Institute of Mathematics and Informatics

Bulgarian Academy of Sciences

MATHEMATICS EDUCATION, WESTERN AND EASTERN TEACHING APPROACHES COMBINED WITH ARTS

Ph.D. Thesis

of

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TABLE OF CONTENTS

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Starting from my very first visit to the Institute of Mathematics and Informatics, Bulgarian Academy of Science, Sofia, Bulgaria, I have found myself immersed in an atmosphere full of a high professional level. These factors were determinative to start the first passes in a new research field.

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Special gratitude to my other supervisor Professor Evgenia Sendova who was the first person who has conducted me through the real experiences combining mathematics and arts as a reinforcement of the teaching and learning processes. Her passion for her job made me more motivated and enthusiastic to reach the expected objectives as well as to know Bulgarian culture better.

Massive thanks also to the Scientific Council of the Institute of Mathematics and Informatics.

INTRODUCTION

According to worldwide surveys, such as PISA[1] and TIMSS[2], Western students often lack mathematical competences, while Asian learners reach high performance.[3] Mathematics subject mean scores, obtained, especially, by the European students, are below the OECD average. Since Math literacy (also known as numeracy) enables students to contribute effectively in actual society, improving their employment prospects, the European Ministers set the target to reduce the average of students with difficulties in reading, mathematics, and science to reach the number fewer than 15% by 2020.

The research intends to exploit the potentiality emerging from a creative integration between Western and Eastern learning approaches combined with the use of the arts. This will allow students to improve and develop their learning skills with a creative and innovative study through appropriate development of the creativity-based approach in learning and problem-solving skills.

The designing of this innovative teaching and learning approach in the field of mathematics was developed starting from the topics already defined in the PISA, IEA[4] TIMSS and national surveys aiming to the development of students' learning skills with a specific focus on the discovery of the connections between mathematics and reality, especially, arts, such as painting, music, dance, theatre, etc.

Moreover, it aimed at approaching the students to the study of scientific subjects through the combination of both meaningful and mastery learning. This allowed the development of a more effective Educational and Training environment for teachers and their students who have benefited from the use of more attractive and fun pedagogical tools in the study of science, mainly mathematics.

The research activities intended to reach the following objectives:

- Exploiting the possibility to find a combination between Western and Eastern approaches (mainly Singapore's method) in mathematics teaching.
- Finding the result of this integration in the arts.

[1] The *Programme for International Student Assessment* (PISA) is a triennial international survey which aims to evaluate education systems worldwide by testing the skills and knowledge of 15-year-old students. In 2015 over half a million students, representing 28 million 15-year-olds in 72 countries and economies, took the internationally agreed two-hour test. Students were assessed in science, mathematics, reading, collaborative problem solving and financial literacy.

[2] *TIMSS* stands for *Trends in International Mathematics and Science Study*.

[3] OECD (2018). PISA, *PISA 2015 results in focus*. Accessible: http://www.oecd.org/pisa/test/ [Retrieved online 20/04/2020]

[4] *IEA* stands for *International Association for the Evaluation of Educational Achievement.*

Demonstrating the objectives above encourage meaningful learning in the development of the research skills to improve the school performance (short-term objective) and a higher interest towards a future perspective in the mathematics field (long-term objective).

This thesis is structured into five chapters.

Chapter 1 exposes the “General presentation of the problem” and explains the rationale which has inspired the definition of the model and the approach.

Moreover, the main tasks to be carried out to reach the research objectives are described.

Chapter 2 – “Learning and Teaching approaches: the state of the art” presents an overview of the current status quo of teaching and learning approaches in mathematics education. It explores the existing learning methods used in both the Western and the Eastern approaches underlining the ones which use the “art” as a teaching tool in mathematics education.

Chapter 3 describes the reference theoretical model which has been used to build a unique approach, combining both the Western and the Eastern approaches, identified in Singapore’s method, through the art.

On the base of the teaching and learning proposed, an experimental phase, involving teachers and students from lower and upper schools, was designed to evaluate and validate the results achieved along the way.

Chapter 4 describes the activities realized and implemented during the research work starting from a deep analysis of literature aiming to study the current status quo of mathematics education and mathematics school curriculum in diverse learning environments. The research teaching/learning approach combining the Western and Eastern approaches with the use of the art through the phases recognized in Singapore's approach for mathematics study is presented.

Moreover, in this Chapter, the research phases implemented for the collection of both qualitative and quantitative data during the experimentation phase are explained and synthesized.

The Chapter 5 analyzes the data obtained, presenting the characteristics of the sample and then underlining some important aspects achieved by referring the two target groups (the first one consist of 11-13 year-old students and the second one of 14-16 year-old).

Besides, the collected data are analyzed also taking into account both age and gender differences.

Finally, a final synthesis has been drawn up by underlining the possible future development of the proposed methods.

CHAPTER 1. GENERAL PRESENTATION OF THE PROBLEM

Nowadays, when Europe is coping with a period of change and with the economic crisis, in particular, which is slowing down the development and the social progress highlighting the structural weaknesses within the European Countries, the role of R&D is difficult to overestimate.[5]

On the other hand, there are other international challenges, such as globalization, resource exploitation, and human ageing. As stated in the European Communication *EUROPE 2020 - A strategy for smart, sustainable and inclusive growth* of 2010[6], the Commission focuses on a stronger strategy that can transform the European Union into a smart, sustainable and inclusive growth for a high level of employment, productivity and social cohesion.

EUROPE 2020 presses on three priorities interdependent and interrelated:

- Smart growth for developing an economy based on knowledge and innovation;
- Sustainable growth to support more efficient and more competitive resources respecting the environment;
- Inclusive growth to promote an economy with increased employment through social and territorial cohesion.

Therefore, research, development, education, and skills are the keywords for Europe of the 21st century despite the obvious national and institutional barriers that limit the development and strengthening of the European Research.

According to EUROSTAT, published in *Science, technology, and innovation in Europe*[7], 1.68% of people in Europe work in the Research & Development (R&D) field, but the situation is changing completely at a National level for some countries as Italy where the percentage of people working as "scientists and engineers" is very low, around 0.61%.[8]

These data (Figure 1) are still confirmed by the following figure representing the intensity of R&D activities in 2016 in Europe where Italy is placed in a low ranking of the average of 2.03% reached by some European countries.[9]

[5] OECD (2002). *The Measurement of Scientific and Technological Activities Frascati Manual 2002: Proposed Standard Practice for Surveys on Research and Experimental Development*. Cambridge University Press, p. 57.
[6] European Commission (EC). (2010). *Europe 2020: a strategy for smart, sustainable and inclusive growth*. Working paper {COM (2010) 2020}.
[7] Eurostat, E. U. (2013). *Science, technology and innovation in Europe*: 2013 edition.
[8] Ibid.
[9] European Commission (EC). (2018) *Una nuova agenda europea per la ricerca e l'innovazione – l'opportunità dell'Europa di plasmare il proprio futuro*. Working paper {COM (2018) 306}.

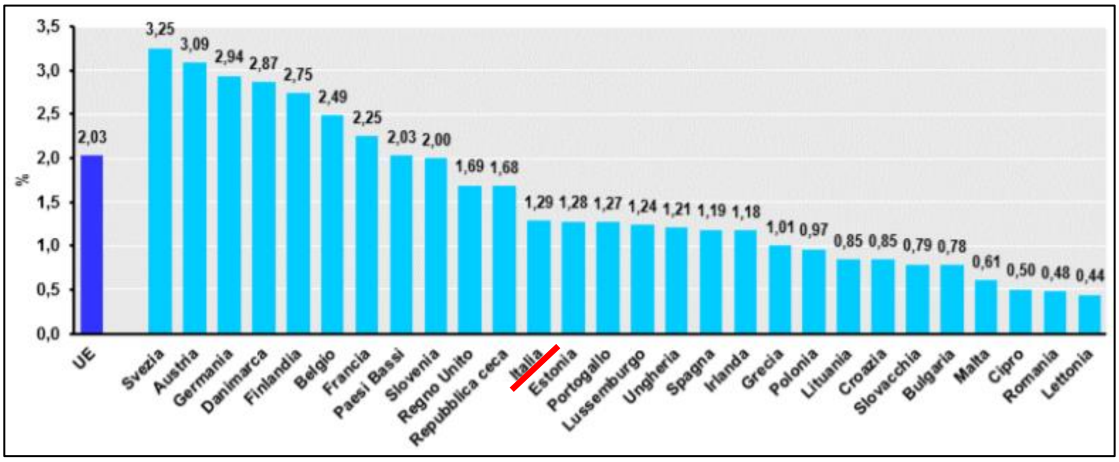

Figure 1: Intensity of activities in R&D in Europe in 2016.
Source: European Commission, Directorate-General for Research and Innovation. Data: Eurostat.

1.1 Scientific work in the 21st century

Schreiner and *Sjøberg* suggest that *it might be that we have now passed the era in which the work of physicists, technicians, and engineers is seen as crucial to people's lives and well-being.*[10] Today's youths are more interested in who they will be rather than what they will do.[11] Negative stereotypes of scientists, engineers, researchers, and other STEM (Science, Technology, Engineering and Mathematics) experts' careers can be found amongst youth in most of the western world, even the United States. The only exception is Japan, where wide investments are made in the educational field by the Government.

School, television, and newspapers feed the people's imagination with the idea that scientist's work is a hard and demanding activity as well as unknown and mysterious. It is enough to think about emblematic figures of scientists like Frankenstein or Doc, the character of the movie *Back to the Future*, or cartoons, which continue to represent *science* in a caricatured and imaginative way. As a result, the scientist's figure appears isolated, immersed in a special work – strange and incomprehensible to everyone at the same time. These stereotypes emerge from research realized in the framework of *2ways* European project – funded by the European Commission under the Seventh Framework Programme. It involved young people from 30 partner countries who expressed their opinions on scientific general subjects and scientists as a profession.[12]

Indeed, to the question *is science interactive?*, only 14% of students answered positively and just 21% of students claim that science stimulates group work.[13] Therefore, science is perceived by many as an individual work, which does not require exchanges or collaborations, making it uninteresting profession (Figure 2).

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[10] Rif. Sjøberg, S., & Schreiner, C. (2010). *The ROSE project: An overview and key findings*. Oslo: University of Oslo, 1-31.
[11] Ibid.
[12] Luraschi, M., & Pellegri G. (2010). *LA SCIENZA APPASSIONA I GIOVANI?* Università della Svizzera Italiana, Science et Cité.
[13] Ibid.

Question: Would a career in science be interesting?

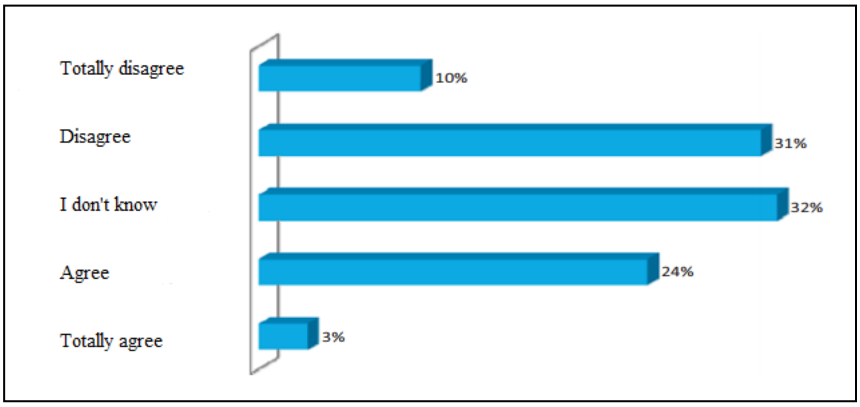

Figure 2: The attitude to science careers in percentage.[14]

This demonstrates that there is a lack of attractive role models and a lack of information and understanding of what careers in STEM are about.

Moreover, another problem, regarding gender differences, is highlighted: girls often are not so encouraged to pursue a scientific career. This is always a result of people's idea fed by movies, TV-talk, and news, which promote a male science where *woman* is rarely present. Indeed, if we consider the same question (from Figure 2) taking into account the gender, the graphic will be as follows:

Question: Would a career in science be interesting?

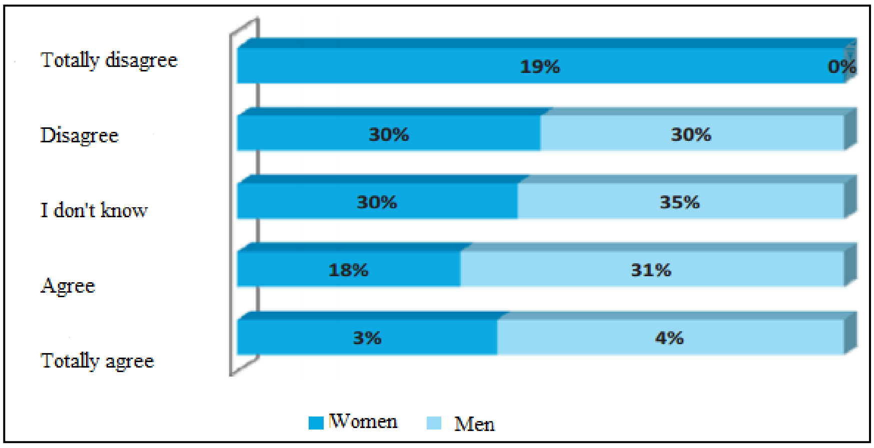

Figure 3: Gender Differences in the Attitudes towards science careers.[15]

Further studies show that women remain under-represented in R&D in every region worldwide. Some of the latest UNESCO data shows that, in North America and Western Europe, the average representation for the women in R&D is 32% (is the lowest average

[14] Luraschi M., Pellegri G., *Op. Cit.*
[15] Ibid.

found in Luxemburg with only a 21% and in the Netherlands with a 24%). A different situation is found in Central and Eastern Europe, where the overall average rises to 40%. In fact, in Latvia, Lithuania and Northern Macedonia, the woman represents more than 50%, and in Albania, Bulgaria, Croatia, Estonia, Moldova and Romania the percentage exceeds 40%. Despite the positive trend occurring in Eastern Europe, a wide array of studies indicates a disparity between the number of women studying science and those who go working as scientists professionally. Overall, women account for a minority of the world's researchers[16].

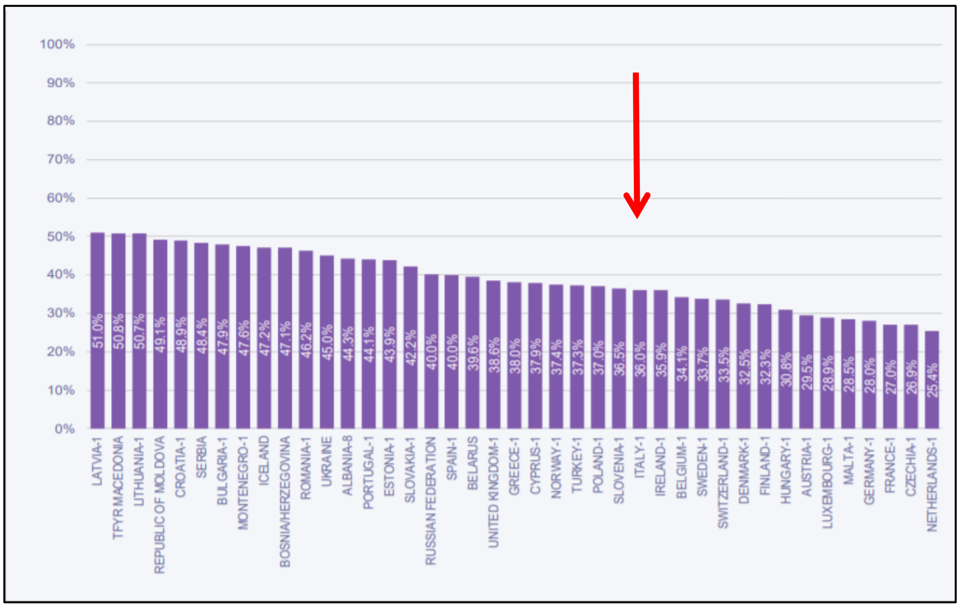

Figure 4: A breakdown of female researchers in Europe.
Female researchers as a percentage of total researchers (HC), 2016 or latest year available.

Source: UNESCO Institute for Statistics, June 2018

The following figure presents the share of women in the total number of researchers by country:

[16] UNESCO – Institute for statistics, *Women in Science*, June 2018. http://uis.unesco.org/sites/default/files/documents/fs51-women-in-science-2018-en.pdf [Retrieved online 13/11/2020]

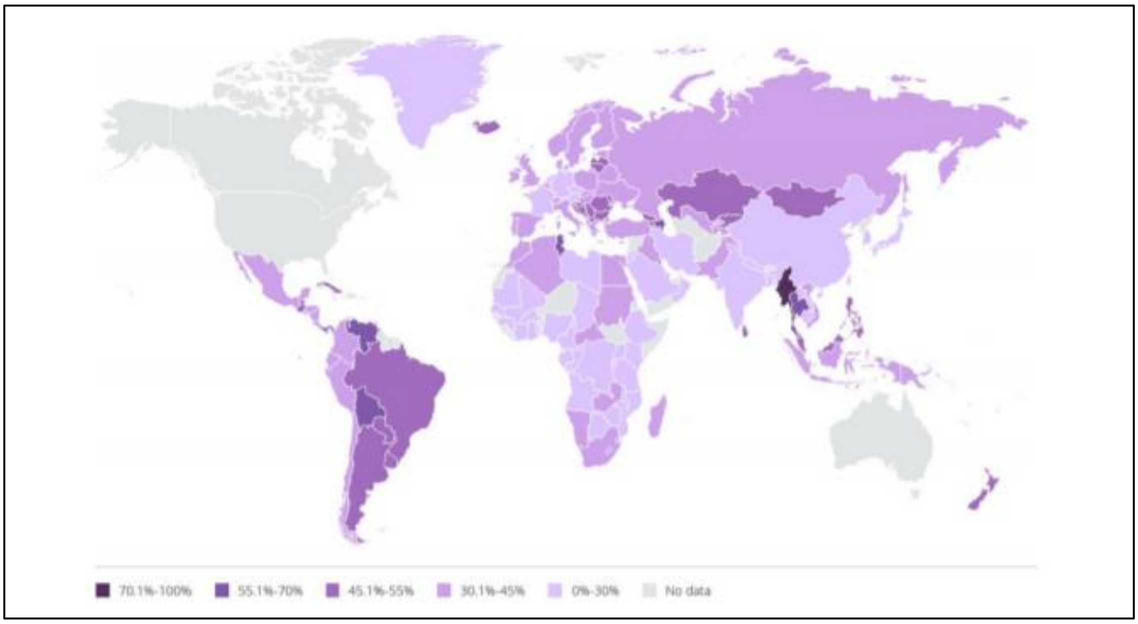

Figure 5: The gender gap in science.
Women as a share of total researchers, 2016 or latest year available.
Data on this map are based on HC, except for Congo and India (based on FTE).

Source: UNESCO Institute for Statistics, June 2018.

Suffice to say that only in 2014 an Iranian woman, *Maryam Mirzakhan*i, was the first woman to win the Fields Medal, the highest honor in mathematics comparable to a Nobel Prize in this field.[17]

Moreover, surveys have shown that, globally, women remain underrepresented in STEM not only as teachers, researchers, and workers but also as students, resulting in a significant gender gap. Worldwide, 2015 PISA results showed that, although a larger share of the top performers are boys at the secondary level, average gender differences in science performance remain minor.

However, it is gender disparities in interest towards STEM that are quite substantial, particularly in lower and/or upper secondary education. Indeed, girls start losing interest in STEM as they get older and in a bigger proportion than boys.[18]

Therefore, there is a significant gender issue in the Mathematics, Science, and Technology (MST) area, with an insufficient number of girls taking up or being encouraged to take up these subjects. Too often teachers and career advisers still call on more traditional thinking that MST is mostly a male domain[19]. Therefore, it is important to create new

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[17] "*Maryam Mirzakhani, la prima donna a vincere il "Nobel" della matematica*" http://www.wired.it/scienza/2014/08/13/medaglia-fields-mirzakhani-donna-nobel-matematica/. [Retrieved online 20/04/2020]

[18] OECD (2018). PISA, *PISA 2015 results in focus*.Op.Cit.

[19] Johansson, L. (2009). Mathematics, science & technology education report. Brussel: *European Round Table of Industrials.*

instruments in order to train and prepare people with the right skill sets for the period when new jobs will come on stream.[20]

Following to OECD[21] report, it is stated, *A network of stakeholders (linking educational resource centers, the business community, science and technology education specialists, and students and teacher communities), should be established to share information on best practices between countries and the various communities involved*. In this context, the European Commission's High-Level Expert Group on Science Education Renewal has made the points that *Teachers are key players…being part of a network allowing them to improve the quality of their teaching and support their motivation* and that *the articulation between national activities and those funded at the European level must be improved*.[22]

In this framework, it is necessary to develop a high-quality teaching and learning environment[23] where students can approach the study of scientific subjects with a higher interest and motivation adequately supported by teachers in their learning, which will be more effective if it is *meaningful*, i.e. active, constructive, intentional, authentic and collaborative.[24]

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[20] Jiménez Iglesias, M., Müller, J., Ruiz-Mallén, I., Kim, E., Cripps, E., Heras, M., ... & Gras-Velázquez, A. (2018). *Gender and innovation in STE (A) M education*. European Schoolnet.

[21] OECD (2008), "*Increasing Student Interest in S&T Studies*", in Encouraging Student Interest in Science and Technology Studies, OECD Publishing, Paris.

[22] Johansson, L., *Op.Cit.*, p.15.

[23] Mayer, R. E. (2010). *Learning with technology* (pp. 179-185). Paris: OECD.

[24] Jonassen, D. H., Howland, J., Marra, R., & Crismond, D. (2008). *Meaningful learning with technology*. Upper Saddle River, NJ: Pearson.

1.2 Student motivation in mathematics study

About 79% of students define mathematics as a big obstacle for their learning process.[25] The difficulties revealed are often related to its being considered more abstract than the others. About 83% of students work using visual memory.[26] This means that if we can imagine a history lesson as a film or a cartoon, it is hardly possible to do the same with a mathematics lesson. For instance, a lesson on inequalities hardly can activate visual memory as a literature subject does. At most, it can stimulate the photographic memory which is only 7% of the visual one.[27]

Besides, the theory should be learned through doing exercises to understand the solution process by activating the procedural memory. This allows students to get skills through learning by doing.

Students often do not perceive a practical utility in mathematics study unless they study specific subjects, such as economics where the utility is directly evident. In short, most students don't have enough stimuli of being interested in mathematics.[28]

Mathematics is not complicated by itself but must be studied differently from other disciplines. The suggested learning approaches usually are the following: *full immersion, an association between image and concept* or using *memory techniques*.[29]

Full immersion is when a student dedicates at least one hour and a half to do the exercises daily.

Association between image and concept: when students try to memorize the mathematics formula or concept, e.g. *function*, through a corresponding image. This is very useful to activate visual memory.

Another method is to take advantage of *memory techniques*. For example, the use of mnemonics can help students remember formulas or demonstrations by looking at them once. These techniques can work well because they stimulate the innate abilities of the student's mind.

However, even if these learning methods can help students in a short-term study, they don't seem effective if we think in long-term learning.

[25] OECD (2018). *Op.Cit.*
[26] Ibid.
[27] Raghubar, K. P., Barnes, M. A. and Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and Individual Differences*, 20 (2), 110-122.
[28] Tramonti, M., & Paneva-Marinova, D. I. (2019). *Maths, Art and Technology: a Combination for an Effective Study*. TEM Journal, 8(1), 82-86.
[29] Raghubar, K. P., Barnes, M. A. and Hecht, S. A. (2010).*Op.cit.*

Another issue, which can impact the student motivation in mathematics study,[30] is directly related to the teaching-learning process.

The traditional one is focused more on content mastery than on the learning abilities development and enhancement of research aptitude. The mainstream education system is oriented towards the teacher: the teacher gives and students receive. Therefore the student evaluation is related to *a correct or incorrect answer*. This education system is focused on school performance and results and it doesn't develop their lifelong learning capacity.

As *Seymour Papert* states: *<<But…life is not about knowing the right answer, it is about getting things to work, about finding the best way to express one's creative ideas....>>.*[31]

The didactic activities in the classroom, especially in lower and upper secondary schools, can be realized in different ways by teachers.[32] For example, by presenting, in the beginning, general principles on the topic to be studied, and then let students deduce particular behaviors or on the contrary, by starting from phenomena, partial information, well-defined known behaviors up to building laws, processes, considerations, and general behavior. This way to proceed is a part of the traditional teaching methodology, focused on the discipline and knowledge and the completeness of the message communicated¸ exploiting the learning deductive mode.[33]

On the other hand, in a context of an active school,[34] deductive teaching is replaced by the inductive modality, centered on the student's educational process involving his/her bio-psychological, socio-cultural and value structure.

The inductive didactics is a teaching operating moment,[35] meant as an intentional and well-organized activity, aiming to manage a training process according to effective strategies that can develop, extend, deepen and modify skills, knowledge, attitudes, and values of

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[30] D'Amore, B., & Frabboni, F. (2005). *Didattica generale e didattica disciplinare. La Matematica*. Pearson Italia Spa.

[31] Papert, S. (1999). Introduction: what is Logo? And who needs it. *Logo philosophy and implementation*.

[32] Tessaro F.(2002). *Metodologia e didattica dell'insegnamento secondario*, Roma, Armando.

[33] Frabboni F.(2006). *Didattica e apprendimento*, Palermo, Sellerio di Giorgianni.

[34] The expression "active school", invented by Bovet becoming later a feature of the modern school, indicates, mainly, the reaction to the traditional school (passive, formalist, unable to respond to students' needs) proposing an innovative education based on some fundamental ideas: 1. Student is the centre of education action and then the educators should take into account his/her needs and interests to be "spontaneously" harmonized with those moral, social and civil of its life environment (puero-centrism); 2. This can be achieved only if the student is free to act in his/her environment. In this way, the educational work consists in stimulating the student to grow through concrete activities (such as manual work and game) always experienced firsthand. Rif . Bovet P., *L'instinct combatif: psychologie, éducation*, Neuchâtel, Delachaux et Niestlé, 1917; *La réforme scolaire à l'université*, Neuchâtel; Genève, Ed. Forum, 1920; *Vingt ans de vie: l'institut J.J. Rousseau de 1912 à 1932*, Neuchâtel; Paris, Delachaux et Niestlé, 1932; Cambi, F. (2014). *Le pedagogie del Novecento*. Gius. Laterza & Figli Spa.

[35] Laneve C.(2003). *La didattica tra teoria e pratica*, Brescia, La Scuola.

students.[36] Consequently, the factors, which can make inductive teaching interesting, regard the increased attention given to students, in terms of effective learning aiming to exploit and develop their motivation.[37]

Johann W. Goethe, in his work *Faust*, affirms that *Wo der Antheil sich verliert, verliert sich auch das Gedächtnis* (When interest is lost, memory is lost).[38]

According to the motivational theories, motivation is a mainspring for the individual to act. From the psychological point of view, the motivation can be defined as the *why* of our actions and as the *end* that pushes the individual to meet their own needs. Therefore, every action done without *motivation* can fail.[39] If the individuals are driven by a strong reason, they will be able to: reach own objectives; develop a positive vision on their work/study; produce new energies needed to change; increase the self-esteem and their abilities; realize their personal and professional development helping others in this process. One of the factors that increase the motivation is the *interest*.[40] Motivating students means to arouse their interest in the study through a process of search and discovery of information that will promote student learning.

This student-centered approach assumes that the teacher is "invisible" by focusing on the learning style and interests of students. In this context, the role of the teacher changes: from the transmitter of knowledge to facilitator of learning. The facilitator's task is to define the right conditions which can develop student learning by providing them with the support in the case of help request, rather than providing categorical or absolute judgments. The aid given is always offered to take into account the characteristics and needs of each student.[41]

For *David Ausubel*,[42] learning is constituted of two stages: the first concerns how information reaches the learner; the second one is related to the learning process where the student incorporates it and reworks it in his/her pre-existing cognitive structure. Therefore, two ways to learn are possible:

- Receiving, if the information is provided by others, and the student must only store it or memorize it;

[36] Calabretta, P.(2018). *Insegnare con "cura" per prendersi cura. La relazione didattica come cura*. Bruno Notarnicola, 15.
[37] Franceschini G.,(2000). *Apprendere, insegnare, dirigere nella scuola riformata. Aspetti metodologici e profili professionali nella nuova scuola di base*, Pisa, Edizioni ETS 2
[38] Goethe J. W.(1990). *Faust*, a cura di Franco Fortini, Mondadori, Milano.
[39] Maslow A.H.(2010). *Motivazione e Personalità*, Armando Editore, Roma.
[40] Bandura, A. (2000). *Autoefficacia: Teoria e applicazioni*.(Presentazione all'edizione italiana di Gian Vittorio Caprara). Edizioni Erickson.
[41] Andrich, S., & Miato, L. (2007). *La didattica positiva: le dieci chiavi per organizzare un contesto sereno e produttivo*. Erickson.
[42] Ausubel, D. P. (2004). *Educazione e processi cognitivi. Guida psicologica per gli insegnanti* (Vol. 25). FrancoAngeli.

- Discovering, if the information is the result of a self-discovery, personal, creative.

The meaningful learning aims at connecting a learning process to relevant concepts already owned, existing in the cognitive structure of the subject. This concept comes from the constructivist theory of knowledge, in other words, there is no knowledge without a process of meaning construction by the learner.

The learners construct their knowledge in *meaningful* contexts through the object's manipulation, tools and through the observation and interpretation of their actions' results. In these terms, meaningful learning becomes contextualized and complex. The students learn more and better if they cope with authentic tasks strictly connected to the real world, where they meet their everyday life's *real problems*.

The same thing occurs when a student approaches math studies. Actually, people often forget that reality, as well as all disciplines, as known by man, is based on mathematical concepts: math could be found in human beings, in architecture, plants, and animals, and as a part of them it regulates also their characteristics.

With the introduction of the Fibonacci series,[43] made by the medieval great mathematician Leonardo da Pisa, it is possible to observe how the growth of flowers, leaves, and branches of a plant follows specific patterns.

Indeed, the leaves grow around the central shaft following a math law closely linked to the Fibonacci sequence: 5 leaves correspond to three laps around the stem, as 8 leaves correspond to 5 laps. Even the number of petals of the flowers follows this rule: the corolla usually has 5 petals but in many flowers such as daisies and chrysanthemums, the numbers of petals are 13, 21 or 34, that are in the Fibonacci series.

In crystals,[44] the symmetry elements regulate the arrangement of the various faces and the various edges, so that these are determined to form solid systems, for example, the cubic form in the minerals such as halite, or the trigonal-rhombohedral form in the dolomite.

The shape and structure of the organism of some animals can be divided specularly into two or more equal parts. This feature is called *symmetry* and can be distinguished in radiata symmetry, such as the *Cnidaria* whose body can be divided into several equal parts, by passing multiple planes orthogonal to the polar axis of the animal, or bilateral, such as the *Triblastici* whose body can be divided into only two equal parts, by passing through his organism only an axis of longitudinal polar symmetry.

[43] Posamentier, A. S., Lehmann, I., & Hauptman, H. A. (2011). *I (favolosi) numeri di Fibonacci*. Emmebi.
[44] Gallo, P., & Vezzani, C. (2007). *Mondi nel mondo: fra gioco e matematica*. Mimesis.

In particular organisms, such as starfish, there is pentameral symmetry due to the development of the five arms around a central pole.

On the base of many structures in nature, some fractals are special geometric figures whose shape is repeated in the same way on different scales.[45] An example of a fractal structure is observable in the wild carrot (inflorescence) in which the basic structure is repeated a finite number of times as well as many organs that are parts of human beings, such as the acinar structure of the liver and nerve fibres.

An interesting question, however, concerns the relationship between fractals and real biological structures. The latter consist of a finite number of cells, thus are not fractals in the strict sense of the word.[46]

Despite this strong presence of mathematics concepts in the reality, their connection doesn't appear so evident during the learning process and often teachers offer to students unduly theoretical approach causing the perception that mathematics is abstract and far away from everyday life. This influences the way to learn mathematics privileging more storage capacity than problem-solving skills.

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[45] Sendova, E. (2017). Constructionism as an educatinal philosophy and a culture–a tribute to Seymour Papert. *Mathematics and Education in Mathematics*, 9-13.

[46] Howland, H. C. (1993). The algorithmic beauty of plants, by Przemyslaw Prusinkiewicz and Aristid Lindenmeyer, Springer-Verlag, New York, 1990, 228 pp., *Color Research & Application, 18*(2), 140-141.

1.3 Problem statement

The globalized labor world requires knowledge based on solid basic skills such as mathematics literacy.

The development of math skills, teaching strategies, and practices becomes primary for educational growth during all school years. An alarming picture of the current situation is provided by PISA and TIMSS international surveys.[47] The major aim of these assessments is to evaluate different features of student learning and to assist policymakers to understand better to what extent their educational systems are measuring up with developments taking place in other countries. Since the mid-1980s many governments have made major investments to equip schools with modern technologies in order to modernize teaching and learning thus providing students with opportunities to learn these technologies and to acquire competencies that they will need in their future life (Figure 6).

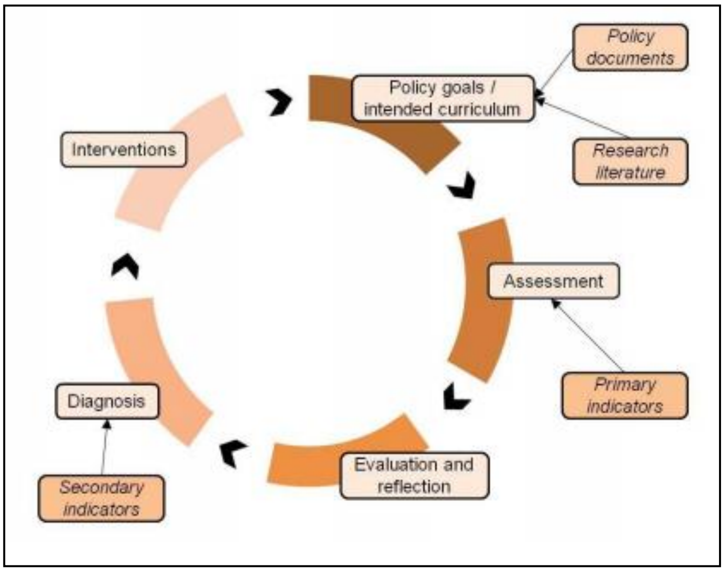

Figure 6: Main steps in the policy cycle.

In the following paragraphs, the results collected by the two international surveys PISA and TIMSS are described.

[47] Scott, E. (2004). *Comparing NAEP, TIMSS and PISA in Mathematics and Science*. National Center for Education Statistics.

1.3.1 The PISA survey

The PISA survey is promoted by the Organization for Economic Cooperation and Development (OECD). It aims at assessing the performances of 15 years-old students in reading, mathematics, and science every three years.[48] It focuses, especially, on how far students, near the end of compulsory education, have acquired knowledge and skills essential for full participation in society. This survey contributes to the assessment of mathematical and science literacy through several proposed questions always set in a real context.

PISA emphasized mathematical literacy and *preparation for life*. After consideration by the science expert group, the following definition of mathematical literacy was adopted by PISA for its testing: *Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen*.[49] This definition includes mathematical thinking and the use of mathematical concepts, procedures, facts and tools for describing, explaining and statement of hypotheses about processes and phenomena. In other words, in mathematics, PISA assesses students' skills to formulate, use and interpret mathematical problems in a variety of situations life (Figure 7).

Stage 1

1. Starting with a problem situated in reality.

2. Organising according to mathematical concepts.

3. Taking into account the mathematical features of the situation and transforming the real-life problem into a mathematical problem.

Stage 2

4. Solving the mathematical problem.

Stage 3

5. Making sense of the mathematical solution in terms of the real situation, identifying the limitations of the situation.

[48] Schleicher, A. (2001). *Knowledge and skills for life: First results from the OECD Programme for International Student Assessment* (PISA) 2000. OECD.

