PLISKA

STUDIA MATHEMATICA
BULGARICA

ПЛИСКА

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СТУДИИ

ON GENERALIZED PETROVSKY PARABOLIC SYSTEMS

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A generalization of the Petrovsky parabolic systems is proposed in this article; "a priori" estimates for them are examined; their characterizing inclusions for some spaces of complex-valued functions with strong generalized derivatives of L. Schwartz-Sobolev type are given.

This article proposes a generalization of the Petrovsky parabolic constant-coefficient systems; examines "a priori" estimates for them, it gives also their characterizing inclusions for some spaces of complex-valued functions with strong generalized derivatives of Laurent Schwartz-Sobolev type.

The generalization of I. G. Petrovsky [1] of the classical parabolicity is a recognized stage in the development of the theory of the differential operators. The Petrovsky's parabolicity is studied by I. G. Petrovsky and his school, by O. A. Oleinik, M. S. Agranovič, S. D. Eidel'man, M. I. Višik, M. I. Ventzel, S. D. Ivasišen. I. M. Gel'fand and G. E. Shilov made a generalization of the Petrovsky's parabolicity in some cases. G. E. Shilov denoted in 1957 in Uspehi Mat. Nauk that a future description of some algebras of type C "may serves as a basis of a specific classification of the second order differential operators". Indeed, elaborating such a description, the author observed the necessity of a supplement to the classical definition of a parabolic operator of second order with constant complex coefficients, so that a same operator not to be simultaneously elliptic and parabolic (see Definition 1′ and below it). Moreover, the description of the algebras of type C, their inclusions and their comparisons with the classical Sobolev spaces W_∞^N open and impose the necessity to extend the definition of Petrovsky's parabolicity. For instance, if A is an elliptic operator of order $N \geqq 2$, then the space $W_\infty^{A,I} \subset W_\infty^{N-1}$. But it is not so if A is a usual parabolic operator, or a parabolic operator after I. G. Petrovsky, or a generalized parabolic operator, introduced in this article. (Here I is the identity operator $If = f$.) So there exists a fnction $f \in W_\infty^{A,I}$, which does not have all continuous partial derivatives till order $N-1$ inclusive. (This is a new result even in the case where A is the classical parabolic operator $\partial^2/\partial x^2 + a\partial/\partial t$, $a \neq 0$.) The same is true also for the space W_∞^α, where α is the linear differential-invariant space, generated by this parabolic operator A. Similar results are true also for the spaces $W_p^{A,I}$, $1 \leqq p < \infty$.

1. Formulation of the results.

Definition 1. *The linear constant-coefficient differential operator $\mathscr{A}$ in $n+1$ variables, $n \geqq 1$, of order $N \geqq 2$, is called parabolic if there exists a real nondegenerated linear transformation L of R^{n+1} such that the operator $\mathscr{A}$ is transformed in an operator A of the kind!*

PLISKA Studia mathematica bulgarica, Vol. 10, 1989, p. 108–115.

(1) $$A = \sum_{|s|+l_s=N} a_s D^s D_t^{l_s} + D_t^{N-k} \sum_{|s|<k} b_s D^s + P(D, D_t), \ \Sigma |b_s| \neq 0,$$

where $0 \leqq l_s < N-k$ *if* $N-k \neq 0$, and $l_s = 0$ *if* $N=k$; $0<k\leqq N$; *P is a polynomial of degree less than N and its degree relatively* D_t *is less than* $N-k$ *if* $N>k$, *and* 0 *if* $N=k$; $D=\partial/\partial x$; $x=(x_1, \dots, x_n)$; $D_t = \partial/\partial t$; $s=(s_1, \dots, s_n)$; $s_1, \dots, s_n$ *are nonnegative integers*, $|s| = s_1 + \dots + s_n$.

Remark. This definition is equivalent for the second-order operators in two variables to the following:

Definition 1′. *The constant-coefficient operator*

$$\mathscr{A} = a_{20}\partial^2/dx^2 + 2a_{11}\partial^2/\partial x \partial y + a_{02}\partial^2/\partial y^2 + \sum_{j+r \leqq 1} a_{jr} \partial^{j+r}/\partial x^j \partial y^r$$

is called parabolic if $a_{20}a_{02} - a_{11}^2 = 0$ *and if* $a_{20} \neq 0$ *then* a_{11}/a_{20} *to be real; if* $a_{02} \neq 0$ *then* a_{11}/a_{02} *to be real.*

The additional condition than usual assures that the operator $\mathscr{A}$ cannot be simultaneously and elliptic. For instance, if $B = (\partial/\partial x + i\partial/\partial y)^2$ then $a_{20}a_{02} - a_{11}^2 = 0$ but simultaneously the second-order homogeneous part of the characteristic polynomial of B is zero on R^2 only at the origin of the plane R^2, i. e., the operator B is elliptic.

