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STUDIA MATHEMATICA
BULGARICA

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# ON THE HOMOGENEOUS-POLYNOMIALLY CONVEX HULL OF UNIONS OF BALLS

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It is constructed a polynomially convex compact $K$ of $\mathbf{C}^n$ consisting of $n$ disjoint closed balls not containing the origin for which $\hat{K}_C$ contains a ball centered at zero.

Let $\mathbf{C}^n$ be the space of $n$ complex variables. Denote by $Q$ a set of homogeneous polynomials of $\mathbf{C}^n$. For any compact $K \subset \mathbf{C}^n$ we call homogeneous-polynomially convex hull of the following set:

$$\hat{K}_Q = \{z \in \mathbf{C}^n : |p(z)| \leqq \sup_K |p(w)| \quad \text{for all} \quad p \in Q\}.$$

$K$ is called a homogeneous-polynomially convex set if $\hat{K}_Q = K$. It is easy to see that $\hat{K}_Q$ is a complete circled compact centered at zero. A similar definition for the class $S$ of homogeneous polynomials in $\mathbf{C}^n$ of the form

$$f(z) = \prod_{j=1}^{m} (\alpha_1^j z_1 + \alpha_2^j z_2 + \dots + \alpha_n^j z_n) \quad \text{for all} \quad m = 1, 2, \dots,$$

was introduced in [1] (the corresponding hull is denoted by $\hat{K}_S$). Note that both definitions coincide in $\mathbf{C}^2$.

The following relations hold true: $\hat{K} \subseteq \hat{K}_Q \subseteq \hat{K}_S$ (here $\hat{K}$ is the usual polynomially convex hull of $K$). In [2] it is proved that if $K$ is a circled compact of $\mathbf{C}^n$, then $\hat{K}_Q = \hat{K}$.

Let $\hat{K}_C$ be the smallest complete circled compact centered at zero that contains $K$ and let $B = \{z \in \mathbf{C}^n : |z| \leqq 1\}$ be a unit ball with boundary $\partial B$. It is obvious that $\hat{K}_C \subset \hat{K}_Q$.

In this article it is constructed a polynomially convex compact $K$ of $\mathbf{C}^n$ consisting of $n$ disjoint closed balls of radii less than 1 with centers on $\partial B$ for which $\hat{K}_C$ contains a ball centered at zero. It is shown also that the number of these balls $n$ can not be restricted. Moreover, it is not possible to replace the balls by linearly convex compacts for to restrict their number (Theorem 1).

A domain $D$ is linearly convex if for every point $\xi \in \partial D$ there is a $(n-1)$-dimensional analytical plane passing through $\xi$ without crossing $\mathcal{D}$.

A compact set is called linearly convex if it can be approximated from outside by linearly convex domains (componentwise).

*PLISKA Studia mathematica bulgarica, Vol.* 10, 1989, *p. 45—49.*

Further, it is proved the existence of a circled compact on $\partial B$ the polynomially convex hull of which contains a closed ball with independent of $n$ radius (Theorem 2).

Theorem 1. *For any sufficiently small number $\delta>0$, of $\mathbf{C}^n$ there exist $n$ disjoint closed balls not containing zero and centered on $\partial B$, the union $K$ of which is polynomially convex while $\hat{K}_C$ (note that $\hat{K}_C\subset\hat{K}_Q$) contains the ball $\{z\in\mathbf{C}^n : |z|\leqq 2-\sqrt{2}-\delta\}$. At the same time, if $K$ is the union of any $n-1$ closed disjoint linearly convex compacts (in particular balls) not containing zero then $\hat{K}_C$ does not contain any neighbourhood of zero.*

To prove this theorem we need the following

Lemma. *In $\mathbf{C}^n$ there exist $n$ disconnected closed balls of radii less than 1 and centered on $\partial B$ such that any complex line passing through zero crosses, at least, one of these balls.*

Proof of the Lemma. Consider $n$ disjoint closed balls of $\mathbf{C}^n$ of the form

$$B_m=\{z\in\mathbf{C}^n ; \sum_{\nu=1}^{n} |z_\nu-a_\nu^{(m)}|^2\leqq R_m^2\},$$

where $a^{(m)}=(0,\dots, 0, \underset{(m)}{1}, 0,\dots, 0)\in\mathbf{C}^n$, $R_m<1$, $m=1,\dots, n$.

Denote

$$L_\lambda=\{z\in\mathbf{C}^n : z_2=\lambda_1 z_1,\ z_3=\lambda_2 z_1,\dots,\ z_n=\lambda_{n-1}z_1\},$$

where $\lambda=(\lambda_1, \lambda_2,\dots, \lambda_{n-1})\in\mathbf{C}^{n-1}$. We have:

$$B_1\cap L_\lambda=\left\{z_1\in\mathbf{C}^1 : |z_1-\frac{1}{1+|\lambda|^2}|^2\leqq\frac{1}{(1+|\lambda|^2)^2}-\frac{1-R_1^2}{1+|\lambda|^2}\right\},$$

where $|\lambda|^2=|\lambda_1|^2+\dots+|\lambda_{n-1}|^2$. It follows that if $|\lambda|\leqq R_1/\sqrt{1-R_1^2}$ then the complex line $L_\lambda$ crosses $B_1$. Further

$$B_m\cap L_\lambda=\left\{z_1\in\mathbf{C}^1 : \left|z_1-\frac{\bar{\lambda}_{m-1}}{1+|\lambda|^2}\right|^2\leqq\frac{|\lambda_{m-1}|^2}{(1+|\lambda|^2)^2}-\frac{1-R_m^2}{1+|\lambda|^2}\right\},$$

for all $m=2, 3,\dots, n$.

