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BOUNDS ON INVERSE SUM INDEG INDEX OF SUBDIVISION GRAPHS

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ABSTRACT. The inverse sum indeg index $ISI(G)$ of a simple graph G is defined as the sum of the terms $\dfrac{d_G(u)d_G(v)}{d_G(u)+d_G(v)}$ over all edges uv of G, where $d_G(u)$ denotes the degree of a vertex u of G. In this paper, we present several upper and lower bounds on the inverse sum indeg index of subdivision graphs and t-subdivision graphs. In addition, we obtain the upper bounds for inverse sum indeg index of S-sum, S_t-sum, S-product, S_t-product of graphs.

1. Introduction. All the graphs considered in this paper are simple and connected. For vertices $x, y \in V(G)$, the distance between x and y in G, denoted by $d_G(x, y)$, is the length of a shortest (x, y)-path in G. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. For a vertex x in G, the eccentricity $\epsilon(x)$ of x is $max\{d_G(x, y)|y \in V(G)\}$. The minimum eccentricity among the vertices

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of G is the radius of G, denoted by $r(G)$, and the maximum eccentricity is its diameter $d(G)$. A vertex x in G is a central vertex if $\epsilon(x) = r(G)$. A graph G is self-centered if $\epsilon(x) = r(G)$ for all vertices $x \in V(G)$. The *subdivision graph* of G, denoted by $S(G)$ is a graph obtained from G by replacing each edge of G by a path of length 2. The *t-subdivision graph* defined by $S_t(G)$ of G is a graph obtained from G by replacing each edge of G by a path of length $t+1$.

Molecular descriptors, that are results of functions mapping molecule's chemical information into a number [16], have found applications in modeling many physicochemical properties in QSAR and QSPR studies [3, 9]. A particularly common type of molecular descriptors are those that are defined as functions of the structure of the underlying molecular graph, such as the Wiener index [19], the Zagreb indices [6], the Randić index [13] or the Balaban J-index [1]. Damir Vukicević and Marija Gasperov [17] observed that many of these descriptors are defined simply as the sum of individual bond contributions.

Among the 148 discrete Adriatic indices studied in [17], whose predictive properties were evaluated against the benchmark datasets of the Internation Academy of Mathematical Chemistry [10], 20 indices were selected as significant predictors of physicochemical properties. In this connection, Sedlar et al. [14] studied the properties of the inverse sum indeg index, the descriptor that was selected in [17] as a significant predictor of total surface area of octane isomers and for which the extremal graphs obtained with the help of Math. Chem. have a particularly simple and elegant structure. The *inverse sum indeg index* is defined as $ISI(G) = \sum\limits_{uv \in E(G)} \frac{1}{\frac{1}{d_G(u)} + \frac{1}{d_G(v)}} = \sum\limits_{uv \in E(G)} \frac{d_G(u)d_G(v)}{d_G(u) + d_G(v)}$.

Extremal values of inverse sum indeg index across several graph classes, including connected graphs, chemical graphs, trees and chemical trees were determined in [14]. The bounds of a descriptor are important information of a molecular graph in the sense that they establish the approximate range of the descriptor in terms of molecular structural parameters. In [4], some sharp bounds for the inverse sum indeg index of connected graphs are given. The inverse sum indeg index of some nanotubes is computed in [5]. Several upper and lower bounds on the inverse sum indeg index in terms of some molecular structural parameters and relate this index to various well-known molecular descriptors are presented in [12]. In this paper, we present several upper and lower bounds on the inverse sum indeg index of subdivision graphs and t-subdivision graphs. In addition, we obtain the upper bounds for inverse sum indeg index of S-sum, S_t-sum, S-product, S_t-product of graphs.

2. Bounds on ISI Index of Subdivision Graphs. In this section, we obtain the upper and lower bounds for the inverse sum indeg index of subdivision graph and t-subdivision graph of a connected graphs. We denote by Δ and δ the maximum and minimum vertex degrees of G, respectively. The graph G is called a (Δ,δ)-*bidegreed* if whose vertices have degree either Δ or δ with $\Delta \neq \delta$.

The Zagreb indices are amoung the oldest topological indices, and were introduced by Gutman and Trinajstić [6] in 1972. These indices have since been used to study molecular complexity, chirality, ZE-ismerism and hetero-systems. The first and second Zagreb indices of G are denoted by $M_1(G)$ and $M_2(G)$, respectively, and defined as $M_1(G) = \sum\limits_{v\in V(G)} (d_G(v))^2$ and $M_2(G) = \sum\limits_{uv\in E(G)} d_G(u)d_G(v)$. The *inverse degree index* of G, denoted by $ID(G)$ is defined as $ID(G) = \sum\limits_{v\in V(G)} \frac{1}{d_G(v)}$. For any even n, the *cocktail party graph* CP_n is the unique regular graph with n vertices of degree $n-2$, it is obtained from K_n by removing $\frac{n}{2}$ disjoint edges.

Let G be a graph with m edges. By definition of the inverse sum indeg index, we have

$$ISI(S(G)) = \sum_{(x,y)\in E(G)} \left(\frac{2d_G(x)}{d_G(x)+2} + \frac{2d_G(y)}{2+d_G(y)} \right). \tag{1}$$

and

$$\begin{aligned} ISI(S_t(G)) &= \sum_{(x,y)\in E(G)} \left(\frac{2d_G(x)}{d_G(x)+2} + \underbrace{1+1+\ldots+1}_{(t-1)\ times} + \frac{2d_G(y)}{2+d_G(y)} \right) \\ &= \sum_{(x,y)\in E(G)} \left(\frac{2d_G(x)}{d_G(x)+2} + \frac{2d_G(y)}{2+d_G(y)} \right) + (t-1)m \\ &= ISI(S(G)) + (t-1)m. \qquad (2) \end{aligned}$$

One can observe that $ISI(G) < ISI(S(G))$ and $ISI(G) < ISI(S_t(G))$.