Definition 2. *The system* $(A_q)_q$ *of linear constant-coefficient differential operators in* $n+1$ *variables,* $n \geqq 1$, *is called parabolic of order* $N \geqq 2$, *if each operator* A_q *is of order* $\leqq N$; *at least one of the operators* A_q *is of order N; and each operator of order N, belonging to the linear hull of the system* $(A_q)_q$ *is parabolic.*

Evidently this is a generalization for the constant-coefficient case of the definition of the Petrovsky's parabolicity (see [1-4]).

Let $\mathscr{D}_\rho$, $0<\rho\leqq\infty$ $(\mathscr{D}_\infty = \mathscr{D})$, be the set of all infinitely differentiable complex-valued functions with compact supports in the closed ball with radius ρ and a center — the point 0. Let $A, A_1, \dots, A_m$ be constant-coefficient linear differential operators. If there exists a constant $\varkappa$ such that $\|Ag\| \leqq \varkappa\{\|A_1 g\| + \dots + \|A_m g\|\}$ for $\forall g \in \mathscr{D}_\rho$, where $\|f\| = \sup_w |f(w)|$, then we shall say that $A_1, \dots, A_m$ jointly dominate A on $\mathscr{D}_\rho$ (in the supremum norm).

Theorem 1. Let $A, B_1, \dots, B_m$ *be linear constant-coefficient differential operators in variables* (x, t), $x=(x_1, \dots, x_n)$, $n \geqq 1$. *Let A be parabolic of order* $N \geqq 2$, *and respectively* $\mathscr{D}_t$ *its order let be* $N-k$, $0<k<N$, *and*

$$A = \sum_{|s|+l_s=N} a_s D^s D_t^{l_s} + D_t^{N-k} \sum_{|s|<k} b_s D^s + P(D, D_t),$$

where $0 \leqq l_s < N-k$; *P is a polynomial of degree less than N and respectively* D_t *is of degree less than* $N-k$. *Let all* B_q *be of orders less than N and respectively* D_t *be of orders less than* $N-k$.

Then the operators $A, B_1, \dots, B_m$ *do not jointly dominate the operator* $D_t^{N-k_1}$, $k \geqq k_1 > 0$, *on* $\mathscr{D}_\rho$, $0<\rho\leqq\infty$.

Corollary. *The operators* $\partial^2/\partial x^2 + \lambda\partial/\partial t$ $(\lambda \in \mathsf{C})$, $\partial/\partial x$, I $(Ig=g)$, *jointly do not dominate the operator* $\partial/\partial t$ *on* $\mathscr{D}_\rho$, $0<\rho\leqq\infty$.

Let the space of complex-valued functions $L_*=L_*(K)\subseteq L_1=L_1(K)$ be the completion of $C^\infty|K$ by a family of seminorms $\{\gamma_j\}_j$, $j=0, 1, 2,\dots$, which induces a locally convex topology. Let this topology be stronger than the weak convergence in the distribution space $\mathscr{D}'$ (and, as $L_*\subseteq L_1$, then the topology in L_* is stronger than the L_1-topology). Here C^∞ is the space of all complex-valued infinitely differentiable functions on R^n, $n\geqq 2$; $K\subseteq \mathrm{R}^n$ with $\overline{\mathring{K}}=\overline{K}$; $\mathscr{D}'$ is the dual of $\mathscr{D}$, i.e., $\mathscr{D}'$ is the space of all linear continuous functionals on $\mathscr{D}$ (the space of all complex-valued infinitely differentiable functions on R^n with compact supports); L_p, $p=1, 2,\dots$, is the space of all complex-valued measurable functions f on K, for which $|f|^p$ is integrable; L_p, $p=1, 2,\dots$, is examined with its usual norm.

Let A be a constant-coefficient linear differential operator in n variables.