[49] Garrido Martos, R. (2016). Assessing Mathematical Literacy: The PISA Experience.

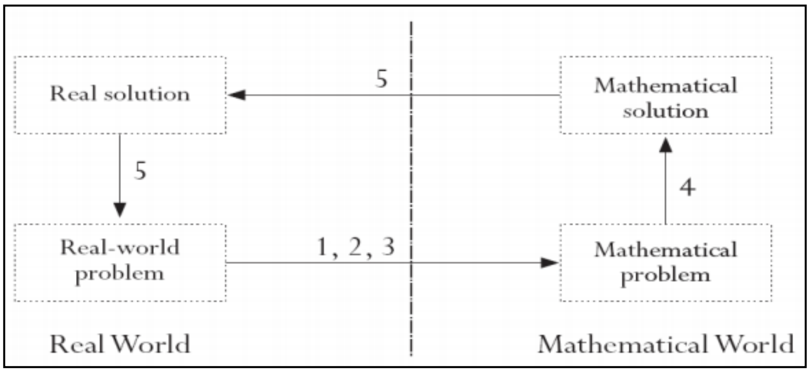

Figure 7: Mathematisation process according to PISA.

Concerning the last survey realized in 2018, the mean scores achieved for the mathematics have registered negative tendency in countries considered to be at the forefront of education, like Finland (where the score has decreased of -12 in last three years), on one hand. While, on the other hand, the results have confirmed the negative ranking (below the OECD average) in other countries, like Greece.

In Italy, the situation is a little bit different. If the survey realized in 2015 have shown that the pupils' performance reached in mathematics a score on the OECD average (490), the last results collected in 2018 present again a negative situation (487, -2) under the OECD average (489) (Fig.8 – Table 1). Probably this inverse tendency is still due to the use of traditional learning-teaching methodologies in the mathematics study.[50]

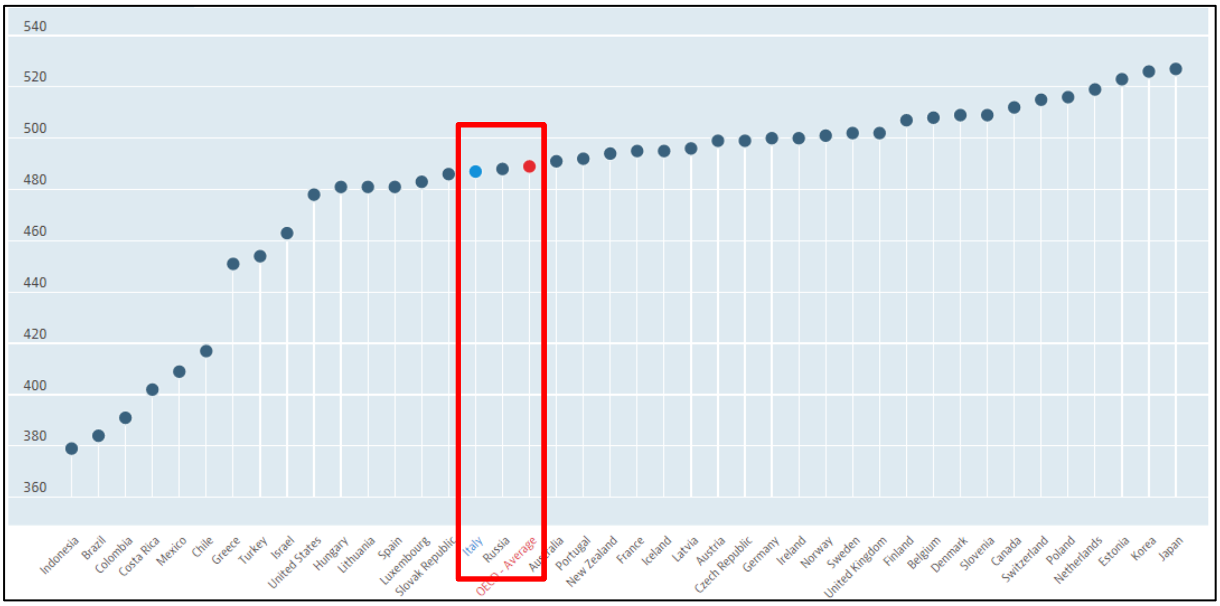

Figure 8: Mathematics performance (PISA).Total, Mean score, 2018.
Source: OECD Education Statistics: PISA2018

[50] OECD (2019), *Mathematics performance* (PISA) (indicator). doi: 10.1787/04711c74-en [Retrieved online 27/04/2020]

Table 1- Snapshot of performance in mathematics, reading and science – PISA 2018 results

Countries/economies with a mean performance/share of top performers above the OECD average
Countries/economies with a share of low achievers below the OECD average

Countries/economies with a mean performance/share of top performers/share of low achievers not significantly different from the OECD average

Countries/economies with a mean performance/share of top performers below the OECD average
Countries/economies with a share of low achievers above the OECD average

	Mean score in PISA 2018			Long-term trend: Average rate of change in performance, per three-year-period			Short-term change in performance (PISA 2015 to PISA 2018)			Top-performing and low-achieving students	
	Reading	Mathematics	Science	Reading	Mathematics	Science	Reading	Mathematics	Science	Share of top performers in at least one subject (Level 5 or 6)	Share of low achievers in all three subjects (below Level 2)
	Mean	Mean	Mean	Score dif.	Score dif.	Score dif.	Score dif.	Score dif.	Score dif.	%	%
OECD average	487	489	489	0	-1	-2	-3	2	-2	15,7	13,4
B-S-J-Z (China)	555	591	590	m	m	m	m	m	m	49,3	1,1
Singapore	549	569	551	6	1	3	14	5	-5	43,3	4,1
Macao (China)	525	558	544	6	6	8	16	14	15	32,8	2,3
Hong Kong (China)	524	551	517	2	0	-8	-2	3	-7	32,3	5,3
Estonia	523	523	530	6	2	0	4	4	-4	22,5	4,2
Canada	520	512	518	-2	-4	-3	-7	-4	-10	24,1	6,4
Finland	520	507	522	-5	-9	-11	-6	-4	-9	21,0	7,0
Ireland	518	500	496	0	0	-3	-3	-4	-6	15,4	7,5
Korea	514	526	519	-3	-4	-3	-3	2	3	26,6	7,5
Poland	512	516	511	5	5	2	6	11	10	21,2	6,7
Sweden	506	502	499	-3	-2	-1	6	8	6	19,4	10,5
New Zealand	506	494	508	-4	-7	-6	-4	-1	-5	20,2	10,9
United States	505	478	502	0	-1	2	8	9	6	17,1	12,6
United Kingdom	504	502	505	2	1	-2	6	9	-5	19,4	9,0
Japan	504	527	529	1	0	-1	-12	-5	-9	23,3	6,4
Australia	503	491	503	-4	-7	-7	0	-3	-7	18,9	11,2
Chinese Taipei	503	531	516	1	-4	-2	6	-11	-17	26,0	9,0
Denmark	501	509	493	1	-1	0	1	-2	-9	15,8	8,1
Norway	499	501	490	1	2	1	-14	-1	-8	17,8	11,3
Germany	498	500	503	3	0	-4	-11	-6	-6	19,1	12,8
Slovenia	495	509	507	2	2	-2	-10	-1	-6	17,3	8,0
Belgium	493	508	499	-2	-4	-3	-6	1	-3	19,4	12,5
France	493	495	493	0	-3	-1	-7	2	-2	15,9	12,5
Portugal	492	492	492	4	6	4	-6	1	-9	15,2	12,6
Czech Republic	490	499	497	0	-4	-4	3	7	4	16,6	10,5
Netherlands	485	519	503	-4	-4	-6	-18	7	-5	21,8	10,8
Austria	484	499	490	-1	-2	-6	0	2	-5	15,7	13,5
Switzerland	484	515	495	-1	-2	-4	-8	-6	-10	19,8	10,7
Croatia	479	464	472	1	0	-5	-8	0	-3	8,5	14,1
Latvia	479	496	487	2	2	-1	-9	14	-3	11,3	9,2
Russia	479	488	478	7	5	0	-16	6	-9	10,8	11,2
Italy	476	487	468	0	5	-2	-8	-3	-13	12,1	13,8
Hungary	476	481	481	-1	-3	-7	6	4	4	11,3	15,5
Lithuania	476	481	482	2	-1	-3	3	3	7	11,1	13,9
Iceland	474	495	475	-4	-5	-5	-8	7	2	13,5	13,7
Belarus	474	472	471	m	m	m	m	m	m	9,0	15,9

Source: OECD, PISA 2018 Database[51]

If the total score achieved in Italy (487) is examined taking into account the gender difference, the final result represents a different scenario.

More precisely, the *boys* have achieved the main score of 494 corresponding to +5 with the respect of the OECD Average 489, but the *girls* - only a score of 479 with a decreasing of -10 below the OECD average.[52] This factor underlines the complexity of the Italian situation where gender differences have a strong effect on the students' performance in mathematics.

[51] OECD (2019), *Snapshot of performance in reading, mathematics and science*, in PISA 2018 Results (Volume I): What Students Know and Can Do, PISA, OECD Publishing, Paris, https://doi.org/10.1787/cbb85a0d-en.

[52] OECD (2019), *Mathematics performance* (PISA) (indicator). doi: 10.1787/04711c74-en [Retrieved online 27/04/2020].

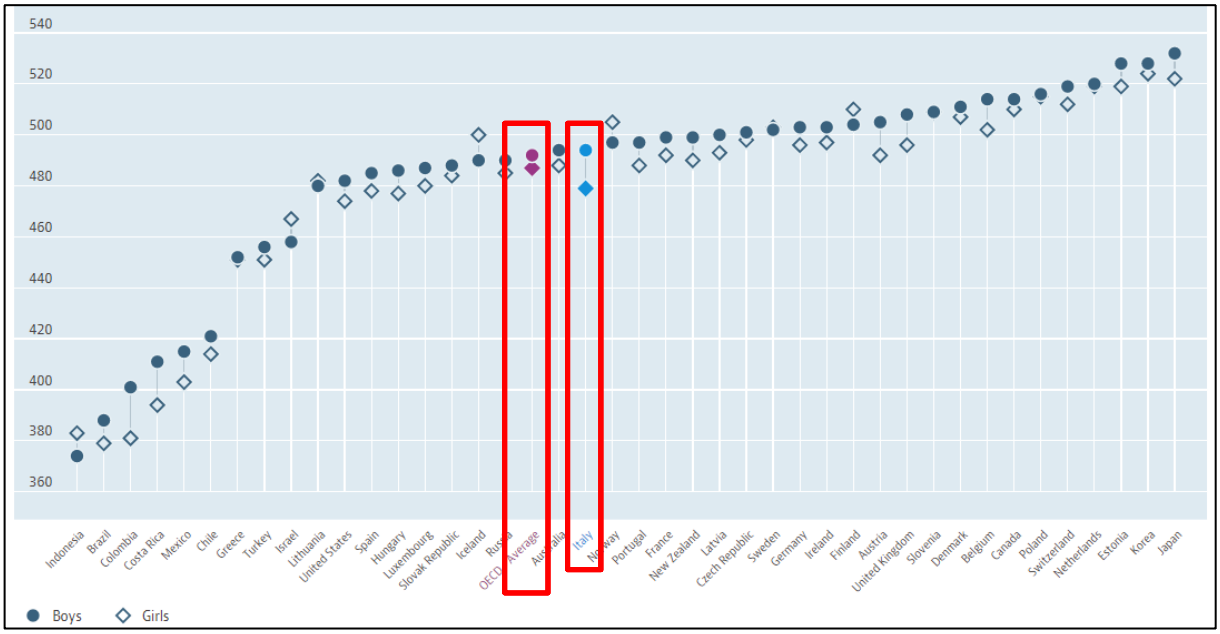

Figure 9: Mathematics performance (PISA).Total Mean score considering gender difference, 2018.

Source: OECD Education Statistics: PISA2018

Taking into account also the *Mathematics, Science & Technology Education Report*,[53] where education has been acknowledged as the cornerstone of Europe's success and will continue to be a determining factor in the prosperity of Europe's citizens and economy for the foreseeable future, the competency in Mathematics, Science and Technology is becoming more and more fundamental.

For Europe as a whole, MST plays a key role in growing adequately the research and development capacity, ensuring economic and productivity growth that are crucial to Europe's future. The learning of mathematics, such as *numeracy* in terms of being able to interpret the behaviour of various mathematic models, and science literacy enable students to contribute effectively in actual society, improving their employment prospects.

[53] Johansson, L. (2009). *Op.Cit.*

1.3.2 The TIMSS survey

Another important international survey is the TIMSS. It is an international assessment of the mathematics and science knowledge of fourth-grade and eighth-grade students.[54] This includes achievement in each of the content and cognitive domains as well as overall mathematics and science achievement. TIMSS was developed by the International Association for the Evaluation of Educational Achievement (IEA) to allow participating nations to compare students' educational achievement.

If we compare this international survey with the previous one, PISA, we notice that the two instruments measure a different kind of elements and features of the same topic.

If one of the major strengths of TIMSS is its measurement of trends over time in mathematics and science performance, PISA has the advantage to collect extensive data on students' attitudes and well-being. In fact, PISA measures the education outcomes, not only as student performance, but also as students' attitudes and believes, and their expectations for future, and how these are related to their personal background.[55]

Moreover, TIMSS sought to find *what students know* and PISA sought to find *what students can do with their knowledge*. These two perspectives are neither *better* nor *worse*. Rather they are different, and each has importance as a learning outcome of the study of science at school. The data gathered in the TIMSS project are related to the intended curriculum (the curriculum specified by the system or other body), the implemented curriculum (the curriculum as taught by teachers, the nature of actual classrooms), and the attained curriculum (what students have learned). The PISA project is not directly focused on any of these aspects of curricula. Rather PISA is concerned with how well 15-year-old students can make use of science knowledge acquired from school and other sources, in everyday life situations that involve science and technology.

The following Table 2 compares the PISA and TIMSS surveys:[56]

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[54] Mullis, I. V., & Martin, M. O. (2017). *TIMSS 2019 Assessment Frameworks. International Association for the Evaluation of Educational Achievement*. Herengracht 487, Amsterdam, 1017 BT, The Netherlands.

[55] Mullis, I. V., Martin, M. O., & Liu, J. Context Questionnaire Framework.

[56] Ho, E. S. C. (2003). Accomplishment and challenges of Hong Kong education system: What we have learned from PISA. *Educational Journal, 31*(2), 1-30.

Table 2 - PISA and TIMSS comparison table[57]

	PISA	TIMSS	
Areas assessed	Scientific literacy Reading literacy Mathematical literacy	Mathematics Science	
Target population	All 15-year-old students regardless of year level – typically this is Year 10, but participating Australian students may be in Years 9 or 11, due to differing school starting ages	Year 4 students	Year 8 students
Focus of assessment	How well students are able to apply understanding and skills in science, reading and mathematics to everyday situations	How well students have mastered the factual and procedural knowledge taught in school mathematics and science curricula	
Situations	Personal Educational Occupational Public Scientific		

TIMSS[58] gathered data from samples of the student population at three levels (middle primary, lower secondary, final year secondary), from the teachers of these students, and their schools and systems. Development of the tests of student learning outcomes for each student population began with an analysis of science curriculum guides and textbooks from many countries to 'identify priority topics' for the tests. An international panel of science curriculum specialists then produced a framework to guide test development.

The data, collected by last TIMSS survey (2015) on mathematics,[59] show that the East Asian countries, in particular, Singapore, are top achievers at fourth and eighth grades in Mathematics. Italy has achieved an invariable average (507 and 494) in both at fourth and eighth grades in mathematics considering trends from 2011 to 2015. While countries like Finland and Germany show a lower average achievement at fourth grade in the same period.[60]

[57] Australian Council for Educational Research, *TIMSS and PISA backgrounder*, ACER. Available at https://www.acer.org/files/TIMSSandPISA-backgrounder.pdf [Retrieved online 27/04/2020].
[58] Martin, M. O., Ina, V. S. M., Foy, P., Olson, J. F., Erberber, E., Preuschoff, C., & Galia, J. (2005). *TIMSS 2007: International science report: findings from IEA's trends in international mathematics and science study at the fourth and eighth Grades*. TIMSS & PIRLS.
[59] Mullis, I. V., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 international results in mathematics.* TIMSS & PIRLS International Study Center at Boston College.
[60] Mullis, I. V., Martin, M. O., Foy, P., & Hooper, M. (2016), *Op. Cit.*

However, if only the last collected data (2015) are considered, the results show two different situations for students at fourth grade and at eighth grade in Italy as represented in the two Figures 10 and 11 below:

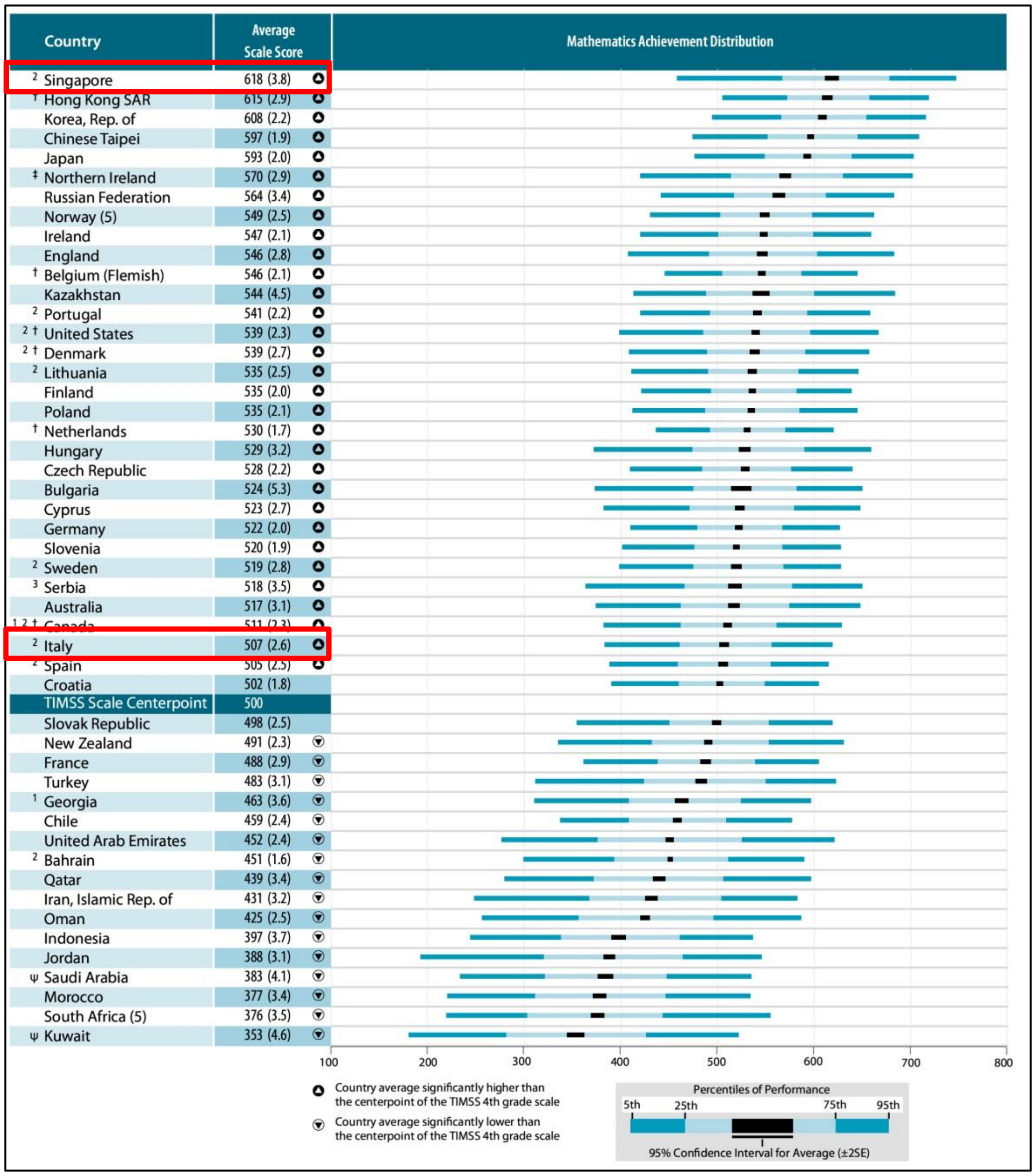

Figure 10: Distribution of Mathematics Achievement at fourth grade.

Source: IEA's Trends in International Mathematics and Science Study – TIMSS 2015

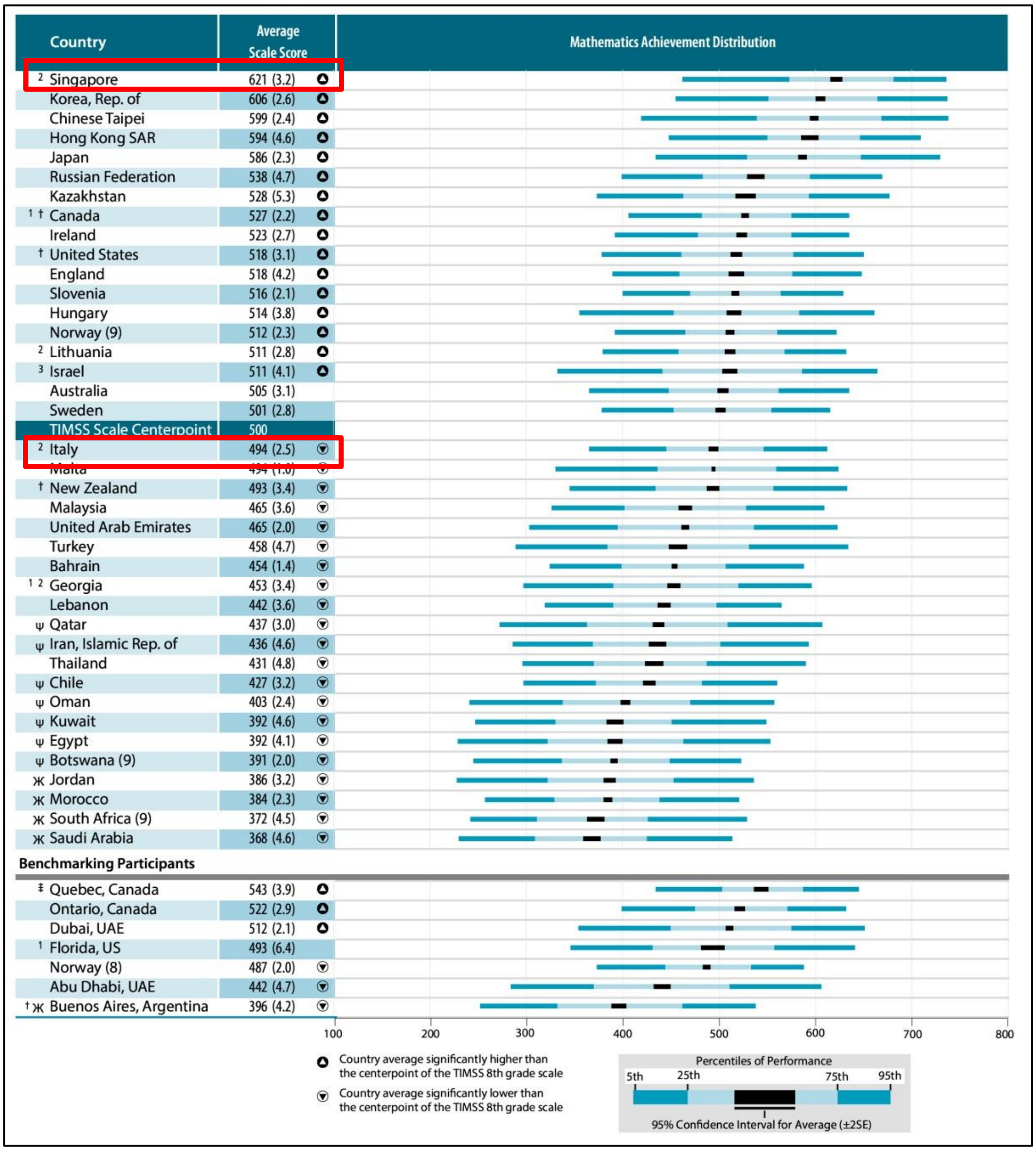

Figure 11: Distribution of Mathematics Achievement at eighth grade.

Source: IEA's Trends in International Mathematics and Science Study – TIMSS 2015

The Figures demonstrate that for fourth-grade students Italy has achieved +2.6 from the TIMSS Scale Centerpoint (500), while for eighth-grade students the results show still a negative situation -2.5 from TIMSS Scale Centerpoint (500). At the same levels, Singapore is placed at the top of the achievement, reaching respectively 618 (+3.8) for fourth grade and 621 (+3.2) for eighth grade.

The gap between Asian countries and the next highest countries was 23 points in 2015, unchanged from 2011 at fourth grade, while at eighth grade the gap between Asian countries and the next highest countries was 48 points in 2015, increasing from 31 in 2011.

However, with a few exceptions, like Norway, Belgium (at fourth grade) and Slovenia, Hungary (at eighth grade), European students, as shown from the graphics, often lack key basic competences in mathematics, science and technology.

In this context, most Ministries of Education share and agree on the educational objectives and results that are the base of these surveys by underlining the importance to encourage the development of the mathematics skills and their assessment through the national exams at the end of the compulsory education.

In fact, on the base of these last results reached by the performances of the students, the aim of the European Ministers is to reduce the average of students with difficulties in literacy, mathematics and science reaching a percentage lower than 15% within 2020 (from the section *News* in EU Commission web site).[61]

In this direction, the Italian Ministry of Education has arranged several seminars to explain the PISA survey, IEA-TIMSS, IEA-PIRLS to schools; to raise their awareness towards these international surveys and to work together for quality school and student performances improvement. These initiatives were funded by the European Commission under PON FSE *Competenze per lo Sviluppo* (Skills for Development).[62]

All these specific strategies are the result of a joint effort, carried out by government institutions with higher education partners or outside the education sector with the goals linked to broader educational aims than just improving student performance, such as:

- Promoting a positive feeling towards STEM subjects;
- Increasing public knowledge of STEM;
- Improving the teaching and learning of STEM in the school environment;
- Increase students' interest in STEM subjects and, consequently, incentivizing the choice of scientific careers at the upper secondary level and higher education level;
- Reinforcing gender equity in studies and professions related to mathematics, science and technology.[63]

Actually, in Italy, in 2007, the Ministry of Education created a committee for the development of scientific and technological culture with the aim of:

- Defining actions and strategies for the scientific and technological culture promotion in all country;

[61] European Commission News no. IP/11/115 Brussels, 1 February 2011. Available at this link: https://ec.europa.eu/commission/presscorner/detail/en/IP_11_115 [Retrieved online 27/04/2020].

[62] Ministero dell'Istruzione, dell'Università e della Ricerca, *Compendio delle disposizioni per l'attuazione degli interventi*, Programmi Operativi Nazionali FSE E FSER Dicembre 2015.

[63] European Commission. (2011). Science education in Europe: National policies, practices and research.

- Suggesting the lines of a development policy which defines the tasks of both public and private organizations;
- Defining projects and plans for schools, adult citizens and society as a whole;
- Planning, in particular, actions and services for teacher training and their support;
- Improving the school curriculum.[64]

Therefore, the main objective of both the politics held and the current work is not only students' increasing scores but improving their performance starting from the implementation of strategies and methods which can improve the process of teaching and learning science by making it more effective through pointing on both intrinsic and extrinsic factors.[65]

[64] European Commission. (2011). *Op.cit.*
[65] Ibid.

1.4 Introduction to the research proposal

Starting from the analysis of the described context, the research project aims to investigate and examine how the combination between meaningful learning and mastery learning produces an increase of motivation and interest for the study of the scientific subject to obtain a meaningful improvement of student final performances.

Firstly, considering the best performances achieved by Asian countries through international mathematics surveys, we proceeded with the analysis of Eastern and Western learning and teaching approaches to underline their strengths and weaknesses. As a representative of the Eastern teaching/ methods we chose and adopted *the Singapore's method* that is spreading for a few years in Europe. Among the Western approaches, the mathematics teaching process implemented mainly in Italian schools was considered.[66] This initial phase was necessary to reach a unique model exploiting the potentialities of both without leaving cultural differences out.

Secondly, starting from Confucius's quote, which claims *"I hear and I forget. I see and I remember. I do and I understand"*, the Arts were introduced in the proposed method to provide lower and upper secondary students with a major adherence to the reality by promoting the development of their creativity. The result is that students improve not only understanding but develop their creativity. As a result, the Confucius' quote is the starting point for the next step: the creativity. Therefore, "I hear and I forget. I see and I remember. I do and I understand and invent".[67]

The research proposal is described in detail in the following paragraphs.

1.4.1 An Eastern learning approach: the Singapore method

The Singapore method is the new trend in the teaching approach which is widespread already in countries like the USA, Australia, and the UK. This method is a highly effective teaching approach originally developed by Singapore's Ministry of Education.

Since the independence of Singapore in 1965, the education system has undergone several reforms. The first one was the New Education System (primary and Secondary) in 1970, aiming at providing students with at least ten years of general education. Mathematics

[66] Cockcroft, W. H. (1982). *Mathematics counts*. London: HM Stationery Office.

[67] Sendova, E. (2014, August). You do–you understand, you explore–you invent: the fourth level of the inquiry-based learning. In *Contructionism and Creativity, Proceedings of the 3rd International Construc-tionism Conference* (pp. 103-112).

was compulsory up to the end of the secondary school and streaming by ability was introduced.

The establishment of the Curriculum Development Institute of Singapore was a relevant milestone in 1980. It was the first time where a project team, led by Dr Kho Tek Hong, prepared instructional materials for the teaching and the learning of mathematics and an effective approach for the professional development of teachers. These instructional materials followed the Concrete-Pictorial-Abstract approach offer students the learning experiences and meaningful contexts, using manipulative and pictorial representations, to help them learn abstract mathematics.[68]

In the 1990s, Singapore's education system was reviewed by emphasizing the importance of both the processes and products in mathematics learning. This resulted in the production of the Mathematics Framework, which is another important feature of Singapore's mathematics education.[69] The framework articulates and explicates the intention of mathematics education, and provides guidance in the teaching, learning and assessment of mathematics. The aim was to develop a mathematics curriculum that will enable all students to learn and apply mathematics. This framework has been updated during the subsequent revisions of the mathematics curriculum in 2000 and 2003 to reflect the changing emphases and needs mainly coming from the integration of three initiatives, *Thinking Skills*, *Information Technology* and *National Education*, into the curriculum.[70]

Actually, the Singapore's method is a combination of the Western pedagogical theories of *Jerome Bruner*, *Richard Skemp* and *Zoltan Diens*, with Eastern teaching approach. The additional value is focused, mainly, on effective teaching of mathematics based on the development of "problem-solving" skills.

It starts from the identification of three fundamental mistakes occurring in the classical learning of mathematics:

1. Considering mathematics as computing – requiring students to learn how to calculate;
2. Thinking of mathematics teaching as delivering specific procedures and expecting the students to apply them at well recognizable setting, for example, requiring students to learn procedures on how to solve e.g. an equation;
3. Focusing on retention – it is important to memorize the formulas.

[68] Yip, J. S. K., & Sim, W. K. (1990). *Evolution of educational excellence: 25 years of education in the Republic of Singapore*. Longman.
[69] Ministry of Education (MOE, 1990, 2000, 2006). Mathematics syllabus (primary). Singapore: Author.
[70] Soh, C. K. (2008). An overview of mathematics education in Singapore. *Mathematics curriculum in Pacific Rim Countries–China, Japan, Korea and Singapore*, 23-36.

The Mathematics definition in Singapore's method becomes a means for the development and improvement of intellectual competences of students.

This approach consists of the introduction of mathematical concepts through process structured into three phases: concrete, pictorial and abstract. Starting from using a symbolic representation to ensure a concrete mathematical experience, students are allowed to reach the abstract representation of the studied concept.

In brief, the *concrete* phase refers to a manipulative experience with real objects to understand how they work. During the *pictorial* phase, students learn how to transfer their knowledge acquired through the real objects into a mental image, into a diagram or drawing. Finally, the *abstract*, they learn to use mathematics symbols, such as *x* for multiplication.

Therefore, teachers let students deal with complex problems, at first shown as real-world problems, and then, they learn that there is no only one right procedure or solution, but they need to construct an appropriate solution every time.

1.4.2 Arts as a learning facilitator tool in mathematics study

The introduction of arts (such as music, painting, dance, theatre), in the proposed approach, helps students connect better the mathematics to reality understanding its actual application in their everyday life.

As *Steiner and Schwarz* claim, in education, it is necessary to have tools and methodologies that can guarantee an active involvement and creative inclusion of learners to let them test the interconnection of different languages, such as visual, sensory, verbal and nonverbal. This favors the development of both cognitive and emotional dimensions, promoting further an intercultural approach. Therefore, art becomes an important element for harmonious development and growing up of the human being.[71]

The mathematician, Piergiorgio Odifreddi, claims that *The universe can be described only in mathematical terms*[72] and, therefore, to understand our world we must know its alphabet, which is mathematics.

The artists of all ages have tried to create works that represent human beings and dimensions. But to do it in the best way, they had also to devote themselves to the studies of geometry and architecture. In arts, we find some mathematics concepts such as *perspective*

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[71] Steiner, R., & Schwarz, L. (1936). *Il mondo dei sensi e il mondo dello spirito*. ITE, Istituto tipografico editoriale.

[72] Odifreddi, P., *Dietro l'arte c'è sempre nascosta la matematica,* <<la Repubblica>> Newspaper, Venerdì 29 Maggio 2015.

and *proportion*. One of the well-known artists, who used mathematics as one of the most important components in his artworks, was the artist *Mauritius Cornelius Escher* (1898-1972) (http://www.mcescher.com/). Due to this combination, he created beautiful works with amazing optical illusions. For example, to imagine how a formation of nano-structural fragments with different symmetries takes place, artistic patterns of metamorphosis,[73] developed by *Escher*, are used. The painting (Figure 12) shows the gradual translational change of symmetry.

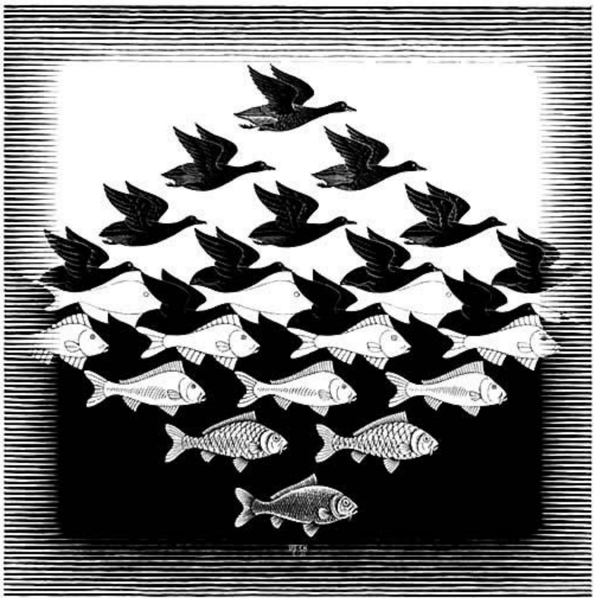

Figure 12: Metamorphosis - Mauritius Cornelius Escher painting.
Source: http://www.mcescher.com/

Thus, the research aims to exploit the potentialities emerging from a creative combination between Western and Eastern pedagogical learning approaches through the use of the arts. This allows students to improve and develop their learning skills with a creative and innovative study through appropriate development of the creativity-based approach in learning and problem-solving skills.

[73] Margolin VI, Zhabrev VA, Tupik VA (2008). *The Physical Basis of Microelectronics: Textbook for Students of High Education Institutions*. Moscow, Publishing Center 'Academy', p. 4.

1.4.3 Art, creativity and student motivation

By exploiting the multidisciplinary approach of science skills development, we identified problem-oriented creativity development as a core concept of our research.

This allows also overcoming the interdisciplinary constraints by paving a way to the transdisciplinary nature of the research where the creative process gets fundamental in a learning process.