Hence, for all $\lambda$ with

$$-|\lambda_1|^2-\dots-|\lambda_{m-2}|^2+\frac{R_m^2}{1-R_m^2}|\lambda_{m-1}|^2-|\lambda_m|^2-\dots-|\lambda_{n-1}|^2\geqq 1,$$

$m=2, 3,\dots, n$, the complex line $L_\lambda$ crosses $B_m$. Therefore, if the radii of the balls satisfies the conditions: $2(1-R_m^2)/(2R_m^2-1)\leqq R_1^2/(1-R_1^2)$, then any complex line $L_\lambda$ crosses at least one of the balls $B_m$, $m=1, 2,\dots, n$.

Note that these conditions are satisfied, for example, in the case when $R_1=\sqrt{2}/3-\delta$, $R_m=2\sqrt{2}/3-\delta$, $m=2, 3,\ldots, n$ (where $\delta>0$ is a sufficiently small number).

It can be easily shown that all complex lines of the form

$$\{z\in \mathbf{C}^n : z_1=0,\ z_2=\alpha_2 t,\ldots,\ z_n=\alpha_n t\},$$

where $t\in \mathbf{C}^1$, $(\alpha_2,\ldots, \alpha_n)\in \mathbf{C}^{n-1}$, also cross $\cup_{m=2}^{n} B_m$.

P r o o f o f T h e o r e m 1. Let $B_m$, $m=1, 2,\ldots, n$, be closed balls as in Lemma's proof. As the centers of the balls are in $\mathbf{R}^n$ their union $K$ is polynomially convex [3, 4].

If one takes the radius $R_1$ of the ball $B_1$ sufficiently close to 1 (namely if $R_1>1-\delta$, $\delta>0$), then the radius $R_m$ of the balls $B_m$, $m=2, 3,\ldots, n$, can be taken close to $\sqrt{2}-1$ and therefore $\hat{K}_C$ contains the whole ball of radius $2-\sqrt{2}-\delta$ (by the cosine law). The first part of the theorem is proved.

Let us prove now the second part of the theorem. Let $K_j \Subset \mathbf{C}^n$, $j=1, 2,\ldots, n-1$, be linearly convex compacts such that $K_j\cap\partial B\neq\emptyset$, $K_j$ does not contain zero and $K_i\cap K_j=\emptyset$, $i\neq j$.

Consider the analytic planes

$$l_j=\{z\in \mathbf{C}^n : a_1^j z_1+a_2^j z_2+\ldots+a_n^j z_n=0\}$$

with $l_j\cap K_j=\emptyset$, $j=1, 2,\ldots, n-1$.

Since $(\mathbf{C}^n\backslash K_j)\supset l_j$, $j=1, 2,\ldots, n-1$, then

$$\mathbf{C}^n\backslash\left(\bigcup_{j=1}^{n-1} K_j\right)=\bigcap_{j=1}^{n-1}(\mathbf{C}^n\backslash K_j)\supset\left(\bigcap_{j=1}^{n-1} l_j\right).$$

Since the system

$$\sum_{k=1}^{n} a_k^j z_k=0,\quad j=1, 2,\ldots, n-1,$$

has the following solution

$$\left.\begin{array}{l} z_1=t_1 z_n\\ z_2=t_2 z_n\\ \cdots\cdots\\ z_{n-1}=t_{n-1} z_n\end{array}\right\}=l\subset \mathbf{C}^n\backslash\left(\bigcup_{j=1}^{n-1} K_j\right),$$

it follows that there is a complex line $l$ passing through the origin without crossing $K=\cup_{j=1}^{n-1} K_j$.

R e m a r k. Similarly it can be shown also that in the space $\mathbf{C}^n$ the minimal number of disjoint balls of radius less than 1 and centered on $\partial B$, such that any complex linear subspace $\mathbf{C}^n$ of dimension $r$ crosses at least one of the balls is equal to $n-r+1$.

Theorem 2. *If* $K=\cup_{j=1}^n B_j$ *is as in Theorem* 1, *then* $(\hat{K}_c\cap\partial B)^\wedge$ *contains the ball* $\{z\in\mathbf{C}^n : |z|\leqq\sqrt{17}/9\}$.

Remark. There are examples showing that $(\hat{K}_c\cap\partial B)^\wedge\neq(\hat{K}_c\cap B)$.