Example 1. *Let G be a r-regular graph with n vertices. Then $ISI(S(G)) = \frac{nr^2}{r+2}$ and $ISI(S_t(G)) = \frac{nr(t(r+2)+r-2)}{2}$.*

Theorem 1. *Let G be a graph with n vertices and m edges. Then* $ISI(S(G)) = 4(m-n) + \sum\limits_{x \in V(G)} \dfrac{8}{d_G(x)+2}$.

P r o o f. For each neighbor of x in G, the term $\dfrac{2d_G(x)}{d_G(x)+2}$ appears exactly once in the sum $\sum\limits_{(x,y)\in E(G)} \Big(\dfrac{2d_G(x)}{d_G(x)+2} + \dfrac{2d_G(y)}{2+d_G(y)}\Big)$. Hence

$$
\begin{aligned}
ISI(S(G)) &= \sum_{x \in V(G)} \Big(\underbrace{\frac{2d_G(x)}{d_G(x)+2} + \frac{2d_G(x)}{d_G(x)+2} + \ldots + \frac{2d_G(x)}{d_G(x)+2}}_{d_G(x) \quad times}\Big) \\
&= \sum_{x \in V(G)} \frac{2(d_G(x))^2}{d_G(x)+2} \\
&= \sum_{x \in V(G)} \Big(2d_G(x) - \frac{4d_G(x)}{d_G(x)+2}\Big) \\
&= 4m - \sum_{x \in V(G)} \Big(\frac{4d_G(x)}{d_G(x)+2}\Big) \\
&= 4m - \sum_{x \in V(G)} \Big(4 - \frac{8}{d_G(x)+2}\Big) \\
&= 4(m-n) + \sum_{x \in V(G)} \Big(\frac{8}{d_G(x)+2}\Big).
\end{aligned}
$$

$\square$

Corollary 1. *Let G be a graph with n vertices and m edges. Then* $ISI(S_t(G)) = (t+3)m - 4n + \sum\limits_{x \in V(G)} \dfrac{8}{d_G(x)+2}$. $\square$

Lemma 1. Schweitzer's inequality*[2, 8] Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that for $1 \le i \le n$ holds $m \le x_i \le M$. Then*

$$\Big(\sum_{i=1}^{n} x_i\Big)\Big(\sum_{i=1}^{n} \frac{1}{x_i}\Big) \le \frac{n^2(m+M)^2}{4mM}.$$

Equality holds if and only if $x_1 = x_2 = \ldots = x_n = m = M$ or n is even, $x_1 = x_2 = \ldots = x_{\frac{n}{2}} = m$ and $x_{\frac{n}{2}+1} == x_{\frac{n}{2}+2} = \ldots = x_n = M$, where $m < M$ and $x_1 \le x_2 \le \ldots \le x_n$.

Using above lemma to obtain the following sharp upper bound for the inverse sum indeg index of subdivision graphs.

Theorem 2. *Let G be a graph with n vertices and m edges. Then $ISI(S(G)) \leq 4(m-n) + \dfrac{n^2(\delta+\Delta+4)^2}{(n+m)(\delta+2)(\Delta+2)}$ with equality if and only if G is regular or a (Δ, δ)-bidegreed graph.*

P r o o f. For any vertex x in $V(G)$, we get $\delta + 2 \leq d_G(x) + 2 \leq \Delta + 2$. Also, $\sum\limits_{x \in V(G)} (d_G(x) + 2) = 2(m+n)$. By Schweitzer's inequality, we obtain

$$\sum_{x \in V(G)} \frac{8}{d_G(x)+2} \leq \frac{n^2(\delta+\Delta+4)^2}{(n+m)(\delta+2)(\Delta+2)}.$$

By Theorem 1, we obtain the required inequality.

By Lemma 1, equality holds if and only if $\delta = \Delta$ or $\dfrac{n}{2}$ vertices of G have degree δ and the remaining $\dfrac{n}{2}$ vertices of G have degree Δ, that is, G is regular or a (Δ, δ)-bidegreed. □

Corollary 2. *Let G be a graph with n vertices and m edges. Then $ISI(S_t(G)) \leq (t+3)m - 4n + \dfrac{n^2(\delta+\Delta+4)^2}{(n+m)(\delta+2)(\Delta+2)}$ with equality if and only if G is regular or a (Δ, δ)-bidegreed graph.*

Lemma 2. *Let a and b be real numbers. Then*

$$\frac{1}{a+b} \leq \frac{1}{4}\Big(\frac{1}{a} + \frac{1}{b}\Big)$$

with equality if and only if $a = b$.

Theorem 3. *Let G be a graph with n vertices and m edges. Then $ISI(S(G)) \leq 4m - 3n + 2ID(G)$ with equality if and only if G is the disjoint union of cycles.*

P r o o f. For each vertex $x \in V(G)$, by Lemma 2, we have $\dfrac{1}{d_G(x)+2} \leq \dfrac{1}{4}\Big(\dfrac{1}{d_G(x)} + \dfrac{1}{2}\Big)$ with equality if and only if $d_G(x) = 2$. Hence

$$\sum_{x \in V(G)} \frac{8}{d_G(x)+2} \leq 2 \sum_{x \in V(G)} \frac{1}{d_G(x)} + n = 2ID(G) + n,$$

where $ID(G)$ is the inverse degree index of G.