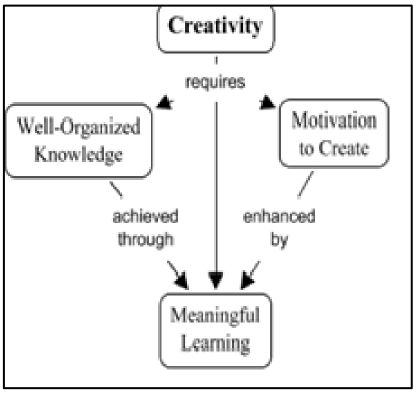

Figure 13: Creativity and meaningful learning.

According to Scardamalia and Bereiter,[74] the didactic process should be active to support and stimulate a holistic process of creative development of new ideas inside a classroom. *Quintana et al.*,[75] underline that this is favored, also, by an inquiry-based approach where the students ask questions, investigate and use empirical data, manipulate them directly through experiments or comparisons, systematization and representations of the data coming out from direct experiments, or from information searched and checked sources. One of the examples is the Virtual School Mathematics Laboratory (*VirMathLab*) developed at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences (IMI-BAS). In particular, it emphasized on the implementation of the Inquiry-Based Learning (IBL) through

[74] Scardamalia, M., & Bereiter, C. (1991). Higher levels of agency for children in knowledge building: A challenge for the design of new knowledge media. *The Journal of the learning sciences*, 1(1), 37-68.
[75] Quintana, C., Reiser, B. J., Davis, E. A., Krajcik, J., Fretz, E., Duncan, R. G., ... & Soloway, E. (2004). A scaffolding design framework for software to support science inquiry. *The journal of the learning sciences*, 13(3), 337-386.

dynamic geometry software.[76] Another example was the implementation of IBL through the creative use of dynamic files to make tessellations and metamorphoses.[77]

Linn defines this process including a problem definition, critical thinking on the available alternatives, planning of possible solutions, on the base of hypothesis, information research, a model construction, a discussion among peers in order to propose shared conclusions.[78] In this context the inquiry-based approach in teaching and learning process is a set of real actions making up a more challenging process where students are motivated to pose their own questions when facing problems or scenarios.

The mathematical inquiry has similarities with inquiry in science.

In fact, both science and mathematical inquiry start from a question or a problem, and the answers are sought through observation and exploration. In many cases, experiments are conducted, mainly in virtual laboratories[79] by using digital tools like GEOMLAND[80], Cabri[81], Sketchpad[82], and GeoGebra[83].

Moreover, connections are made to questions offering interesting reasoning in both science and mathematical inquiry. This inquiry process is led by or leads to, hypothetical answers – often called conjectures – that are subject to validation. This is rarely a linear process. Quite often, initial conjectures are found to be true only under specific conditions, which may lead to their modification, or even to questioning the definition of the mathematical objects involved. Further, the process may lead to new questions and problems

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[76] Kenderov, P., Chehlarova, T., & Sendova, E. (2015). A Virtual Mathematics Laboratory in support of educating educators in inquiry-based style. In *Educating the educators: international approaches to scaling-up professional development in mathematics and science education. Proceedings of the conference hosted jointly by project mascil (mathematics and science for life) and the German Centre for Mathematics Education (DZLM), 15-16 December 2014 in Essen, Germany* (p. 167).

[77] Chehlarova, T., Sendova, E., & Stefanova, E. (2012). Dynamic tessellations in support of the inquiry-based learning of mathematics and arts. *Theory, Practice and Impact-Proceedings of Constructionism*, 21-25.

[78] Linn, M. C. (2013). *Internet environments for science education*. Routledge.

[79] Sendova, E. (2016). INQUIRY-BASED LEARNING AS A NATURAL VEHICLE FOR CROSS-CURRICULAR INTEGRATION: THE BULGARIAN EXPERIENCE. In *PROGRAM KONFERENCE 13 3. MEDNARODNI KONFERENCI O UČENJU IN POUČEVANJU MATEMATIKE KUPM 2016 NA POT ALI V TRETJE GRE RADO 20* (p. 33).

[80] *GEOMLAND* - Laboratory for Mathematics Explorations in Logo style is the learning environment used in the teacher training and in secondary schools. A web version was developed to provide teachers with the software, research and methodology papers as well as demonstrations and examples. Nikolova, I. (2001). Teacher development in ICT: Vision and Implementation. In *Information and Communication Technologies in Education* (pp. 71-82). Springer, Boston, MA.

[81] *Cabri* is the dynamic math and geometry software to create geometric and numerical constructions such as transformations, measurements and calculus, tables and graphical representations, expressions and equations. Available at link https://cabri.com/en/student/cabri-ii-plus/ [Retrieved online 27/04/2020].

[82] *Sketchpad* to create digital artwork to share online and export to popular image format.Available at link https://sketch.io/sketchpad/ [Retrieved online 27/04/2020].

[83] GeoGebra is a software for learning and teaching mathematics providing tools for the study of geometry, algebra and analysis. Available at link https://www.geogebra.org/?lang=it. [Retrieved online 27/04/2020].

whose solution may affect the answers to the initial question, or even the formulation of the question itself. [84]

The productive use of the learning implies different factors: a defined context, a frame, a focal point for the questions at different levels. Therefore teaching overshoots the simple data and information transmission by improving the production of useful and applicable knowledge. This process is encouraged by the inquiry-based approach application.[85]

From the teacher's point of view, adopting the inquiry methodology means to aim at developing the students' investigative capacities and attitudes and mindset that allow them to increase their knowledge for all their life. They learn to learn.

In this learning process, students are involved actively in the production of new knowledge and their evaluation has to determine their progress in the development of their capacity more than the knowledge level of contents. Thus, the issue, that motivates the students and increases their curiosity, is to know *how to know* and not only knowledge in itself. As a result, we can define the mainstream didactic method as *thinking of what* or declarative knowledge, while the inquiry-based learning approach is *thinking how* or procedural.[86]

In this context, art is a way for students to express their creativity and to find harmony in the development of both cognitive and emotional dimensions, but, at the same time, it obeys to mathematics rules and propositions. Similarly, mathematics reconstructs reality according to its laws getting an idealized replica of the subject. Consequently, art and mathematics are closely related.

Let us consider some examples of the interconnection between art and mathematics through the artworks by Escher:

[84] Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *Zdm*, *45*(6), 797-810.
[85] Kuster, G., Johnson, E., Keene, K., & Andrews-Larson, C. (2018). Inquiry-oriented instruction: A conceptualization of the instructional principles. *PRIMUS, 28*(1), 13-30.
[86] Rosenshine, B. (2012). Principles of Instruction: Research-Based Strategies That All Teachers Should Know. *American educator*, *36*(1), 12.

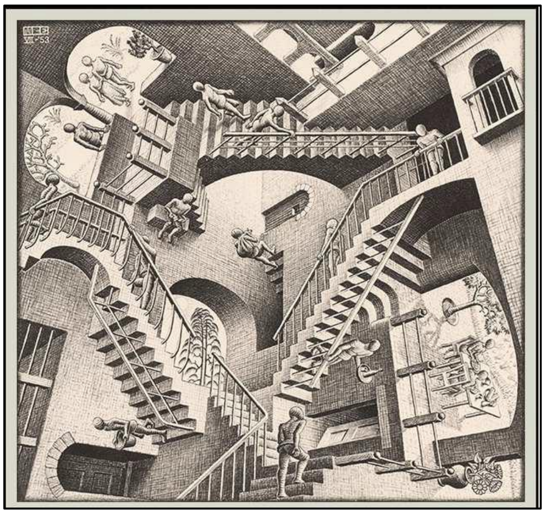

Figure 14: Relativity.

Figure 15: Gravity.

Source: http://www.mcescher.com/

Other examples of the interconnection between art and mathematics are the perspective in painting, the harmonic series and Fibonacci numbers in music, the symmetry of the body or the balance in dance allowing to find the *center of gravity* and to maintain the stability.[87] It is enough to think, also, of the concept of *beauty* in human beings and the *golden ratio* that is a geometrical proportion, a reference standard for the artists, used, especially, in figurative arts to create harmony, perfection that is in a one-word is *beauty*.[88]

In the representation of Vitruvian Man realized by Leonardo Da Vinci,[89] we can observe that the human figure follows very precise symmetrical relationships: the man's height is equal to the distance between the ends of the hands with arms extended determining the possibility to inscribe the whole figure in a square (Figure 16).

[87] Maletic, V. (2011). *Body-space-expression: The development of Rudolf Laban's movement and dance concepts* (Vol. 75). Walter de Gruyter.

[88] Severini, G., & Pacini, P. (1972). *Dal cubismo al classicismo e altri saggi sulla divina proporzione e sul numero d'oro*: A cura di Piero Pacini. Marchi & Bertolli.

[89] Livio, M. (2003). *La sezione aurea. Storia di un numero e di un mistero che dura da tremila anni.* Rizzoli , Milano.

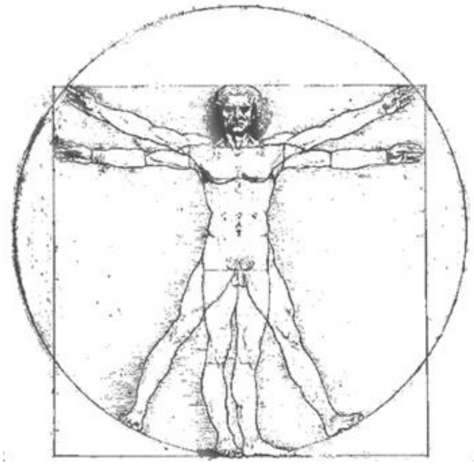

Figure 16: Vitruvian Man of Leonardo Da Vinci.
Source: https://www.leonardodavinci.net/the-vitruvian-man.jsp

Exploiting the potentialities of the interconnections between mathematics and art, students can discover how mathematics and scientific rules have an impact on all the aspects of reality.

1.4.4 Objectives

The research objectives can be synthesized as follows:

- Exploiting the possibility to find a combination between Western and Eastern approaches (mainly Singapore's method) in mathematics teaching.
- Finding the result of this combination in the arts.

Demonstrating the objectives above encourage meaningful learning in the development of the research skills to improve the school performance (short-term objective) and increase interest towards a future perspective in the mathematics field (long-term objective).

The objective of the study has been reached in three research years and was divided into different tasks: 1) Study of the problem, 2) Development of a research teaching approach combined with the arts, 3) Experimentation phase and 4) Data processing and evaluation of the achieved results. Each of them is described in the following sections.

1.4.4.1 Task 1: Study of the problem

This task has started from the literature review aiming at defining the current status quo in mathematics education. The aim was to investigate how math topics are defined in PISA, TIMSS, and national surveys and to compare the school curriculum on math in diverse learning environments.

This task was accomplished through the following steps:

- Analysis of the existing practices of the school approaches for mathematics topics in lower and upper secondary schools;
- Analysis of relevant factors such as current pedagogies mathematics-related subjects at the lower and upper secondary level, number of mathematics students at the university level;
- Gender Analysis of a different approach to the mathematics subjects, of the existing technologies applied to improve student learning and skills, of the different use of creativity;
- Analysis of the teachers and students attitudes towards the introduction of innovation into school and in learning and teaching approaches;

- Comparison of learning and teaching approaches applied to the mathematical subject between Eastern (Singapore's method) and Western methods;
- Collection of a relevant bibliography, such as books, articles, and papers through public databases or specialized libraries.

1.4.4.2 Task 2: Development of a research teaching/learning approach combined with the arts

This task aimed to define the research model approach for mathematics education as a result of the combination of Western and Eastern teaching approaches with the arts. This allowed collecting and analyzing all the information necessary to define the experimental path in terms of contents and structure that can match the teaching and learning needs of teachers and students.

The methodological assumption driving the entire research proposal is based on the design-based approach introduced in 1992 by *Brown*[90] and *Collins*[91] to overcome the limits established by the strictly experimental methodologies and by ethnographic ones. This research method is suitable for the complex dynamics typical of real educational situations.[92] The choice of the methodology is justified by the fact that the researcher's action interfered purposely with the education context because it was participatory. The researcher and teachers involved worked closely together collaborating actively in the design and the analysis. Due to the design-based research, in terms of methodology, a double analysis at theoretical (cognitive) and practical (didactic) levels was performed.

As to the definition of the research design, the steps identified by Kenneth D. Bailey were followed[93], involving directly teachers, such as:

- Definition of the *mathematization* concept to define a suitable training path adjusted to the knowledge and skills of students;
- Analysis of the exercises and mathematics concepts from school curricula;
- Selection of the mathematics exercises and concepts to be used during the experimental activities with the students following the teachers;

[90] Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The journal of the learning sciences*, 2(2), 141-178.
[91] Collins, A. (1992). Toward a design science of education. In *New directions in educational technology (pp. 15-22). Springer, Berlin, Heidelberg.*
[92] Pellerey, M. (2005). Verso una nuova metodologia di ricerca educativa: la Ricerca basata su progetti (Design-Based Research). *Orientamenti pedagogici*, 52(5), 721-737.
[93] Bailey, K. D., & Rossi, M. (1995). *Metodi della ricerca sociale*. Bologna: Il Mulino.

- Selection of the artworks containing mathematics patterns or concepts to be shown to students.
- Definition and selection of reference sample (teachers and students).
- Defining the contents and methodologies as well as the topics and the pedagogical tools to be used during the experimentation phase on the base of the results achieved during the previous tasks.
- Drawing up the guidelines for the experimentation activity management.
- Definition of data collection tools (qualitative and quantitative):

For the quantitative data

a) Definition of the indicators and variables;
b) Drawing up a questionnaire (semi-structured) divided into two parts submitted to the students, respectively before and after the experimentation;
c) Selection of mathematics exercises to be delivered to the students.

For qualitative data

a) Drawing up a semi-structured grid for the participatory observation;
b) Preparation of a grid for the qualitative data collection by the teacher;
c) Organization of a discussion group to collect the qualitative data coming out from students' feedback.

A better description of the experimentation activities is introduced later in Chapter 4 and Chapter 5 together with the achieved results.

1.4.4.3 Task 3: Experimentation phase

The aim was to carry out the experimentation activities to collect data and to start the evaluation of the research results.

This task required the following steps:

- Arrangement of the first meeting between teachers and students aiming at:

a) Showing different types of real connections between the artistic expression/creation and math patterns.
b) Submitting a questionnaire to the students related to their possible problems with the study of mathematics.

c) Submitting mathematics exercises selected previously.

- Realization of the experimentation activities with teachers and students involved.
- Organization of several meetings with teachers and students aiming at:

a) Permitting students to introduce their work and results achieved.
b) Submitting to students a second questionnaire to collect new information.
c) Establishing a discussion to gather informal feedback from students.
d) Submitting a set of the mathematics exercises related to the topics studied.

A full description of the research model and tools tested by teachers with their students is introduced in Chapter 3.

1.4.4.4 Task 4: Data processing and evaluation of the achieved results

The aim was to start the processing and evaluation of the research results.

This task required the following steps:

- Coding and analysis phases;
- Data interpretation (both qualitative and quantitative);
- Drawing up the final report with a discussion of the results.

For a more specific discussion, refer to Chapter 5 relating to the testing phase.

CHAPTER 2. LEARNING AND TEACHING APPROACHES: THE STATE-OF-THE-ART

The national and international investigations report that the learning process in mathematics becomes thwarting, exhausting and non-immediate for many students.[94] This sense of discomfort gets worse by the fact that, at the social level, people think being good at math is synonymous to be smart. As a result, failures in mathematics can affect a student's self-esteem. If success in mathematics requires special skills, then already first failures wouldn't foster in younger students the certainty of being able to improve. The vicious circle made by failures, low self-esteem, decreased motivation and reduced care can't be solved without the support of a teacher, as an expert who can facilitate the reconstruction of a positive relationship with mathematics at school.[95]

Another factor, which influences mathematics learning, is what is called *error* and the attitude of both teachers and students towards it. When people think of *error*, immediately, they often refer to a binary conception of knowledge: knowledge *right* or *wrong*.

In the pedagogical field, the concept of *error* is something different from what is simply *wrong*. The error is usually due to the lack of knowledge, and the mistake is accidental and could be self-corrected or corrected by their teacher. In particular, in lower and upper secondary schools the errors are often confused with mistakes determining, consequently, frustration in young students.[96]

Therefore, also the mathematics difficulties derive from this misunderstanding turning it into a strictly didactic problem.[97] The traditional approach of teachers towards a student with mathematics difficulties foresees, usually, the correction of errors or the explanation of the correct procedure and, possibly, the repetition of those topics considered important by the teacher that students answer correctly.

Unfortunately, these targeted actions don't produce the desired effects, but, paradoxically, they increase the differences between students with high and low performances compounding the original problem.[98]

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[94] Zan, R. (2007). L'interpretazione degli errori: prime osservazioni. *Difficoltà in matematica: Osservare, interpretare, intervenire*, 69-112.

[95] Rif. Bartolini Bussi, M. G., Ramploud, A., & Baccaglini-Frank, A. (2013). *Aritmetica in pratica. Strumenti e strategie dalla tradizione cinese per l'inizio della scuola primaria*, Erickson, Trento.

[96] Binanti, L. (Ed.). (2001). *Pedagogia, epistemologia e didattica dell'errore*. Rubbettino Editore.

[97] Zan, R., & Di Martino, P. (2004). Io e la matematica»: una, cento, mille storie. *La didattica della matematica: una scienza per la scuola*.

[98] Zan R. (2007), *Op.cit.*

In fact, the attention towards the typical errors widens the gap among students because only the 'good' students seem they take advantage of it.

This is what *Rosetta Zan* calls *teacher antinomy*. Actually, the author claims that even the benefits for a “good” student are purely fictitious so that sometimes they turn into damage because learners think that mathematics learning doesn’t require a specific study but it is enough to pay attention in class.[99]

According to the constructivist theory,[100] knowledge is largely constructed by learners, who don’t only add new information to their previous knowledge, but they create new links building new relationships with new information. This allows explaining many mathematical errors in an alternative manner than the traditional one, for which the error is simply due to a lack of knowledge and skills. Therefore, it is necessary to apply a methodology focused more on: (i) *problem-solving* overcoming the difficulties in its utilization in the classroom and (ii) finding suitable problems to be discussed with students.

However, there are significant differences when considering Western and Eastern students.

[99] Zan R. (2007), *Op.cit.*
[100] Waite-Stupiansky, S. (2017). Jean Piaget’s constructivist theory of learning. In *Theories of Early Childhood Education* (pp. 3-17). Routledge.

2.1 Substantial differences in mathematics performances

The last OECD-PISA (international survey on mathematics, science and reading) was organized in 2018, enabling thus the comparisons in student performance over time from 2009. The PISA 2018 mathematics framework is organized into several major sections. The first section, *Definition of Mathematical Literacy*, explains the theoretical underpinnings of the PISA mathematics assessment, including the formal definition of the mathematical literacy - construct. The second section, *Organisation of the Domain*, describes three aspects: the mathematical processes and the fundamental mathematical capabilities (e.g. *competencies*); the way mathematical content knowledge is organized and the contexts in which students deal with mathematical challenges.[101]

On the base of the results achieved, the students from the following Asian countries were top-ranked: Singapore reached a score of 569 above the OECD average 489 followed by Hong Kong – China (551), Macau-China (558), Beijing-Shanghai-Jiangsu-Zhejiang (Regions of China participating in testing) (591).[102]

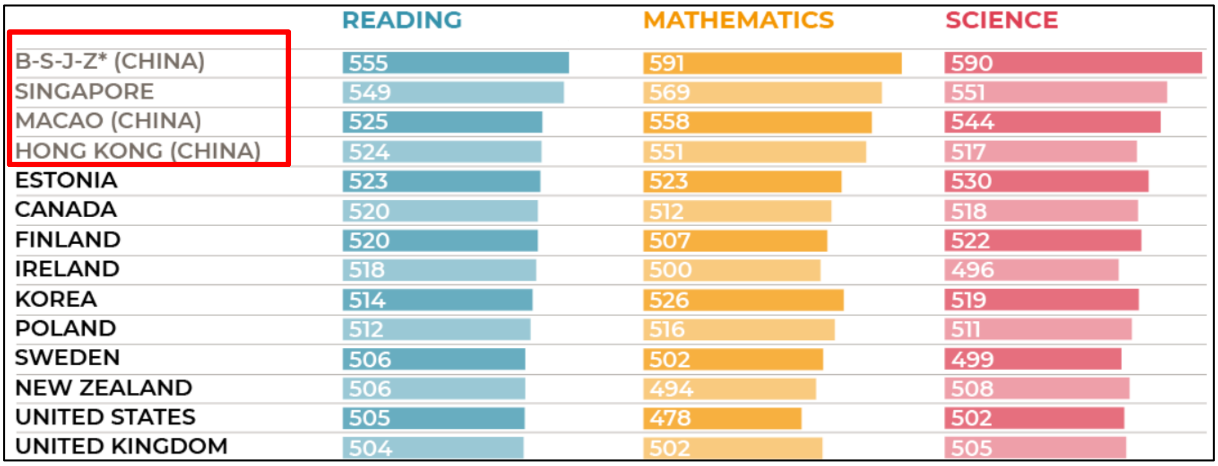


Figure 17: Snapshot of students' performance in reading, mathematics and science.

Source: OECD Education Statistics: PISA Results -2018

With the comparison of the previous PISA results, these countries confirmed their higher positions with reporting no differences between boys and girls.

On the other hand, Italian students' last performances in mathematics demonstrate a negative trend. They don't overcome the OECD average score by ranking at 31° place. Also in other countries, like Finland, even if still placed over the OECD average, the final ranking list

[101] OECD (2019), *PISA 2018 Assessment and Analytical Framework*, PISA, OECD Publishing, Paris, https://doi.org/10.1787/b25efab8-en.

[102] OECD (2019), Snapshot of performance in reading, mathematics and science, *Op.Cit.*

underlines a negative trend concerning the previous results. In particular, in Finland, PISA data in 2006 indicates a negative relation between the performance mean score and the students' interest score: highest in science score, lowest in interest. This shows that the negative score in the students' interest has, over the years, led to a lowering of the performance score, losing in PISA survey of 2018 points.

Besides, even countries as the United States show a main score substantially below the OECD average by ranking at 37° place with a score of 478 as shown in Figure 18.[103]

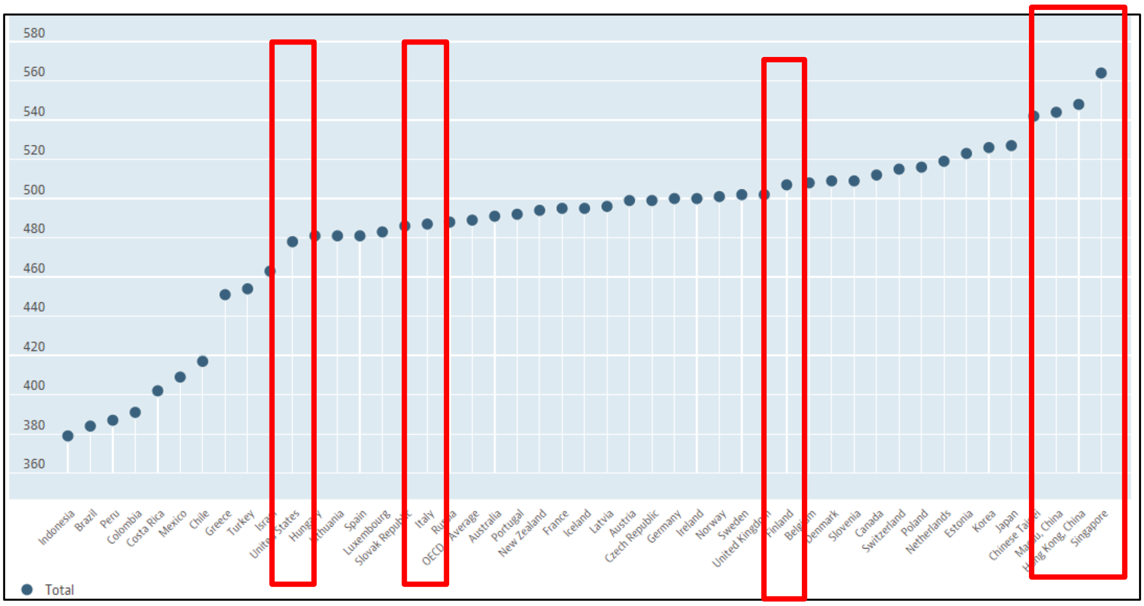


Figure 18: Comparison among countries in the average score in mathematics.

Source: OECD Education Statistics: PISA: Programme for International Student Assessment -2018

Moreover, the last results achieved by OECD PISA 2015 show, again, that Chinese performances are better than in the other countries, as the following Table 3 describes.

[103] OECD (2019), *Op.cit.*

Table 3 - Snapshot of performance in mathematics, reading, and science

Countries/economies with a mean performance/share of top performers **above** the OECD average
Countries/economies with a share of low achievers **below** the OECD average

Countries/economies with a mean performance/share of top performers/share of low achievers not significantly different from the OECD average

Countries/economies with a mean performance/share of top performers **below** the OECD average
Countries/economies with a share of low achievers **above** the OECD average

	Science		Reading		Mathematics		Science, reading and mathematics	
	Mean score in PISA 2015	Average three-year trend	Mean score in PISA 2015	Average three-year trend	Mean score in PISA 2015	Average three-year trend	Share of top performers in at least one subject (Level 5 or 6)	Share of low achievers in all three subjects (below Level 2)
	Mean	Score dif.	Mean	Score dif.	Mean	Score dif.	%	%
OECD average	493	-1	493	-1	490	-1	15.3	13.0
Singapore	556	7	535	5	564	1	39.1	4.8
Japan	538	3	516	-2	[illegible]	[illegible]	25.8	5.6
Estonia	534	2	519	9	520	2	20.4	4.7
Chinese Taipei	532	0	497	1	542	0	29.9	8.3
Finland	531	-11	526	-5	511	-10	21.4	6.3
Macao (China)	529	6	509	11	544	5	23.9	3.5
Canada	528	-2	527	1	516	-4	22.7	5.9
Viet Nam	525	-4	487	-21	495	-17	12.0	4.5
Hong Kong (China)	523	-5	527	-3	548	1	29.3	4.5
B-S-J-G (China)	518	m	494	m	531	m	27.7	10.9
Korea	516	-2	517	-11	524	-3	25.6	7.7
New Zealand	513	-7	509	-6	495	-8	20.5	10.6
Slovenia	513	-2	505	11	510	2	18.1	8.2
Australia	510	-6	503	-6	494	-8	18.4	11.1
United Kingdom	509	-1	498	2	492	-1	16.9	10.1
Germany	509	-2	509	6	506	2	19.2	9.8
Netherlands	509	-5	503	-3	512	-6	20.0	10.9
Switzerland	506	-2	492	-4	521	-1	22.2	10.1
Ireland	503	0	521	13	504	0	15.5	6.8
Belgium	502	-3	499	-4	507	-5	19.7	12.7
Denmark	502	2	500	3	511	-2	14.9	7.5
Poland	501	3	506	3	504	5	15.8	8.3
Portugal	501	8	498	4	492	7	15.6	10.7
Norway	498	3	513	5	502	1	17.6	8.9
United States	496	2	497	-1	470	-2	13.3	13.6
Austria	495	-5	485	-5	497	-2	16.2	13.5
France	495	0	499	2	493	-4	18.4	14.8
Sweden	493	-4	500	1	494	-5	16.7	11.4
Czech Republic	493	-5	487	5	492	-6	14.0	13.7
Spain	493	2	496	7	486	1	10.9	10.3
Latvia	490	1	488	2	482	0	8.3	10.5
Russia	487	3	495	17	494	6	13.0	7.7
Luxembourg	483	0	481	5	486	-2	14.1	17.0
Italy	481	2	485	0	490	7	13.5	12.2
Hungary	477	-9	470	-12	477	-4	10.3	18.5
Lithuania	475	-3	472	2	478	-2	9.5	15.3
Croatia	475	-5	487	5	464	0	9.3	14.5
CABA (Argentina)	475	51	475	46	456	38	7.5	14.5
Iceland	473	-7	482	-9	488	-7	13.2	13.2
Israel	467	5	479	2	470	10	13.9	20.2
Malta	465	2	447	3	479	9	15.3	21.9
Slovak Republic	461	-10	453	-12	475	-6	9.7	20.1
Greece	455	-6	467	-8	454	1	6.8	20.7
Chile	447	2	459	5	423	4	3.3	23.3
Bulgaria	446	4	432	1	441	9	6.9	29.6

Source: OECD processing based on database PISA 2015

This was judged as absolutely surprising by both Chinese and Western researchers, who, on the other hand, point out two paradoxes:[104]

- *Student paradox*: Chinese students may be thought to use low-level strategies and learning approach based on rote, but, in reality, they prefer high-level strategies and based on the meaning getting better results in the international surveys.
- *Teacher paradox*: Chinese teachers can reach results of meaningful learning even working in contexts considered as unfit by Western teachers.

2.1.1 Factors influencing mathematics performance

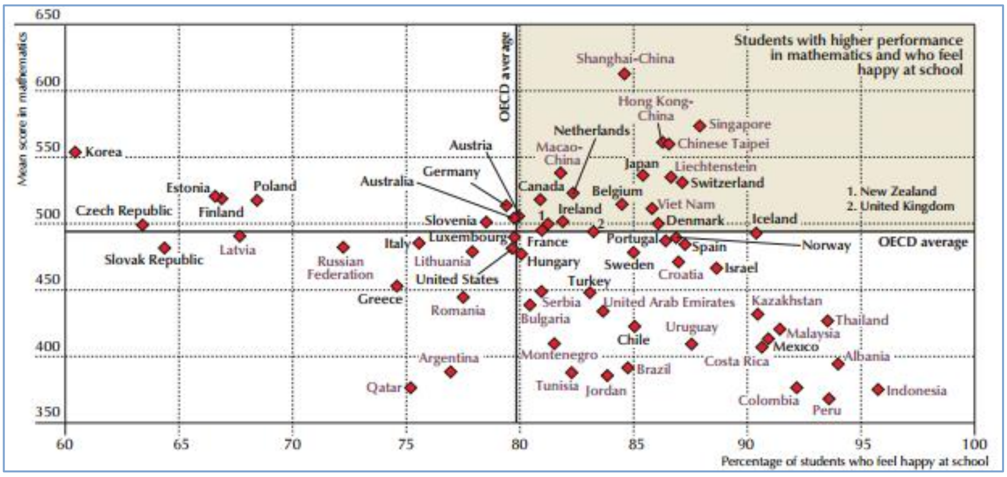
An interesting factor, underlining another difference between Western and Asian students, emerging from these administered surveys, is the quality of relationships between teacher and student analyzed in the framework of survey *Do teacher-student relations affect student's*

[104] Bartolini Bussi, M. G., Ramploud, A., & Baccaglini-Frank, A. (2013). Aritmetica in pratica. Strumenti e strategie dalla tradizione cinese per l'inizio della scuola primaria.

well-being at school.[105] For the first time starting from the survey realized in 2012, student happiness and their well-being at school were evaluated.[106]

As a result, students in Belgium, Hong Kong China, Japan, Liechtenstein, Shanghai-China, Singapore, Switzerland, and Chinese Taipei have achieved a high score (above-average OECD) as well as above-average performances in mathematics as shown in Table 4 here below:

Table 4 - Where students are both happy and high-achieving



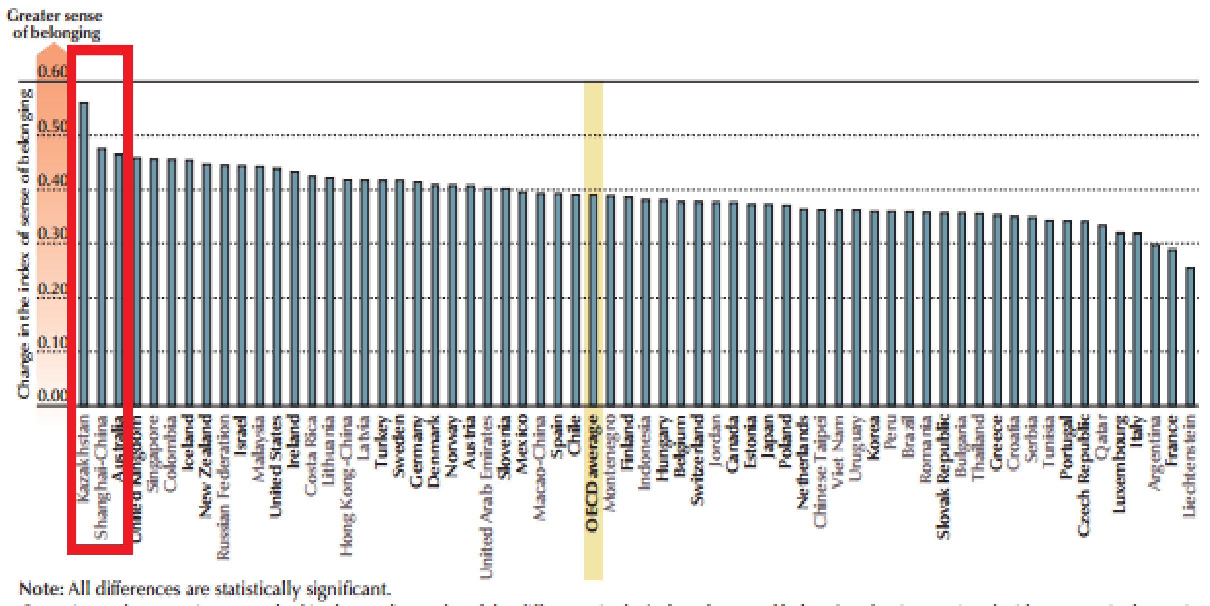
Source: OECD processing based on database PISA 2012[107]

The collected data show a strict relation between student well-being and their school performances. Indeed, the countries, such as China, at the top of the ranking, are still those where young students get the best scores in mathematics and language understanding tests as described in Table 4.

The results of the PISA survey show that positive and constructive teacher-student relations are good ground for better performance in mathematics. Therefore this could be a key factor for fostering the social and emotional development of students at school. The results achieved by students show that "sense of belonging among students" ensures a good quality of teacher-student relation (Table 5).

[105] Mikk, J., Krips, H., Säälik, Ü., & Kalk, K. (2016). Relationships between student perception of teacher-student relations and PISA results in mathematics and science. *International journal of science and mathematics education*, 14(8), 1437-1454.
[106] Borgonovi, F., & Pál, J. (2016). *A framework for the analysis of student well-being in the PISA 2015 study.*
[107] Ibid.

Table 5 - Good teacher-student relations foster a sense of belonging among students



Note: All differences are statistically significant.
Countries and economies are ranked in descending order of the difference in the index of sense of belonging that is associated with a one-unit change in the index of teacher-student relations after accounting for differences in students' socio-economic status and performance in mathematics.

Source: OECD processing based on database PISA 2012[108]

Comparing countries and economies with two indicators, *teacher-student relation and mathematics performance*, shows that in some countries, where the development in a competitive, knowledge-based global economy is crucial, like Shangai-China, Singapore, Hong Kong-China and Chinese Taipei, students show high performance. On the contrary, countries, like Italy, France, Hungary, Bulgaria, Greece, are still below the OECD average, as the following table describes:[109]

[108] Borgonovi, F., & Pál, J. (2016). *Op. Cit.*
[109] OECD, *Do teacher-student relations affect students' well-being at school?*, <<PISA in Focus>>, OECD 04 April, (50) 2015.

