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ON JACOBI SERIES HAVING POLAR SINGULARITIES ON THE BOUNDARIES OF THEIR CONVERGENCE REGIONS

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We consider some properties of series in Jacobi polynomials which have polar singularities on the boundaries of their convergence regions.

Let α and β be complex numbers such that $\alpha, \beta, \alpha+\beta+1 \neq -1, -2, \dots$ The polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ defined by the equalities

$$P_n^{(\alpha,\beta)}(z)=\binom{n+\alpha}{n}F(-n,\ n+\alpha+\beta+1,\ \alpha+1\,;\ \frac{1-z}{2}),$$

where $F(a,\ b,\ c\,;\ z)$ is Gauss hypergeometric function, are called Jacobi polynomials with parameters α and β [2 (1.20)].

The series of type

$$\sum_{n=0}^{+\infty} a_n P_n^{(\alpha,\beta)}(z) \tag{1}$$

we shall call Jacobi series.

Let $w(z)$ be that branch of the inverse of Zhukowski transformation $z=2^{-1}(w+w^{-1})$ for which $|w(z)|>1$. Then in the region $G=\mathbf{C}-[-1,\ 1]$ the following asymptotic formula holds true:

$$P_n^{(\alpha,\beta)}(z)=n^{-1/2}P^{(\alpha,\beta)}(z)[w(z)]^n\{1+p_n^{(\alpha,\beta)}(z)\}, \tag{2}$$

where $P^{(\alpha,\beta)}(z)\neq 0$, $\{p_n^{(\alpha,\beta)}(z)\}_{n=1}^{+\infty}$ are holomorphic functions in the region G and $p_n^{(\alpha,\beta)}(z)=O(n^{-1})$ $(n\to+\infty)$ uniformly on every compact subset of this region [2, (3.1)].

If

$$0<r^{-1}=\limsup_{n\to+\infty}\sqrt[n]{|a_n|}<1 \tag{3}$$

then the series (1) is absolutely uniformly convergent on every compact subset of the region $E(r)=\{z\in\mathbf{C}:|z+1|+|z-1|<r+r^{-1}\}$ and divergent in $\mathbf{C}-\overline{E}(r)$. This property is based on the asymptotic behaviour (2) of Jacobi polynomials. Let us denote $\gamma(r)=\partial E(r)$.

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This paper is a continuation of [4]. We shall consider some statements connected with Jacobi series which have polar singularities on the boundaries of their convergence regions. These statements are similar to some theorems for power series with polar singularities on their convergence circles ([3], [1, T. 3.3.11]).

Now we formulate two statements for further use.

Theorem 1 [2, Th. 6.1]. *Let* $1<R<+\infty$ *and* $f(z)$ *be a complex function holomorphic in* $E(R)$. *Then* f *can be represented in* $E(R)$ *by Jacobi series* (1) *with coefficients*

$$a_n=\frac{1}{2\pi i J_n^{(\alpha,\beta)}}\int_{\gamma(r)} f(\zeta)Q_n^{(\alpha,\beta)}(\zeta)d\zeta,\quad n=0,\ 1,\ 2,\dots,$$

where $1<r<R$ *and* $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ *are Jacobi functions of second kind* [2, p. 16] *and*

$$J_n^{(\alpha,\beta)}=\begin{cases}\dfrac{2\Gamma(\alpha+2)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & n=0,\\[2mm] \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, & n\geqq 1.\end{cases}$$

Lemma 1 [4]. *Let* $k\geqq 1$ *be an integer and let* K_1 *and* K_2 *be compact subsets of* G *such that*

$$\sup_{z\in K_1}|w(z)|<\sup_{\zeta\in K_2}|w(\zeta)|.$$

Then

$$\frac{1}{(\zeta-z)^k}=\sum_{n=0}^{+\infty}c_{n,k}(\zeta)P_n^{(\alpha,\beta)}(z),$$

where $\zeta\in K_2$, $z\in K_1$ *and*

$$c_{n,k}(\zeta)=\varphi_k(\zeta)n^{k-1/2}[w(\zeta)]^{-n}[1+O(n^{-1})]\quad (n\to+\infty) \tag{4}$$

uniformly on K_2. *Here* $\varphi_k(\zeta)\neq 0$ *for* $\zeta\in K_2$ *and* $k=1,\ 2,\dots$

In [4] we established the following

Theorem 2. *Let the coefficients* $\{a_n\}_{n=0}^{+\infty}$ *of Jacobi series* (1) *satisfy the condition* (3) *and let* $f(z)$ *be the sum of* (1) *in* $E(r)$. *If all singular points* $f(z)$ *on the ellipse* $\gamma(r)$ *are simple poles then the following asymptotic formula holds*

$$\alpha_n r^n n^{-1/2}=O(1)\ (n\to+\infty). \tag{5}$$

The following statement is converse to Theorem 2.