By Theorem 1, we obtain the required inequality.

Equality holds if and only if each vertex $x \in V(G)$, $d_G(x) = 2$, that is, G is the disjoint union of cycles. $\square$

Corollary 3. *Let G be a graph with n vertices and m edges. Then $ISI(S_t(G)) \leq (t+3)m - 3n + 2ID(G)$ with equality if and only if G is the disjoint union of cycles.*

Theorem 4. *Let G be a graph with n vertices and m edges. If p is the number of pendant vertices of G, then $ISI(S(G)) \geq 4(m-n) + 8\Big(\frac{p}{3} + \frac{n-p}{\Delta+2}\Big)$ with equality if and only if G is regular or a $(\Delta, 1)$-bidegreed graph.*

Proof. One can see that

$$\begin{aligned}
\sum_{x \in V(G)} \frac{8}{d_G(x)+2} &= 8\Big(\underbrace{\frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{3}}_{p \ \ times} + \sum_{x \in V(G),\, d_G(x) > 1} \frac{1}{d_G(x)+2}\Big) \\
&= 8\Big(\frac{p}{3} + \sum_{x \in V(G),\, d_G(x) > 1} \frac{1}{d_G(x)+2}\Big) \\
&\geq 8\Big(\frac{p}{3} + \underbrace{\frac{1}{\Delta+2} + \frac{1}{\Delta+2} + \ldots + \frac{1}{\Delta+2}}_{n-p \ \ times}\Big) \\
&= \frac{8p}{3} + \frac{8(n-p)}{\Delta+2}.
\end{aligned}$$

By Theorem 1, we obtain the required inequality.

Equality holds if and only if for every non-pendant vertex $x \in V(G)$, $d_G(x) = \Delta$. If $p = 0$, then for every vertex $x \in V(G)$, $d_G(x) = \Delta$, that is, G is regular, where $2 \leq \Delta \leq n-1$. Assume $p > 0$. If there is no non-pendant vertex in G, then $G \cong K_2$ and otherwise, G is $(\Delta, 1)$-bidegreed. $\square$

Corollary 4. *Let G be a graph with n vertices and m edges. If p is the number of pendant vertices of G, then $ISI(S_t(G)) \geq (t+3)m - 4n + 8\Big(\frac{p}{3} + \frac{n-p}{\Delta+2}\Big)$ with equality if and only if G is regular or a $(\Delta, 1)$-bidegreed graph.*

Corollary 5. *Let G be a graph with n vertices and m edges. If G has no pendant vertices, then $ISI(S(G)) \geq 4(m-n) + \frac{8n}{\Delta+2}$ and $ISI(S_t(G)) \geq (t+3)m - 4n + \frac{8n}{\Delta+2}$. The equality holds for both cases if and only if G is Δ-regular, where $2 \leq \Delta \leq n-1$.*

Let $d_i(x)$ be the number of vertices at distance i from the vertex x in G, that is, $d_i(x) = |\{y \,|\, d_G(x,y) = i\}|$.

Theorem 5. *Let G be a graph with n vertices and m edges. Then*

$$ISI(S(G)) \geq 4\Big(m - \frac{n(n-r(G))}{n-r(G)+2}\Big) \tag{3}$$

with equality if and only if $G \cong K_n$ or $G \cong CP_n$.

Proof. Since $d_i(x)$ is the number of vertices at distance i from the vertex x in G. One can observe that $d_G(x) \leq n - \epsilon(x)$ with equality if and only if $\epsilon(x) = 1$ and $d_G(x) = n-1$ or $\epsilon(x) \geq 2$ and $d_2(x) = d_3(x) = \ldots = d_{\epsilon(x)}(x) = 1$. Thus for every vertex $x \in V(G)$, we obtain

$$\frac{8}{d_G(x)+2} \geq \frac{8}{n-\epsilon(x)+2} \geq \frac{8}{n-r(G)+2}.$$

By Theorem 1, we obtain the required result.

Suppose that equality holds in (3). Then G is self-centered and for every vertex $x \in V(G)$, equality holds in $d_G(x) \leq n - \epsilon(x)$. If $\epsilon(x) = 1$ for some vertex $x \in V(G)$, then $d_G(x) = n-1$ and $\epsilon(y) \leq 2$ for all vertices $x \neq y$. Since G is self-centered, $\epsilon(x) = 1$ for all vertices $x \in V(G)$. Thus $G \cong K_n$.

Now, suppose that $\epsilon(x) \geq 2$ for all vertices $x \in V(G)$. If $\epsilon(x) \geq 3$ for some vertex y, then $d(G) = 3$ (otherwise, there exist at least two neighbors at distance 2 for the central vertex) and $G \cong P_4$, a path on 4 vertices. This contradicts that G is self-centered. So, $\epsilon(x) = 2$ for all vertices $x \in V(G)$ and then $d_G(x) = n-2$ for all vertices $x \in V(G)$. It gives $G \cong CP_n$. □

Theorem 6. *Let G be a graph with m edges. Then $ISI(S(G)) \leq \dfrac{M_1(G)}{4} + m$ with equality if and only if G is the disjoint union of cycles.*

Proof. For any vertex $x \in V(G)$, we obtain

$$\frac{2d_G(x)}{d_G(x)+2} \leq \frac{2+d_G(x)}{4}$$

with equality if and only if $d_G(x) = 2$. Thus by equation (1), we have

$$ISI(S(G)) \leq \sum_{(x,y)\in E(G)} \Big(\frac{2+d_G(x)}{4} + \frac{2+d_G(y)}{4}\Big) = \frac{1}{4}(M_1(G) + 4m).$$