Table 6 - Comparing countries' and economies' performance in mathematics

Statistically significantly **above** the OECD average
Not statistically significantly different from the OECD average
Statistically significantly **below** the OECD average

Mean score	Comparison country/economy	Countries and economies whose mean score is not statistically significantly different from the comparison country's/economy's score
591	B-S-J-Z (China)	
569	Singapore	
558	Macao (China)	Hong Kong (China)[1]
551	Hong Kong (China)[1]	Macao (China)
531	Chinese Taipei	Japan, Korea
527	Japan	Chinese Taipei, Korea, Estonia
526	Korea	Chinese Taipei, Japan, Estonia, Netherlands[1]
523	Estonia	Japan, Korea, Netherlands[1]
519	Netherlands[1]	Korea, Estonia, Poland, Switzerland
516	Poland	Netherlands,[1] Switzerland, Canada
515	Switzerland	Netherlands,[1] Poland, Canada, Denmark
512	Canada	Poland, Switzerland, Denmark, Slovenia, Belgium, Finland
509	Denmark	Switzerland, Canada, Slovenia, Belgium, Finland
509	Slovenia	Canada, Denmark, Belgium, Finland
508	Belgium	Canada, Denmark, Slovenia, Finland, Sweden, United Kingdom
507	Finland	Canada, Denmark, Slovenia, Belgium, Sweden, United Kingdom
502	Sweden	Belgium, Finland, United Kingdom, Norway, Germany, Ireland, Czech Republic, Austria, Latvia
502	United Kingdom	Belgium, Finland, Sweden, Norway, Germany, Ireland, Czech Republic, Austria, Latvia, France
501	Norway	Sweden, United Kingdom, Germany, Ireland, Czech Republic, Austria, Latvia, France, Iceland
500	Germany	Sweden, United Kingdom, Norway, Ireland, Czech Republic, Austria, Latvia, France, Iceland, New Zealand
500	Ireland	Sweden, United Kingdom, Norway, Germany, Czech Republic, Austria, Latvia, France, Iceland, New Zealand
499	Czech Republic	Sweden, United Kingdom, Norway, Germany, Ireland, Austria, Latvia, France, Iceland, New Zealand, Portugal[1]
499	Austria	Sweden, United Kingdom, Norway, Germany, Ireland, Czech Republic, Latvia, France, Iceland, New Zealand, Portugal[1]
496	Latvia	Sweden, United Kingdom, Norway, Germany, Ireland, Czech Republic, Austria, France, Iceland, New Zealand, Portugal,[1] Australia
495	France	United Kingdom, Norway, Germany, Ireland, Czech Republic, Austria, Latvia, Iceland, New Zealand, Portugal,[1] Australia
495	Iceland	Norway, Germany, Ireland, Czech Republic, Austria, Latvia, France, New Zealand, Portugal,[1] Australia
494	New Zealand	Germany, Ireland, Czech Republic, Austria, Latvia, France, Iceland, Portugal,[1] Australia
492	Portugal[1]	Czech Republic, Austria, Latvia, France, Iceland, New Zealand, Australia, Russia, Italy, Slovak Republic
491	Australia	Latvia, France, Iceland, New Zealand, Portugal,[1] Russia, Italy, Slovak Republic
488	Russia	Portugal,[1] Australia, Italy, Slovak Republic, Luxembourg, Spain, Lithuania, Hungary
487	Italy	Portugal,[1] Australia, Russia, Slovak Republic, Luxembourg, Spain, Lithuania, Hungary, United States[1]
486	Slovak Republic	Portugal,[1] Australia, Russia, Italy, Luxembourg, Spain, Lithuania, Hungary, United States[1]

Source: OECD processing based on database PISA 2018[110]

Looking through the PISA results in 2015, another relevant factor comes out. In particular, the results related to students' science beliefs, engagement and motivation are interesting if the achievements in Asian countries are compared with some Western countries, such as Italy, as follows in Table 7:

[110] Borgonovi, F., & Pál, J. (2016). *Op. Cit.*

Table 7- Snapshot of students' science beliefs, engagement, and motivation

Countries/economies with values above the OECD average

Countries/economies with values not significantly different from the OECD average

Countries/economies with values below the OECD average

	Mean science score	Beliefs about the nature and origin of scientific knowledge		Share of students with science-related career expectations				Motivation for learning science		
		Index of epistemic beliefs (support for scientific methods of enquiry)	Score-point difference per unit on the index of epistemic beliefs	All students	Boys	Girls	Increased likelihood of boys expecting a career in science	Index of enjoyment of learning science	Score-point difference per unit on the index of enjoyment of learning science	Gender gap in enjoyment of learning science (Boys - Girls)
	Mean	Mean index	Score dif.	%	%	%	Relative risk	Mean index	Score dif.	Dif.
OECD average	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Singapore	556	0,22	34	28,0	31,8	23,9	1,3	0,59	35	0,17
Japan	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Estonia	534	0,01	36	24,7	28,9	20,3	1,4	0,16	24	0,05
Chinese Taipei	532	0,31	38	20,9	25,6	16,0	1,6	-0,06	28	0,39
Finland	531	-0,07	38	17,0	15,4	18,7	0,8	-0,07	30	0,04
Macao (China)	529	-0,06	26	20,8	22,0	19,6	1,1	0,20	21	0,16
Canada	528	0,30	29	33,9	31,2	36,5	0,9	0,40	26	0,15
Viet Nam	525	-0,15	31	19,6	21,2	18,1	1,2	0,65	14	0,06
Hong Kong (China)	523	0,04	23	23,6	22,9	24,2	0,9	0,28	20	0,26
B-S-J-G (China)	518	-0,08	37	16,8	17,1	16,5	1,0	0,37	28	0,14
Korea	516	0,02	38	19,3	21,7	16,7	1,3	-0,14	31	0,32
New Zealand	513	0,22	40	24,8	21,7	27,9	0,8	0,20	32	0,03
Slovenia	513	0,07	33	30,8	34,6	26,8	1,3	-0,36	22	-0,03
Australia	510	0,26	39	29,2	30,3	28,2	1,1	0,12	33	0,16
United Kingdom	509	0,22	37	29,1	28,7	29,6	1,0	0,15	30	0,18
Germany	509	-0,16	34	15,3	17,4	13,2	1,3	-0,18	29	0,43
Netherlands	509	-0,19	46	16,3	16,9	15,7	1,1	-0,52	30	0,25
Switzerland	506	-0,07	34	19,5	19,8	19,1	1,0	-0,02	30	0,17
Ireland	503	0,21	36	27,3	28,0	26,6	1,1	0,20	32	0,09
Belgium	502	0,00	34	24,5	25,3	23,6	1,1	-0,03	28	0,20
Denmark	502	0,17	32	14,8	11,8	17,7	0,7	0,12	26	0,09
Poland	501	-0,08	27	21,0	15,4	26,8	0,6	0,02	18	-0,10
Portugal	501	0,28	33	27,5	26,7	28,3	0,9	0,32	23	0,08
Norway	498	-0,01	35	28,6	28,9	28,4	1,0	0,12	29	0,27
United States	496	0,25	32	38,0	33,0	43,0	0,8	0,23	26	0,21
Austria	495	-0,14	36	22,3	26,6	18,0	1,5	-0,32	25	0,23
France	495	0,01	30	21,2	23,6	18,7	1,3	-0,03	30	0,31
Sweden	493	0,14	38	20,2	21,8	18,5	1,2	0,08	27	0,22
Czech Republic	493	-0,23	41	16,9	18,6	15,0	1,2	-0,34	27	-0,06
Spain	493	0,11	30	28,6	29,5	27,8	1,1	0,03	28	0,11
Latvia	490	-0,26	27	21,3	21,1	21,5	1,0	0,09	18	0,03
Russia	487	-0,26	27	23,5	23,2	23,8	1,0	0,00	16	0,07
Luxembourg	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Italy	481	-0,10	34	22,6	24,7	20,6	1,2	0,00	22	0,24
Hungary	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Lithuania	475	0,11	22	23,9	22,5	25,4	0,9	0,36	20	-0,14
Croatia	475	0,03	32	24,2	26,8	21,8	1,2	-0,11	22	0,05
CABA (Argentina)	475	0,09	28	27,8	26,2	29,3	0,9	-0,20	15	-0,14
Iceland	473	0,29	28	23,8	20,1	27,3	0,7	0,15	24	0,26
Israel	467	0,18	38	27,8	26,1	29,5	0,9	0,09	20	0,06
Malta	465	0,09	54	25,4	30,2	20,4	1,5	0,18	48	0,11
Slovak Republic	461	-0,35	36	18,8	18,5	19,0	1,0	-0,24	25	-0,02
Greece	455	-0,19	36	25,3	25,7	24,9	1,0	0,13	27	0,12
Chile	447	-0,15	23	37,9	36,9	39,0	0,9	0,08	15	-0,09
Bulgaria	446	-0,18	34	27,5	28,8	25,9	1,1	0,28	17	-0,16
United Arab Emirates	437	0,04	33	41,3	39,9	42,6	0,9	0,47	22	-0,02
Uruguay	435	-0,13	27	28,1	23,8	31,9	0,7	-0,10	16	-0,07
Romania	435	-0,38	27	23,1	23,3	23,0	1,0	-0,03	17	-0,05
Cyprus[1,2]	433	-0,15	33	29,9	29,3	30,5	1,0	0,15	29	0,06
Moldova	428	-0,14	37	22,0	22,5	21,3	1,1	0,33	22	-0,17
Albania	427	-0,03	m	24,8	m	m	m	0,72	m	m
Turkey	425	-0,17	18	29,7	34,5	24,9	1,4	0,15	12	0,01
Trinidad and Tobago	425	-0,02	28	27,8	24,6	31,0	0,8	0,19	24	-0,01
Thailand	421	-0,07	35	19,7	12,4	25,2	0,5	0,42	18	-0,05
Costa Rica	420	-0,15	16	44,0	43,8	44,2	1,0	0,35	4	-0,03
Qatar	418	-0,10	33	38,0	36,3	39,9	0,9	0,36	25	0,00
Colombia	416	-0,19	21	39,7	37,1	42,0	0,9	0,32	7	-0,02
Mexico	416	-0,17	17	40,7	45,4	35,8	1,3	0,42	12	0,01
Montenegro	411	-0,32	23	21,2	20,1	22,4	0,9	0,09	14	-0,07
Georgia	411	0,05	42	17,0	16,4	17,7	0,9	0,34	23	-0,13
Jordan	409	-0,13	28	43,7	44,6	42,8	1,0	0,53	23	-0,25
Indonesia	403	-0,30	16	15,3	8,6	22,1	0,4	0,65	6	-0,06
Brazil	401	-0,07	27	38,8	34,4	42,8	0,8	0,23	19	-0,04
Peru	397	-0,16	23	38,7	42,7	34,6	1,2	0,40	9	0,01
Lebanon	386	-0,24	35	39,7	41,0	38,5	1,1	0,38	32	-0,04
Tunisia	386	-0,31	18	34,4	28,5	39,5	0,7	0,52	15	-0,12
FYROM	384	-0,18	30	24,2	20,0	28,8	0,7	0,48	17	-0,29
Kosovo	378	0,03	22	26,4	24,7	28,1	0,9	0,92	14	-0,16
Algeria	376	-0,31	16	26,0	23,1	29,2	0,8	0,46	14	-0,12
Dominican Republic	332	-0,10	13	45,7	44,7	46,8	1,0	0,54	6	-0,05

Source: OECD, PISA 2015 Database

High performances in Asian countries, e.g. Singapore, express a more positive and inclusive image of science in students. This enables a positive attitude towards scientific careers and learning motivation enhanced.

On the other side, low performances, as in Italy, are strictly related to the creation of a mental negative image, stereotyped on who are the scientists and the engineers and on who are the people choosing this kind of career. Since science knowledge and understanding are useful even beyond the scientists' work and are necessary for full participation in a science-based technology world, school scientific topics should be promoted more positively by enhancing new resources and methodologies to increase student interest and enjoyment.[111]

[111] Gurria, A. (2016). *PISA 2015 results in focus*. PISA in Focus, (67), 1.

2.2 Substantial factors influencing survey results

The latest OECD PISA results show which countries are the highest performers and which are getting better in science mathematics and reading. The main aim of this inquiry is not to test the rote learning but how well students can solve a problem and apply their knowledge and skills to real-world situations.

As in the previous paragraphs described, the results show that Singapore has been the highest-scoring country in all areas since it joined testing in 2009.

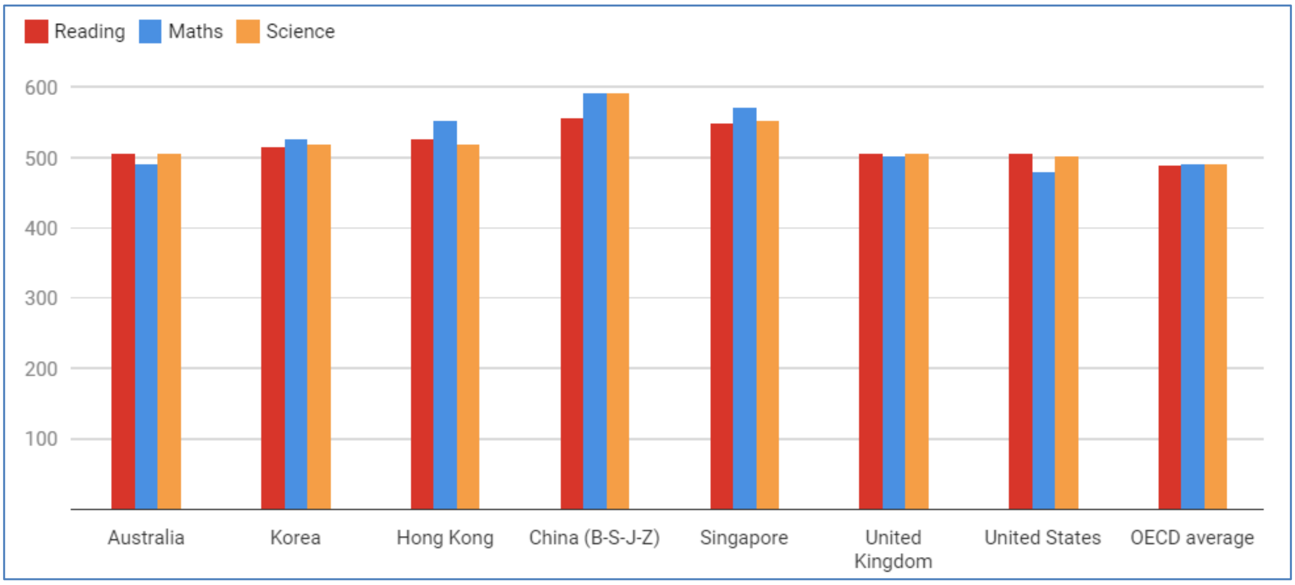


Figure 19: PISA 2018 scores in reading, mathematics, and science.

Beijing, Shanghai, Jiangsu, and Zhejiang (B-S-J-Z) are the regions of China that participated in testing.

Source: OECD Education Statistics - PISA 2018

The education systems in Singapore invest relentlessly in teachers: recruiting the best, training teachers well and giving them practical support and development as some other countries do. However, the results have revealed that they have followed a much more intensive process involving the whole teaching and learning process. In Singapore's case, for example, this high-performing is due to the results achieved by intentional government policies addressed to treat teachers as professional partners.

Actually, Singapore invests heavily in recruiting bright people to teach by encouraging many young people to apply to the selection process despite the difficulties of passing. To incentivize the applications, the Ministry of Education pays student teachers a relative salary during their initial teacher training.[112]

Singapore's government also reformed the teacher career structure in 2000 to help make it a more attractive choice by improving the salaries, creating a new system of teacher performance management, and developing a professional learning process.

In this context, the position of the *master teacher* was introduced in the school system. This professional figure is a teacher, who, having reached the outstanding performance in their subject field, becomes the teaching leader in these subjects, helping set directions and connecting schools to the best research.

In addition, they are also mentoring senior teachers in schools, who in turn, help others improve through regular classroom observations, research groups, professional learning communities and mentoring.[113]

Moreover, Singapore's school system strongly emphasizes practical skills and student learning, mainly in teacher initial education, managed and provided only by the National Institute of Education (NIE).

The strength of this system is due to the close relationship between the Ministry of Education and schools manifested in allowing the facilitated collection of feedback, suggestions and joint meetings to improve school learning continuously.[114]

[112] Jensen, B., Hunter, A., Sonnemann, J., & Burns, T. (2012). *Catching up: learning from the best school systems in East Asia*, Grattan Institute, viewed 21 April 2013.
[113] Ibid.
[114] Ibid.

2.3 Western current approaches to improve mathematics learning

During recent years, researchers have been identifying different approaches for mathematics learning and teaching to improve student performance. In a multiplicity of theories and methods, we will take into account only some of them, especially, those considered functional to the research objective, such as cooperative learning, problem-based learning, inquiry-based learning, and technology-enhanced learning.

2.3.1 Cooperative learning

Cooperative learning (CL) is a method involving students divided into small groups working on an assignment or project under conditions in which certain criteria are satisfied, by including that the group members be held individually accountable for the complete content of the assessment or project.[115]

This approach is useful to minimize the occurrence of those unpleasant situations and to maximize the learning and satisfaction that result from working on a high-performance team. The main idea is that students learn more by doing something actively through collective work with individual responsibility. Therefore, cooperative learning is an active and constructive method enhancing learning in different ways: students are "involved in doing things and thinking about the things they are doing".[116] Actually, this process engages students in the building, understanding of facts, ideas and skills through the completion of specific tasks and activities by involving them directly in their learning process.[117]

The working group for cooperative learning must have the following features:

1. <u>Positive interdependence</u>: the group members rely on each other to achieve the goal. Students should feel responsible for their personal learning as well as for the learning of the other group members.
2. <u>Individual responsibility</u>: all students in a group have to be responsible for what the whole group has learned. Each student, during the tests, should demonstrate individually what he/she has learned.
3. <u>Face-to-face interaction</u>: even if a part of the work can be done individually, all group members must work interacting, checking together with the reasoning process, the

[115] Felder, R. M., & Brent, R. (2007). *Cooperative learning*. ACS Publications.
[116] Eison, J. (2010). *Using active learning instructional strategies to create excitement and enhance learning*.
[117] Keengwe, J. (Ed.). (2015). *Handbook of research on educational technology integration and active learning*. IGI Global.

conclusions, overcoming the difficulties and providing with feedback and comments. In this way, students become like teachers for each other.

4. Collaborative: students in the group are encouraged to develop some abilities such as leadership, communication, decision making and how to defend their ideas, conflict resolutions in interpersonal relationships.
5. Evaluation of the work: the members, periodically, evaluate the effectiveness of their work and the functioning of the group, and identify the changes needed to improve its efficiency.[118]

Cooperative learning is not simply a synonym for students working in groups, but it has to include the five listed elements that are present. It is used for any type of assignment that can be given to students in lecture classes, laboratories or project-based training.

In the cooperative learning approach, the teacher will be a supervisor: he/she will have to work out of class by choosing the task to be done, forming groups, preparing the materials, while, in the classrooms; he/she will have to be attentive to interpersonal relationships, not making suggestions, coordinate and adjust the final discussion.

This technique has been successfully used in all STEM (Science, Technology, Engineering and Mathematics) disciplines.

For an effective application of cooperative learning approach, first of all, teachers have to know the educational background of their students related to the subject studied, their relationship with the other students in the class, their cognitive abilities, attitude towards school, teachers, subject, family situation, and extracurricular interests.[119]

This allows contextualizing the learning process by favoring the creations of meaningful patterns for students and which connect school contents to the context of their real-life experience.

If on one hand, this element helps students to make connections between the knowledge and their lives, on the other hand, it helps to store long-term memory which will support them to apply what they have learned in their professional life.

[118] Baldrighi, A., Pesci, A., & Torresani, M. (2003). Relazioni disciplinari e sociali nell'apprendimento cooperativo. Esperienze didattiche e spunti di riflessione. *Atti Matematica e Difficoltà n. 12 "Osservare, valutare, orientare gli alunni in difficoltà*, 170-178.

[119] Johnson, D. W., Johnson, R. T., & Holubec, E. J. (1996). *Apprendimento cooperativo in classe: migliorare il clima emotivo e il rendimento*. Edizioni Erickson.

In fact, contextualized learning states that learning occurs only when students process new information or knowledge in such a way that it makes sense in their reference framework.[120]

The teachers will make two types of control actions and review: the first–during the activities with the use of observation forms to monitor behavior and participation level of each student in the assigned work (monitoring phase); the second, at the end of the activity, comparing their observations and those from the student group to understand what needs to be improved (processing phase).

In summary, cooperative learning refers to work done by students delivering a product, such as a set of problem solutions, a laboratory or project report, or the design of a product or a process, under conditions that satisfy the five criteria mentioned above.

To implement the cooperative learning approach, teachers will have to do the following tasks:

1. Making a good analysis of the existing situation;
2. Identifying lessons to be performed;
3. Setting goals and tasks;
4. Taking organizational decisions;
5. Defining both monitoring and processing phases.

In addition, among the advantages of cooperative learning approach implementation, as demonstrated by extensive research, is that it leads to greater management and superior development of communication and teamwork skills, for example, leadership, project management, and conflict resolution skills.[121]

2.3.2 Problem-based learning

The problem-based learning (PBL) is an instructional methodology aiming at learning to solve problems. It is based on the assumption that *when we solve the many problems we face every day, learning occurs.*[122] The focus is on the problem which leads the student learning.

[120] Davtyan, R. (2014, April). Contextual learning. In *ASEE 2014 Zone I Conference* (pp. 3-5).
[121] Laal, M., & Ghodsi, S. M. (2012). Benefits of collaborative learning. *Procedia-social and behavioral sciences, 31*, 486-490.
[122] Barrows, H. S., & Tamblyn, R. M. (1980). *Problem-based learning: An approach to medical education*. Springer Publishing Company, p.1

The problem is posed, at the first step, so that the students can discover that they need to acquire a new knowledge before solving it.[123]

PBL's basic feature is that the students develop the ability to use knowledge to solve problems. They don't learn about problems but they are challenged to find out any possible solution to solve them.

Therefore, knowledge becomes a means not the final achievement. The teaching approach changes the perspective on how a student can deal with new knowledge and its construction.

PBL favors the development of general skills such as critical thinking and abstract reasoning because it becomes the ideal way for students to apply their theoretical knowledge authentically. This approach is used often to exploit student experience in a process of problem-solving, rather than simply seeking a *correct* solution.[124]

In fact, it supports the development of problem-solving, meant as the set of all the methods and techniques for the solution of the problems and the implementation of the related strategies.

The word *problem* stands for: 1. a matter to be solved starting from the elements known through reasoning; 2. a difficult or complex issue for which a solution is sought.

The attitude to problem-solving is often considered as an intelligence indicator. Solving problems is a matter of skill which can be acquired by imitation and practice. The best way to learn how to solve problems is through the observation and imitation of other people who know how to solve them and then making practice.

The teaching and learning approach based on a problem can be intuitive, systematic, algorithmic, and partial, by attempts, by exclusion. However, for a student can be suitable to use some approaches and no other, in relation to the specific problem, because a solving method may be more appropriate and fruitful than another. The application of one method depends also on the personal features and abilities.

In general, all these approaches are alternative ways to be used according to the problem met and their learning will guarantee the needed flexibility to deal with several types of problems or situations.[125]

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[123] Barrows, H. S. (1994). *Practice-based Learning: Problem-based Learning Applied to Medical Education*. Southern Illinois University, School of Medicine, PO Box 19230, Springfield, IL 62794-9230.

[124] Glover, I., Hepplestone, S., Parkin, H. J., Rodger, H., & Irwin, B. (2016). Pedagogy first: Realising technology enhanced learning by focusing on teaching practice. *British Journal of Educational Technology, 47*(5), 993-1002.

[125] Spagnolo, F. (1998). Insegnare le matematiche nella scuola secondaria (Manuale di Didattica delle Matematiche per la formazione post-universitaria). *La Nuova Italia Editrice*.

At a didactic level, therefore, it is important, on the one hand, to enhance and strengthen the styles and individual attitudes and, on the other hand, to enrich and diversify the problem-solving strategies, helping students discover and use different approaches. For example, when teachers administer to students a problem to be solved, they should take into account of two main purposes: a) to help the student solve the problem submitted; b) to develop students' skills so that they can use them in any situation.[126]

In this approach, the teacher should support students, providing them with needed inputs and suggestions to proceed independently in the search for the solution, and stimulating in them the *right* questions. *George Polya*, in his work *How to solve it*,[127] indicates a possible question-driving schedule that is divided into phases applicable to *proof problems* and not to *demonstration problems*.[128]

In summary, the problem-based learning approach can be applied in several disciplines from the practically-focused to more theoretical. It works well as an individual activity but is very effective when is implemented in mini-groups by encouraging the development of interpersonal, team-working, creativity and influencing skills in students.

2.3.3 Inquiry-based learning

The National Research Council (U.S.) has described the inquiry as *a set of related processes by which scientists and students asking questions about the natural world and investigate the phenomena; so that students acquire knowledge and develop a rich understanding of concepts, principles, models, and theories.*[129]

The first studies about the *inquiry* and on a method called *Inquiry-Based Science Education* (IBSE) are due to Rosalind Driver in the 70s-80s.[130]

However, in this research work, we will focus on this learning approach in mathematics education (IBME). IBME refers to student-centred teaching, in which students are asked to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, look for mathematical and scientific ways of how to answer these questions, such as carrying out experiments, systematically controlling

[126] Polya, G., Pólya, G., Pólya, G., Mathématicien, H., Suisse, E. U., Pólya, G., & Mathematician, H. (1970). *La scoperta matematica: capire, imparare e insegnare a risolvere i problemi*. Feltrinelli.
[127] Pólya, G. (1945). How to solve it. Princeton. *New Jersey: Princeton University*.
[128] Voltolini, F. (1969). POLYA, G. Come risolvere i problemi di matematica. Logica ed euristica nel metodo matematico.
[129] National Research Council. (1996). *National science education standards*. National Academies Press.
[130] Driver, R. (1983). *Pupil as scientist*. McGraw-Hill Education (UK).

variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalizations, interpreting and evaluating their solutions, communicate and discuss with the others (normally with their school mates) their solutions effectively.[131]

This method is considered by most researchers in science and mathematics education as the most effective teaching strategy.[132] Actually, in IBME, teachers use questions, problems and scenarios to help students learn through individual thought and investigation. Instead of simply presenting facts, the teacher encourages students to talk about a problem and draw on their intuition to understand it. Inquiry-based learning also focuses on letting students ask their questions — essentially providing their own inquiry. Student-led questions follow teacher-guided inquiry. Instead of lecturing about learning goals, the teacher cultivates a learning environment and helps students explore it through questions and experiences. [133]

Some characteristics can be underlined, such as: a. *process focus* - this approach valorizes the process over the product that means if students solve problems themselves, they internalize better the conceptual processes; b. *investigation* - the teacher may pose a problem or scenario derived from the class content or students' questions and the students then investigate to find an answer; c. *group learning* - working in pairs or in small groups to explore a problem, students help each other throughout their learning process to share and build upon ideas as well as articulate how they arrived at a solution; d. *discussion monitoring* - teachers can arise the discussion among students by posing questions to gauge their understanding and correct any misconceptions; e. *real-life application* - students learn how to solve math problems that have a meaningful life application.

In the IBME teaching, the laboratories are a fundamental part of the course focusing on student learning rather than teacher's speeches. An example is represented by GEOMLAND — a mathematical laboratory in Logo style where the searching process for the solution provides students with a deeper insight about some mathematical phenomena by letting them apply different methods of investigations.[134]

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[131] Dorier JL., Maass K. (2014) Inquiry-Based Mathematics Education. In: Lerman S. (eds) Encyclopedia of Mathematics Education. Springer, Dordrecht.

[132] Towns, R., & Sweetland, J. (2008). Inspired issue brief: Inquiry-based teaching. *Center for Inspired Teaching. Retrieved from http://www. inspiredteaching. org/wp-content/uploads/impact-researchbriefs-inquiry-based-teaching. pdf*.

[133] European Commission. High Level Group on Science Education, European Commission. Science, & Economy. (2007). *Science education now: A renewed pedagogy for the future of Europe* (Vol. 22845). Office for Official Publications of the European Communities.

[134] Sendov B., Sendova E. (1995) East orWest—GEOMLAND is Best, or Does the Answer Depend on the Angle?. In: diSessa A.A., Hoyles C., Noss R., Edwards L.D. (eds) Computers and Exploratory Learning. NATO ASI Series (Series F: Computer and Systems Sciences), vol 146. Springer, Berlin, Heidelberg.

Generally in inquiry-based learning approach, a four-level continuum[135] in classifying the levels of inquiry in activity is identified: confirmation, structured, guided and open as quoted in Sendova, E. (2014, August). *You do–you understand, you explore–you invent*: the fourth level of the inquiry-based learning.[136] The difference is based on how much information, in terms of guided questions, procedures, and expected results, is provided to students and how much guidance is given by teachers.[137]

The first level, *confirmation inquiry*, is when students are provided with the questions and method (procedure) and the results are known in advance. This level of inquiry is useful when the teacher aims to reinforce a previously introduced idea, to introduce students to the experience on how to conduct investigations or to have students practice a specific inquiry skill, such as collecting and recording data.

In the second level, *structured inquiry*, the question, and procedure are still given by the teachers, but students can generate an explanation supported by the evidence they have collected. While confirmation and structured inquiry are considered as lower-level inquiries, they are often used in elementary curricula. These allow students, step by step, to develop their abilities to conduct a more open-ended inquiry.

In the third level, *guided inquiry*, the teachers provide students only with the research question and then they should design the method (procedure) to test their question and the resulting explanations. This level is more efficient if students have numerous opportunities to learn and practice different ways to plan experiments and record data.

Finally, in the last level, *open inquiry*, students act like real scientists, posing questions, designing and carrying out investigations and communicating their results. It is based on more scientific reasoning from the students.[138]

In the case of mathematics, inquiry presents evident similarities with scientific inquiry. Actually, like scientific inquiry, mathematical inquiry starts from a question or a problem, and answers are sought through observation and exploration. However, despite the similarities with scientific inquiry, the mathematical inquiry has some specificity, both regarding the type of questions and the processes to answer them.

As in scientific inquiry, the mathematical inquiry is often motivated by questions arising from the natural world or the made-world around us.

[135] Banchi, H., & Bell, R. (2008). The many levels of inquiry. *Science and children*, *46*(2), 26.
[136] Sendova, E. (2014, August). *Op.cit.*
[137] Banchi, H., & Bell, R. (2008). *Op.Cit.*
[138] Ibid.

Thus the sources of mathematical inquiry in IBME and the associated questions may be very diverse. On one hand, they can emerge from natural phenomena, technical problems, human artefacts, art, and daily life problems. From the other hand, mathematical objects themselves can be an essential source of mathematical inquiry (e.g. what is the greatest product that can be obtained by decomposing a positive integer into a sum of positive integers and multiplying the terms of the sum?).[139]

Consequently, the nature of the question has an impact on the inquiry process. In the first case, when the questions come from an external source, they should be transformed to be accessible to mathematical work. This transformation implies a mathematisation process and the mathematical modelling construction, which are an important part of the process of inquiry in mathematics.

This modelling is generally presented as a cyclic process, which creates some similarity with the model for IBSE as shown in Figure 20, even if at a surface level.[140]

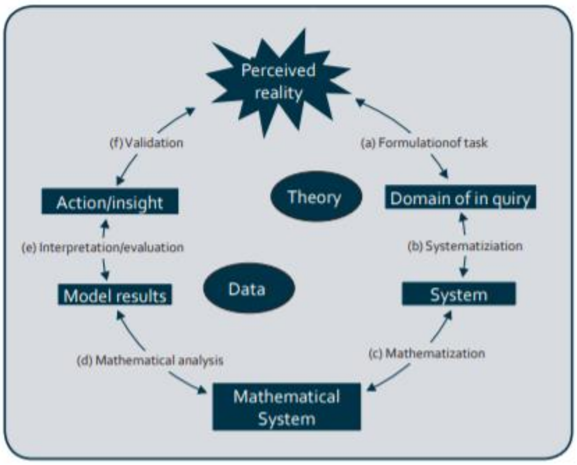

Figure 20: Modelling cycle.

Source: A visual representation of the mathematical modelling process (adapted from Blomhøj & Jensen, 2003)

1. *Task formulation* (more or less explicit) related to perceived reality by the subject. Through this process, the object is the result of the modelling process.

However, both the object and the formulated task lead the identification and construction of an inquiry's domain.

2. *Selection and construction* of the relevant objects, relations etc. from the inquiry's domain to make a mathematical representation possible.

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[139] Blomhøj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: Conceptual clarification and educational planning. *Teaching mathematics and its applications*, *22*(3), 123-139.

[140] Ibid.

3. *Transformation and translation* of the identified objects and relations from their initial state to the mathematical formulation by further abstraction and idealisation process.

4. *Using mathematical methods* to get results and conclusions.

5. *Interpretation* of these as results and conclusions regarding the system or the related inquiry's domain.

6. *Validity evaluation* of the model by comparing the data (observed or predicted) and the already established knowledge (theoretically based or shared/personal experience).

This modelling cycle thus organises the relationship and interaction between two systems: an extra-mathematical system and a mathematical system.[141]

In the IBME application, teachers should introduce to their students the mathematics in a context where the problems are carefully chosen because in the problem-solving process the students will be gradually immersed into typical mathematical activities.

According to Paul Halmos' quote *Don't preach facts, stimulate acts* for both teaching and learning.[142]

This means that teachers should help students understand the mathematical concepts, not just the mechanics of how to solve a specific problem. *Stimulating acts* stands for encouraging students to develop their informal methods for doing mathematics through different actions: to question, to explore, to observe, to discover, to assume, to explain, to prove.

Therefore, the role of the teacher in inquiry-based learning is quite different from that of a teacher in a conventional lesson. Instead of providing direct instruction to students' teachers help students generate theory based on their content-related questions and guide them to assume a researcher role.

However, the teacher's role is not passive: students need guidance as to whether their investigation plans make sense. Actually, teachers, through inquiry-based experiences, should provoke students' thinking and curiosity; to construct carefully the problem scenario or mathematical context; to manage multiple student investigations at the same time; to continuously assess the progress of each student as they work toward their solution or final product, and to respond at the moment to students' emerging queries and discoveries.

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141 Artigue, M., & Blomhøj, M. (2013), *Op.Cit.*

142 Chehlarova, T., & Sendova, E. (2011). Enhancing the inquiry-based learning via reformulating classical problems and dynamic software. *Науковий часопис Національного педагогічного університету імені МП Драгоманова. Серія 3: Фізика і математика у вищій і середній школі*, (8), 125-132.

This teaching approach is based on some very specific research results for teaching which can be summarized as follows:

- The understanding in mathematics is much more than the knowledge of problem-solving mechanics;
- Everyone, especially the students, know and structure new knowledge changing and redefining already possessed concepts and adding new ones to those already known and deemed reliable.
- The social context is fundamental to learning mediation;
- Effective learning requires that students are aware of and responsible for their own learning.[143]

One example of the inquiry-based learning implementation is the research projects realized by students in the classroom. In this case, the activities to be carried out on research projects in secondary schools can be structured into four phases:

- Preparation aiming to support students in exploring the topic through the selection of appropriate lectures, problems, and resources to study.
- Research phase focusing on the involvement of students in investigation activities by designing a specific project, that can be short (lasting up to 2 weeks) or long (from 6 weeks to 2 years).
- Presentation aiming to develop skills for written and oral presentation of these projects.
- "Passing on the torch" – teaching students to act as mentors.[144]

Moving towards IBME generally starts with the implementation of small inquiries in which teachers value their students' questions, act jointly with them on the basis of their questions and productions, create the conditions for students to make connections within mathematics and with the external world.

However, today, creating the conditions for using IBME in an effective way is very difficult in many educational systems due to the curriculum structure, classroom organization and mathematics schedules, teacher training, and many other factors.[145]

[143] Solbes, J., Fernandez-Sanchez, J., Dominguez-Sales, M. C., Canto, J., & Guisasola, J. (2018). Influence of teacher training and science education research in the teaching practice of science in-service teachers. *ENSENANZA DE LAS CIENCIAS, 36*(1), 25-44.
[144] Sendova, E. (2014, August).*Op.Cit.*

2.3.4 Technology-enhanced learning

Continuous and complex transformations of our society generate new values and lifestyles, which lead to changes related to the cognitive and communicative aspects of an individual. Thanks to the structured vision of knowledge, the individuals become the main responsible for their personal and social development throughout the whole life.