Lemma 2. *For the function $h\in L_*$ let there exist such a sequence (φ_m), $\varphi_m\in C^\infty|K$, that $(\varphi_m)\overset{*}{\to}h$ and $(A\varphi_m)\overset{*}{\to}H$. Here $(g_m)\overset{*}{\to}g$ denotes that $\gamma_j(g_m-g)\to 0$ as $m\to\infty$ for $\forall$ j. If for another sequence (ψ_m), $\psi_m\in C^\infty|K$, we have $(\psi_m)\overset{*}{\to}h$, $(A\psi_m)\overset{*}{\to}M$, then $H=M$ in L_*.*

Proof. The properties of the space L_* assure that $(\varphi_m-\psi_m)\to 0$ in $\mathscr{D}'$. Therefore $A(\varphi_m-\psi_m)\to 0$ *in* $\mathscr{D}'$. So

$$\int A(\varphi_m-\psi_m)(w)\varphi(w)dw=\int(\varphi_m-\psi_m)(w)A^\bullet\varphi(w)dw$$

for $\forall\,\varphi\in\mathscr{D}$, where $A^\bullet=\Sigma(-1)^k a_k D^k$ if $A=\Sigma a_k D^k$. Thus $H=M$ in L_1. However H, $M\in L_*\subseteq L_1$. That is why $H=M$ in L_*. ■

Hence we can give

Definition 3. *For the function $h\in L_*$ let there exist such a sequence (φ_m), $\varphi_m\in C^\infty|K$ with $(\varphi_m)\overset{*}{\to}h$, $(A\varphi_m)\overset{*}{\to}H$. Then H will be called a generalized strong A_* derivative of the function h of L. Schwartz-Sobolev type and it will be denoted by A_*h.*

According to Lemma 2 if there exists such a derivative A_*h for the function $h\in L_*$ it would be unique.

Let the space $W_*^{A_1,\dots,A_m}=W_*^{A_1,\dots,A_m}(K)$ be the completion of $C^\infty|K$ by the family of seminorms $(\pi_j)_j$, $j=0, 1,\dots$, $\pi_j f=\gamma_j f+\Sigma_q\gamma_j A_q f$, where $A_1,\dots,A_m$ are linear constant-coefficient differential operators.

Remarks. 1. Thus $W_*^{A_1,\dots,A_m}$ is the space of all complex-valued functions $\in L_*$ with strong generalized $A_{q,*}$ derivatives in L_*.

2. Further, if $u_q\in L_*$, $q=1,\dots,m$, each solution $h\in L_*$ of the system $A_q h=u_q$, $q=1,\dots, m$, belongs to $W_*^{A_1,\dots,A_m}$.

3. In the case $L_*=C_0$ (R^2) and $A_1,\dots, A_q,\dots, A_m$ with $A_1,\dots, A_q$ – homogeneous of order N in 2 variables and $A_{q+1},\dots, A_m$ – equal to $D^s=\partial^{|s|}/\partial x^s$, $\forall s=(s_1, s_2)$, $|s|<N$, $(x=(x_1, x_2))$, such spaces $W_*^{A_1,\dots,A_m}$, completion in the supnorm, are proposed by K. de Leeuw and H. Mirkil [5]. A particular case in considered also in [6].

Let $W_*^N=W_*^N(K)$ be the Sobolev space of the restrictions on of all complex-valued functions with partial derivatives of order not larger than N in L_*.

Further, let $L_*=L_p=L_p(K)$, $p=1, 2,\dots,\infty$. Here for the importance of the continuity and for conciseness, L_∞ is examined with the supremum norm (see [7]).

Theorem 3. 1. *Let A be a constant-coefficient linear differential operator of order $N \geqq 1$ in $n \geqq 2$ variables. For $1 < p < \infty$ we have $W_p^{A,I} = W_p^N$ if and only if A is elliptic; $W_\infty^{A,I} \subset W_\infty^{N-1}$ if A is elliptic (and only if for $n \geqq 3$).*

2. *Let A be a linear generalized Petrovsky parabolic constant-coefficient operator of order $N \geqq 2$ in $n \geqq 2$ variables. Then $W_\infty^A \not\subset W_\infty^{N-1}$ and moreover $W_\infty^{A,\dots,A^{(s)},\dots,I} \not\subset W_\infty^{N-1}$, where $A^{(s)}$ is an operator derivative of the operator A, $s = (s_1, \dots, s_n)$ (i.e., if $A = \Sigma c_l D^l$, then $A^{(s)} = \Sigma_{l \geqq s} c_l s! \binom{l}{s} D^{l-s}$).*