Proof. Let the points $(\beta,\ldots,\beta,\underset{(j)}{\alpha},\beta,\ldots,\beta)\in(B_j\cap\partial B)$, $j=1,2,\ldots,n$, be with real coordinates. This means that $\alpha^2+(n-1)\beta^2=1$ and $(1-\alpha)^2+(n-1)\beta^2\leqq 2/9$. From this it follows that: $8/9\leqq\alpha<1$, $\beta=(1-\alpha^2)^{1/2}/(n-1)^{1/2}$.

Consider the torus

$$T_j=\{z\in\mathbf{C}^n : |z_1|=\ldots=|z_{j-1}|=|z_{j+1}|=\ldots=|z_n|=\beta,\ |z_j|=\alpha\},$$

$j=1,2,\ldots,n$. It is obvious that $T_j\subset(\hat{B}_{jc}\cap\partial B)$, $j=1,2,\ldots,n$.

Denote $F_\gamma=\{z\in\mathbf{C}^n : |z_1\ldots z_n|=\gamma\}$. We have:

$$\min_{F_\gamma}\sqrt{|z_1|^2+\ldots+|z_n|^2}=\sqrt{n}\,|z_1|$$

$$=\sqrt{n}\,\alpha^{1/n}\beta^{(n-1)/n}=n^{1/2}\alpha^{1/n}(1-\alpha^2)^{(n-1)/2n}/(n-1)^{(n-1)/2n}=\rho_n(\alpha).$$

Further, $\rho(\alpha)=\lim_{n\to\infty}\rho_n(\alpha)=(1-\alpha^2)^{1/2}>0$. Consequently $\rho=\max_\alpha\rho(\alpha)=\sqrt{17}/9$. If we denote $E=\cup_{j=1}^n T_j$, then $\hat{E}\supset\hat{E}_Q\supset\{|z|\leqq\sqrt{17}/9\}$, where $\hat{E}_Q$ is the holomorphic convex hull $E$. According to our construction $E\subset(\hat{K}_c\cap\partial B)$.

The notion of projective capacity we use here was introduced in [5].

A homogeneous polynomial $f$ of degree $\mu$ of $\mathbf{C}^n$ is called normalized if the following equality holds true:

$$\int_{\partial B}\ln|f|\,d\sigma=\mu\int_{\partial B}\ln|z_n|\,d\sigma,$$

where $\sigma$ is the normalized "surface" measure on $\partial B$, i.e. $\int_{\partial B}d\sigma=1$.

For any arbitrary circled compact $K\subset\partial B$ we denote $m_j=m_j(K)=\inf\{\sup_K|f|$, for all normalized polynomials $f$ of degree $j\}$, $j=1,2,\ldots$

The projective capacity $\mathrm{Cap}(K)$ is defined by the equality

$$\mathrm{Cap}(K)=\lim_{j\to\infty}m_j^{1/j}=\inf_j m_j^{1/j}.$$

Since $z_1^j$ is a normalized polynomial, then

$$0\leqq m_j\leqq\sup_K|z_1^j|\leqq\sup_{\partial B}|z_1^j|=1.$$

From this it follows that $0\leqq\mathrm{Cap}(K)\leqq 1$.

Theorem (Sibony-Wong-Alexander) [5]. *If* $K\subset\partial B$ *is a circled compact such that* $\mathrm{Cap}(K)>0$ *then the polynomially convex hull* $\hat{K}$ *contains the ball* $\{z\in\mathbf{C}^n : |z|\leqq\mathrm{Cap}(K)\}$.

For the projective capacity $K=\partial B$ the following estimate holds true:

$$n^{-1/2}e^{-(\gamma+\varepsilon_n)/2}\leqq \mathrm{Cap}(\partial B)\leqq n^{-1/2},$$

where $\gamma$ is the Euler's constant, $\lim_{n\to\infty}\varepsilon_n=0$. From this it follows that $\mathrm{Cap}(\partial B)\to 0$ as $n\to\infty$ and therefore in the given theorem the radius of the ball which is contained in $\hat{K}$ tends to zero when $n\to\infty$.

Note that, as it can be seen from Theorem 2, there are circled compacts $K\subset\partial B$, $K\neq\partial B$, such that $\hat{K}$ contains a ball with independent from $n$ radius.

## REFERENCES


1. H. Alexander. A note on projective capacity. *Can. J. Math.*, **34**, 1982, No 6, 1319-1329.
2. G. Khudajberganov. A note on the homogeneous-polynomially convexity and on the projective capacity. — In: Multidimensional Complex Analysis, Krasnoyarsk, 1985, 261-264.
3. G. Khudajberganov. About one problem of Grauert's. *Doklady Akad. Nauk Uzbek. SSR*, 1975, No 3, 7-8.
4. G. Khudajberganov. On the one problem of Grauert. — In: On the Holomorphic Functions of Several Complex Variables. Krasnoyarsk, 1976, 123-130.
5. H. Alexander. Projective capacity. *Ann. Math. Studies*, **100**, 1981, 3-27.


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*Received 12.8.1985*