Theorem 3. *Let the coefficients* $\{a_n\}_{n=0}^{+\infty}$ *of Jacobi series* (1) *satisfy condition* (5) *and let* $f(z)$ *be the sum of* (1) *in* $E(r)$. *If all singular points of* $f(z)$ *on* $\gamma(r)$ *are poles then they are simple poles.*

Proof. Let the poles of $f(z)$ on $\gamma(r)$ be $z_1, z_2, \dots, z_m$, $s>1$ be their maximal order and let us suppose that $z_1, z_2, \dots, z_p$ $(1 \leqq p \leqq m)$ are of order s. Then

$$f(z) = \sum_{j=1}^{m} \sum_{k=1}^{k_j} \frac{A_{jk}}{(z-z_j)^k} + g(z) = h(z) + g(z), \tag{6}$$

where $k_1, k_2, \dots, k_m$ are the orders of poles $z_1, z_2, \dots, z_m$, respectively $(k_1 = k_2 = \dots = k_p = s)$ and $g(z)$ is a complex function holomorphic in $E(R)$ for some $R \in (r, +\infty)$.

In the region $E(R)$ the function $g(z)$ can be represented by the following Jacobi series (see Th. 1)

$$g(z) = \sum_{n=0}^{+\infty} a_n'' P_n^{(\alpha,\beta)}(z).$$

Jacobi functions of second kind have the following representation $(n \geqq 1)$

$$Q_n^{(\alpha,\beta)}(z) = n^{-1/2} Q^{(\alpha,\beta)}(z) [w(z)]^{-n-1} \{1 + q_n^{(\alpha,\beta)}(z)\}, \tag{7}$$

where $Q^{(\alpha,\beta)}(z) \neq 0$, $\{q_n^{(\alpha,\beta)}(z)\}_{n=1}^{+\infty}$ are holomorphic functions in the region G and $q_n^{(\alpha,\beta)}(z) = O(n^{-1})$ $(n \to +\infty)$ uniformly on every compact subset of this region [2, (3.11)]. Using this asymptotic formula and the integral representation of the coefficients $\{a_n''\}_{n=0}^{+\infty}$ (see Theorem 1) we obtain that $\lim_{n\to+\infty} a_n'' r^n = 0$. Since the function $h(z)$ is holomorphic in the region $E(r)$ we have

$$h(z) = \sum_{n=0}^{+\infty} a_n' P_n^{(\alpha,\beta)}(z)$$

for any $z \in E(r)$.

Using Lemma 1 one can obtain that

$$a_n' = \sum_{j=1}^{m} \sum_{k=1}^{k_j} A_{jk} c_{n,k}(z_j).$$

From the asymptotic formula (4) it follows that when $n \to +\infty$

$$a_n' = \sum_{j=1}^{m} A_{jk_j} n^{k_j - 1/2} \varphi_{k_j}(z_j) [w(z_j)]^{-n} [1 + O(n^{-1})].$$

Using that $z_1, z_2, \dots, z_p$ are poles of order $s>1$ we get for $n \to +\infty$,

$$a_n' r^n = (b_1 \alpha_1^n + \dots + b_p \alpha_p^n) n^{s-1/2} + O(n^{s-3/2}),$$

where $b_1, b_2, \dots, b_p$ are non zero constants, $|\alpha_j| = 1$ $(j = 1, 2, \dots, p)$ and $\alpha_j \neq \alpha_k$ for $j \neq k$. Then

$$a_n r^n n^{-1/2} = c_n n^{s-1} + O(n^{s-2}),$$

where $c_n = b_1\alpha_1^n + b_2\alpha_2^n + \ldots + b_p\alpha_p^n$. It follows that $\lim_{n\to+\infty} c_n = 0$.