3. *Let the set α of the linear constant-coefficient differential operators $A_1, \dots, A_m$ be such that if $A \in \alpha$ then each operator derivative $A^{(s)} \in \alpha$. (Such a system α will be called differential-invariant.) In this case, provided K is a compact hypercube, $\overset{\circ}{K} \neq \emptyset$, the space $W_\infty^\alpha = W_\infty^\alpha(K) = W_\infty^{A_1,\dots,A_m}(K)$ is an algebra of type C on K respectively the pointwise multiplication. If $\alpha = (A_q)$ is a parabolic differential-invariant system of order $N \geqq 2$, in variables $n \geqq 2$, $q \geqq 1, \dots, m$, then $W_\infty^\alpha \not\subset W_\infty^{N-1}$.*

If $\alpha = (A_q)$, $q = 1, \dots, m$, is a differential-invariant system of second order in $n \geqq 2$ variables and if $W_\infty^\alpha \not\subset W_\infty^1$, then α is parabolic.

The proof of Theorem 3, point 1, is an almost immediate cosequence from some well-known a priori estimates (see [7]). However this point reveals the difference between the spaces $W_p^{A_1,\dots,A_m}$ and the Sobolev spaces W_p^N.

2. Proofs. We need the following Proposition 4:

Proposition 4. *Let A, F, $B_1, \dots, B_m$ be linear constant-coefficient differential operators in variables (x, t), $x = (x_1, \dots, x_n)$, $n \geqq 1$, of order not larger than N, $N \geqq 2$, with the following properties: the order of A is N and its order respectively $\partial/\partial t$ is $N-k$, $0 < k < N$; the orders of B_q are less than N and their orders respectively $\partial/\partial t$ are less than $N-k$; the order of F is less than N and its order respectively $\partial/\partial t$ is $N - k_1 \geqq N - k$.*

Then the operators A, $B_1, \dots, B_m$ jointly do not dominate F on D.

In its proof we shall use:

Theorem 5 (K. de Leeuw, H. Mirkil [7]). *Let A, $A_1, \dots, A_m$ be linear constant-coefficient differential operators. $A_1, \dots, A_m$ jointly dominate A on $\mathscr{D}$ if and only if there exist such integrable (i.e., with finite total mass), measures $\mu_1, \dots, \mu_m$ such that for their Fourier-Stieltjes transforms $M_1, \dots, M_m$ we have*

$$\sigma A = M_1 \sigma A_1 + \dots + M_m \sigma A_m. \tag{2}$$

Here if B is a linear differential operator $B = \Sigma b_s D^s$, then σB is the full characteristic polynomial of B, i.e., $\sigma B = \Sigma b_s (iX)^s$, $iX = (iX_1, \dots, iX_n)$. Furthermore if (2) holds and if the orders of $A_1, \dots, A_m$ are non larger than N, then the order of A is also not larger than N and

$$A^N = c_1 A_1^N + \dots + c_m A_m^N, \tag{3}$$

where $A_1^N, \ldots, A_m^N$, A^N *are the homogeneous parts of order* N *of the operators* $A_1, \ldots, A_m$, A, *respectively, and where* $c_1, \ldots, c_m$ *are the masses assigned to the origin by* $\mu_1, \ldots, \mu_m$ *respectively.*

Theorem 6 (W. F. Eberlein [8, 7]). *Let* μ *be an integrable measure, let* c *be the assigned mass of* μ *at the origin and* M *be the Fourier-Stieltjes transform of* μ. *Then the constant function* $f(x) \equiv c$ *can be approximated uniformly by* $\pi * M$ *with* π *a probability measure of finite support.*