Equality holds if and only if for every vertex $x \in V(G)$, $d_G(x) = 2$. This implies G is a disjoint union of cycles. □

Corollary 6. *Let G be a graph with m edges. Then $ISI(S_t(G)) \leq \frac{M_1(G)}{4} + tm$ with equality if and only if G is the disjoint union of cycles.*

Lemma 3. *(Cauchy-Schwarz inequality)*

Let $X = (x_1, x_2 \dots x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be two sequences of real numbers. Then $\Big(\sum_{i=1}^{n} x_i y_i\Big)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$ with equality if and only if the sequences X and Y are proportional, i. e., there exists a constant c such that $x_i = cy_i$, for each $1 \leq i \leq n$.

As a special case of the Cauchy-Schwarz inequality, when $y_1 = y_2 = \dots = y_n$, we get the following result.

Corollary 7. *Let $x_1, x_2, \dots, x_n$ be real numbers. Then $\Big(\sum_{i=1}^{n} x_i\Big)^2 \leq n \sum_{i=1}^{n} x_i^2$ with equality if and only if $x_1 = x_2 = \dots = x_n$.*

Theorem 7. *Let G be a graph with n vertices and m edges. Then $ISI(S(G)) \geq \frac{4(m^2 - n^2) + 4n^2}{m + n}$ with equality if and only if G is regular.*

Proof. By Cauchy-Schwarz inequality, we get

$$\left(\sum_{x \in V(G)} \big(d_G(x) + 2\big)\right)\left(\sum_{x \in V(G)} \frac{1}{d_G(x) + 2}\right) \geq \left(\sum_{x \in V(G)} \sqrt{d_G(x) + 2}\frac{1}{\sqrt{d_G(x) + 2}}\right)^2$$

with equality if and only if all the $d_G(x)$'s are equal.

Moreover, $\sum_{x \in V(G)} \big(d_G(x) + 2\big) = 2(m + n)$. Thus

$$\sum_{x \in V(G)} \frac{1}{d_G(x) + 2} \geq \frac{n^2}{2(m + n)}.$$

By Theorem 1, we obtain the required inequality.

Equality holds if and only if all the $d_G(x)$'s are equal. This implies G is regular. □

Corollary 8. *Let G be a graph with n vertices and m edges. Then $ISI(S_t(G)) \geq \frac{4(m^2 - n^2) + 4n^2}{m + n} + (t + 1)m$ with equality if and only if G is regular.*

Let G be a graph with n vertices and m edges. If $m = n-1$, n and $n+1$ then G is called a tree, unicyclic and bicyclic graphs, respectively.

Corollary 9. *Let G be a tree on n vertices. Then $ISI(S(G)) \geq \dfrac{4(n-1)^2}{2n-1}$ and $ISI(S_t(G)) \geq \dfrac{4(n-1)^2}{2n-1} + (n-1)(t+1)$.*

Corollary 10. *Let G be a unicyclic graph on n vertices. Then $ISI(S(G)) \geq 2n$ and $ISI(S_t(G)) \geq n(t+3)$.*

Corollary 11. *Let G be a bicyclic graph on n vertices. Then $ISI(S(G)) \geq \dfrac{4(n+1)^2}{2n+1}$ and $ISI(S_t(G)) \geq \dfrac{4(n+1)^2}{2n+1} + (n+1)(t+1)$.*

Lemma 4. *[11] Let f be a convex function on the interval I and $x_1, x_2, \ldots, x_n \in I$. Then $\dfrac{x_1 + x_2 + \ldots + x_n}{n} \leq \dfrac{f(x_1) + f(x_2) + \ldots, f(x_n)}{n}$ with equality if and only if $x_1 = x_2 = \ldots = x_n$.*

Theorem 8. *Let G be a graph on m edges. Then $ISI(S(G)) > \dfrac{4\delta\, m - \delta\, M_1(G)}{2}$.*

P r o o f. For any vertex x in G, $d_G(x) \geq \delta$. By the definition of inverse sum indeg index of the subdivision graph of G, we have

$$\begin{aligned}
ISI(S(G)) &= \sum_{xy \in E(G)} \left(\frac{2d_G(x)}{d_G(x)+2} + \frac{2d_G(y)}{2+d_G(y)} \right) \\
&\geq \sum_{xy \in E(G)} \left(\frac{2\delta}{d_G(x)+2} + \frac{2\delta}{2+d_G(y)} \right).
\end{aligned}$$

Let $f(x) = \dfrac{1}{x}$. Since f is a convex function on $(0, +\infty)$, by Jensen's inequality, for any edge $xy \in V(G)$, we obtain

$$\frac{2}{d_G(x)+2} + \frac{2}{2+d_G(y)} \geq \frac{8}{4+d_G(x)+d_G(y)}$$

with equality if and only if $d_G(x) = d_G(y)$. Hence

$$ISI(S(G)) \geq \sum_{xy \in E(G)} \left(\frac{8\delta}{4+d_G(x)+d_G(y)} \right)$$

$$= 2\delta \sum_{xy\in E(G)} \left(1+\frac{d_G(x)+d_G(y)}{4}\right)^{-1}$$

By Bernoulli's inequality, we have

$$\begin{aligned} ISI(S(G)) &> 2\delta \sum_{xy\in E(G)} \left(1-\frac{d_G(x)+d_G(y)}{4}\right) \\ &= 2\delta\, m - \frac{\delta}{2}\sum_{xy\in E(G)} (d_G(x)+d_G(y)) \end{aligned}$$