The key competence to be acquired is *learning to learn* in a learner-centered environment. The class becomes a knowledge-building community, where members are involved in processes of co-construction and sharing of knowledge and meaning from formal, non-formal and informal knowledge. The climate of cooperation and positive complicity among the members, supported by the intentional use of technologies, helps to promote learning to learn that becomes a key to build digitally aware citizenship and reduce the digital divide. An example is the use of virtual collaborative learning environment in the learning and teaching process where participants can work together as a group to construct and share knowledge with the support of technology.

Such kind of settings provides students with the opportunity for collaborative knowledge building, particularly through peer-to-peer learning.

Besides, both students and teachers can communicate synchronously and asynchronously in virtual collaborative learning through different electronic tools (e.g. group and discussion chats, wikis, and blogs, virtual class).[146]

If, on one hand, the technology is rapidly developing, on the other hand, the teaching practices often are still far from its effective use in a learning process. Actually, some problems emerged during the last months with schools closure due to the coronavirus emergency. All European schools have been dealing with e-learning and, consequently, with the use of digital tools. The current situation has underlined a deep gap existing between teachers and students in terms of digital expertise and among students themselves in terms of equipment lack (e.g. PC, Internet connection). As a result, most schools and teachers, mainly in Italy, are still bounded to the traditional management and vision of their class.

Technology-enhanced learning (TEL) refers to the support of teaching and learning through the use of technology and sometimes this expression can be used synonymously with e-learning but can also be used to refer to technology-enhanced classrooms and learning with technology, rather than just through technology. For example, in Bulgaria in most of the schools, the technology was used with an old curriculum, just adding

[145] Artigue, M., & Blomhøj, M. (2013), *Op.Cit.*
[146] Ghaoui, C. (Ed.). (2003). *Usability evaluation of online learning programs*. IGI Global.

new school subjects, like IT and informatics (with technology). Another experiment (RGI) from 1978, started with a completely new curriculum with computers being a natural part of it (through technology), e.g. there was a subject: Language and Mathematics (English, Bulgarian, Russian, Mathematics and Logo) integrated into a single subject.[147]

In this context, the traditional class gets to change into an environment without borders and constraints.

The technologies can become meaningful learning tools[148] if they provide students with the opportunity to learn *with* and not *from* technologies. This means that technology is not a process but a tool through which educational practices are mediated. Therefore, students learn meaningfully if they can consciously: master the use of technologies; creatively use them, organize and represent what they know and learn; create products, solve real-life problems , to reflect on contents and processes.

One of the differences between the virtual and physical classes is associated with the ways of communication and the tools to be used. Mainly, there are two communication modalities - synchronous and asynchronous. The former is referred to as the face-to-face meeting places or the communication occurring via telephone, video conferences, text messages, shared fibre links for a more remote modality. The latter regards communication through exhibitions, whiteboards, but also accessed via the Internet, etc. These new modalities to communicate encourage learners to interact and collaborate with other peers on a much broader scale.

In the last years, new technology has been emerging (http://icampus.mit.edu/projects/teal/) which transforms the concept of "class". It is a model where technology enables active learning environment, in brief TEAL (*Technology Enabled Active Learning*). This favors the development and the increasing of informal learning spaces for students, both in and out of the institutions. This means that there is no unique place available and dedicated to learning, but with the integration of technology in the class, learners can study everywhere, anytime.[149]

The new methodology expands the competences to be learned by students including critical thinking, communicating to and among peers and a wider community, working in

[147] Sendova, E., Boytchev, P., Stefanova, E., Nikolova, N., & Kovatcheva, E. (2009, September). Creating a natural environment for synergy of disciplines. In *European Conference on Technology Enhanced Learning* (pp. 549-555). Springer, Berlin, Heidelberg.

[148] Sánchez, O. R. M., Ostaiza, D. S. C., & Granda, B. A. S. (2018). An analysis about the role of the technology in the teaching-learning process. *RECIAMUC*, *2*(1), 554-561.

[149] Gordy, X. Z., Zhang, L., Sullivan, A. L., Haynie, L., Richards-Moore, L., & Bailey, J. H. (2018). A multi-disciplinary empirical investigation of active learning classroom’s effects on student learning. *Interdisc Educ Psychol*, *2*(1), 3.

multidisciplinary teams, problem-solving, active learning, knowledge construction, inquiry, and exploration. It is possible to define a technology framework based on *3E* (standing for *Enhance*, *Extend* and *Empower*) for using technology to effectively support learning, teaching and also assessment across different disciplines. *Enhance* means adopting technology simply and effectively in order to support actively students and increase their activities. *Extend* is referred to the future use of technologies facilitating the key aspects of students' individual and collaborative learning and assessment through increasing their choice and control. *Empower* is the developed use of technology that requires higher individual and collaborative learning to create new knowledge to be used in different job environments.[150]

The *3E* principles are underpinned by a socio-constructivist approach[151] with a focus on active learning, frequent assessment, and personalized learning. The aim is to increase student engagement and performance. This *3E* framework considers how activities can be incorporated as a minimum (*Enhance*), through the use of the technology giving students more responsibility for their learning process (*Extend*), and to underpin more sophisticated, authentic activities related to the environments for which they are preparing for the future (*Empower*). The theoretical approach described above encourages the use of technology as a pedagogical tool.

Teachers can make the best use of technology in their classrooms by developing their awareness of a range of digital technologies. This means that it should be clear how and why the technology use supports students' learning and then in which way it can be integrated into the classes. When the introduction of technology is explored, a useful reference can be considered in its implementation in the classroom. The SAMR model (SAMR standing for *Substitution*, *Augmentation*, *Modification*, and *Redefinition*) describes different stages of technology integration into the classroom.[152] The technology can be regarded as *substitution* or *augmentation*. In the first case, according to this model, technology acts as a substitute with no functional change. For example, student types their report using Microsoft Word, instead of producing a handwritten report. The second term is referred to the technology acting as a substitute with functional improvement. This is the case when teachers add comments and

[150] Goodyear, P., & Retalis, S. (2010). Learning, technology and design. In *Technology-Enhanced Learning* (pp. 1-27). Brill Sense.

[151] Pass, S. (2007). When constructivists Jean Piaget and Lev Vygotsky were pedagogical collaborators: A viewpoint from a study of their communications. *Journal of Constructivist Psychology*, *20*(3), 277-282.

[152] Hambrock, H. B., & Richter, R. G. (2019). A Pedagogical Approach Towards Curating Mobile Apps in an Educational Context. In *Ubiquitous Inclusive Learning in a Digital Era* (pp. 81-106). IGI Global.

feedbacks electronically and then mail to students. Both the *substitution* and *augmentation* aim to enhance the learning process of the students.[153]

However, when the introduction of technology leads towards a transformation of the learning environment, the model refers to other two phases - modification and redefinition. In the first case, technology allows for relevant task redesign. For example, students load their reports onto a blog and other peers view and add comments. In the second case, *redefinition*, technology allows for the creation of new tasks. For instance, students can produce online work including multimedia contents (images, audio, and video) instead of writing a report.

There are several examples of how to integrate technology in the classroom, such as a *serious game*, *virtual reality*, *augmented reality*, *educational robotics* and, especially for mathematical topics, *virtual laboratories*.

2.3.4.1 Serious games

Playing games is an important activity where students develop skills for life regardless of age or level of development; they can discover concepts from real-world and fundamental relationships among them. Games motivate learning, thus increasing the chance to achieve the desired outcomes. However, motivation remains just a prerequisite for learning.

Games have always interested people, providing them with entertainment and joy. Since they have to mimic reality, some developers started to use these games with an additional goal: not only to have fun but to train, educate. These new applications are named *serious games*.[154]

Introducing games thematic in subjects can help people improve their skills in a specific field. However, to define a game as *serious*, it must have well-defined learning goals and promote the development of important strategies and skills to increase the cognitive and intellectual abilities of learners. Other important elements contribute to increasing the educational value of the game, such as the use of sensual stimuli, fantasy, challenge, and curiosity (desire to know or learn) in the game. In serious games, instructional content is included within game characteristics where students play the game and have fun, forgetting

[153] Hambrock, H. B., & Richter, R. G. (2019).*Op.Cit.*
[154] Djaouti, D., Alvarez, J., Jessel, J. P., & Rampnoux, O. (2011). Origins of serious games. In *Serious games and edutainment applications* (pp. 25-43). Springer, London.

about the *learning* part of the experience while the teacher defines learning goals and didactical approaches.[155]

The serious games are not just simulators, but they can immerse learners into a world where they need to invest themselves mentally or intellectually to go ahead, to deal with challenges, to reach the goals.

They can be defined as applications of interactive technology that extend far beyond the traditional video-game market including training, visualization, simulations, etc. These games are used in a wide variety of areas such as education, project management, military, healthcare, etc.[156] Just some examples of the application of serious games: in health-related issues, they include physical fitness, education in health/self-directed care, recovery and rehabilitation, diagnosis and treatment of mental illness/mental conditions.

Other examples of games are classified according to their purpose:

- *Edutainment* refers to entertainment games that are designed to be educational. One industry that is loaded with edutainment products is kids' software. An example in mathematics education is *Math Blasters* (Figure 21) aiming to teach kids mathematics while keeping them entertained.[157]

Figure 21: Example of an online game in *Math Blaster* for kids.

[155] Arnab, S., Berta, R., Earp, J., De Freitas, S., Popescu, M., Romero, M., ... & Usart, M. (2012). Framing the adoption of serious games in formal education. *Electronic Journal of e-Learning*, 10(2), 159-171.
[156] Ferreira, N. (2002). Serious Games. *Distributed Computer Graphics*. Universidade do Minho. Portugal.
[157] The *Math Blaster* is available here http://www.mathblaster.com/. [Retrieved online 10/05/2020].

- *Game-based learning* deals with applications that have defined learning outcomes. They are designed to balance the subject matter with the game-play and the ability of the player to retain and apply the subject matter to the real world.
- *News-game* is an application of journalism in video game form that conveys some kind of interactive news or editorial content. They are usually created in response to current events.
- *Training and Simulation Games* - Games used for the acquisition or exercise of different skills, to teach effective behaviour in the context of simulated conditions or situations.
- *Persuasive Games* influence players to take action through game-play. They are designed to change attitudes or behaviours of the users through persuasion.
- *Organizational-dynamic* is usually designed for the specific purpose of furthering personal development and character building, particularly in addressing complex organizational situations, such as managing change and innovation diffusion in a company, helping people in the organization to introduce productive collaboration patterns, managing difficult meeting situations, etc.

The use of games in the learning processes has several positive aspects for the student. It can be meaningful, experiential, and social. It creates new environments where people learn by integrating thinking, social interaction, and technology. The players, in this case, students, are directly and actively involved in their learning process, by experiencing the world in new ways.[158] The educational game called *Math-City* (Figure 22), for instance, is a simulation-based game in which students can create and maintain their city with residential, commercial, and industrial buildings, as well as renewable and non-renewable power sources while practising different mathematical concepts.[159]

[158] Luchev, D., Paneva-Marinova, D., Pavlov, R., Kaposi, G., Márkus, Z. L., Szántó, G., ... & Veres, M. (2016). *Game-based learning of Bulgarian iconographical art on smart phone application*.
[159] Polycarpou, I., Krausea, J., Rader, C., Kembel, C., Poupore, C., & Chiu, E. (2010). Math-City: An educational game for K-12 mathematics. *Procedia-Social and Behavioral Sciences*, *9*, 845-850.

Figure 22: Example Collaborative Math-City game board and math question.[160]

Another example is *MathCityMap* which allows students of all ages to find interesting objects in the schoolyard, in the city park or anywhere else, using their math skills. Within *MathCityMap* (Figure 23), students can apply their mathematical knowledge outside the classroom and can discover their environment from a mathematical perspective.[161]

Figure 23: A math trail (polygon) with all the tasks symbolized by pins in *MathCityMap*.[162]

[160] Stone, K., Polycarpou, I., Krause, J., & Rader, C. (2011). Electronic Collaborative Learning in Math-City. In *Proceedings of the International Conference on Frontiers in Education: Computer Science and Computer Engineering (FECS)* (p. 1). The Steering Committee of The World Congress in Computer Science, Computer Engineering and Applied Computing (WorldComp).

[161] The MathCityMap application is available here https://mathcitymap.eu/ . [Retrieved online 10/05/2020].

[162] Ludwig, M., & Jablonski, S. (2019, July). Doing Math Modelling Outdoors-A Special Math Class Activity designed with MathCityMap. In *HEAD'19. 5th International Conference on Higher Education Advances* (pp. 901-909). Editorial Universitat Politècnica de València.

Besides, games empower learners to acquire problem-solving skills by developing understanding.

The use of games in education can offer virtual environments, ideal for simulating different situations and contexts in which students can develop their future professional skills.

One of the relevant features of the serious games is that students learn by doing: they learn while playing without being consciously aware of it.[163] An example is the games developed in the 3D virtual environment in the framework of the TALETE project.[164] This 3D environment is a virtual kitchen, the everyday context for students, where math mini-games have been embedded in several objects as shown in Figure 24.

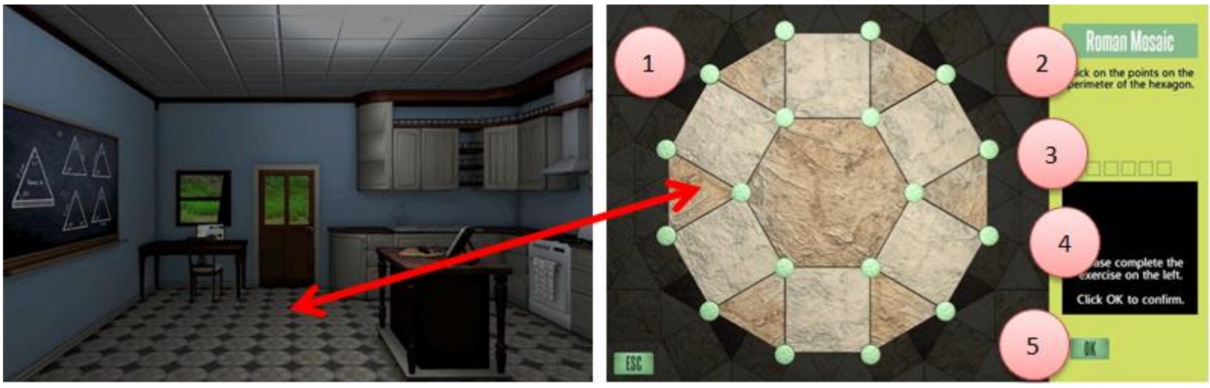

Figure 24: TALETE start screen + screenshot of a scenario with the following areas: 1 - Area for visualization of the context and visual feedback, enabling learning by doing and exploring techniques; 2 - Area for the textual representation of the context, problem, tasks and questions; 3 – Area for answering; 4 – Feedback area and 5 – Navigation area.

As a result, on one hand, students can play the developed eight mini-games directly from their web-browser by discovering the real connections between the mathematics patterns and objects commonly used in a kitchen. On the other hand, the teachers can evaluate their performances by accessing a database where the students' scores are recorded.[165]

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[163] Measles, S., & Abu-Dawood, S. (2015, March). Gamification: Game–based methods and strategies to increase engagement and motivation within an elearning environment. In *Society for Information Technology & Teacher Education International Conference* (pp. 809-814). Association for the Advancement of Computing in Education (AACE).

[164] The TALETE (Teaching mAths through innovative LEarning approach and conTEnts) project was funded with support from the European Commission in the framework of COMINIUS Programme from 2011-2013. Website: http://project.unimarconi.it/talete/index.php/en/. [Retrieved online 10/05/2020].

[165] Monova-Zheleva, M. H., & Tramonti, M. (2015). Uses of the Virtual World for Educational Purposes. *Компютърни науки и комуникации*, *4*(2), 106-125.

2.3.4.2 Virtual reality

Virtual reality (VR) can be referred to as immersive multimedia[166] or computer-simulated reality that replicates an environment to simulate a physical presence in places in the real or imagined world, allowing the user to interact with it .

Standard virtual reality systems use VR-headsets to generate realistic images, sounds and other sensations that simulate a user's physical presence in a virtual environment. This allows a person to look around the virtual environment, move around in it, and interact with virtual characters or objects.[167]

The most innovative headsets are the Oculus Quest (Figure 25), which is the first system designed for virtual reality that does not need cables and PC. It is composed of a viewer that is well integrated with the audio and two controllers (Oculus touch) that allow interaction in the virtual environment through hands and gestures.[168] All movements will be visible in VR with realistic precision. Furthermore, it offers the possibility to view and monitor, through an external device, everything the student does within the virtual environment.

Figure 25: Oculus Quest viewer and touch controller.

Very similar to the Oculus Quest, is Oculus Rift (Figure 26) which requires a constant connection to the PC through cables. Rift offers high-quality graphics, thanks to technological innovations such as the asynchronous Spacewarp (ASW) through a PC. For this reason, the PC's features should be compatible with the VR otherwise the experience could fail.

[166] Biocca, F., & Delaney, B. (1995). Immersive virtual reality technology. *Communication in the age of virtual reality*, 15, 32.

[167] Wikipedia, with help from Bart Pursel, *Information, People, and Technology*. Open text Book. Available https://psu.pb.unizin.org/ist110/ [Retrieved online 16/05/2020].

[168] Parong, J., & Mayer, R. E. (2018). Learning science in immersive virtual reality. *Journal of Educational Psychology*, 110(6), 785.

Figure 26: Oculus Rift viewer and touch controller.

HTC Vive Cosmo (Figure 27) is another viewer that includes two different typologies: the first is stand-alone, with integrated processing hardware and battery, and the second one requires external power from a smartphone or PC.

Figure 27: HTC Vive Cosmo viewer and touch controller.

There are other examples, like Gear VR (Figure 28), which is the first mobile virtual reality viewer developed by Samsung in collaboration with Oculus.[169]

Samsung's approach to VR was different right from the beginning, and the Gear VR was an impressive piece of tech when it was first released in November of 2015. Samsung's VR HMD delivers a stripped-down VR experience, using Oculus head-tracking technology in combination with Android smartphones to power mobile VR experiences. Instead of dedicated display technology, lenses allow the phone's screen to act as a stereoscopic display, making the device simpler and less expensive than other options. Samsung has added hand controllers to the Gear VR experience, bringing it more in line with current VR content.

[169] Olmos, E., Cavalcanti, J. F., Soler, J. L., Contero, M., & Alcañiz, M. (2018). Mobile virtual reality: A promising technology to change the way we learn and teach. In *Mobile and ubiquitous learning* (pp. 95-106). Springer, Singapore.

Figure 28: GEAR VR viewer including a mobile device.

Similar to Gear VR, but very cheap is Google Cardboard (Figure 29), a simple platform dedicated to virtual reality. It is designed and developed by Google for use with special glasses and a Smartphone. This tool is very simple to build by using the templates available on the Internet, and the Cardboard applications can be developed with the three software development kits available.[170]

Figure 29: Google Cardboard viewer including a mobile device.

The usage of Virtual Reality in education is quite popular. In mathematics, there are different applications, like Heromask Mathematics where 5-12 years old students with the support of a special viewer (Figure 30) can learn mathematics by playing.

[170] Education, S. T. (2018). *IMMERSIVE COMPUTING@ GOOGLE*, INC.

Figure 30: Heromask viewer for learning language and mathematics for 5-12 years old students.

The games developed to be used in this platform concern mathematical operations and mental calculation as shown in the following picture (Figure 31). However, the learning activities set in the VR should be well-planned and designed to be effective. Otherwise, the result risks to become mere entertainment instead of supporting students in their learning process.

Figure 31: Games in VR in Heromask Mathematical operations.

An interesting project combining virtual reality and mathematics realized in class is *Alice among the wonders of mathematics*. Students, divided into small groups, have started to analyse the story *Alice in Wonderland* with the scope to find any possible mathematical connections in the text. During this path, students have designed patterns and discussed on some main points, such as the numbers relating to the clock measuring the time; Alice's fall has introduced the concept of speed and acceleration.[171] Afterwards, the students have been asked to create a virtual world to be explored with the cardboard glasses. This was designed from a virtual narration and the mathematics connections found.

[171] Corsaro, G. (2017). Didattica nella realtà virtuale e nella realtà aumentata.

To do that, students used *Cospaces*[172], a platform to easily build 3D kid-friendly creation tools and environments (Figure 32). Therefore, students of any grade and subject build their 3D creations, animate them with a specific code, and explore them in VR or, sometimes also in AR (augmented reality - described in the next paragraph).

Figure 32: Using CoSpaces Edu to Create Virtual Reality Experiences.

In mathematics education, another interesting work is the one realized by the research team at the University of Oklahoma in Stillwater led by mathematician Henry Segerman. They have created the first software capable of simulating worlds in which the rules of traditional geometry no longer exist by allowing simplifying the study of non-Euclidean geometries. The project, born from the collaboration between mathematicians and physicists, is called *Hyperbolic VR* and is conducted in collaboration with the group of mathematician-artists of San Francisco called *eleVR*. [173]

[172] Available at https://cospaces.io/edu/ [Retrieved online 17/05/2020].

[173] Hart, V., Hawksley, A., Matsumoto, E. A., & Segerman, H. (2017, July). Non-euclidean virtual reality I: explorations of H^3. In *Bridges 2017 Conference Proceedings.*

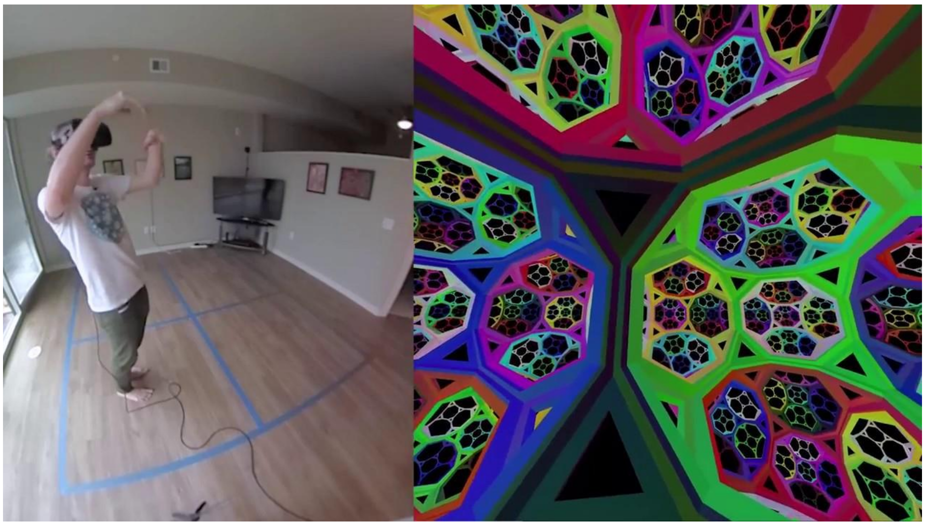

Figure 33: Virtual simulations of three-dimensional non-Euclidean spaces.
Source: h3.hypernom.com and h2xe. hypernom.com

The use of virtual reality in the classroom has several advantages as follows:

1. Providing outstanding visualizations that aren't possible in the traditional classroom. By wearing a VR headset, students are immersed in high-quality visualizations that let them explore different realities from their experiences.
2. Creating interest - VR technology can create amazing experiences that could never be "lived" in real life by increasing students' motivation in their learning.
3. Increasing students' engagement as this experience involves all their senses.
4. Learning through fun is more pleasant for students. This doesn't mean that the tasks for the students have to be easy, but it should challenge them to overcome any difficulty by using their creativity to solve them.

However, there are some negative aspects to be taken into account when students study through virtual reality:

1. Deteriorating human connections - traditional education is based on personal human communication and interpersonal relationships. This means that if the lessons are mainly focused on virtual reality, this could damage the relationships between students and the overall human communication.

   However, some current versions of the serious games in VR try to fix this issue. There are some applications where the interaction between the students is required: while one student is immersed in Virtual Reality, the others can provide him/her information or suggestions to complete the assigned task.

2. Lack of flexibility - Whereas in-class teachers can be flexibleask questions, receive answers, using a virtual reality headset is a different experience because it depends on the specific software used.
3. Functionality issues - Like with any programmed software, functionality issues can occur due to, for example, the Internet connection and PCs incompatibility by compromising the learning process of the student.
4. Even the students' senses can be disturbed. In fact, after 30 minutes in VR, students can lose spatial awareness of the room around them; they can feel the disorientation, mainly if they are prone to motion sickness or vertigo and this can make the VR experience uncomfortable.

   If exposed to VR for a long time, in some individuals it can cause nausea, eye strain mostly as it's related to motion sickness and the speed of the objects moving inside.[174]

For this reason, Mixed Reality technologies are often preferred, because they combine real and virtual elements.

One example is 360° videos. They are video recordings of a real-world scene, where the view in every direction is recorded at the same time. 360° video is a version of VR created with only real-world content. Even if these 360 videos are not the same with *real Virtual Reality*, they possess some features of Virtual reality, for example, the immersive experience.[175]

2.3.4.3 Augmented reality

Besides virtual reality, there are other types of interactive technologies. One of them is augmented reality (AR). It is the alive, direct or indirect view of a physical, real-world environment whose elements are augmented (or supplemented) by computer-generated sensory input such as sound, video, graphics or GPS data. Augmented Reality is an overlay of content on the real world, but that content is not anchored to or part of it.[176]

[174] Clark H., Duckworth S., Heil J., Hotler D., Piercey D., Thumann L. *The Google Cardboard Book – Explore, Engage and educate with virtual reality*. EdTechTeam.
[175] Thompson, L. J., Krienke, B., Ferguson, R. B., & Leck, J. D. (2018). Using 360-Degree Video for Immersive Learner Engagement. *Journal of Extension*, 56(5), n5.
[176] Hockly, N. (2019). Augmented reality. *ELT Journal*, 73(3), 328-334.

Augmented reality is the integration of digital information with the user's environment in real-time. Unlike virtual reality, which creates an artificial environment, augmented reality uses the existing environment and overlays new information on top of it.

Some examples of this technology have been developed by Ingvar Kamprad Elmtaryd Agunnaryd (IKEA) and Google. Ingvar Kamprad Elmtaryd Agunnaryd has developed a table as part of its concept kitchen that suggests recipes based on the ingredients on the table, which is a great example of Augmented Reality working in the real world, potentially. However, Google Glass, produced by Google, was the first attempt to bring the augmented reality to consumers.[177]

The technology of augmented reality is used in different sectors and with different objectives: in training for skills development, in gaming, in modelling objects and books. Another example is based on discovery-based learning as in *Geotag*, which is a GPS coordinate that associates content such as videos, textual information, audio or any user-generated content to a specific location. The augmented reality applications draw on specific tags created by companies but also depend on the content that everyday users add through Geotagging.

An application that uses the GPS geolocation is Pokémon Go (Figure 34). It is a free-to-play video game based on augmented reality developed by Niantic. It gives users the chance to explore real-world locations while looking for Pokémon (game characters). While the users are outside, walking around, the smartphone vibrates to let them know when there are Pokémon nearby. Once they meet a Pokémon, they need to catch them by throwing a Poké Ball using the touch screen of their smartphone but paying attention that they could escape.

Nevertheless, the misuse of this game can have negative effects. In fact, the players are so concentrated to look at their mobile screen that they don't mind the dangers around them. For example, the distraction has led, in several cases, to injuries while walking in search of Pokemon, as well as to road accidents caused by the use of the app by people driving means of transport.[178]

[177] Kim, L. (2018, July). The Lessons of Google Glass: Aligning Key Benefits and Sociability. In *International Conference on Human Interface and the Management of Information* (pp. 371-380). Springer, Cham.

[178] Serino, M., Cordrey, K., McLaughlin, L., & Milanaik, R. L. (2016). Pokémon Go and augmented virtual reality games: a cautionary commentary for parents and pediatricians. *Current opinion in pediatrics*, *28*(5), 673-677.

This mix between involvement in the game and distraction has led several people to engage in very dangerous behaviors, such as crossing streets, going beyond the safety line of railways and subways, being in unsafe places or isolated.[179]

Figure 34: Pokémon application during the game.

In mathematics education, there is a Math VR application (Figure 35), produced by ACE-Learning, which enables visual learners to see-through augmented reality how Math concepts work in a three-dimensional (3D) environment. Actually, by viewing augmented models, students gain a better understanding of the concepts they are studying. It is simply to download Math VR App and print the marker to see math models come alive.

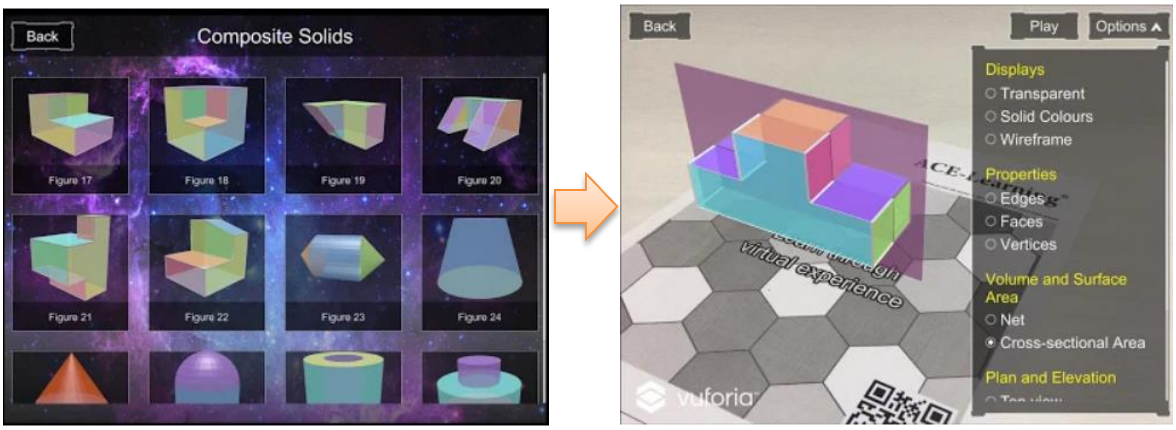

Figure 35: An example of mathematics exercise with the polygon in Math VR App.

Another example is provided by GeoGebra, the software combing geometry and algebra and now using also the augmented reality (AR) technology.

[179] Kari, T. (2016). Pokémon GO 2016: exploring situational contexts of critical incidents in augmented reality. *Journal of Virtual Worlds Research*, *9*.

GeoGebra places math objects in real-world (Figure 36) so that you can explore them from different angles. Students can place solid objects, such as pyramids, prisms and cubes by exploring and learning their features. More advanced students can look at the impossible Penrose triangle, 3D functions, spiral staircase and Sierpinski pyramid among others. They will have to physically move around the virtual objects to complete the tasks.[180]

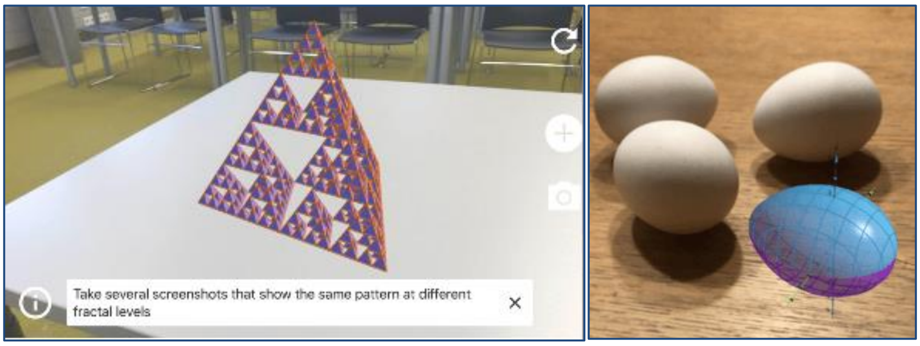

Figure 36: Examples of objects exploration and tasks to be carried out in GeoGebra AR.

There are several advantages to the use of augmented reality in the classroom, mainly to promote and motivate learning in students.

First of all, Augmented Reality can support experiential learning in a classroom by providing contextual clues or information on learning. Secondly, Augmented Reality can promote a deeper understanding between the real and virtual world by associating learning information with reality. Finally, Augmented Reality can enable students to lead active learning by constructing and manipulating 3D objects or clues in person.

Here are some examples of using AR in a classroom. Teachers can produce *markers*, which include information about a topic or content and post them on the board, walls, or anywhere in the classroom. Then, students can scan the markers or QR code (IC Quick Tips) with mobile devices to get additional information. Teachers could engage in classroom discussions about the topic shown.[181]

Besides, students can conduct class projects or homework assignments using Augmented Reality software or applications with a tablet or mobile device. Students actively engaged in learning activities with Augmented Reality are expected to accomplish them more successfully. All of this provides more options for students with different learning profiles and ways to present information. One possibility of Augmented Reality is to promote

[180] https://www.geogebra.org/m/R8Qd7U8y
[181] Kan, T. W., Teng, C. H., & Chou, W. S. (2009, December). Applying QR code in augmented reality applications. In *Proceedings of the 8th International Conference on Virtual Reality Continuum and its Applications in Industry* (pp. 253-257). ACM.

collaboration through social interaction among students in the AR environment because multiple students can share virtual objects. This kind of virtual objects can be a means to communicate with students or to collaborate remotely.

There are four benefits of Augmented Reality: (a) multi-sensory immersion, (b) transitional interface, (c) tangible user interface, and (d) synergy with mobile devices.

- *Multi-sensory immersion:* AR leads to the sensory immersion of information or knowledge by augmenting human perceptions with 3-D objects or materials.
- *Transitional interface:* It provides a seamless transitional interface between the real-world and a virtual world.
- *Tangible user interface:* It offers a tangible user interface through which digital objects or information can be touchable in Augmented Reality.
- *Synergy with mobile devices:* As mobile devices and their applications are advancing, mobile users can experience more gestures and touch.[182]

However, there are three disadvantages of Augmented Reality: (a) high level of reliance on digital information, (b) privacy concerns, and (c) a need for extra wearable devices.[183]

- *High level of reliance on digital information:* Too much reliance on digital information may cause a decrease of working memory in the brain which in turn hinders the development of brain functions.
- *Privacy concerns:* As AR software and applications are developing, it will become easy to gather information on AR stuff from social network services (SNS) and post them and thereby an unwelcome situation such as private information may occur.
- *A need for extra wearable devices:* Since users may want to feel more authenticity in Augmented Reality, wearable devices such as Google Glass and Apple's iWatch may be provided for users to offer authentic experiences more conveniently and expansively.

In this context, while in virtual reality, the information added or subtracted electronically are dominant, to the point that people are immersed in a virtual situation; in Augmented

[182] Ibáñez, M. B., & Delgado-Kloos, C. (2018). Augmented reality for STEM learning: A systematic review. *Computers & Education*, 123, 109-123.

[183] Civil, D. H. K. C., & Horsham, W. S. (2017). *The application of Augmented Reality and Virtual Reality in the construction industry using wearable devices.*

Reality, instead, the person continues to live the common physical reality, but benefits from additional information or manipulation of virtual reality as well . Both VR and AR can be considered part of a continuum and not simply two opposite concepts. This definition follows the famous *Reality-Virtuality continuum* theorized by Milgram and Kishino, representing a spectrum of technologies ranging from pure real reality to pure virtual reality (Figure 37).[184]

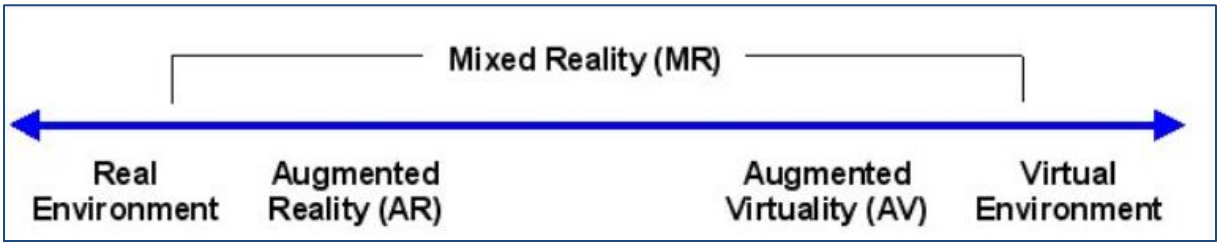

Figure 37: Virtuality Continuum, Milgram and Kishino in 1994.