Proof of Proposition 4. Let assume the contrary. Then from the Leeuw-Mirkil's Theorem 5 it follows that

$$\sigma F = M_0 \sigma A + M_1 \sigma B_1 + \ldots + M_m \sigma B_m \tag{4}$$

for the Fourier-Stieltjes transforms $M_0, M_1, \ldots, M_m$ of suitable integrable measures $\mu_0, \mu_1, \ldots, \mu_m$. In (4) let fix $X = C = (C_1, \ldots, C_n)$, C_q be constants. Let divide the obtained equation by $T^{N-k} \neq 0$ and let $|T| \to \infty$. $M_0, \ldots, M_m$ are Fourier-Stieltjes transforms of integrable measures and hence they are bounded as $|T| \to \infty$ for fixed $X = C$. Thus we receive that $\lim_{|T| \to \infty} |M_0(C, T)|$ exists (may be it is equal to ∞ if $k > k_1$ and the contradiction, so obtained, proves again our assertion in this case $k > k_1$). Moreover, $\lim_{|T| \to \infty} |M_0(C, T)|$ is strictly positive for each C with eventual exceptions of the zeros of a polynomial in variables $C = (C_1, \ldots, C_n)$, which polynomial is $\not\equiv 0$.

The Leeuw-Mirkil's Theorem 5 and the equation (4), yield that $0 = \alpha A^N$, where A^N is the homogeneous part of A of order N and α is the mass, assigned at the origin by the measure μ_0. Thus $\alpha = 0$. The Eberlein's Theorem 6 interprets the constant function α in terms of the Fourier-Stieltjes transform M_0. The function $\alpha \equiv 0$ can be approximated uniformly by $\pi * M_0$, with π — a probability measure of finite support. Hence we obtain a contradiction since $\alpha \equiv 0$, but $\lim_{|T| \to \infty} |M_0(C, T)| > 0$ almost everywhere. Therefore the Proposition 4 is true.

Proof of Theorem 1. An immediate consequence from the Proposition 4 is that the operators $A, B_1, \ldots, B_m$ from Theorem 1 do not jointly dominate $F = D_t^{N-k_1}$ on $\mathscr{D}$. Let now prove that $A, B_1, \ldots, B_m$ do not jointly dominate F on $\mathscr{D}_\rho$, $0 < \rho < \infty$. Evidently if $A, B_1, \ldots, B_m$ jointly dominate F on some $\mathscr{D}_{\rho_0}$, $0 < \rho_0 < \infty$, then $A, B_1, \ldots, B_m$ jointly dominate F on each $\mathscr{D}_\rho$, $0 < \rho < \infty$. Thus let suppose that $A, B_1, \ldots, B_m$ jointly dominate F on each $\mathscr{D}_\rho$, $0 < \rho < \infty$:

If $k > k_1$, the proof of Theorem 1 might be simpler (and nonusing Theorems 5, 6) by choosing any $\varphi(u, v) \in \mathscr{D}_1$, $u = (u_1, \ldots, u_n)$, $\varphi \not\equiv 0$, and examining the assumed inequality for the functions $g(x, t) = \varphi(\alpha x, \beta t)$, $\alpha \geqq 1$, $\beta \geqq 1$, $\alpha x = (\alpha x_1, \ldots, \alpha x_n)$, i.e. examining the inequality

$$\| F g(x, t) \| \leqq \varkappa \{ \| A g(x, t) \| + \Sigma_j \| B_j g(x, t) \| \}, \tag{5}$$

where $\varkappa$ is a corresponding constant for $\mathscr{D}_1$. For a fixed φ, (5) is, respectively α and β, a "polynomial" inequality with coefficients $D_u^k D_v^l \varphi(u, v)$. α and β can increase to infinity remaining g in $\mathscr{D}_1$. Then a necessary requirement for (5) is that the order of F respectively D_t to be not larger than the order of A respectively D_t. Thus, a contradiction is obtained in the case $k > k_1$.

Furthermore, as B_q are of orders less than N and their orders relatively D_t are less than $N-k$, a similar argument proves that it is sufficient to carry out the proof of Theorem 1 for the case $B_q \equiv 0$, $P \equiv 0$.