By the definition of the firat Zagreb index of G, we get

$$ISI(S(G)) > \frac{4\delta\, m - \delta\, M_1(G)}{2}.$$

□

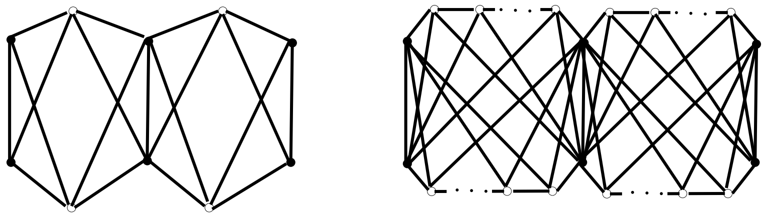
Corollary 12. *Let G be a graph with m edges. Then*

$$ISI(S_t(G)) > (2\delta + t - 1)m - \frac{\delta\, M_1(G)}{2}.$$

3. ISI Index of S and S_t-products of Graphs.

The *S-product* of G_1 and G_2, denoted by $G_1[G_2]_S$, is defined by $S(G_1)[G_2] - E^*$, where $E^* = \{(x,y_1)(x,y_2) \in E(S(G_1)[G_2]) | x \in V(S(G_1)) - V(G_1),\, y_1y_2 \in E(G_2)\}$, that is, $G_1[G_2]_S$ is a graph with the set of vertices either $[x_1 = x_2 \in V(G_1)$ and $y_1y_2 \in E(G_2)]$ or $[x_1x_2 \in E(G_1)$ and $y_1, y_2 \in V(G_2)]$. The *S_t-product* of G_1 and G_2, denoted by $G_1[G_2]_{S_t}$, is defined by $S_t(G_1)[G_2] - E^*$, where $E^* = \{(x,y_1)(x,y_2) \in E(S_t(G_1)[G_2]) \mid x \in V(S_t(G_1)) - V(G_1),\, y_1y_2 \in E(G_2)\}$, that is, $G_1[G_2]_{S_t}$ is a graph with the set of vertices either $[x_1 = x_2 \in V(G_1)$ and $y_1y_2 \in E(G_2)]$ or $[x_1x_2 \in E(G_1)$ and $y_1, y_2 \in V(G_2)]$. One can observe that $G_1[G_2]_{S_t}$ has $|V(G_2)|$ copies of the graph $S_t(G_1)$ and we can label these copies by vertices of G_2. The vertices in each copy we denote two types of vertices, such as the vertices in $V(G_1)$ (black vertices) and the vertices in $V(S_t(G_1)) - V(G_1)$ (white vertices). The S and S_t-products of P_3 and P_2 are shown in Figure 1.

Theorem 9. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $ISI(G_1[G_2]_S) \leq \frac{n_1 ISI(G_2)}{4} + \frac{M_1(G_1)}{2}\left(\frac{n_2^2 H(G_2)}{4} + \frac{n_2^3}{8} + n_2^3 ID(G_2)\right) + \frac{n_1 M_1(G_2)}{8} + \frac{M_2(G_2) ID(G_2)}{8n_2} + \frac{n_2(4m_1m_2 + n_1m_2 + m_1n_2^2)}{4}$.



$P_3[P_2]_S$ $\qquad$ $P_3[P_2]_{S_t}$

Fig. 1. The S and S_t-products of P_3 and P_2

P r o o f. Let $\{x_1, x_2, \ldots, x_{n_1}\}$ and $\{y_1, y_2, \ldots, y_{n_2}\}$ be the vertex sets of G_1 and G_2, respectively. From the definition of inverse sum indeg index and the structure of the graph $G_1[G_2]_S$, we have

$$\begin{aligned} ISI(G_1[G_2]_S) &= \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1[G_2]_s)} \frac{d_{G_1[G_2]_S}((x_1,y_1))d_{G_1[G_2]_S}((x_2,y_2))}{d_{G_1[G_2]_S}((x_1,y_1))+d_{G_1[G_2]_S}((x_2,y_2))} \\ &= \sum_{x_1=x_2\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \frac{d_{G_1[G_2]_S}((x_1,y_1))d_{G_1[G_2]_S}((x_2,y_2))}{d_{G_1[G_2]_S}((x_1,y_1))+d_{G_1[G_2]_S}((x_2,y_2))} \\ &+ \sum_{x_1x_2\in E(S(G_1))} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \frac{d_{G_1[G_2]_S}((x_1,y_1))d_{G_1[G_2]_S}((x_2,y_2))}{d_{G_1[G_2]_S}((x_1,y_1))+d_{G_1[G_2]_S}((x_2,y_2))} \\ &= A_1 + A_2, \end{aligned} \tag{4}$$

where A_1 and A_2 are the sums of the terms, in order.

We shall calculate A_1 and A_2 of (4) separately.

