we obtain immediately $(\mu-\mu_0)(1+\varphi)=0$ a.e. However $1+\varphi=f_z\neq 0$ a.e. (see [7, p. 37]), where $f$ is a normal solution of Beltrami equation with the characteristic $\mu$. Thus $\mu_0(z)=\mu(z)$ a.e. and theorem is proved.

We observe that the weak convergence of Hilbert transformations $T(\mu_n)\to T(\mu)$ is not a necessary and sufficient condition of the characteristic convergence $\mu_n \xrightarrow{\text{ch.}} \mu$. Indeed according to the isometric relation for Hilbert transformation the weak convergence $T(\mu_n)\to T(\mu)$ is equivalent to the weak convergence $\mu_n\to\mu$. That the weak convergence $\mu_n\to\mu$ is not a necessary and sufficient condition for the characteristic convergence $\mu_n \xrightarrow{\text{ch.}} \mu$ has been established before (see [6, p. 25]).

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