Thus, let $k=k_1$, $B_q \equiv 0$, $q=1,\dots, m$,

$$A = \sum_{|s|+l_s=N} a_s D^s D_t^{l_s} + D_t^{N-k} \sum_{|s|<k} b_s D^s, \quad l_s < N-k, \quad \Sigma |a_s| \neq 0, \quad \Sigma |b_s| \neq 0.$$

The contrary of Theorem 1 is assumed on each $\mathscr{D}_\rho$, $0<\rho<\infty$. Let $\varkappa_\rho = \min \{\varkappa : \| Ff \| \leqq \varkappa \| Af \|, \forall f \in \mathscr{D}_\rho\}$. Hence for $\varkappa - \varepsilon > 0$, $\varepsilon > 0$, there exists such a function $f_\rho \in \mathscr{D}_\rho$ that

$$\| Ff_\rho \| \leqq \varkappa_\rho \| Af_\rho \|, \quad (\varkappa_\rho - \varepsilon) \| Af_\rho \| < \| Ff_\rho \|. \tag{6}$$

Let the functions $g_\rho(u_1,\dots,u_n, v)$ be determined by $g_\rho(\alpha_1 x_1,\dots,\alpha_n x_n, \beta t) = f_\rho(x_1,\dots,x_n, t)$, where the constants α_j, $\beta>0$, $j=1,\dots, n$.

If $\alpha_j \leqq \rho_0/\rho$, $\forall j$, $\beta \leqq \rho_0/\rho$, we have $g_\rho \in \mathscr{D}_{\rho_0}$ for $\forall \rho$. In this case the last of the inequalities (6) might be transformed in

$$(\varkappa_\rho - \varepsilon) \| A_{x,t} f_\rho \| < \| F_{x,t} f_\rho \| = \beta^{N-k_1} \| F_{u,v} g_\rho \| \leqq \varkappa_{\rho_0} \beta^{N-k_1} \| A_{u,v} g_\rho \|. \tag{7}$$

If $\varphi_\rho \in \mathscr{D}_\rho$ satisfies the inequalities (6), then each function $C_\rho \varphi_\rho$ ($C_\rho \neq 0$ – a constant), also satisfies (6). So we can assume that $\| A_{u,v} g_\rho \| = 1$ for $\forall \rho$. Then it follows from (7) that

$$(\varkappa_\rho - \varepsilon) \| \sum_{|s|+l_s=N} a_s \alpha^s \beta^{l_s - N + k} D_u^s D_v^l g_\rho + D_v^{N-k} \sum_{|s|<k} b_s \alpha^s D_u^s g_\rho \| < \varkappa_{\rho_0}, \tag{8}$$

where $\alpha = (a_1,\dots, \alpha_n)$.

The Theorem 1 is already proved on $\mathscr{D} = \mathscr{D}_\infty$, therefore $\varkappa_\rho \to \infty$ as $\rho \to \infty$.

Since $N-k-l_s > 0$, then we get the wanted contradiction from (8) as $\rho \to \infty$ for $\alpha_j = \rho_0/\rho^{\delta_j}$, $\beta = \rho_0/\rho^\gamma$, with $\delta_j \geqq 1$, $\forall j$, $\gamma > 1$ (and eventually $\delta_j = \delta_j(\rho)$, $\gamma = \gamma(\rho)$)), ρ_0 – fixed.

Proof of Theorem 3. The point 1 is an immediate corollary from the well-known a priori estimates for the elliptic operators (see [7, 9]);

Point 2 is a consequence from Theorem 1: As A is parabolic of order N in $n \geqq 2$ variables, then there exists a linear nondegenerated real transform L of $\mathbb{R}^n$, such that A is transformed in LA in the form (1) in variables $(x_1,\dots, x_{n-1}, t)$. Theorem 1 yields that $W_\infty^{LA} \nsubseteq W_\infty^{N-1}$. Hence $W_\infty^A \nsubseteq W_\infty^{N-1}$ also.

Furthermore, it is evident that each operator derivative $(LA)^{(s)}$ of LA in the transform L does not dominate D_t^{N-1}. Thus $W_\infty^{LA,\dots,LA^{(s)},\dots,I} \nsubseteq W_\infty^{N-1}$. Hence $W_\infty^{A,\dots,A^{(s)},\dots,I} \nsubseteq W_\infty^{N-1}$;

For the proof of point 3 let remind

Definition. *An algebra R of type C of complex-valued functions on a compact K is a Banach algebra of complex-valued functions on K for which* 1) *The norm $\|f\|$ in R of $f \in R$ is equivalent to the norm* $\sup_{w \in K} \{\inf_{g \in R} \{ \|g\|$, *with $g=f$ in some neighbourhood of w*$\}\}$. 2) *R is an algebra without radical.*

The algebras of type C are introduced by G. E. Shilov in [10].