First we calculate the sum

$$A_1 = \sum_{x_1=x_2\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \frac{d_{G_1[G_2]_S}((x_1,y_1))d_{G_1[G_2]_S}((x_2,y_2))}{d_{G_1[G_2]_S}((x_1,y_1))+d_{G_1[G_2]_S}((x_2,y_2))}.$$

For each vertex (x_i, y_j) in $G_1[G_2]_S$, the degree of (x_i, y_j) is $n_2 d_{G_1}(x_i) + d_{G_2}(y_j)$. Thus

$$A_1 = \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \frac{(n_2 d_{G_1}(x_1) + d_{G_2}(y_1))(n_2 d_{G_1}(x_1) + d_{G_2}(y_2))}{2n_2 d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2))}.$$

By Jensen's inequality, we have

$$\frac{1}{2n_2 d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2))} \leq \Big(\frac{1}{8n_2 d_{G_1}(x_1)} + \frac{1}{4d_{G_2}(y_1) + d_{G_2}(y_2)}\Big)$$

with equality if and only if $2n_2d_{G_1}(x_1) = d_{G_2}(y_1) + d_{G_2}(y_2)$. Thus

$$\begin{aligned} A_1 &\leq \frac{1}{4}\sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\left(\frac{n_2d_{G_1}(x_1)}{2} + \frac{d_{G_2}(y_1)+d_{G_2}(y_2)}{2} + \frac{d_{G_2}(y_1)+d_{G_2}(y_2)}{2n_2d_{G_1}(x_1)}\right) \\ &\quad + \frac{1}{4}\sum_{x_1\in V(G_1)}\sum_{y_1y_2\in E(G_2)}\left(\frac{n_2^2d_{G_1}(x_1)^2}{d_{G_2}(y_1)+d_{G_2}(y_2)} + n_2d_{G_1}(x_1) + \frac{d_{G_2}(y_1)d_{G_2}(y_2)}{d_{G_2}(y_1)+d_{G_2}(y_2)}\right) \\ &= \frac{1}{4}\left(3n_2m_1m_2 + \frac{n_1M_1(G_2)}{2} + \frac{M_2(G_2)ID(G_1)}{2n_2} + \frac{n_2^2M_1(G_1)H(G_2)}{2} + n_1ISI(G_2)\right). \end{aligned}$$

Next we find the value of the sum A_2.

$$\begin{aligned} A_2 &= \sum_{x_1x_2\in E(S(G_1))}\sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\frac{d_{G_1[G_2]_s}((x_1,y_1))d_{G_1[G_2]_s}((x_2,y_2))}{d_{G_1[G_2]_s}((x_1,y_1))+d_{G_1[G_2]_s}((x_2,y_2))} \\ &= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1\in V(G_1),e\in E(G_1)\\ x_1 \text{ and } e \text{ are incident in } G_1}}\frac{d((x_1,y_1))d((e,y_2))}{d((x_1,y_1))+d((e,y_2))} \\ &= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{\substack{x_1\in V(G_1),e\in E(G_1)\\ x_1 \text{ and } e \text{ are incident in } G_1}}\frac{(n_2d_{G_1}(x_1)+d_{G_2}(y_1))2n_2}{n_2d_{G_1}(x_1)+d_{G_2}(y_1)+2n_2} \\ &= \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{x\in V(G_1)} d_{G_1}(x_1)\frac{\left(2n_2^2d_{G_1}(x_1)+2n_2d_{G_2}(y_1)\right)}{n_2(d_{G_1}(x_1)+2)+d_{G_2}(y_1)} \end{aligned}$$

One can see that

$$\frac{1}{n_2(d_{G_1}(x_1)+2)+d_{G_2}(y_1)} \leq \frac{1}{16n_2d_{G_1}(x_1)} + \frac{1}{32n_2} + \frac{1}{4d_{G_2}(y_1)}.$$

Thus

$$\begin{aligned} A_2 &\leq \sum_{y_1\in V(G_2)}\sum_{y_2\in V(G_2)}\sum_{x\in V(G_1)}\left(\begin{array}{r}\frac{n_2d_{G_1}(x_1)}{8} + \frac{n_2d_{G_1}(x_1)^2}{16} + \frac{d_{G_1}(x_1)^2}{2d_{G_2}(y_1)} \\ + \frac{d_{G_2}(y_1)}{8} + \frac{d_{G_1}(x_1)d_{G_2}(y_1)}{16}\end{array}\right) \\ &= \frac{n_2^3m_1}{4} + \frac{n_2^3M_1(G_1)}{16} + \frac{n_2^3M_1(G_1)ID(G_2)}{2} + \frac{n_1n_2m_2}{4} + \frac{n_2m_1m_2}{4}. \end{aligned}$$

From A_1 and A_2, we get the desired result. $\square$

Theorem 10. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $ISI(G_1[G_2]_{S_t}) \leq \frac{n_1 ISI(G_2)}{4} + \frac{M_1(G_1)}{2}\Big(\frac{n_2^2 H(G_2)}{4} + \frac{n_2^3}{8} + n_2^3 ID(G_2)\Big) + \frac{n_1 M_1(G_2)}{8} + \frac{M_2(G_2) ID(G_2)}{8n_2} + \frac{n_2(4m_1m_2 + n_1m_2 + m_1n_2^2)}{4} + n_2^2(t-1)m_1.$

Proof. Let $\{x_1, x_2, \ldots, x_{n_1}\}$ and $\{y_1, y_2, \ldots, y_{n_2}\}$ be the vertex sets of G_1 and G_2, respectively. From the definition of ISI index and the structure of the graph $G_1[G_2]_{S_t}$, we have

$$\begin{aligned}
ISI(G_1[G_2]_{S_t}) &= \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1[G_2]_{S_t})} \frac{d_{G_1[G_2]_{S_t}}((x_1,y_1))d_{G_1[G_2]_{S_t}}((x_2,y_2))}{d_{G_1[G_2]_{S_t}}((x_1,y_1)) + d_{G_1[G_2]_{S_t}}((x_2,y_2))} \\
&= \sum_{x_1=x_2\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \frac{d_{G_1[G_2]_{S_t}}((x_1,y_1))d_{G_1[G_2]_{S_t}}((x_2,y_2))}{d_{G_1[G_2]_{S_t}}((x_1,y_1)) + d_{G_1[G_2]_{S_t}}((x_2,y_2))} \\
&+ \sum_{x_1x_2\in E(S(G_1))} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \frac{d_{G_1[G_2]_{S_t}}((x_1,y_1))d_{G_1[G_2]_{S_t}}((x_2,y_2))}{d_{G_1[G_2]_{S_t}}((x_1,y_1)) + d_{G_1[G_2]_{S_t}}((x_2,y_2))} \\
&= A_1 + A_2, \qquad (5)
\end{aligned}$$

where A_1 and A_2 are the sums of the terms, in order.