Obviously

$$D = \begin{vmatrix} 1 & \cdot & \cdot & \cdot & 1 \\ \alpha_1 & \cdot & \cdot & \cdot & \alpha_p \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_1^{p-1} & \cdot & \cdot & \cdot & \alpha_p^{p-1} \end{vmatrix} \neq 0.$$

For every n we have

$$\begin{aligned} &b_1\alpha_1^n + \ldots + b_p\alpha_p^n = c_n, \\ &\cdots\cdots\cdots\cdots\cdots \\ &b_1\alpha_1^{n+p-1} + \ldots + b_p\alpha_p^{n+p-1} = c_{n+p-1}. \end{aligned}$$

From this system we obtain

$$\alpha_j^n b_j D = \begin{vmatrix} 1 & \ldots 1 & c_n & 1 & \ldots & 1 \\ \alpha_1 & \ldots \alpha_{j-1} & c_{n+1} & \alpha_{j+1} & \ldots & \alpha_p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{p-1} & \ldots \alpha_{j-1}^{p-1} & c_{n+p-1} & \alpha_{j+1}^{p-1} & \ldots & \alpha_p^{p-1} \end{vmatrix} = D_{j,n}$$

for every $j = 1, 2, \ldots, p$. Obviously, $\lim_{n\to+\infty} D_{j,n} = 0$ $(j = 1, \ldots, p)$. From this it follows that $b_j = 0$ for $j = 1, 2, \ldots, p$ which contradicts to the fact that $b_j \neq 0$ $(j = 1, 2, \ldots, p)$. Therefore $s = 1$ and thus Theorem 3 is proved.

Theorem 4. *Let the coefficients* $\{a_n\}_{n=0}^{+\infty}$ *of Jacobi series* (1) *satisfy the conditions* (3) *and* (5). *Then every pole of the sum* $f(z)$ *of* (1) *on the ellipse* $\gamma(r)$ *is simple.*

Proof. Let us suppose that the function $f(z)$ has a pole $z_0 \in \gamma(r)$ of order $s > 1$. For $z \in E(r)$ and for m sufficiently large we have:

$$(z-z_0)^s f(z) = (z-z_0)^s \sum_{n=0}^{+\infty} a_n P_n^{(\alpha,\beta)}(z) = (z-z_0)^s \sum_{n=0}^{m} a_n P_n^{(\alpha,\beta)}(z)$$

$$+ (z-z_0)^s \sum_{n=m+1}^{+\infty} a_n P_n^{(\alpha,\beta)}(z_0) \frac{P_n^{(\alpha,\beta)}(z)}{P_n^{(\alpha,\beta)}(z_0)} = (z-z_0)^s f_1(z) + (z-z_0)^s f_2(z).$$

Let $\varepsilon > 0$ be arbitrary. From (5) and (2) it follows that $a_n P_n^{(\alpha,\beta)}(z_0) = O(1)$ when $n \to +\infty$. Let $\mu > 0$ be such that the intersection of the set $U(z_0\mu) = \{z : |z-z_0| < \mu\}$ and the interval $[-1, 1]$ is empty. We fix μ. Using the asymptotic formula (2) we get

$$|(z-z_0)^s f_2(z)|=O(|z-z_0|^s(r-|w(z)|))$$

for $z\in V(z_0, \mu)=U(z_0, \mu)\cap E(r)$. Let $V_1(z_0, \mu)\subset V(r_0, \mu)$ be such that $|z-z_0| =O(r-|w(z)|)$ as $z\to z_0$ and $z\in V_1(z_0, \mu)$. Then $|(z-z_0)^s f_2(z)|=O(|z-z_0|^{s-1})$ for $z\in V_1(z_0, \mu)$ and m sufficiently large. Using the inequality $s>1$ we obtain that for m large enough and for $\delta>0$ small enough the following inequality holds:

$$|(z-z_0)^s f_2(z)|<\varepsilon/2, \tag{8}$$

where $|z-z_0|<\delta$ and $z\in V_1(z_0, \mu)$. We fix m. Obviously for $z\in \bar{E}(r)$ we have:

$$|(z-z_0)^s f_1(z)|\leqq K|z-z_0|^s,$$

where K is a constant independent from z. Then for $\delta>0$ sufficiently small we obtain that

$$|(z-z_0)^s f_1(z)|<\varepsilon/2$$

for $|z-z_0|<\delta$. From this inequality and from (8) it follows that $(z-z_0)^s f(z)=0\,(1)$ as $z\to z_0$ and $z\in V_1(z_0, \mu)$. This contradicts to our assumption that z_0 is a pole of order $s>1$. Therefore, $s\leqq 1$ and thus Theorem 4 is proved.