It is proved in [11-14] that the spaces $W^{\alpha}_{\infty}(T)$ of complex-valued functions on the torus $T=\mathsf{R}^n/Z^n$ are algebras of type C on T, where α is differential-invariant, as in Theorem 3, the point 3; Z is the integer lattice.

[11] proves the case $N=1$, $n=2$; [12] proves particular cases when $N=2$, $n=2$; [13] proves the case when $N=1$, $\forall n$. There is no difficulty (see [15]) to extend the method of the proof in the case of $W^{\alpha}_{\infty}(K)$, where K is a compact hypercube in R^n with $\mathring{K}\neq\emptyset$. Thus $W^{\alpha}_{\infty}(K)$ are algebras of type C of complex-valued functions on K.

Remark. Since $\alpha\neq\emptyset$ is differential-invariant, then $I\in\alpha$. Hence the topology in W^{α}_{∞} is stronger than the pointwise convergence. This permits to apply the Closed Graph Theorem to the inclusions $W^{\alpha}_{\infty}\subset W^{N}_{\infty}$.

Now, let α be a parabolic system of order N. Then there exists such a variable x_{j_0} that all operators $A_1,\dots,A_m$ do not dominate jointly the operator $(\partial/\partial x_{j_0})^{N-1}$. Using The Closed Graph Theorem, this involves that $W^{\alpha}_{\infty}\not\subseteq W^{N-1}_{\infty}$.

Finally, let $W^{\alpha}_{\infty}\not\subset W^{1}_{\infty}$, where the differential-invariant space $\alpha=(A_1,\dots,A_m)$ in $n\geqq 2$ variables is of second order. It follows for each linear real nondegenerated transform L of R^n from $W^{\alpha}_{\infty}\not\subset W^{1}_{\infty}$ that still more $W^{L_\alpha}_{\infty}\not\subset W^{1}_{\infty}$, where $L_\alpha=(LA_1,\dots,LA_m)$. Now, let the second order operator A belong to the linear hull of α. We must prove that A is parabolic according to the Definition 1: It follows from $W^{LA}_{\infty}\not\subset W^{1}_{\infty}$ and from the Closed Graph Theorem that the operator LA (for each fixed L) with its operator derivatives jointly do not dominate all $\partial/\partial u_j$, $j=1,\dots,n$, where $(L\mathsf{R}^n)$ $(u_1,\dots,u_n)$. If B is a linear differential operator, let $B^{[2]}$ be the homogeneous part of second order of B. Let L_1 be a real linear non-degenerated transform on R^n, such that $A^{[2]}$ is transformed in

$$L_1A^{[2]}=(L_1A)^{[2]}=\sum_j \varepsilon_j\partial^2/\partial u_j^2+i\sum_{j,l} b_{jl}\partial^2/\partial u_j\partial u_l, \tag{9}$$

where $\varepsilon_j=\pm 1$, 0, and $\beta_{jl}\in\mathsf{R}$. Such L_1 exists: Since $A^{[2]}=\Sigma a_{jl}\partial^2/\partial x_j\partial x_l$ with $a_{jl}=a'_{jl}+ia''_{jl}$, a'_{jl}, $a''_{jl}\in\mathsf{R}$, hence L_1 is a canonical Lagrange transform for the quadratic real form $\Sigma a'_{jl}x_jx_l$.

Since $W^{L_1\alpha}_{\infty}(K)\not\subset W^{1}_{\infty}(K)$ and K may be a compact set $\bar{\mathring{K}}=K$, hence there exists a function $\varphi\in W^{L_1\alpha}_{\infty}(K)$, $\varphi\notin W^{1}_{\infty}(K)$ which φ has not all first partial derivatives continuous on some compact $K_0\subset K$. Further, evidently we may suppose φ to be a real-valued function and that there does not exist its continuous $\partial/\partial u_n\varphi$ on K_0. Some of ε_j in (9) are equal to zero: Since there exist all generalized $(L_1A_q)^{(s)}$, $s\in Z^n_+$, $|s|\leqslant 2$, $q=1,\dots,m$, derivatives in the supremum norm for the functions of $W^{L_1\alpha}_{\infty}$, and since

$$(L_1A)^{(J)}=(\varepsilon_j\partial/\partial u_j+i\Sigma_l\beta_{jl}\partial/\partial u_l), \tag{10}$$

where $J=(0,\dots,0,1,0,\dots,0)$ with 1 on the j-place, hence there exist all $(\varepsilon_j\partial/\partial u_j\varphi)$ which are continuous. Therefore $\varepsilon_n=0$. Further, let r be the number of $\varepsilon_j\neq 0$, so $r<n$. Moreover, since φ is real, hence (9), (10) involve the existence of all