Similarly to the proof of Theorem 9, we get

$$A_1 \leq \frac{1}{4}\left(\begin{array}{r} 3n_2m_1m_2 + \frac{n_1M_1(G_2)}{2} + \frac{M_2(G_2)ID(G_1)}{2n_2} \\ + \frac{n_2^2 M_1(G_1)H(G_2)}{2} + n_1 ISI(G_2) \end{array}\right)$$

$$\begin{aligned}
A_2 &= \sum_{\substack{x_1x_2\in E(S_t(G_1)),\\ x_1\in V(G_1),\\ x_2\in V(S_t(G_1))-V(G_1)}} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \frac{d_{G_1[G_2]_s}((x_1,y_1))d_{G_1[G_2]_s}((x_2,y_2))}{d_{G_1[G_2]_s}((x_1,y_1)) + d_{G_1[G_2]_s}((x_2,y_2))} \\
&+ \sum_{\substack{x_1x_2\in E(S_t(G_1)),\\ x_1,x_2\in V(S_t(G_1))-V(G_1)}} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} \frac{d_{G_1[G_2]_s}((x_1,y_1))d_{G_1[G_2]_s}((x_2,y_2))}{d_{G_1[G_2]_s}((x_1,y_1)) + d_{G_1[G_2]_s}((x_2,y_2))} \\
&= A_2' + A_2'',
\end{aligned}$$

where A_2' and A_2'' are the sums of the terms, in order.

By a similar argument of Theorem 9, we get

$$A_2' \leq \frac{n_2^3 m_1}{4} + \frac{n_2^3 M_1(G_1)}{16} + \frac{n_2^3 M_1(G_1) ID(G_2)}{2} + \frac{n_1n_2m_2}{4} + \frac{n_2m_1m_2}{4}.$$

In addition,

$$\begin{aligned} A_2'' &= \sum_{\substack{x_1x_2\in E(S_t(G_1)),\\ x_1,x_2\in V(S_t(G_1))-V(G_1)}} \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} (1) \\ &= \sum_{y_1\in V(G_2)} \sum_{y_2\in V(G_2)} (m_1(t-1)) \\ &= m_1n_2^2(t-1). \end{aligned}$$

From A_1 and A_2, we obtain the desired result. □

4. ISI **Index of** S **and** S_t**-sums of Graphs.** Let G_1 and G_2 be two graphs. The *S-sum* $G_1+_SG_2$ is a graph with vertex set $(V(G_1)\bigcup E(G_1))\times V(G_2)$ in which two vertices (u_1,v_2) and (u_2,v_2) of $G_1+_SG_2$ are adjacent if and only if $[u_1=u_2\in V(G_1)\wedge v_1v_2\in E(G_2)]$ or $[v_1=v_2\in V(G_1)\wedge u_1u_2\in E(S(G))]$. The *$S_t$-sum* $G_1+_{S_t}G_2$ is a graph with vertex set $(V(G_1)\bigcup E(G_1))\times V(G_2)$ in which two vertices (u_1,v_2) and (u_2,v_2) of $G_1+_{S_t}G_2$ are adjacent if and only if $[u_1=u_2\in V(G_1)\wedge v_1v_2\in E(G_2)]$ or $[v_1=v_2\in V(G_1)\wedge u_1u_2\in E(S_t(G))]$. The S and S_t sums of the graphs P_3 and P_2 are shown in Figure 2.

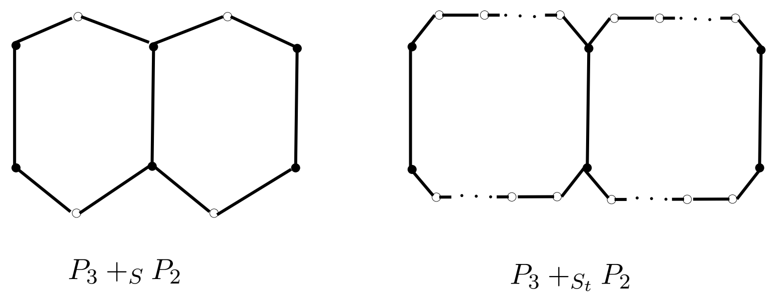


Fig. 2. The S and S_t-sums of P_3 and P_2

Theorem 11. *Let G_i be a graph with n_i vertices and m_i edges, $i=1,2$. Then* $ISI(G_1+_SG_2)\leq \dfrac{n_1ISI(G_2)}{4}+\dfrac{M_1(G_1)(H(G_2)+8ID(G_2)+8n_2)}{8}+\dfrac{n_1M_1(G_2)}{8}+ m_1M_2(G_2)+\dfrac{19m_1m_2+4n_1m_2+8m_1n_2}{4}$.