2.3.4.4 Educational robotics

During the last years, school systems have recognized the relevance of algorithm thinking or *computational thinking*, and computer coding to be taught starting from primary school. The main idea is that knowledge and informatics skills can't be restricted to the use of devices or software only, but they should be integrated into the teaching and learning processes to let students approach the informatics and coding principles.[185] The latter, in turn, permits students to develop logical and analytical thinking aiming to solve problems in different contexts of such an approach as educational robotics. The methodology favours the development of students' potentialities because provides immediate and concrete applications. This contributes to the competences and knowledge construction in mathematics, science, and technology from one side and in entrepreneurship and language from the other.

An immediate application of *computational thinking* can be found in educational robotics, which can be regarded as a teaching approach to several subjects through different cognitive artefacts. This means that "educational robotics" is not a synonym of teaching robotics as a pure discipline, and therefore neither just a teaching coding applied to robots nor

[184] Milgram, P., & Kishino, F. (1994). A taxonomy of mixed reality visual displays. *IEICE TRANSACTIONS on Information and Systems*, *77*(12), 1321-1329.
[185] Bosciani M., Beri M. (2016). *Imparare a programmare con Scratch. Il manuale per programmatori dai 9 anni in su*. Adria (RO): Apogeo.

the study of how an android works.[186] In general, it is a teaching methodology, engaging students into problem-solving, learning by doing, that favours the development of "computational thinking" by including a constructive approach to the error.[187]

Being mainly driven by the basic learning and teaching strategies implemented in this sector, i.e. discovery and inquiry-based learning, team working, problem-solving that significantly favour the investigation, a positive role of the error is recognized.

In this context, learning becomes more effective because student knowledge construction is supported by the realization of a concrete, meaningful project, when every mistake made or challenge faced continuously stimulates students' curiosity. Through the use of this methodology, it is possible to introduce students to both specific and transversal skills and relate them to school disciplines either directly or indirectly. For this reason, taken in this perspective, educational robotics is not a single discipline, but rather a tool, deepening the comprehension and disciplines perception in general. Based on the constructionist approach, as defined by *Seymour Papert*[188], where the learning process is student-centered whose active role determines knowledge construction,[189] the methodology supports students in the creation and designing of a didactic pathway focused on the concrete objects manipulations.[190]

The idea to let children manage computers, providing them with objects that they can easily manipulate and make experimentations, arises from the Massachusetts Institute of Technology (MIT) and the revolutionary ideas of *Seymour Papert*[191], *Wally Feurzeig* and *Cynthia Solomon* with their programming language LOGO. LOGO is more than a simple programming language, which as many Logoists defined a LOGO spirit on how doing the things: *Logo is a programming language plus a philosophy of education*.

Students can explore mathematical patterns and concepts by creating their projects by exploiting the LOGO culture's attitude to *getting it to happen*.[192]

It is designed upon Jean Piaget's constructivism and Marvin Minsky's artificial intelligence research at MIT.

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186 Taylor, R. Y. (2019). Book Review: Robotics in STEM Education: Redesigning the Learning Experience. *Journal of STEM Education: Innovations and Research*, 19(5).

187 Denning, P. J., & Tedre, M. (2019). *Computational Thinking*. Mit Press.

188 Papert, S. (1999). *Op.cit.*

189 Perdana, R., & Atmojo, I. R. W. (2019, March). A conceptual of teaching models inquiry-based social constructivism (IbSC). In *IOP Conference Series: Earth and Environmental Science* (Vol. 243, No. 1, p. 012110). IOP Publishing.

190 A. Dochshanov (2017) "TINKERING" as Learning Reinforcement towards Multidisciplinarity in Research-oriented Education, *EDULEARN17 Proceedings*, pp. 9855-9859.

191 CNRITD di Genova, C. I. T. D., CNRI di Scienze, C. I., & della Cognizione-Roma, T. *La robotica educativa come strumento di apprendimento e creatività.*

192 Papert, S. (1999). *Op.cit.*

Initially, the LOGO was used to move a simple robot, which could be given the following commands: FORWARD 50 to GO FORWARD 50 steps or RIGHT 90 to TURN RIGHT 90 degrees (Figure 38). The output was a line graphics produced by the combination of the commands for movement and a small retractable pen. This result was drawn on the screen by a small "cursor" which was similar to a turtle (derived originally from a robot of the same name), which later became a small triangle.[193]

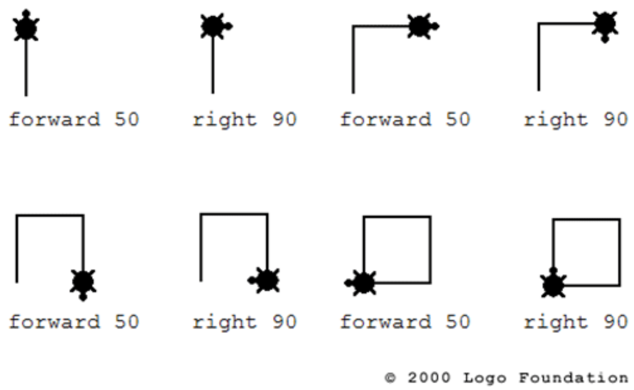

Figure 38: LOGO commands: turtle language.

Afterwards, LOGO became very important for mathematics education by providing new ways of teaching and learning through different tools, such as natural language, music, graphics, animation, storytelling, turtle geometry, robots, and other physical devices.[194]

As stated by Gary Stager (2007) in his PhD dissertation, *An Investigation of Constructionism in the Maine Youth Center*, these are focused on the eight big ideas of *Seymour Papert*:

1. *Learning by doing* – students learn better when they do something they find interesting.
2. *Technology as a building material* - the technology can make things more interesting and attractive, and students can learn more by making them.
3. *Hard fun* - people learn and work best when they enjoy what they are doing. This doesn't mean that the task has to be easy. In fact, for example, a successful carpenter enjoys doing carpentry or a successful businessman enjoys working hard at making deals.

[193] Sendova, E., Nikolov, R., & Boytchev, P. (2018). A Glance Backward With Nostalgy and Forward With Optimism (or Do We Need to Start From Scratch When Introducing Children to Programming).
[194] Solomon, C., HARVEY, B., Kahn, K., Lieberman, H., Miller, M. L., Minsky, M., ... & Silverman, B. (2020). History of Logo.

4. *Learning to learn* - the idea that "the only way to learn is by being taught" can cause failures in school and life. Therefore, students should become more responsible for their learning process.
5. *Taking time, the proper time for the job* - students should learn how to manage their time without someone, at school, telling them every five minutes or hours what to do.
6. *You can't get it right without getting it wrong* - students shouldn't be afraid to get their tasks wrong, because the only way to get them right is to look carefully at what went wrong when happened.
7. *Do unto ourselves what we do unto our students* - people are learning all the time from the different experience occurred and every difficulty met is an opportunity to learn more.
8. *Entering a digital world* – where knowing about digital technology (e.g. how to use computers) is important, as well as knowing how to use them to learn about everything else.[195]

From the first experiment with the programmable mechanical turtle to the virtual turtle which moves on a virtual stage and finally the prototype of LEGO/Logo was built by Mitchel Resnick.[196] This new combination joined the programming language LOGO and the LEGO bricks. As a result, the LEGO bricks became programmable (Figure 39).

Figure 39: Lego Mindstorms, an example of programmable bricks.

[195] Martinez, S. L., & Stager, G. (2013). Invent to learn. *Making, Tinkering, and Engineering in the Classroom. Torrance, Canada: Construting Modern Knowledge*.
[196] Resnick, M., Ocko, S., & Papert, S. (1988). *LEGO, Logo, and design. Children's Environments Quarterly*, 14-18.

An example, involving 29 pilot schools (about 2% of the Bulgarian K-12 schools), was the research project carried out by the Bulgarian Academy of Sciences and the Ministry of Education formed a Research Group for Education (RGE) who were in charge of the development of a new curriculum aimed to incorporate computers as its natural component.[197]

The Lego-Logo-based computer environments were developed and tuned to specific subject domains and oriented to the students' exploratory activities. This allowed students to move from constructing controllable models to fully programmable microworlds by participating in the construction of their knowledge.[198]

The educational robotics activities have been designed with the aim of transversal skills improvements, such as programming, mechanics (robot construction) or electronics.[199] However, some educators have expanded the possible applications of robotics in school by creating activities involving other disciplines such as mathematics, languages or the arts.

About 80% of educational robotics experiences take place in the field of mathematics, as stated by Benitti, in a review on robotics studies.[200]

In fact, in recent years, several educational paths have appeared that use robots to develop mathematical skills. In particular, several educational robotics activities have been designed to develop skills related, for example, to the concept of distance[201] or angle[202] and concepts of space-time relationship.[203]

For example, *Mitnik et al.*,[204] in their study conducted with 70 students divided into two groups (the first who used robots for the activities related to the angle and distance study and the second one who carried out the same activities without), declare that the students, who have used robots, achieved results better in both topics.

Furthermore, they observed how in the group that used a robot there was more collaboration and more motivation to carry out the activities, while in the second group the students got bored more easily.

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197 Asenova, P. (2015). Supporting the Bulgarian Young Talent in the Field of Informatics. *Serdica Journal of Computing*, *9*(3-4), 269-280.

198 Nikolov, R., & Sendova, E. (1991, August). Informatics for all school ages. In *Proceedings of the Third European Logo Conference, Parma, Italy* (pp. 83-96).

199 Benitti, F. B. V. (2012). Exploring the educational potential of robotics in schools: A systematic review. *Computers & Education*, *58*(3), 978-988.

200 Benitti, F. B. V. (2012). *Op.Cit.*

201 Mitnik, R., Nussbaum, M., & Soto, A. (2008). An autonomous educational mobile robot mediator. *Autonomous Robots*, *25*(4), 367-382.

202 Grimaldi, R. (2015). A scuola con i robot. Innovazione didattica, sviluppo delle competenze e inclusione sociale.

203 Julià, C., & Antolí, J. Ò. (2016). Spatial ability learning through educational robotics. *International Journal of Technology and Design Education*, *26*(2), 185-203.

204 Mitnik, R., Nussbaum, M., & Soto, A. (2008). *Op.Cit.*

Also, Grimaldi reached the same conclusions for the concept of angle and the space-time relation.[205] The author also noted that students with more learning difficulties have obtained more satisfactory results with the use of robots. Julià and Antolí also came to the same conclusions for the learning of some spatial concepts, always comparing traditional courses with the ones that used educational robots.[206]

The results of these experiences are certainly not generalizable, but they highlight the current experiments that are taking place in this field.

The robots or robotics elements used vary in correspondence with the school level and students' age.[207] Starting from primary school, young students can deal with more complex tasks addressed to the manipulation and construction of cognitive artefacts by using coding software.

In the first cycle of compulsory school (kindergarten and first two years of primary school) BeeBot (Figure 40) is often used and its updated version BlueBot, which also allows a connection to the tablet via Bluetooth (in reality BeeBot and BlueBot are not real robots because they don't have sensors).[208]

Figure 40: Lego Mindstorms, an example of programmable bricks.

For the second cycle of primary school, Lego WeDo 2.0 (Figure 41), a robot to be built with Lego bricks and programmable with iconic procedural languages, is the most used.

[205] Grimaldi, R. (2015). *Op. Cit.*
[206] Julià, C., & Antolí, J. Ò. (2016). *Op.Cit.*
[207] Davcev, K., Koceska, N., & Koceski, S. (2019). A Review of Robotic Kits Used for Education Purposes.
[208] Beltrametti, M., Campolucci, L., Maori, D., Negrini, L., & Sbaragli, S. (2017). La robotica educativa per l'apprendimento della matematica. Un'esperienza nella scuola elementare. *Didattica della matematica. Dalla ricerca alle pratiche d'aula*, *1*, 123-144.

Figure 41: Kit for LEGO Education WeDo 2.0.

Also from Lego, but for middle school students, the Mindstorm EV3 robot (Figure 42) is very appreciated by teaching community.

Figure 42: Basic kit for EV3 - LEGO MINDSTORMS Education.

Finally, the Thymio II robot (Figure 43), created at the Polytechnic University of Lausanne and programmable with an iconic language for events, is spreading as a versatile educational robot that can be used by a wide age group ranging from 4 to 14 years.[209]

[209] Beltrametti, M., Campolucci, L., Maori, D., Negrini, L., & Sbaragli, S. (2017). *Op.Cit.*

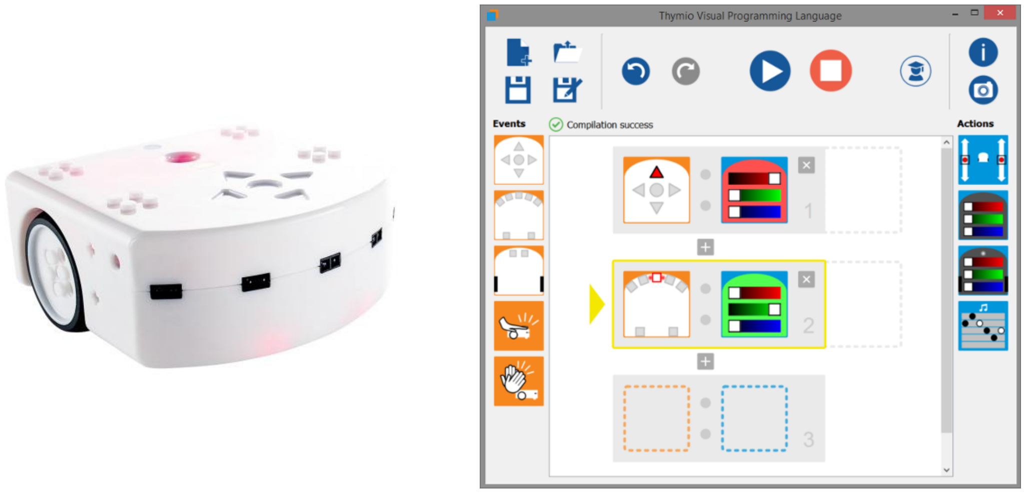

Figure 43: Thymio II robot with the screenshot showing the iconic language for programming.

2.3.4.5 Virtual Laboratories

A Virtual Laboratory is an interactive environment for creating and conducting simulated experiments where students can interact with a digital real lab resembling environment or perform other similar activities which are mainly far from the student or which have no immediate physical reality. The virtual laboratory includes interactive multimedia objects composed of various formats, such as text, hypertext, sound, images, animations, video and graphics.

This educational tool is designed on the base of constructivism principles.[210] This means that the virtual laboratory environment is considered as an opportunity for students to construct their knowledge regarding the topic to be studied. It is enough to think of MatLab where computer graphics are invaluable in providing dynamic visualizations by generating animations to support scientific concept development.

There are different categories of Virtual Laboratory environment as follows: 1. Simulations, 2. Applets, 3. Virtual labs, 4. Virtual Reality Laboratories (VRL).

Simulations are imitations of real-life systems in time, via computers. These represent a process based on a model that is cheaper, faster, less risky and more affordable than the real process. The computer simulations are applications that can be inserted into lessons as movies and/or played directly.

OnlineLabs[211] aims to collect online labs simulations in different subjects, mainly mathematics and it provides online resources allowing the students to get hands-on math experiences without being in a physical classroom or laboratory environment (Figure 44).

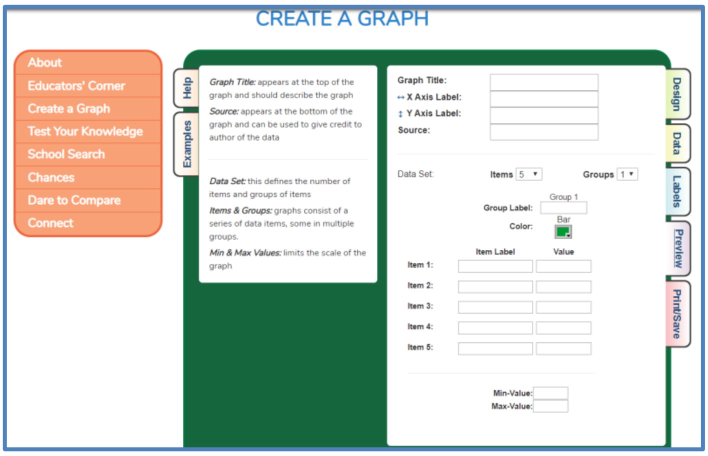

Figure 44: An example of a Virtual Laboratory Simulation on creating a graph.

[210] Kintsch, W. (2009). Learning and constructivism. In *Constructivist instruction* (pp. 235-253). Routledge.
[211] http://onlinelabs.in/math

The researchers demonstrate that how the information is presented is very important in the learning process.[212] The use of the visual representation in education is based on the *dual-coding* theory, especially if we talk about highly visual interactive simulations.

This theory suggests a model of human cognition divided into two dominant processing systems, one verbal and one non-verbal. The verbal system is used mainly in linguistics. While the non-verbal system concerns the processing of all nonverbal phenomena, including emotional reactions.

Dual coding theory predicts three distinctive levels of processing within and between the verbal and non-verbal systems, such as representational, associative and referential.[213] Representational processing describes the connections between incoming stimuli of informational units within the two systems, whereas referential processing is the building of connections between the verbal and non-verbal systems. In other words, dual coding theory predicts that words and pictures provided by instruction will activate these coding systems in different ways.[214]

Therefore, the dual coding theory becomes the base to explain and describe the educational meaning of *simulation*.

There are two different areas related to the simulations: *Real-life applications* and *Simulation-based laboratories*.[215]

The simulations provided by real-life applications are very useful in scientific courses because they simplify the learning process by showing the real-life applications of mathematical concepts. For example, in the framework of a mathematics course, teachers can show to students a real example of the integers. This will help them to learn this fundamental concept emphasizing the connection between the mathematics concept and the real situation.

One example is represented by the application named *Oriented straight line: integers*, that is a part of the free database created by the University of Colorado (Figure 45).[216]

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[212] Rosenshine, B. (2012). Principles of Instruction: Research-Based Strategies That All Teachers Should Know. *American educator*, 36(1), 12.
[213] Sadoski, M., & Paivio, A. (2013). *Imagery and text: A dual coding theory of reading and writing*. Routledge.
[214] Rieber, L. P. (2009). Supporting discovery-based learning within simulations. In *Cognitive effects of multimedia learning* (pp. 217-236). IGI Global.
[215] Finkelstein, N. D., Adams, W. K., Keller, C. J., Kohl, P. B., Perkins, K. K., Podolefsky, N. S., ... & LeMaster, R. (2005). When learning about the real world is better done virtually: A study of substituting computer simulations for laboratory equipment. *Physical review special topics-physics education research*, 1(1), 010103.
[216] Martini, M. (2014). *How modern technologies solve laboratory's dilemma in distance learning.* Digital Universities, 1.

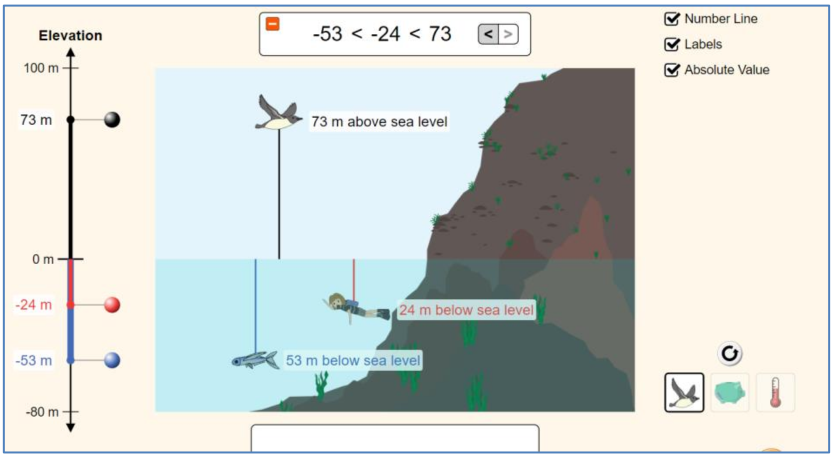

Figure 45: Number line – integers created by the University of Colorado.

This type of simulation is highly customizable and interactive. Students can make several changes in the parameters by showing different cases and then observe the consequences.

The *Simulations-based laboratory* is another way to use computer simulations to reproduce real instruments and prepare experiences in a simulated virtual laboratory.

Some researchers show that students learn from simulation using inquiry-based learning and learning by doing to intensify their understanding.[217]

An additional example is COMSOL that is software for the numerical solution of partial differential equations as shown in the following figure:

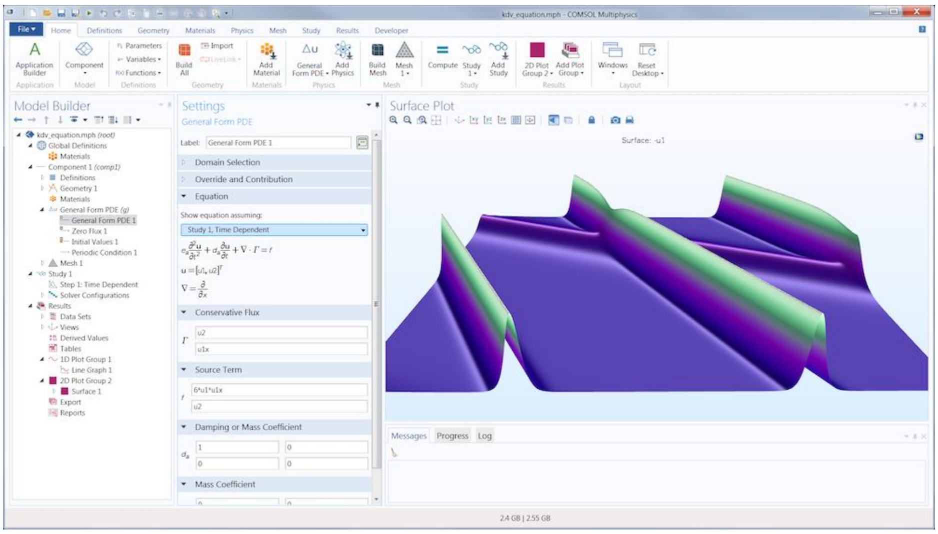

Figure 46: Example of customizing the coefficients of partial differential equation in the COMSOL.

[217] Kukkonen, J. E., Kärkkäinen, S., Dillon, P., & Keinonen, T. (2014). The effects of scaffolded simulation-based inquiry learning on fifth-graders' representations of the greenhouse effect. *International Journal of Science Education*, 36(3), 406-424.

The applets (Figure 47) are experimental devices in small virtual laboratories and are quite popular in science subjects. In general, they are small in size and can be found online, therefore are easily transported.

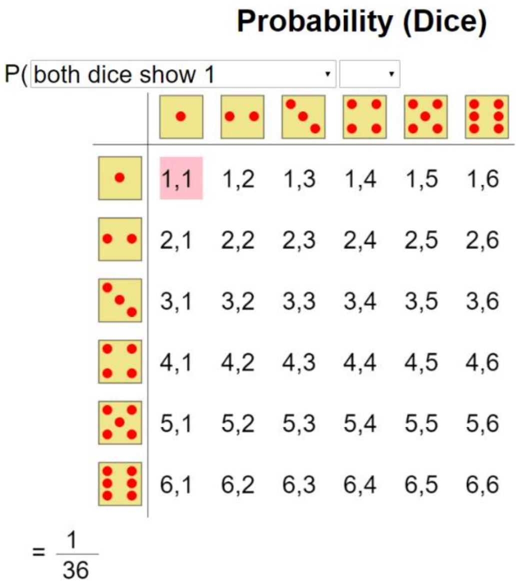

Figure 47: An example of a Mathematics Applets Virtual lab on probability by using the dice.[218]

The *Virtual labs* simulate a virtual system, the computer screen, mathematics laboratories, exploiting the potential offered by modern media technology key feature-technical interaction and direct and plausible manipulation of objects and parameters. The following figure shows the Virtual math Laboratory (Figure 48) and in particular, a K-Surface as a surface of constant negative Gaussian curvature. The initial Gauss map may be edited on the left. On the right-hand side, one may investigate the surface and its Gauss map. The Gauss map may be seen as the evolution of massive balls on the sphere connected by rubber bands.

[218] https://h2maths.webs.com/ - https://h2maths.webs.com/Statistics/ProbabilityDice.html

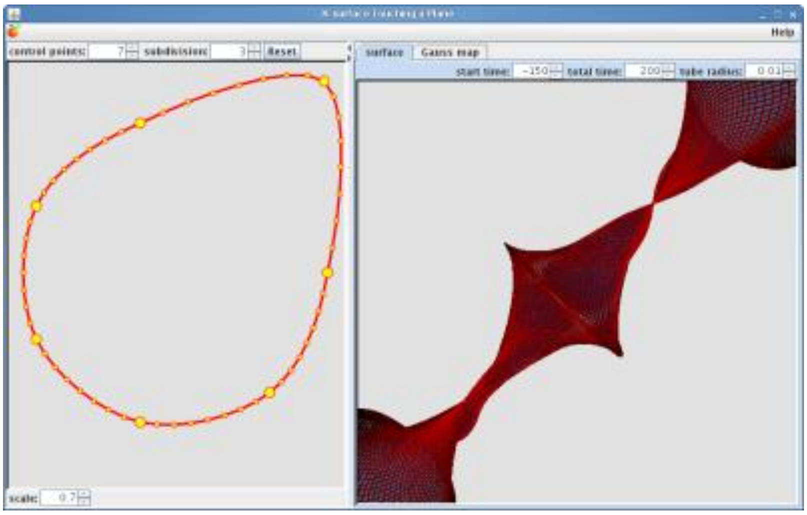

Figure 48: K-surfaces with a cone point in the Virtual math Laboratory[219].

Finally, Virtual Reality Laboratories are computer-based and highly interactive. The user becomes a participant in a *virtually real* world, in an artificial three-dimensional optical environment.[220]

At the educational level, even if some students can't easily understand some concepts in a traditional lab class, several studies show that it is easier to learn difficult concepts through the virtual labs. Students are more active in learning in a virtual environment than in a real lab class because it provides a user-friendly interface and easy to understand concepts.[221]

Exercises can easily be repeated by changing the parameters and variables. While, in a real lab, that can be time-consuming for both instructors and students.

Among the virtual lab, it is important to mention *Geomland*, a mathematical laboratory in LOGO style,[222] designed, developed and experimented by Bulgarian researchers from the Faculty of Mathematics and Informatics at Sofia University, supervised by Prof. Bozhidar Sendov.[223]

Geomland is an environment supporting mathematics explorations in Euclidean geometry and allowing students to construct and investigate the properties of geometrical objects, to formulate hypotheses and to construct proofs (Figure 49).

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[219] math.tu-berlin.de

[220] Hernández-de-Menéndez, M., Guevara, A. V., & Morales-Menendez, R. (2019). Virtual reality laboratories: a review of experiences. *International Journal on Interactive Design and* Manufacturing (IJIDeM), 1-20.

[221] Tüysüz, C. (2010). The Effect of the Virtual Laboratory on Students' Achievement and Attitude in Chemistry. *International Online Journal of Educational Sciences*, 2(1).

[222] Sendov, B., & Sendova, E. (1995). East orWest—GEOMLAND is Best, or Does the Answer Depend on the Angle?. In *Computers and exploratory learning* (pp. 59-78). Springer, Berlin, Heidelberg.

[223] Sendov, B., & Sendova, E. (1997). Tuning a Logo-like environment to a knowledge domain. *Eurologo'97*.

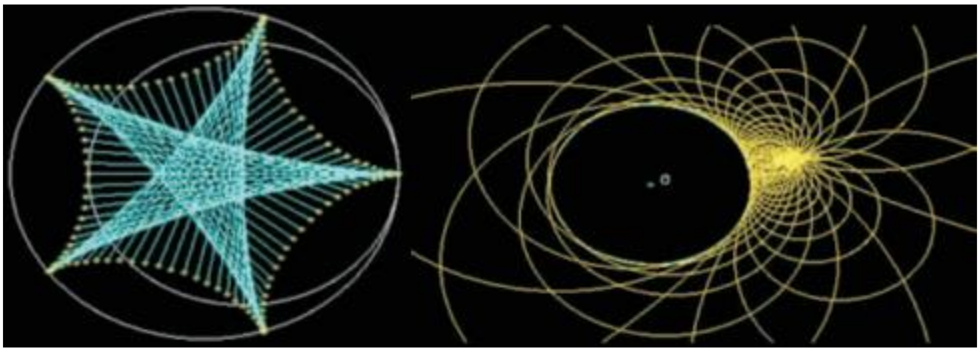

Figure 49: Explorations with Geomland.[224]

This learning environment provides flexible tools for defining notions as well as expressing, articulating and experimenting with new ideas. This can facilitate problem-solving, promote student and teacher individualization and support clear communications.[225]

Another example of a virtual lab is the VirMathLab (Virtual School Mathematics Laboratory) developed by the *Institute of Mathematics and Informatics* at the *Bulgarian Academy of Science* (IMI-BAS).[226]

This is a platform collecting numerous didactical resources (such as files with dynamic constructions, didactic scenarios on various topics, video files and publications) created on the base of the inquiry-based approach. These resources are represented by computer models for solving real-life problems both for evaluation and self-evaluation. Most of the dynamic files have been developed using the GeoGebra software and can be used online or in .ggb format.[227]

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224 Asenova, P. (2015).

225 Sendova, E., & Sendov, B. (1999). Harnessing the power of programming to support explorations in Euclidean geometry. *International Journal of Continuing Engineering Education and Life Long Learning*, *9*(3-4), 183-200.

226 Kenderov, P., Chehlarova, T., & Sendova, E. (2014, December). A Virtual Mathematics Laboratory in support of educating educators in inquiry-based style. In *Educating the educators: international approaches to scaling-up professional development in mathematics and science education. Proceedings of the conference hosted jointly by project mascil (mathematics and science for life) and the German Centre for Mathematics Education* (DZLM), 15-16 December 2014 in Essen, Germany (p. 167).
Ref. http://www.math.bas.bg/omi/cabinet/index.php?appletid=22 *VirMathLab (Virtual School Mathematics Laboratory)*[Accessed 10/11/2019]

227 Zengin, Y., Furkan, H., & Kutluca, T. (2012). The effect of dynamic mathematics software geogebra on student achievement in teaching of trigonometry. *Procedia-Social and Behavioral Sciences*, 31, 183-187.

2.4 Western and Eastern (Chinese) learning processes

The work[228] of *Di Paola B.* and *Spagnolo F.*, on learning styles of mathematics comparing Italian and Chinese students, underlines how mathematics teaching/learning can't exclude a multicultural analysis regarding social, cultural, anthropological and geographical aspects of students.

They point out as it is often difficult to understand and interpret, at both national and international levels, the results obtained, especially, in the presence of Chinese cultural features. Firstly, this research has investigated an epistemological path of Chinese culture compared with Western one concerning logical argumentation schemes used by lower and upper secondary school students in algebraic and geometry problem-solving tasks.

The difference of learning processes and teaching approaches of two countries, China and Italy, derived from two different philosophical lines: Chinese students are referred to a philosophical line of Confucius/Tao/Buddha, while Italians to Socrates/Platone/Aristotle.[229] In addition, according to another related research [230] on the analysis of cognitive processes with a specific mathematics context, from arithmetical thought to algebraic reasoning, in both Chinese and Italian students come from primary and secondary schools.

According to the collected data,[231] Chinese students have highly pragmatic and concrete behavior highlighted by procedural algorithmic reasoning to holistic thinking about coding and decoding of information in various situations/problems. Actually, during the mathematical argumentation and conjecturing, they often manifest a heuristic approach by trial and error, aimed at finding a *fundamental algorithm* as a proof tool.

The same type of arithmetical reasoning is used by Italian students, but this tends more towards exploration and numerical conjecturing.

For Chinese students, the argument and organization of reasoning take place through the hierarchization of reasoning models (models and sub-models assets and sub-sets) that don't seem to refer to bivalent logic, but, by extensive use of the *variability* concept.[232]

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[228] Spagnolo, F., & Paola, B. (2010). *European and Chinese cognitive styles and their impact on teaching mathematics.* Springer Verlag, pp.139-153.

[229] Spagnolo, F., & Paola, B. (2010), *Op. cit.*

[230] Di Paola, B. (2009). *Pensiero aritmetico e pensiero algebrico in ambienti multiculturali: il caso cinese*.

[231] Rif. Ajello, M., Spagnolo, F., & Xiaogui, Z. (2005). Reasoning patterns and logical-linguistic questions in European and Chinese cultures: Cultural differences in scholastic and non-scholastic environments. *Mediterranean Journal for Mathematics Education, Cyprus Mathematical Society*, 27-65.

[232] Ajello, M., Spagnolo, F., & Xiaogui, Z. (2005).*Op.cit.*

On the other side, the typical reasoning of Italian students is Aristotelian-Euclidean, which is hypothetical-deductive, expressed through a finite chain of conjunctions included in a bivalent logic process.

This means that Chinese students, from primary and secondary school, are more skilful than the Italian peers as *solvers* in algebraic contexts formalized and feel more comfortable in finding the possible factors of a relation, expressed verbally, graphically and in tabular form, among variables. In pre-algebraic contexts, these students demonstrate a kind of arithmetic reasoning, closely linked to a strong memorizing of "arithmetic facts" that help them in the definition of relational holistic thinking.

The arguments in a generalization process refer to algorithmic-procedural reasoning. Their good manipulative skills on the algebraic symbolism could derive from the features of their ideographic written language that helps them in algebraic syntax control formalization .

This shows that the difference in behavior between the two student styles come out, especially, from their language differences. Indeed, in Chinese students there is a constant balance between a serial, local thinking and global, holistic one, favoring cognitive categorizations and generalizations.[233]

As a result, the constructive elements of the Chinese ideographic script may favor an easier *access* to Algebra.

In addition, the Chinese school system highlights a strong coherence between the math standards (*international curriculum*), school textbooks and guides for teachers (*implemented curriculum*) and practical classes (*curriculum realized*).[234]

This consistency is guaranteed by the highly centralized system of control over the classes. In Chinese schools, there are no figures like teacher-researcher, who, in Western schools, can choose to explore new methods to search for teaching-learning effective tools. However, there are figures of experienced teachers who lead the faithful implementation of the school curriculum through in-service training based largely on the collective study of lessons identified as the best practices.

Accordingly, also the relationship between teaching and learning changes[235]: from a more independent activity of students in the West to an activity strictly led by teachers in East and, in particular, in the Confucian area.

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[233] Spagnolo, F., & Paola, B. (2010). *European and Chinese cognitive styles and their impact on teaching mathematics.* Springer Verlag.

[234] Bussi, M. G. B.(2009) Valori, tradizioni, modelli culturali: tracce nei curricoli di matematica <<Pedagogia più Didattica>>.

In general, every math class in China has a precise and hard script, but this is functional to classrooms involving also more than 50-60 students: revision, homework correction, presentation of the daily topic, formulation of the daily problem, its illustration, analysis of sub-problems, individual work or in a small group, presentation of possible solutions from the students, discussion of solution methods, exercises, focusing and summary of the key points, homework assigning, the announcement of the subject for a next lesson.[236]

2.4.1 Singapore's method to study mathematics

In 1980, the Institute for Curriculum Development (CDI) created the mathematics curriculum for primary school, later recognized as the "Singapore method". This system is based on learning studies of *Richard Skemp*, *Zoltan Dienes* and *Jerome Bruner*.

Singapore mathematics is not actually different mathematics, it is simply a method created by collecting the teacher experience and good practices realized during their professional development path. The added value of Singaporean pedagogics lies in having put these general principles together and having developed them in a systematic method.

Singapore mathematics supports the students in perceiving this subject not as a set of arbitrary rules or procedures but emphasizes the relational understanding among the parts. This concept derives from the study of *Richard Skemp* on the instrumental and relational learning described in his paper *Relational Understanding and Instrumental Understanding*.[237]

In this paper, *Skemp* distinguishes between the ability to perform a procedure (instrumental) and the ability to explain the procedure (relational). These two abilities favor the development of two corresponding learning methods – relational and instrumental.