$(\Sigma_l \beta_{jl}\, \partial/\partial u_j\, \varphi)$ generalized derivatives in the supremum norm and still more of $(\sum_{l,(\varepsilon_l=0)} \beta_{jl}\, \partial/\partial u_l)\varphi$. But at least there does not exist a continuous derivative $(\partial/\partial u_n)\,\varphi$ on K_0. Therefore rang $(\beta_{jl})_{(\varepsilon_l=0)} < n-r$. That is why the operator $L_1 A^{[2]}$ can be reduced by a nondegenerated linear real transform $\mathscr{L}$ of $(u_l)_{(\varepsilon_l=0)}$ into the form

$$\mathscr{L} L_1 A^{[2]} = (\mathscr{L} L_1 A)^{[2]} = \Sigma_j\, \varepsilon_j\, \partial^2/\partial u_j^2 + iQ(\partial/\partial u_{j(\varepsilon_j \neq 0)},\quad \partial/\partial w_l),$$

where Q is a quadratic form in less than n variables. Thus the operator A is parabolic. ■

Acknowledgements. [3] has been pointed out to the author by N. Popivanov.

The author is grateful to Professor Raymond G. Ayoub, Professor Frank T. Birtel and Professor Paul T. Bateman for introducing parts of these results to Abstracts of papers presented to the Abstracts of American Mathematical Society, vol. 3, 1982, No 3, 4, 5. This was a strong stimulus for further mathematical investigations.

REFERENCES

1. И. Г. Петровский. О проблеме Коши для систем линейных уравнений с частными производными в области неаналитических функций. *Бюлл. МГУ*, А, **1**, 1938, № 7, 1–72; О некоторых проблемах теории уравнений с частными производными. *Успехи мат. наук*, **1**, 1946, № 3–4, 44–70.
2. С. Д. Эйдельман. Параболические системы. Москва, 1964.
3. М. С. Агранович, М. И. Вишик. Эллиптические граничные задачи с параметром и параболические задачи общего вида. *Успехи мат. наук*, **19**, 1964, № 3, 53–161.
4. С. Д. Ивасишен. Матрицы Грина граничных задач для параболических по И. Г. Петровскому систем общего вида. *Матем. сборник*, **114**, 1981, № 1, 110–166; **114**, 1981, № 4, 523–265.
5. K. de Leeuw, H. Mirkil. Algebras of differentiable functions. *Bull. Amer. Math. Soc.*, **68**, 1962, 411–415.
6. А. С. Мадгерова. Пространства функций с сильными обобщенными производными типа Соболева. *Доклады БАН*, **34**, 1981, № 7, 923–926.
7. K. de Leeuw, H. Mirkil. A priori estimates for differential operators in L_∞ norm. *Illinois J. Math.*, **8**, 1964, No 1, 112–124.
8. W. F. Eberlein. Abstract ergodic theorems and weak almost periodic functions. *Trans. Amer. Math. Soc.*, **67**, 1949, 217–240.
9. H. Triebel. Interpolation theory, function spaces, differential operators. Berlin, 1978.
10. Г. Е. Шилов. О регулярных нормированных кольцах. *Труды мат. ин-та им. Стеклова, АН СССР*, **21**, 1947, 1–99.
11. Г. Е. Шилов. Об однородных кольцах на торе. *Доклады АН СССР*, **82**, 1952, № 5, 681–684.
12. Б. С. Митягин. О некоторых свойствах функций двух переменных. *Вестник МГУ*, **5**, 1959, 137–152.
13. Г. Е. Шилов. О кольце функций на *n*-мерном торе. *Киев. Научные зап. ун-та*, **11**, 1952, № 7, 17–23.
14. А. С. Мадгерова. Об однородных алгебрах функций на торе и их примарных идеалах. – В: Материалы III научной конференции болгарских аспирантов, обучающихся в СССР. Москва, 1978, 532–541.
15. А. С. Мадгерова. О некоторых классах однородных банаховых алгебр. (Диссертация. Москва, МГУ „Ломоносов", 1978.)

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Received 4.5.1986