Theorem 5. *Let the coefficients* $\{a_n\}_{n=0}^{+\infty}$ *of Jacobi series* (1) *satisfy condition* (3) *and let* $f(z)$ *be the sum of* (1) *in* $E(r)$. *If*

$$a_n r^n n^{-1/2}=o(1)\quad (n\to+\infty) \tag{9}$$

then the function $f(z)$ *has no poles on the ellipse* $\gamma(r)$.

Proof. According to Theorem 4 if $f(z)$ has a pole z_0 on $\gamma(r)$ then it is simple. Using (9) and the technics from the proof of Theorem 3 one can establish that $\lim_{z\to z_0}(z-z_0)f(z)=0$. This contradicts to the simplicity of z_0. This proves Theorem 5.

Theorem 6. *Let the coefficients of Jacobi series* (1) *satisfy condition* (3), *let* $f(z)$ *be the sum of* (1) *in* $E(r)$ *and let for* $z_0\in\gamma(r)$

$$\lim_{n\to+\infty} a_n P_n^{(\alpha,\beta)}(z_0)=A\neq 0.$$

Then the unique pole of $f(z)$ *on* $\gamma(r)$ *can be only the point* z_0.

Proof. Using the asymptotic formulae (2), (7) as well as Stirling formula we get

$$\lim_{n\to+\infty} Q_n^{(\alpha,\beta)}(z_0)P_n^{(\alpha,\beta)}(z_0)/J_n^{(\alpha,\beta)}=B\neq 0. \tag{10}$$

Let us consider the function

$$g(z)=f(z)-\frac{A}{B}\,\frac{1}{z_0-z}$$

holomorphic in $E(r)$. Using Lemma 1 and the equality $c_{n,1}(z_0)=Q_n^{(\alpha,\beta)}(z_0)/J_n^{(\alpha,\beta)}$ (see [4]) we obtain that

$$g(z)=\sum_{n=0}^{+\infty}\left\{a_n-\frac{A}{BJ_n^{(\alpha,\beta)}}Q_n^{(\alpha,\beta)}(z_0)\right\}P_n^{(\alpha,\beta)}(z)=\sum_{n=0}^{+\infty}b_nP_n^{(\alpha,\beta)}(z).$$

From (9) and (10) it follows that $\lim_{n\to+\infty}b_nP_n^{(\alpha,\beta)}(z_0)=0$. Using this limit it can be easily seen that $\lim_{n\to+\infty}b_nr^nn^{-1/2}=0$. Then according to Theorem 5 the function $g(z)$ has no poles on the ellipse $\gamma(r)$. Thus Theorem 6 is proved.

Theorem 7. *Let the coefficients of Jacobi series satisfy condition* (3) *and* $f(z)$ *be the sum of* (1) *in* $E(r)$. *If all singular points of* $f((z)$ *on the ellipse* $\gamma(r)$ *are poles and if* p *among them are of maximal order* s *then if* n *is sufficiently large at least one member from the finite sequence of coefficients of* (1)

$$a_{n+1},\ a_{n+2},\dots,\ a_{n+p}$$

is different from zero.

Proof. In the same manner as in Theorem 3 we obtain that when $n\to+\infty$

$$a_nr^nn^{-1/2}=(b_1\alpha_1^n+\dots b_p\alpha_p^n)n^{s-1}+O(n^{s-2}),$$

where $b_j\neq 0$, $|\alpha_j|=1$ $(j=1,\dots,p)$ and $\alpha_j\neq\alpha_k$ $(j\neq k)$.

Further, as in the proof of Theorem 3.3.11 from [1] we get that if n is sufficiently large at least one element from the finite sequence of coefficients

$$a_{n+1}r^{n+1}(n+1)^{-1/2},\quad a_{n+2}r^{n+2}(n+2)^{-1/2},\dots,\ a_{n+p}r^{n+p}(n+p)^{-1/2}$$

is different from zero. Thus Theorem 7 is proved.

A corollary of this statement is the following:

Theorem 8. *Let the coefficients of Jacobi series* (1) *satisfy the condition* (3) *and let there exist an integer* $p>0$ *and a sequence* $\{\lambda_\nu\}_{\nu=1}^{+\infty}$ *of positive integers such that* $p\leqq\lambda_{\nu+1}-\lambda_\nu$ $(\nu=0,1,2,\dots)$ *and* $a_n=0$ *for* $\lambda_\nu<n<\lambda_{\nu+1}$. *If the sum* $f(z)$ *of* (1) *in* $E(r)$ *has only polar singularities on* $\gamma(r)$ *then the number of poles of maximal order on* $\gamma(r)$ *is* $\geqq p$.

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