Based on *Zoltan Dienes'* ideas (1960), systematic variation becomes a relevant element in Singapore's method applied to the mathematics study. The idea is that you vary the lesson through a series of examples that deal with the same problem or topic. Variation can take the form of mathematical variability and perceptual variability. The first implies that students need to experience many variations linked to the concept structure, in order to single out the general mathematical concept which is constant to all manipulations. The second suggests that conceptual learning is maximized when children are exposed to a concept through a

[235] Bartolini Bussi, M. G. (2011). Culture lontane come risorsa: la Cina. *L. Cerrocchi e A. Contini (a cura di), Culture migranti. luoghi fisici e mentali d'incontro,* 281-299.

[236] Manca, C. (2012). L'italiano per capire e per studiare. Il XVII Convegno Nazionale Giscel. *Italiano LinguaDue*, *4*(1), 514-524.

[237] Skemp, R. R. (2006). *Relational understanding and instrumental understanding. Mathematics teaching in the middle school*, 12(2), 88-95.

variety of physical contexts or embodiment.[238] The final result of the problem's variations inclusion in a systematic way in the mathematics study is that the students become more aware of what they are learning.

Singapore mathematics is based on a learning process based on three phases: concrete phase, pictorial phase, and abstract phase. This path is linked to the studies made by *Jerome Bruner* in 1966, who – starting from the psychological point of view - concluded by describing the learning processes by dividing them into three fundamental steps, which move from the awareness of real objects (concrete or action-based - enactive representation) to proceed, then, towards the pictorial representations (image-based - iconic representation), and finally to the symbols (abstraction).[239]

Actually, Bruner studied how children learn and put forward the Concrete Pictorial Abstract (CPA) approach to learning (used in the Singapore's method) by coining the word *scaffolding* to describe how children build on the information they have already mastered.

Another important theoretical concept of Bruner inspiring the Singapore method is the spiral approach of the curriculum. A spiral curriculum is built on the idea that when teaching new topics a constant revisiting of the basic ideas that were previously taught allows students to activate previously formed neural pathways. This, in turn, facilitates more effective understanding. Relating the new concepts and skills to prior learning allows students to understand and retain them more effectively. This means that teachers help children call back to the core idea of the main mathematical topic to be studied but each time at a higher level in order to reach, in the best way, the abstract representation. Consequently, it is not important to teach the same topic again and again in the same manner but to bring the students to a higher level as a spiral.[240]

The adoption of Singapore's method invites the teacher to slow down the transition towards symbols during the teaching process, above all to avoid that the learner develops a gap among the knowledge stages, and then finds difficulty in the assimilation process. Given the multidimensional nature of the proposed learning approach, the link between Bruner's study and the experience of mathematical education in Singapore is believed to bringing the benefits, so far reached mainly in the mathematics field, even in other sectors.[241]

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[238] Sriraman, B., & Lesh, R. (2007). *A conversation with Zoltan P. Dienes. Mathematical thinking and learning*, 9(1), 59-75.
[239] Bruner, J. S., & Riverso, E. (1972). *Studi sullo sviluppo cognitivo*. A. Armando.
[240] Resurreccion, J. A., & Adanza, J. (2015, March). *Spiral progression approach in teaching Science in selected private and public schools in Cavite*. In Proceedings of the DLSU Research Congress (Vol. 3, pp. 1-12).
[241] Wong, K. Y. (Ed.). (2009). *Mathematics education: The Singapore journey* (Vol. 2). World Scientific.

Another important element of Singapore's method is to pursue the best possible result without excessively extending the number of notions to be learned mechanically by the students.

In this direction, Singapore's national guidelines of 2006 invite teachers to develop the capacity of the learner in a relatively limited number of skills. This ensures more attention to the qualitative aspect over the quantitative in the teaching process.[242]

These skills are developed and improved through a problem-based approach and not only with the exercises' mechanical repetitions. The main reason is that this last method favors only mnemonic learning, ineffective for the knowledge and skills consolidation in the medium–long term, particularly, in students with reduced memory capacity. The problem-based approach, on the other hand, affects positively the student's cognitive and emotional abilities, arouses attention, and involves them in solving the problem by encouraging them to use their resources and to develop new skills.[243]

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[242] Arispe, I. (2008). *National Mathematics Advisory Panel: Foundations for Success*, p.21
[243] Ibid.

2.5 Art in mathematics

Artists, musicians, painters, and dancers have often found the inspiration for their artworks from scientific concepts or models, such as the Fibonacci algorithms sequence (Figure 50) used in music as *J.S. Bach* in "*Das Wohltemperierte Clavier I*" – BWV 846, body symmetry in dance to find the so-called *center of gravity* and to maintain stability in balance position, perspective and geometry objects in the painting, such as the dodecahedron model represented by *Jacopo de' Barbari* in his work "Luca Pacioli's Portrait" (1495) (Figure 51).

Figure 50: Fibonacci sequence on the face of a church in Pisa.[244]

Figure 51:Luca Pacioli's Portrait" (1495).

Roger Penrose, a Nobel laureate in Physics 2020 and Emeritus Professor at Oxford University, is known among physicists for his contributions to the theory of relativity and cosmology. He was the first to discover *impossible objects*[245] such as Penrose's staircase and triangle (Figure 52).

[244] A marble intarsia on the main entrance of the church of San Nicola in Pisa reveals the influence of the great Pisan mathematician Leonardo Fibonacci due to the presence of circles whose radii represent the first nine elements of the Fibonacci's sequence and which were arranged to depict some properties of the sequence. In addition, the tiles can be used as an abacus to draw sequences of regular polygons inscribed in a circle of given radius. (In Armienti, P. (2016). The medieval roots of modern scientific thought. A Fibonacci abacus on the facade of the church of San Nicola in Pisa. *Journal of Cultural Heritage*, *17*, 1-6).

[245] Penrose, L. S., & Penrose, R. (1958). Impossible objects: A special type of visual illusion. *British Journal of Psychology*, *49*(1), 31-33.

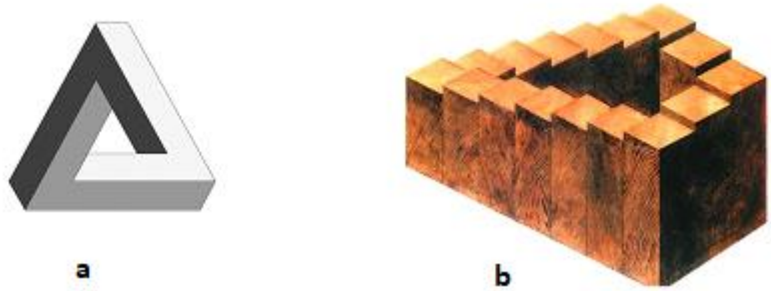

Figure 52: Penrose’s impossible objects: a) Triangle b) Stairs.

Penrose’s staircase, known also as an infinite or impossible staircase, is an example of an optical illusion.

This is the two-dimensional representation of a staircase that changes its direction by 90 degrees four times while one climbs or descends, to return to the starting point in around infinity. Although it is not possible to create this object in three dimensions, the image of Penrose can give the illusion falsified perspective. Penrose’s famous staircase was used by *Escher* in his lithograph *Ascending and Descending* (Figure 53).

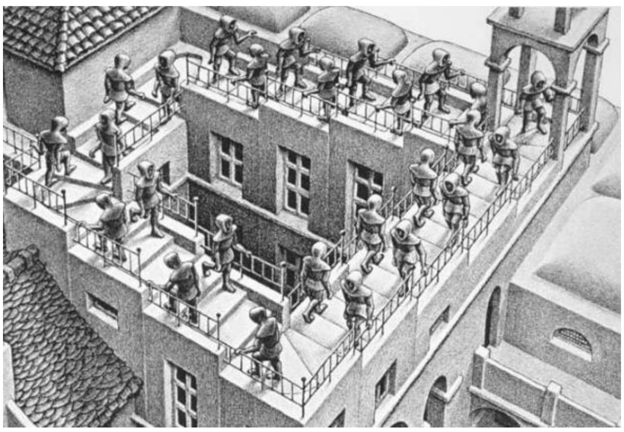

Figure 53: Escher's Ascending and descending stairs Optical Illusion.

Penrose’s triangle is another *impossible object*. This object can exist only as a two-dimensional representation and cannot be built in the space, because it presents an impossible overlap of parallel lines with different perspective constructions. It appears as a solid consisting of three square-based prisms joined together with three right angles to form a triangle (Figure 54). The result is an optical illusion of a closed triangle that occurs only by looking at it from a specific point of view (Figure 55).

Figure 54: Sculpture of the Penrose Triangle in East Perth, Australia.

Figure 55: Pictures of the Penrose Triangle taken from different perspectives.

In geometry, the Penrose tiling is a pattern of geometric shapes based on the golden section (Figure 56), which allows obtaining a tessellation of infinite surfaces in an aperiodic manner. Scientists have found these patterns also in the atoms and quasicrystals that are the shapes that do not occur naturally among crystals, such as polygons with 5 or 10 sides.

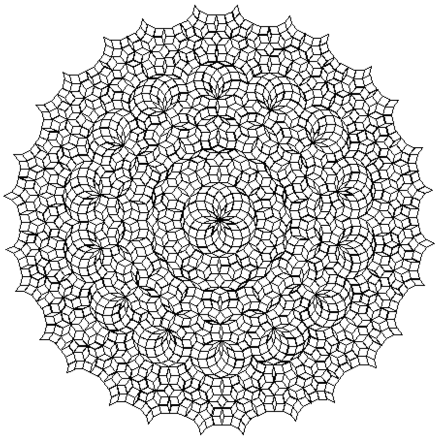

Figure 56: Penrose octagonal tiling with seed 14x1.

Scientists have recognized Penrose patterns in the beauty and geometric complexity of Arabic tile mosaics (Figure 57). A new study shows that the Islamic pattern-making process refers to more advanced math of quasi-crystals.[246]

Some of the most complex patterns, called *girih* in Persian, consist of sets of contiguous polygons fitted together with little distortion without gaps. Running through each polygon (a decagon, pentagon, diamond, bowtie or hexagon) is a decorative line. *Mr Lu* found that the interlocking tiles were arranged in predictable ways to create a pattern that never repeats — that is, quasi-crystals.

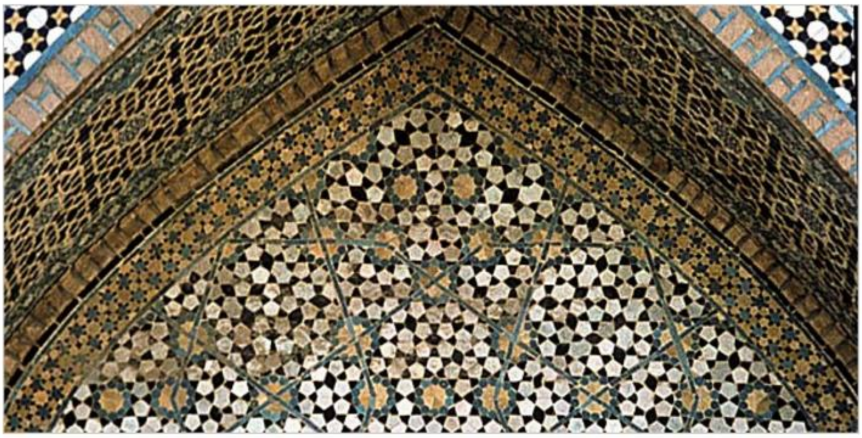

Figure 57: Mosaic Sophistication - A quasi-crystalline Penrose pattern at the Darb-i Imam shrine in Isfahan, Iran.Credit. Dudley and M. Elliff.

Mathematics doesn't refer only to figurative arts, but also to literature. *Gabriele Lolli*, with his work *Discorso sulla Matematica*, intends to show that methods and attitudes behind mathematics creativity activities are similar to those found in literature and poetry through reviewing the Calvin paradigm: Lightness, Quickness, Exactitude, Visibility, Multiplicity.[247]

He shows, further, that *Pons Asinorum*, a very selective logic operation, is assimilated to the composition of a medieval poem or the fairy tale telling; while the cone used to represent the set-theoretical universe "V" becomes the shape of Dante's Inferno in *Divine Comedy*.

Besides, there are several teaching examples which show art, such as painting, dance, theatre and music as a significant means to explain and describe science and math model and concepts to students of all ages. Therefore, mathematics education finds in the art representation a very effective teaching and learning approach showing a strong connection with reality and everyday life.

[246] Lu, P. J., & Steinhardt, P. J. (2007). Decagonal and quasi-crystalline tilings in medieval Islamic architecture. *science*, *315*(5815), 1106-1110.
[247] Calvino, I. (2012). *Lezioni americane*. Edizioni Mondadori.

In fact, according to some research studies,[248] a real pedagogical and didactic approach has been highlighted: Art-based learning is stated which favors higher learning performances thanks to the application and the integration of the artistic process into the learning and teaching methods.

One of such examples is the *Discovering the Art of Mathematics* project, supported by the National Science Foundation in Virginia.[249] It provides resources to support the teaching of Mathematics for Liberal Arts as an innovative approach for students at the college level. The provided resources and books enable the classrooms to be transformed into more responsive environments for conducting current research on learning and for addressing the needs and interests of the students.

The internal survey involved 460 students and the questionnaires were administered before and after the realization of the training courses.

The results show that the applied methodologies report relevant changes in the learning process of mathematics. The courses, so structured, have a positive impact on students' beliefs and attitudes, especially, on the relation between math and creativity.[250]

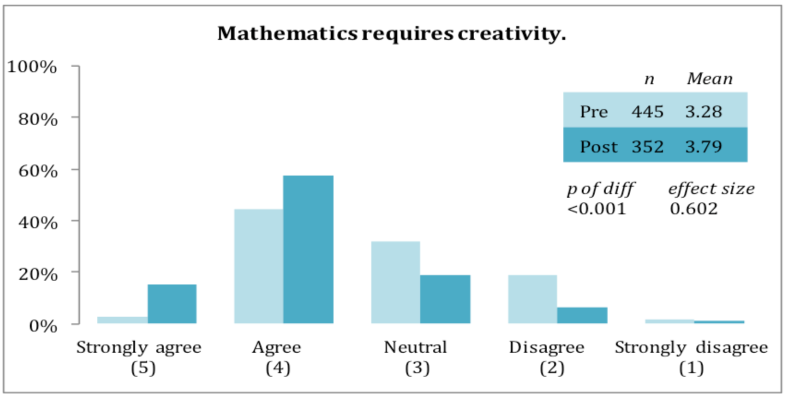

Figure 58: Internal survey administered to n. 460 students on “Mathematics requires creativity”.

[248] Rooney, R. (2004). *Arts-based teaching and learning. Review of the literature*. VSA Arts, Washington, DC: WESTAT Rockville, MD.
[249] https://www.artofmathematics.org/.
[250] https://www.artofmathematics.org/blogs/vecke/the-story-of-our-evaluation

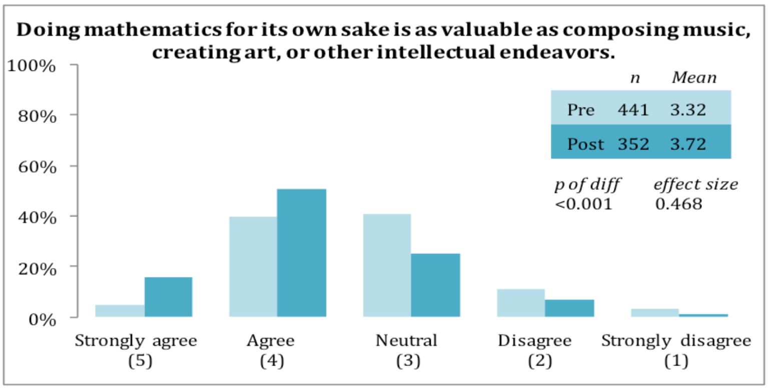

Figure 59: Internal survey administered to n. 460 students on the comparison between “doing mathematics” and “creating arts”.

In addition, another interesting factor is that no gender differences in the results were achieved.

These results are also supported by *Howard Gardner*’s multiple intelligences theory which is focused on the concept that people can exploit the potential of their eight basic intelligences (verbal-linguistic intelligence, logical-mathematical intelligence, visual-spatial intelligence, bodily-kinesthetic intelligence, musical intelligence, interpersonal intelligence, intrapersonal intelligence, naturalistic intelligence),[251] meant as a *bio-psychological potential to process information that can be activated in a cultural setting to solve problems or create products that are of value in a culture*.[252]

The use of the different thinking strategies and abilities allows students to understand the relations among structures, activities, and bits of intelligence (as defined by Garder). Indeed, the different representations of the same object study in both math and art can encourage students in the use of their abilities by developing the whole intelligence spectrum.[253]

In this direction, research work was realized with learners of different ages (pupils, pre-service and in-service teachers) in Bulgaria under the *InnoMathEd* project funded by the European Commission. The main objective was to design and to create objects on the base of the geometrical transformations such as congruence (translations, rotations, reflections, and their possible compositions). The achieved results showed this learning approach supported

[251] Niroo, M., Nejhad, G. H. H., & Haghani, M. (2012). The effect of Gardner theory application on mathematical/logical intelligence and student's mathematical functioning relationship. *Procedia-Social and Behavioral Sciences*, *47*, 2169-2175.
[252] Gardner, H. E. (2000). *Intelligence reframed: Multiple intelligences for the 21st century*. Hachette UK.
[253] Chehlarova, T., Sendova, E., & Stefanova, E. (2012). Dynamic tessellations in support of the inquiry-based learning of mathematics and arts. *Theory, Practice and Impact-Proceedings of Cosntructionism*, 21-25.

students to understand clearly geometry topics through their integration with the design and reconstruction of artistic artifacts.[254]

There are several examples of the integration between art and math (especially, physics and geometry) already existing at the national level, such as the organization of guided tours in museums. At the European level, there are several teaching activities supporting mathematics education of students with the use of games or even robotics.

However, there are only a few teaching examples where mathematics education and art are combined and integrated. The current cases of combining mathematics education with art are rather isolated as well as exploratory and have weak links to official school curricula. Therefore, the research has addressed this gap by strengthening the links between mathematics, arts and school math curricula.

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[254] Chehlarova, T., & Sendova, E. (2010, August). Stimulating different intelligences in a congruence context. In *Proceedings for Constructionism 2010. The 12th EuroLogo conference* (pp. 16-20).

CHAPTER 3. PRESENTATION OF RESEARCH MODEL/ APPROACH DEVELOPMENT

This Chapter aims to present the model and the approach proposed in this research work and how they were developed as indicated in the Chapter 1, Paragraph 1.4.

Firstly, the reference theoretical framework is described. This has been mainly identified in the *Theory of Didactical Situations in Mathematics*[255] of *Guy Brousseau*, an educator, specialized in Mathematics Didactics and founder and promoter of the research in the field of mathematical didactics. Actually, he founded a specific research center in Bordeaux and the COREM (Center for Observation and Research on Mathematics Teaching).

Besides, it is shown how this model has inspired the proposed one, which is the combination of Western and Eastern (Singapore's method) teaching and learning approaches using the *art*.

Finally, the research approach is explained, that is shown, step by step, how the proposed model is turned into a new didactic approach which valorizes mainly the problem-solving skills, the creativity and the imagination in the students involved without losing knowledge value, in particular, related to the school curriculum.

3.1 Theoretical introduction of the research model

The research model, that is the reference theoretical framework of this proposal, had a heuristic function and practice orientation by allowing guiding the tasks to be carried out up to the achievement of the results.[256]

The main model was the *Theory of Didactical Situations in Mathematics*, elaborated by *Guy Brousseau*,[257] aiming to define the conditions under which a *learner* is led to *do* mathematics, how to use it and how to invent it.

On the base of the constructivist approach, Brousseau defines three situations in the teaching process:

1. *A-didactical* situation is a context containing all the conditions which allow students to establish a relationship with a specific knowledge regardless of their teacher. This means that

[255] Brousseau, G. (2006). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990* (Vol. 19). Springer Science & Business Media.
[256] Dal Prà Ponticelli, M. (1985). I modelli teorici del servizio sociale. *Astrolabio, Roma*.
[257] Brousseau, G. (2006). *Op.cit*.

student's actions and answers depend on such relationship with the problem or the difficulty to be solved or overcome, even if this relation is no so explicit.

2. *Non-didactical* situation is the context that is not organized in order to allow students to learn specific knowledge. For example, all operations with natural numbers may be considered as a non-didactical situation.

3. *Didactical* situation is the one in which the situation is designed and well-organized in order to favor knowledge acquisition. It is enough to consider as didactical all the activities done in the classrooms where teacher intentionally "teaches" and students are forced to learn.[258]

Using the words of *Guy Brousseau*: *The modern conception of teaching, therefore, requires the teacher to provoke the expected adaptation in her/his students by a judicious choice of "problems" that she/he puts before them. These problems [...] must make the students act, speak, think and evolve by their own motivation [...] The student knows very well that the problem was chosen to help her/his to acquire a new piece of knowledge, but she/he must also know that this knowledge is entirely justified by the internal logic of the situation and she can construct it without appealing to didactical reasoning. Not only can she do it, but she must do it because she will have truly acquire this knowledge only when she is able to put it to use by herself in situations which she will come across outside any teaching context and in the absence of any intentional direction. Such a situation is called an a-didactical situation [...]*

This situation or problem chosen by the teacher is an essential part of the broader situation in which the teacher seeks to devolve to the student an a-didactical situation which provides her with the most independent and most fruitful interaction possible... She is thus involved in a game with the system of interaction of the student with the problems she gives her. This game, or broader situation, is the didactical situation.[259]

In other words, Brosseau defines the didactical contract, between the teacher and student, as the results of the game rules and strategies of the didactical situation. This is the way that the teacher presents the didactic situation depending closely on the specific topic to be taught. The setting up of this didactical contract allows the presentation and the development of didactical situations formed by a-didactical situations.

Students, facing the a-didactical situation, construct, step by step, a piece of knowledge mediated by teachers.

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[258] Brousseau G., *Op cit.*

[259] Cit. Ibid., pp. 30-31.

However, this knowledge should situate the students' production and performances in the real context. [260]

The three situations identified can be imagined like interaction systems between one or more subjects with a *milieu* that is a context or means. According to the theory of didactical situations, teachers have to encourage in their students a *behavior*, which they should take independently in order to demonstrate their knowledge. For this reason, an important element is the *milieu* that teachers know well or have prepared by themselves for students. Actually, the teacher's art is in the organization of a relationship between a student and the *milieu* including two main tasks:

- leaving a reasonable uncertainty about the knowledge on the subject to be reduced;
- making sure that this reduction can take place.

To synthesize the different situations occurring, Brousseau refers to *Polygons* of didactics as *Didactics triangle* proposed by *Yves Chevallard*, where the three apices are constituted by Knowledge (in this case Mathematics), Teacher and Learner/Student as follows:

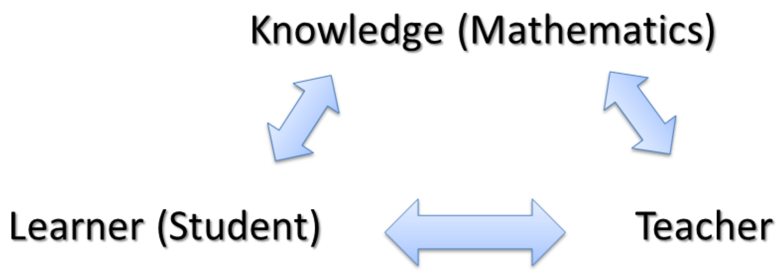

Figure 60: *Didactics triangle* proposed by *Yves Chevallard*[261].

However, by introducing in this graphics the *milieu* as another crucial element in a didactical situation, the figure turns into a didactical quadrangle as shown below:

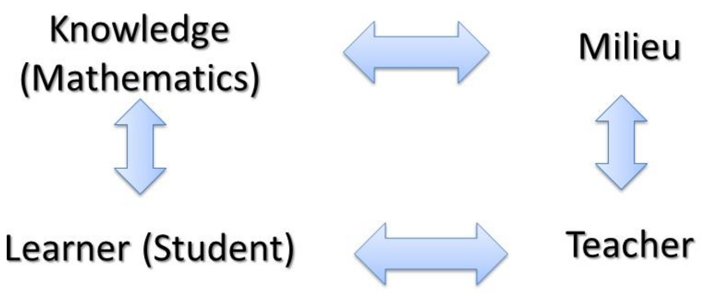

Figure 61: The introduction of *milieu* in the *didactics triangle* of *Yves Chevallard*.

[260] Samaniego, A., & Barrera, S. V. (1999). Brousseau in Action: Didactical Situation for Learning How To Graph Functions.
[261] Chevallard, Y. (1982). Pourquoi la transposition didactique? Communication au. *Séminaire de didactique et de pédagogie des mathématiques*, 1981-1982.

Moreover, if we consider also the difference between school knowledge and student knowledge coming from his/her background, context and learning approaches, the graphics will turn into a *didactical hexagon* as proposed by *Guy Br*ousseau where the functional meaning of these relationships among different types of knowledge are underlined (Figure 62):

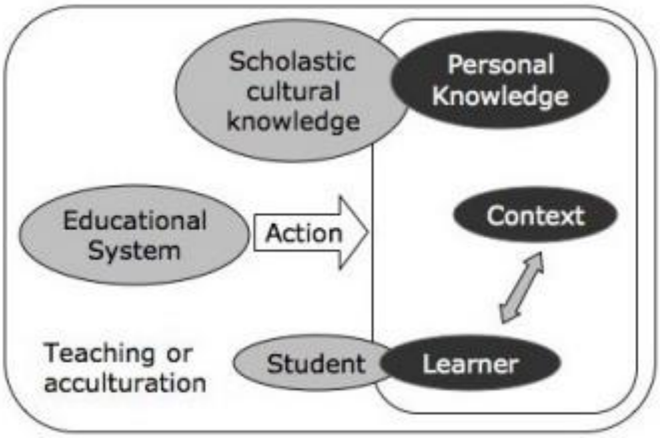

Figure 62: *Didactics Hexagon* of Guy Brousseau.

The main point of the *Theory of didactical situations in mathematics* of *Guy Brousseau* is that knowledge is not just something transferable from one person to another, but is gained by people through constructing it, step by step. According to the analysis of the thinking act elaborated by *John Dewey*, five phases are identified as follows: a) a felt difficulty; b) definition of the difficulty; c) a tentative solution; d) development of the consequences for the solution proposed, and e) further observations and searches towards the definitive acceptance or rejection of the proposal.[262]

Without neglecting *Edouard Claparède* who claimed the actions are all intentionally aimed to readapt the individual to the environment with the scope to find a balance, broken by the environmental changes or by the subjects themselves. The re-found balance, after the breaking, can only lead the subject to a superior level of knowledge. Actually, for Claparède, the individuals start their reasoning only through a deep need which turns into a problem or a question motivating and giving on interest to the activity or its content.[263]

In this constructivist theoretical framework, problem-solving is a fundamental tool to learn mathematics. This means that student's need to reach a solution in a problematic situation gives rise to a reflection action which, accordingly, becomes knowledge. Therefore,

[262] Aebli, H. (1958). Una didáctica fundada en la psicología de Jean Piaget.
[263] Aebli H., *Op. Cit*., pp. 31-32.

the encouragement, in mathematics teaching, of student's problem-solving ability can be a good means to favor learner's reflection and motivation.

3.1.1 The didactical situation in the model proposed

The model proposed in this research work is the result of the integration of the three types of didactical situations identified by Brousseau. Therefore, the learning/teaching situation developed according to the three phases of the Eastern learning/teaching approach (concrete, pictorial and abstract), was *A-didactical*, *Non-didactical*, *Didactical*.

In detail, the final situation was:

1. *A-didactical*, because the students learned mathematical topics by discovering that different relationships exist among things or mathematical concepts (even if they cannot be so explicit) and by developing, accordingly, problem-solving skills avoiding the memorization of the solution procedure only.
2. *Didactical*, because several worksheets were prepared for the students before starting the experimentation phase. These contained instructions to lead the student from the concrete phase to pictorial up to the abstract one as it is explained above in Paragraph 3.3.
3. *Non-didactical*, because the teachers had the function to mediate and support the learning process through the creativity and the imagination of their students. The use of creativity from students was let free, especially when they produced their art-works on the base of mathematics concepts studied.

Therefore, re-considering the figure of *Didactics Hexagon* of *Guy Brousseau* (Figure 61) from the point view of the proposed model, the art was identified as the *context* or better the *milieu* to be used in reaching knowledge. Knowledge achieved by students was *constructed*, meaning that students went step by step through the concrete phase to pictorial up to reach the abstract of a mathematical concept.

Besides, the possibilities to develop problem-solving skills in a didactic situation, aiming to discover the relationships among things or concepts stimulated the creativity of students and motivated them in their learning path.

3.2 West and East towards a unique teaching approach

The research approach was focused, mainly, on the combination of the mathematics approaches set in West and East, between Western analytical methods and Eastern holistic approaches, namely the Singapore's method.

Citing *Nisbett*, we can sum up: *[…] as a result, as Asian thought is "holistic", drawn of the perceptual field as a whole and to relations among objects and events within that field. By comparison to Western modes of reasoning, East Asian thought relies far less on categories or on formal logic; it is fundamentally dialectic, seeking a "middle way" between opposing thoughts. By contrast, Westerners focus on salient objects or people, use attributes to assign them to categories, and apply rules of formal logic to understand their behavior*. [264]

The theoretical reference system under consideration could be schematized by referring to *Aristoteles* with the organization of bivalent logic characterizing the way of arguing in Western culture and therefore of our students, and to Confucius with Confucian method transmission of the Tao, and the book *I Ching*.[265]

As *Leung* stated: *Aristoteles, more than any other thinker, determined the orientation and the content of Western intellectual history. He was the author of a philosophical and scientific system that through the centuries became the support and vehicle for both medieval Christian and Islamic scholastic thought: until the end of the 17th century, western culture was Aristotelian. And, even after the intellectual revolutions of centuries to follow, Aristotelian concepts and ideas remained embedded in Western thinking*.[266] For example, Euclidean geometry, the first structured language in mathematics history, is a model of Aristotelian bivalent logic.

In fact, Eastern researchers prefer, in general, to determine the relationships among things, such as among phenomena or objects following a different approach from Aristotelian hypothetical deductive one.[267]

In other words, the aim is to favor a relational thought connected to a concrete arithmetic concept and to develop skills for the recognition of relationships among variables, to be able

[264] Peng, K., & Nisbett, R. E. (1999). Culture, dialectics, and reasoning about contradiction. *American psychologist*, *54*(9), 741.

[265] I Ching (Libro dei Cambiamenti) is fundamental text for the understanding of Chinese metaphysics of *yīn* e *yáng* and distinctive element for the understanding of some of China's reasoning patterns easily found in the history of mathematics

[266] Leung, F. K. S., *Mathematics Education in East Asia and the West: Does Culture Matter?* in Leung, F. K. S., Graf, K. D., & Lopez-Real, F. J. (Eds.). (2006). *Mathematics education in different cultural traditions-A comparative study of East Asia and the West: The 13th ICMI study* (Vol. 9) Springer Science &Business Media.

[267] Spagnolo F., Di Paola B., *Op.Cit.*, 2010.

to work on them dynamically. This approach, mediated by the teacher, can promote in the student a stronger recognition of the relationship between syntax and semantics of arithmetic writing. For example, the theory of the elements is defined on the base of their function in a situation. This is analogous to the method mostly used in mathematics, according to which an object is determined from the relationships established with other objects.

Therefore, the mathematical reasoning results can be assimilated to a set of analysis and synthesis processes that evolve in the development of both abstraction and generalization phases through so-called *variations* as well as the observation and recognition of invariant elements can give rise to generalizations through the search of algorithmic procedures unifying and contextualized in more fields. This process has been called by some mathematics teachers the *trilogy of problem-solving*.[268]

In this meta-cognition *independent* process, the definitions, patterns of hypothetical-deductive reasoning demonstrations are not used. However, this is focused on the ability to review, rationally, the information and data provided by the problematic situation met; to learn how to categorize them in relation to knowledge previously achieved and to choose the best ways for their representation and use to draw up concrete conclusions.[269]

3.2.1 Teaching/Learning approach based on three phases of the Singapore approach

The graphic representation of Singapore approach is a pentagonal figure (Figure 63) where mathematical problem-solving skills are at the center and five sides are represented by (i) attitudes (such as beliefs, interest, appreciation, confidence and perseverance); (ii) metacognition (such as self-regulation of learning); (iii) processes (reasoning, communication, connections and heuristics skills, application and modelling); (iv) concepts (numerical, algebraic, geometrical, analytical) and (v) skills (numerical calculation, algebraic manipulation, spatial visualization, data analysis, measurement, use of mathematical tools, estimation).[270]

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[268] Gu, L., HUNAG, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In *How Chinese learn mathematics: Perspectives from insiders* (pp. 309-347).

[269] Stephens, M., & Keqiang, R. X. (2014). Using a framework of 21st century competencies to examine changes between China's 2001 and 2011 mathematics curriculum standards for basic education. *Journal of Mathematics Education at Teachers College*, *5*(2).

[270] Setiawan, A. R. (2019). A Brief Overview of Singapore Mathematics Syllabus.

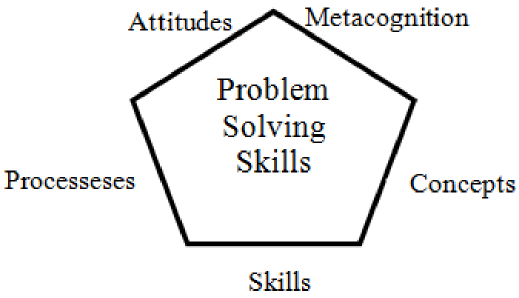

Figure 63: Graphic representation of the Singapore approach.

The main feature of this approach is the use of visual and model-drawing strategies which underline mathematics and word problems and leave out the memorization occurring through repetitive exercises.[271] In effect, Singaporean textbooks use very often the visual aids like *strip models*, which are pictorial representations of mathematical problems requiring a basic arithmetic operation to be solved. These tools are used to help a young student learn to choose more appropriate arithmetic operations to solve a certain problem.[272]

The Singapore approach is defined as the *Concrete-Pictorial-Abstract approach* (CPA) applying the so-called *Concrete-Representational-Abstract* (CRA) *strategies*.[273]

The approach aims to teach the student how to represent mathematics concepts visually before applying the abstract symbols like numbers and equations.[274] This helps students make their connections and draw generalizations about the concepts learnt and not simply memorization of disconnected and isolated facts.[275]

In this context, students experience different types of representations of the same concept by finding correspondences and relationships among them. This step conducts students to consolidate the conceptual understanding.[276]

Thanks to this transition, starting from concrete objects to pictures arriving then to symbols, the Singapore approach offers many and various opportunities to learn mathematical

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[271] Sami, F. (2012). The Singapore System: An Example of How the US Can Improve Its Mathematics Education System. *MathAMATYC Educator*, *3*(2), 9-10.
[272] Beckmann, S. (2004). Solving algebra and other story problems with simple diagrams: A method demonstrated in grade 4-6 texts used in Singapore. *The Mathematics Educator*, *14*(1).
[273] Witzel, B. S., Riccomini, P. J., & Schneider, E. (2008). Implementing CRA with secondary students with learning disabilities in mathematics. *Intervention in School and Clinic*, *43*(5), 270-276.
[274] Ginsburg, A., Leinwand, S., Anstrom, T., & Pollock, E. (2005). What the United States Can Learn From Singapore's World-Class Mathematics System (and What Singapore Can Learn from the United States): An Exploratory Study. *American Institutes for Research*.
[275] Hazelton, M., & Brearley, D. (2008). Singapore Math: Challenging and Relevant Curriculum for the Gifted Learner. *Understanding Our Gifted*, *21*(1), 10-12.
[276] Clements, D. H. (2000). 'Concrete'manipulatives, concrete ideas. *Contemporary issues in early childhood*, *1*(1), 45-60.

concepts, especially for those students who have difficulties.[277] In this way, students have time to face mathematics concepts by testing many and different approaches to solve similar problems.