Proof. Let $\{x_1,x_2,\ldots,x_{n_1}\}$ and $\{y_1,y_2,\ldots,y_{n_2}\}$ be the vertex sets of G_1 and G_2, respectively. From the definition of ISI index and the structure of the graph $G_1 +_S G_2$, we have

$$
\begin{aligned}
ISI(G_1 +_S G_2) &= \sum_{(x_1,y_1)(x_2,y_2)\in E(G_1+_S G_2)} \frac{d_{G_1+_S G_2}((x_1,y_1))d_{G_1+_S G_2}((x_2,y_2))}{d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_2,y_2))} \\
&= \sum_{x_1=x_2\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \frac{d_{G_1+_S G_2}((x_1,y_1))d_{G_1+_S G_2}((x_2,y_2))}{d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_2,y_2))} \\
&\quad + \sum_{x_1x_2\in E(S(G_1))} \sum_{y_1\in V(G_2)} \frac{d_{G_1+_S G_2}((x_1,y_1))d_{G_1+_S G_2}((x_2,y_2))}{d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_2,y_2))} \\
&= A_1 + A_2, \qquad (6)
\end{aligned}
$$

where A_1 and A_2 are the sums of the terms, in order.

We shall calculate A_1 and A_2 of (6) separately.

First we calculate the sum A_1. For each vertex (x_i, y_j) in $G_1 +_S G_2$, the degree of (x_i, y_j) is $d_{G_1}(x_i) + d_{G_2}(y_j)$. Thus

$$
\begin{aligned}
A_1 &= \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \frac{(d_{G_1}(x_1) + d_{G_2}(y_1))(d_{G_1}(x_1) + d_{G_2}(y_2))}{2d_{G_1}(x_1) + (d_{G_2}(y_1) + d_{G_2}(y_2))} \\
&\le \frac{1}{4} \sum_{x_1\in V(G_1)} \sum_{y_1y_2\in E(G_2)} \Big((d_{G_1}(x_1) + d_{G_2}(y_1))(d_{G_1}(x_1) + d_{G_2}(y_2))\Big) \\
&\quad \left(\frac{1}{2d_{G_1}(x_1)} + \frac{1}{(d_{G_2}(y_1) + d_{G_2}(y_2))}\right) \\
&= \frac{3m_1m_2}{4} + \frac{n_1M_1(G_2)}{8} + m_1M_2(G_2) + \frac{M_1(G_1)H(G_2)}{8} + \frac{n_1ISI(G_2)}{4}.
\end{aligned}
$$

Next we find the value of the sum A_2.

$$
\begin{aligned}
A_2 &= \sum_{x_1x_2\in E(S(G_1))} \sum_{y_1\in V(G_2)} \frac{d_{G_1+_S G_2}((x_1,y_1))d_{G_1+_S G_2}((x_2,y_2))}{d_{G_1+_S G_2}((x_1,y_1)) + d_{G_1+_S G_2}((x_2,y_2))} \\
&= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1\in V(G_1),\, e\in E(G_1) \\ x_1 \text{ and } e \text{ are incident in } G_1}} \frac{(d_{G_1}(x_1) + d_{G_2}(y_1))d_{S(G_1)}(x_2)}{d_{S(G_1)}(x_1) + d_{S(G_1)}(x_2) + d_{G_2}(y_1)} \\
&= \sum_{y_1\in V(G_2)} \sum_{\substack{x_1\in V(G_1),\, e\in E(G_1) \\ x_1 \text{ and } e \text{ are incident in } G_1}} \frac{2(d_{G_1}(x_1) + d_{G_2}(y_1))}{2 + d_{G_1}(x_1) + d_{G_2}(y_1)} \\
&= \sum_{y_1\in V(G_2)} \sum_{x_1\in V(G_1)} \frac{2d_{G_1}(x_1)(d_{G_1}(x_1) + d_{G_2}(y_1))}{2 + d_{G_1}(x_1) + d_{G_2}(y_1)}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{y_1\in V(G_2)}\sum_{x_1\in V(G_1)} 2d_{G_1}(x_1)(d_{G_1}(x_1)+d_{G_2}(y_1))\left(\frac{1}{d_{G_1}(x_1)+1}+\frac{1}{d_{G_2}(y_1)+1}\right)\\
&\leq \frac{1}{4}\sum_{y_1\in V(G_2)}\sum_{x_1\in V(G_1)} 2d_{G_1}(x_1)(d_{G_1}(x_1)+d_{G_2}(y_1))\left(\frac{1}{d_{G_1}(x_1)}+1\right)\\
&\quad+\frac{1}{4}\sum_{y_1\in V(G_2)}\sum_{x_1\in V(G_1)} 2d_{G_1}(x_1)(d_{G_1}(x_1)+d_{G_2}(y_1))\left(\frac{1}{d_{G_2}(y_1)}+1\right)\\
&= M_1(G_1)(n_2+ID(G_2))+2m_1n_2+m_2n_1+4m_1m_2.
\end{aligned}
$$

From A_1 and A_2 we get the desired result. □

A similar proof of Theorem 11, we obtain the following theorem.

Theorem 12. *Let G_i be a graph with n_i vertices and m_i edges, $i = 1, 2$. Then* $ISI(G_1 +_{S_t} G_2) \leq \frac{n_1 ISI(G_2)}{4} + \frac{M_1(G_1)(H(G_2)+8ID(G_2)+8n_2)}{8} + \frac{n_1M_1(G_2)}{8} + m_1M_2(G_2) + \frac{19m_1m_2+4n_1m_2+8m_1n_2}{4} + n_2(t-1)m_1.$

5. Conclusion. In this article, several number of upper and lower bounds for inverse sum indeg index of subdivision of some class of graphs are investigated.

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