However, some researchers have identified some gaps in both Eastern and Western teaching approaches.[278]

The results, emerging from this research, state that western math teachers focus on student understanding through concrete examples, while Eastern teachers on abstract thinking after the use of concrete examples. In addition, Eastern teachers are more concentrated on the mathematics content, process, and student's learning when they plan their lessons and teaching to their students through *rote learning* or *repetition*.[279]

Leung, indeed, identifies 6 dichotomies to show the key factors in Eastern and Western teaching/learning approaches: *product (content) versus process, rote learning versus meaningful learning, studying hard versus pleasurable learning, extrinsic versus intrinsic motivations, whole-class teaching versus individualized learning, competence of teachers: subject matter versus pedagogy*.[280]

Therefore, combining the main features of the Eastern teaching /learning as identified by Leung with the major characteristics of the Western one, the results can be synthesized in the following figure:

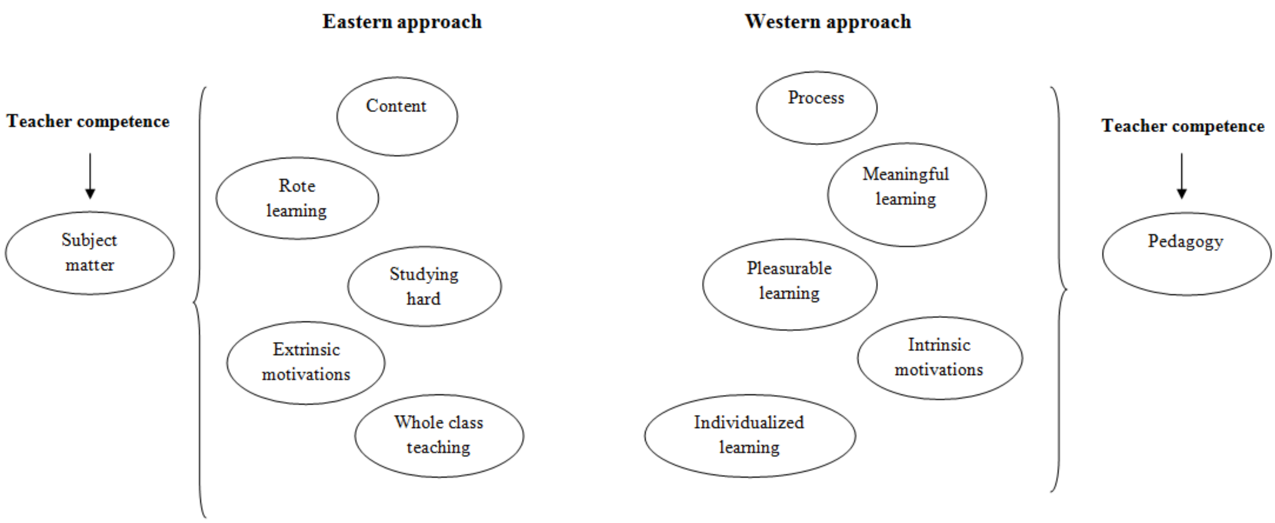

Figure 64: Synthesis of Eastern and Western approaches.

[277] Jordan, L., Miller, M. D., & Mercer, C. D. (1999). The Effects of Concrete to Semiconcrete to Abstract Instruction in the Acquisition and Retention of Fraction Concepts and Skills. *Learning Disabilities: A Multidisciplinary Journal*, *9*(3), 115-22.
[278] Stigler, J. W., & Hiebert, J. (1999). The Teaching Gap The Free Press New York.
[279] Leung F.K. S., *Op. cit.*
[280] Ibid.

Starting from this point of view, the research approach proposed intends to combine the features detailed in Figure 64 to create a unique synthesized approach as follows:

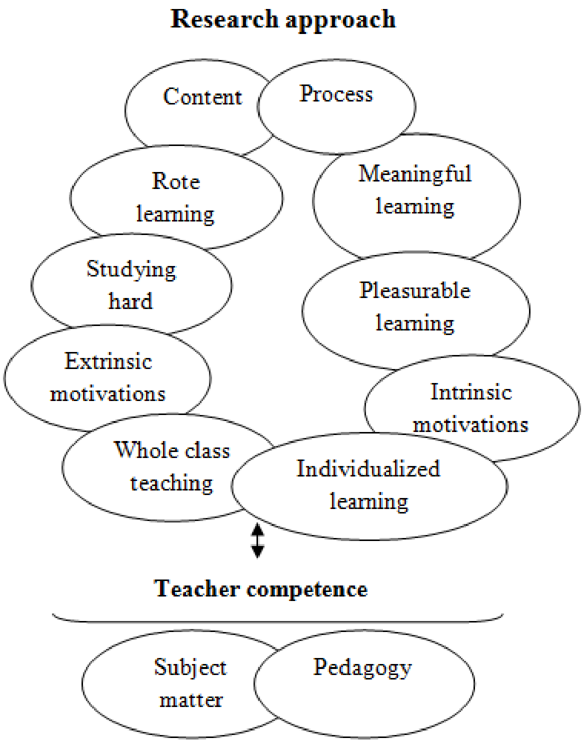

Figure 65: Synthesis proposed by the research work.

Therefore, to reach the synthesis proposed in Figure 65, inspired by the Singapore method, the three phases (concrete, pictorial and abstract) with the Aristotelian method of logic deductive (Figure 66) were combined by using a particular tool which will be described in the next paragraph.

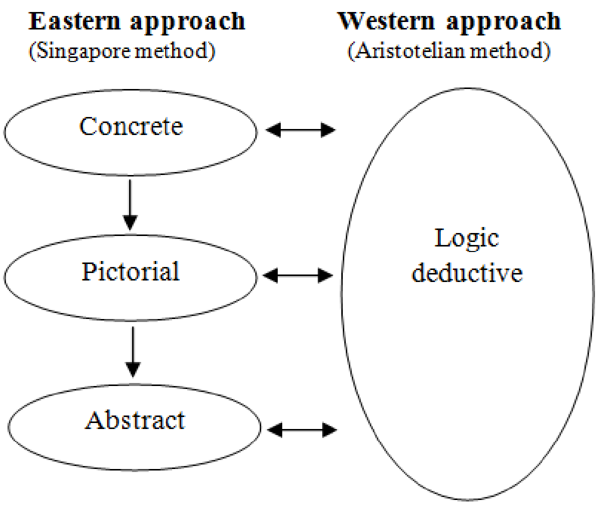

Figure 66: Combination between Singapore method and the Aristotelian method.

3.2.2 Art as combining element

Art was chosen as the fundamental element to combine the two approaches.

In the traditional Chinese approach, especially in the concrete phase, where the manipulative act is very important, tools and artifacts are properly created and used.

The tools and artifacts used in the research approach were specific art-works selected on purpose according to mathematical concepts to be learned by students. This allowed them to develop system thinking based on applicable knowledge.

In this case, learning was focused on experience-centered approaches and, at the same time, the learning process became a more joyful and motivating experience.

Reporting the words of a Romanian mathematician, *Grigore Moisil*, who claimed: *A mathematician is concerned with mathematics because she/he sees something beautiful in it, something interesting which she/he likes, which makes her/his think and which carries her away*. *Imagination is a source of information itself*.

Using the art-works as tools, students developed, further, their imagination and creativity through mathematics learning and study.

The introduction of the *art* to combine and join eastern and Western learning/teaching approaches turns Figure 64 into the following:

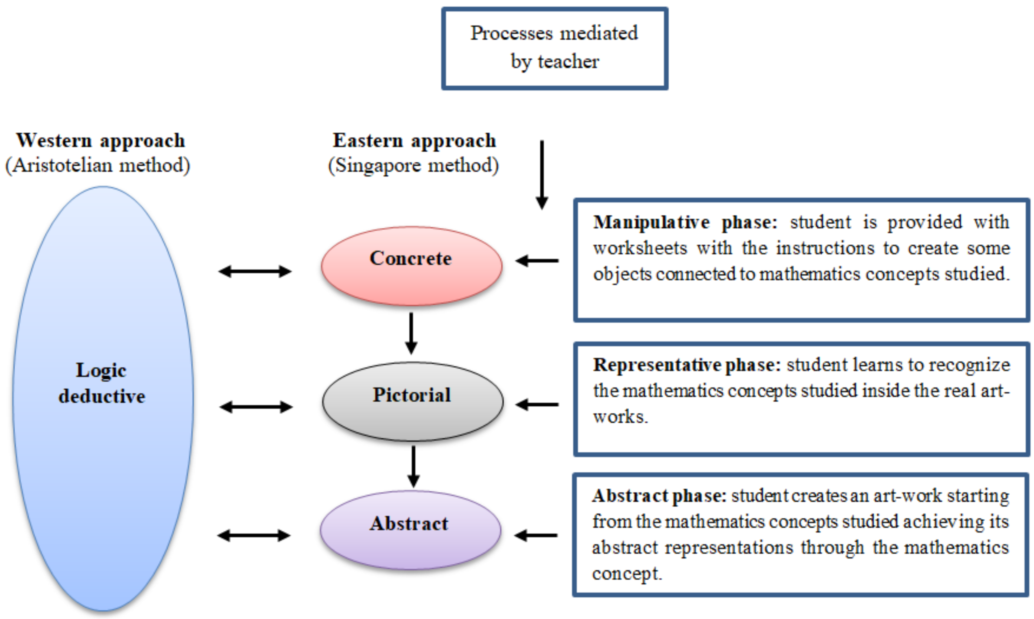

Figure 67: Representation of Western and Eastern approaches (Singapore's method) combined with art.

This procedure lets students face mathematics problems with variations, a fundamental element used systematically in Chinese mathematics teaching. In these types of problems, we have a constant element that can be represented by Arabic numbers as usually they are specified in the textbook or by the general mathematics concept to be studied while the context or other surrounded elements change. For instance, a student can learn to recognize the different symmetries working on several friezes and plan ornaments. In this example, the symmetry is the constant element and the ornament is the context changing.

This allows students to learn, analyze and to see the problem from different point views. Moreover, identifying the relationships among the different mathematics problems, they don't only memorize the resolution procedure, but also develop and reinforce problem-solving skills.

Taking into account Figure 65, the approach proposed in this research work can be synthesized as follows:

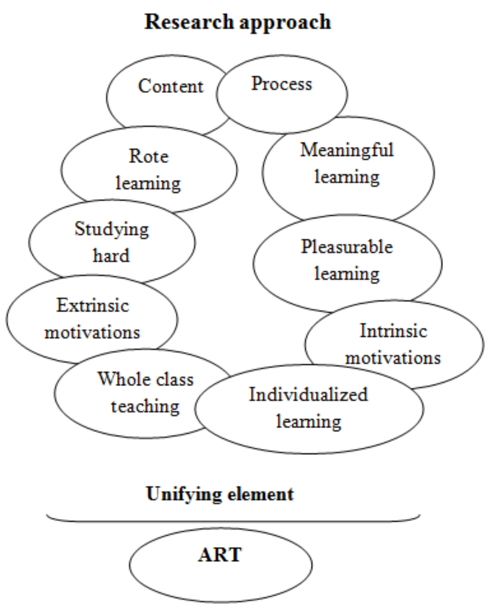

Figure 68: Synthesis of the research proposed approach.

The research proposed work is based on the assumption that art can be used as a unifying element of the Eastern and Western learning approaches by filling up the gaps described in the previous paragraph.

Art-works contain mathematics concepts (**content**) but the student needs to be led inside the creative **process** to be able to recognize them (Chapter 2, Paragraph 2.4). The capacity of students to find the mathematics concept inside the art-works is the result of a **hard study** through a research work mediated by their teachers through the application of learning tools produced and found on purpose. In the meantime, art becomes a **pleasure**, especially, when students are involved directly in the artistic process touching their cognitive and emotional dimensions through the use of different languages such as visual, sensory, verbal and non-verbal.

The use of the art-works in mathematics learning can be considered **rote learning** (repetition), inside the theory of *variation problems* applied to a mathematics concept. This means that even if the mathematics formula/concept to be studied remains constant, the art-works referring to them can be different and then the surrounded elements can change.

However, at the same time, as described in Chapter 1, Paragraph 1.4.3, it becomes **meaningful learning**, because students are involved actively in the production of their knowledge increasing their curiosity on *how to know* and not only on knowledge in itself.

Moreover, it offers to the students both **intrinsic motivations** (coming from the single individual) and **extrinsic motivations** (coming from culture and society, since the art-works are part of mankind's cultural heritage).

Finally, the art can be used in both **individualized learning**, especially when students have some specific difficulties in mathematics study, and **whole-class teaching** for the development of sharing and exchanging moments by favoring different learning approaches such as cooperative, problem-solving, inquiry based-learning as described in Chapter 2.2.

In the following chapters, the concrete development of the research proposed approach, in terms of activities carried out, is described.

CHAPTER 4. PRESENTATION OF RESEARCH REALIZATION AND IMPLEMENTATION

This chapter aims to describe the activities realized and implemented in the research work as indicated in Chapter 1, Paragraph 1.4.

After a deep analysis of literature aiming to study the current status quo of mathematics education, how mathematics topics are defined in the PISA, TIMSS and national surveys, the comparison of school curriculum on mathematics in diverse learning environments, the research teaching/learning approach by combining the Western and Eastern approaches with the use of the art was developed.

This analysis was realized also to verify the status quo of mathematics education, in particular related to gender issues and the student attitude towards scientific careers.

It started from the comparison between the definition of the *mathematics representation*, assumed in the international (PISA, TIMSS) and national surveys, and school curriculum regarding mathematics study. This step was useful to construct didactic materials used, afterwards, in the experimentation activity, the reference sample, and the evaluation and validation tools suitable to verify the effectiveness and efficiency of the proposed approach as well as the relevance of results achieved.

The teachers together with the researcher defined the mathematics concepts/formula to be investigated on the base of the school curriculum and the art-works related.

In the end, three worksheets, as guidelines, were constructed to provide the teachers with the instructions to proceed through the three phases: concrete, pictorial and abstract. The instructions described the materials (such as paper with specific shape, pencils, and colours) and/or software applications (e.g. GeoGebra) to be used, the objective, the vocabulary, activity sequence structured on the base of the three phases (concrete, pictorial and abstract).

For the concrete phase, students were invited, for example, to build some objects (e.g. pentagram to study geometrical figures), figures (e.g. friezes, ornaments to study the symmetry) with instructions (provided by the teacher or the researcher) and the equipment provided. The aim was to let students make the experience and become familiar with the mathematical concept studied through object manipulation.

As agreed with teachers, GeoGebra application was utilized for this purpose for several reasons.

First of all, Italian teachers consider it (GeoGebra) user-friendly when used with low ICT skilled students on PC, Mobile, and Tablet.

Secondly, due to students' previous experiences with the software, they were able to concentrate directly on the subject to be explored without first being worried about a new tool to be learnt.

Besides, the software was used for manipulating both 2D and 3D objects with different types of animation.

Figure 69: Experimentation time - Concrete phase with students at *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

The use of the modelling program, during the concrete phase, has allowed students to explore and understand mathematical concepts through the help of visualization and virtual object manipulation (Figure 69). Therefore, students learned and familiarized themselves with the specific objects' construction on the base of, for example, symmetry concept. This helped students to reinforce their visualization skills, modelling the real-world problems and making connections between the real world and mathematics (Figure 70).

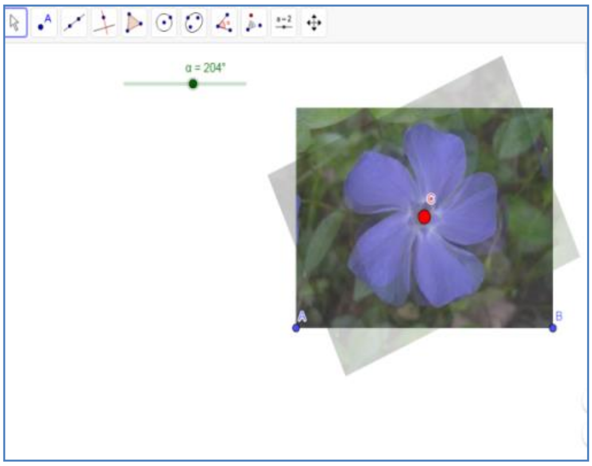

Figure 70: Visual Modelling Programming – Example of symmetry study – GeoGebra.

Afterwards, in the pictorial (or also representative) phase, students started their research through the finding of the art-works or the real-life application containing the mathematical concept or formula studied of the previous phase both individually and in team working. This helped students recognize the same concept in different contexts (and then in different art-works – Figure 71). This put their attention on the existing different representations of mathematics concepts.

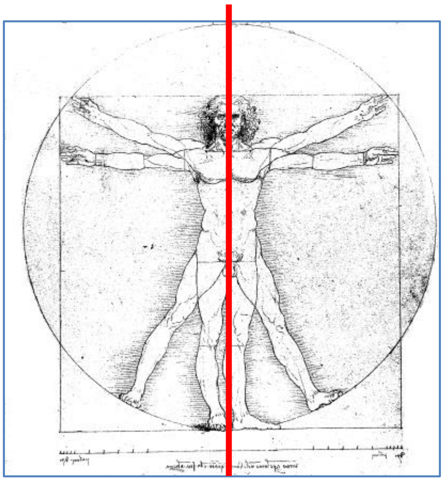

Figure 71: Experimentation time - Pictorial phase – artwork selected by students *at Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

The aim was to help students find in the habitual surrounding objects the mathematical concept studied, that, in this case, was the *symmetry*. During this phase, they used not only what is strictly known as the art-works but also they have found the mathematical concept in other contexts as well, e.g. in nature (Figure 72).

Figure 72: Experimentation time - Pictorial phase – object from nature selected by students at *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

Finally, students were invited to create their art-work starting from the mathematics concept studied in the two previous phases by achieving the abstraction representation of the mathematical concept (Figure 73).

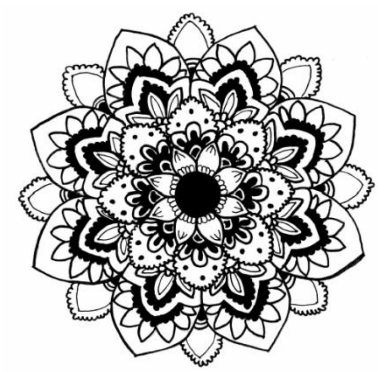

Figure 73: Experimentation time - Abstract phase – artwork realized by a group of students from *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

All three phases were mediated by teachers and/or the researcher who supported the students' study by introducing every phase and by stimulating them with questions and observations (Figure 74).

Figure 74: Preparatory activity for each phase with the students from *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

Finally, a part of these works was uploaded in a 3D virtual museum,[281] developed by *Institute for Computer Science and Control, Hungarian Academy of Science* in collaboration with *Institute of Mathematics and Informatics, Bulgarian Academy of Sciences*.[282]

[281] Márkus, Z. L., Kaposi, G., Veres, M., Weisz, Z., Szántó, G., Szkaliczki, T., ... & Pavlova, L. (2018). Interactive game development to assist cultural heritage. *Digital Presentation and Preservation of Cultural and Scientific Heritage*, 8, 71-82.

[282] Tramonti, M., & Paneva-Marinova, D. I. (2019). Maths, Art and Technology: a Combination for an Effective Study. *TEM Journal*, 8(1), 82-86.

4.1 The 3D Virtual Museum description

The virtual museum, *Mathematics and Arts*,[283] was realized to show some of the final outputs of the research work.

First of all, the objectives and model proposed in this research work are presented by underlining the structure and the tasks followed during the three-phase (concrete-pictorial-abstract).

Afterwards, some of the selected *art-works* created by students[284] are presented (Figure 75). These art-works are related to the study of the symmetry and were realized by two different target groups from Italian secondary schools – first cycle (11-13 years old) and second cycle (14-16 years old). The aim was to show both hand-made and digital art-works produced by students during this stage.

The user can navigate inside the virtual museum by following some instructions described in the homepage:

- To look around, you can click and hold the left mouse button and move the mouse;
- To turn left, you can use "left" arrow key or "A" key;
- To turn right – you can use "right" arrow key or "D" key;
- To move forwards – you can use the "up" arrow key or "W" key;
- To move backwards, you can use "down" arrow key or "S" key;
- And finally, to interact with the hotspots, you can use the left click of the mouse on them.

[283] The virtual museum "Mathematics and Arts", developed by Institute for Computer Science and Control, Hungarian Academy of Science in collaboration with Institute of Mathematics and Informatics - Bulgarian Academy of Sciences from Bulgaria, is available here: http://files.elearning.sztaki.hu/Escape3D/Mathematics_and_Arts/Intro_HTML/index.html

[284] Paneva-Marinova, D. I., Stoikov, J. S., Pavlova, L. R., & Luchev, D. M. (2019). System architecture and intelligent data curation of virtual museum for ancient history. *Труды СПИИРАН*, 18(2), 444-470.

Figure 75: An example of the art-works produced by secondary school students in the virtual museum.

The virtual museum is composed of two rooms where these *art-works* can be seen. Every frame on the wall is an interactive object (Figure 76).

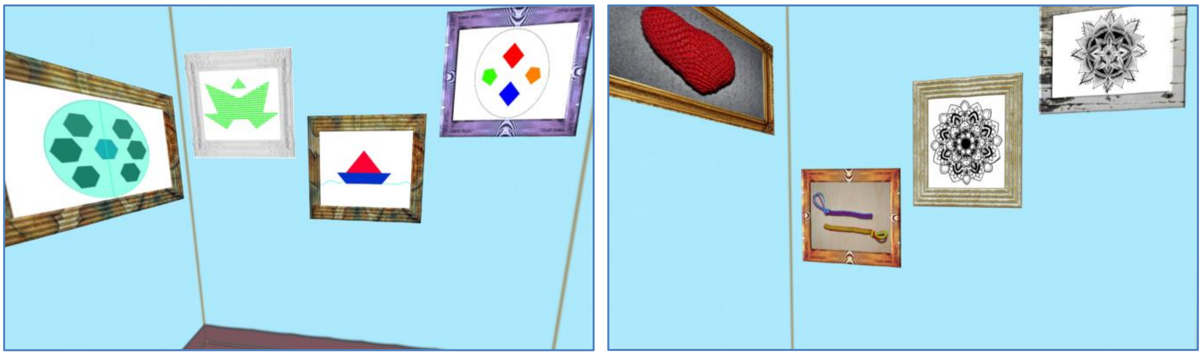

Figure 76: First and second rooms where the art-works are showed in the virtual museum.

In fact, by clicking them, users can get some information about the author(s), the object created and its relation to the real world, as well as the mathematics concept represented (Figure 77).

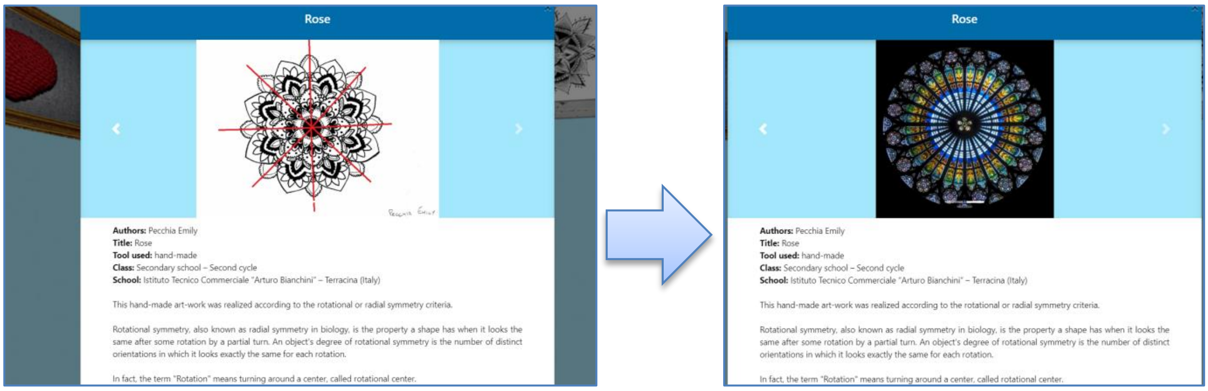

Figure 77: An example of the information card prepared per each art-work created by students and published in the virtual 3D Museum.

Moreover, by clicking the frame next to the door of the second room, entitled *Mathematics and Art*, the user launches a video showing the key moments of the experimentation phase realized with the students during the study of symmetry (Figure 78).

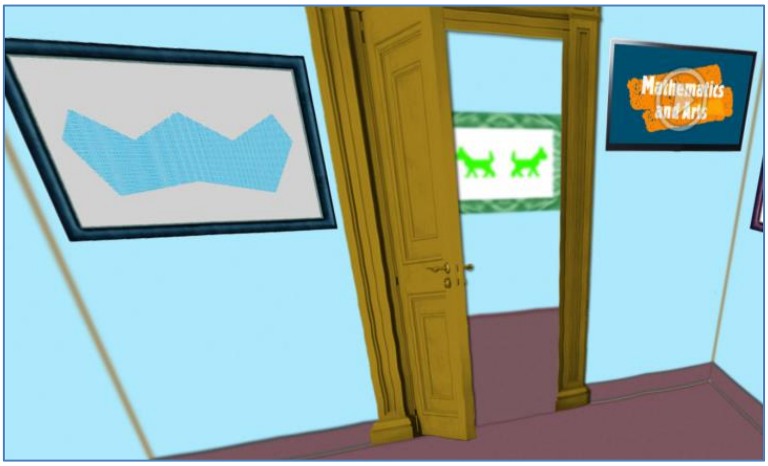

Figure 78: Frame where the user can launch the video, “Mathematics and Art”, showing some key moments during the experimentation phase.

4.2 The research sample formation

The research involved volunteer teachers and students from upper and lower secondary schools. At first, a reference sample was constructed by involving 3 mathematics teachers of lower and upper secondary school and 3 art teachers.

Given the interdisciplinary character of the research work, combining mathematics and art, it was important to involve teachers from both the subjects to show the transversality of the topics studied.

Afterwards, the research sample included 130 secondary school students of different ages, 11-13 for the first cycle and 14-16 - the second one.

The choice of the number of teachers to be involved was referred to as a regional territorial limit.

While the choice of the students' age was motivated by the fact that generally during this age boys/girls may both ripen or lack their interest in science subjects, considering these studies as abstract and boring (Ref. Chapter 1.1).

The sampling method chosen for the teacher selection was a non-probability sampling (*judgmental sampling or convenience sampling*) due to the necessity to have two important factors for the realization of the entire experimental phase and to avoid its interruption.[285]

The first factor is the teacher's motivation to complete all the expected activities autonomously and/or together with the researcher because they were carried out as the school curriculum activities. The second one was determined by the authorization of the headmaster to realize the experimental activities in the school and to allow the presence of the *researcher* (external individual to the school staff) in the classrooms.

Therefore, working together with the teachers in the classes, even if the students were underage, no authorization from their parents to participate in the experimental activities was necessary.

[285] Bailey, K. D., & Rossi, M. (1995), *Op.Cit.*, pp.149-150.

4.3Data collection tools

The tools for both qualitative and quantitative data collection during the experimentation phase realization were prepared.

For the quantitative data – one questionnaire divided into two parts was drawn up to gather data from students respectively before and after the experimentation phase. In the meanwhile, the involved teachers selected the mathematics exercises to be submitted to the students before and after the application of the proposed approach. The choice of these exercises was done on the base of the mathematics concept/formula to be studied using the new method and on the base of the school mathematics curriculum and knowledge level of their students on the topic (Figure 79).

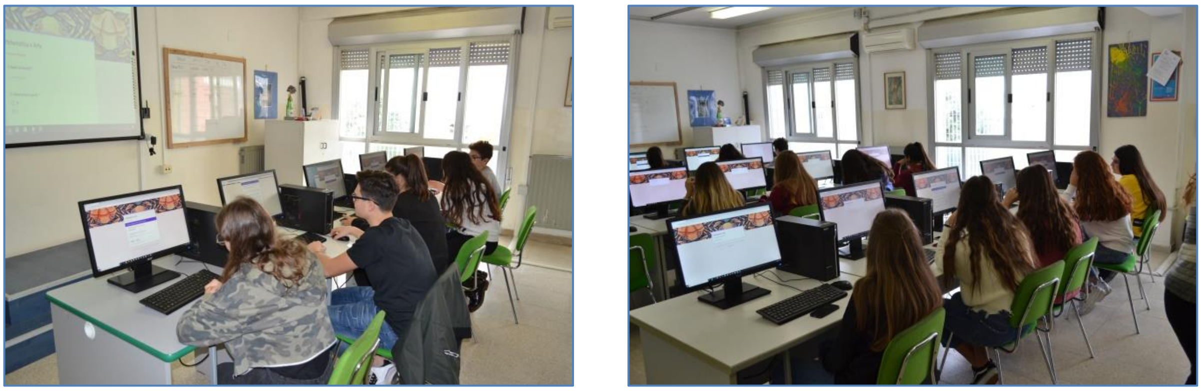

Figure 79: Two moments during the questionnaires' submission for the quantitative data collection at *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

For the qualitative data – a semi-structured grid was drawn up to gather the data through the observation participatory method by teacher and researcher. The aim was to reveal more information on possible attitudes and behavior from students towards the proposed approach during the experimentation phase.

In addition, a discussion group was organized to collect feedback data from students (Figure 80).

Figure 80: Two moments during the qualitative data collection at *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

From the analysis and comparison of the achieved results gathered through these tools, it was possible to explore and comprehend a clear overview of the effectiveness of the proposed method.

4.3.1 The quantitative data collection

The main data collection tool was one semi-structured questionnaire, divided into two parts, submitted to the students with open[286] and closed[287] questions. The open ones have been used to specify better the structured responses giving more space to the thinking of the respondent subject, thus overcoming any limits set by the expected answers; investigate actions of a certain interest (the method's attractiveness and/or motivation in the mathematics study); allow responding comprehensively by also making some clarifications and, lastly, to avoid that the response methods listed in the questionnaire were redundant.

There are also open and structured multiresponse questions; *control questions* to verify the respondent's veracity and consistency of some answers given to other questions and, finally, *filter questions* that allow avoiding questions that do not pertain to them.

For some questions, in particular, for multiple-choice ones, the number of answers which could be provided was limited (a maximum of 2).

[286] The open questions are those for which the answer modalities are not specified, leaving a certain freedom to the interviewees in expressing their opinion regarding the topic dealt with (Bailey, K. D., & Rossi, M. (1995), *Op.Cit.*, pp.149-150). In this research, the open form was also used for questions that had more than one alternative in order to better understand the respondents' attitudes, opinions and motivations on the phenomenon studied.

[287] Closed questions are those for which the respondent must choose between one or more answers. There are several advantages: 1) standardized responses facilitate comparison among the subjects; 2) the coding and analysis of the answers are simplified; 3) they can somehow clarify the meaning of the question to the respondents if they are not sure; 4) the reduction of irrelevant responses (Bailey, K. D., & Rossi, M. (1995), *Op.Cit.*, pp.146-149).

Several questions contain a rating scale to allow the measurement of the respondents' opinions and behaviors in a quantitative way by using qualitative nominal variables.[288] The rating scale used is from *strongly agree* to *strongly disagree* considering the other two intermediate and one neutral position.

Every scale used was constituted of different items (indicators) which helped to understand the attitude and behavior to be studied.

In fact, the technique of rating scale is used above all in the measurement of attitudes (general concept, basic beliefs not directly detectable) through specific concepts, that is, the opinions expressed by an individual (empirically detectable expression of an attitude).[289]

This kind of scale has provided a list of scored categories from which the respondent could choose. However, the scale, in our case, used a description nominal that indicated what each category represented. Besides its simplicity, this method has the advantage to facilitate the processing execution of statistical analyses during the data processing phase.

Moreover, this kind of questions allowed to catch different aspects related to the variables to be analyzed.

The first part of the questionnaire (*APPENDIX 1 – Data Collection Tools*) submitted before starting the experimentation phase was structured into five knowledge areas. The first *student profile* (Q1-Q2) aimed to collect basic information on students, the second *student habits in relation with technology* (Q3-Q6) to understand the student behavior towards the technology and its use in his/her everyday life.

The third area *attitude and perception towards mathematics study* (Q7.1-Q7.6) intended to reveal the motivation and interest of the students for the mathematics study. The fourth *knowledge and use habits of technology support in mathematics study* (Q8, Q9.1-Q9.4) aimed to understand if students perceive the technology, not only as entertainment, but also as supporting tool for learning and, in this case, for the mathematics study. The last knowledge area was related to *student perception of combination between art and mathematics* (Q10.1-Q10.4) to verify if students can easily comprehend this combination.

The second part of the questionnaire (*APPENDIX 1 - Data Collection Tools*) was submitted after the experimentation of the research approach based on the three phases of implementation (concrete-pictorial-abstract). It was divided into four knowledge areas. The first *student profile* (Q1-Q2) aimed, as in the initial questionnaire, to collect profile information of the student. The second *student attitude and perception of the realized*

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[288] Stevens, S. S. (1951). *Mathematics, measurement, and psychophysics*.
[289] Krech, D., Crutchfield, R. S., & Livson, N. (1970). *Elements of psychology: A briefer course*. Alfred A. Knopf.

experience (Q3.1-Q3.3, Q4.1-Q4.3, Q5.1-Q5.5) was focused to understand the impact of the realized experience on students in terms of initial motivation, technology application and art as supporting tool in mathematics study. The third area *perception of the effectiveness level in understanding study contents* (Q6.1-Q6.3) aimed to reveal the level of comprehension of mathematics concepts in students by following the research proposed approach.

Finally, the last fourth area *general overview towards this new synthesis between mathematics, technology, and art* was useful to comprehend a possible change in student's attitude or perception to scientific subjects (Q7, Q8, and Q9.1-Q9.5).

Therefore, the questionnaire was the result of an articulated work realized in two phases, as follows:

1. In the first phase – after a preliminary study, the questions, concerning the different areas of knowledge, were elaborated. At this stage the questions were very numerous, sometimes taken up from other questionnaires and research. Using the detection tools already tested in other studies has allowed the researcher to validate the measurement tool. This procedure is called *criterion validity* (or *criterion-related validity*) and allows one to validate an instrument using another previously used in other research contexts, providing empirical results that have not been proven as wrong;
2. The second phase – all the questions were classified by dividing them by knowledge areas as described above.

4.3.2 The qualitative data collection

In order to collect the qualitative data, a grid was prepared (*APPENDIX 1 – Data Collection Tools*). This tool was used during the experimentation phase with the students for participatory observation. It was useful for gathering opinions, understanding the attitudes and behaviors of the students.[290]

[290] Trinchero, R. (2002). *Manuale di ricerca educativa* (pp. 1-432). Franco Angeli.

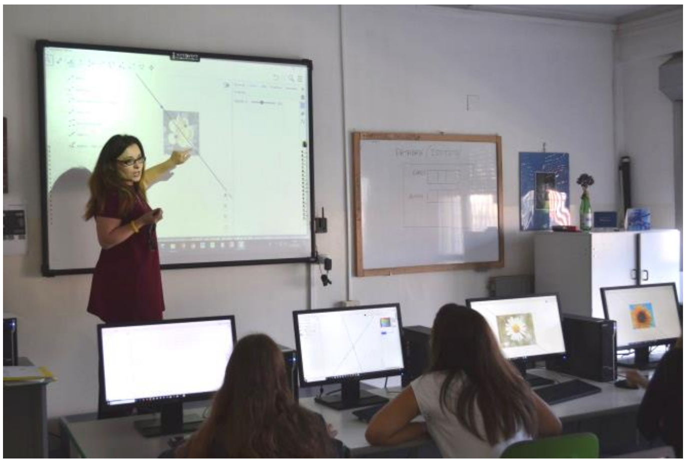

Figure 81: Experimentation time - Concrete phase with students at *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

Observation is the method of collecting information used in social research to gather the most significant and relevant data in relation to the research objectives, in particular, those that are related to a particular context.

The observed behaviors are indicators of latent factors, such as opinions and attitudes, reconstructed by relating the observed behaviors to each other, and the environmental conditions in which they occur.[291]

Therefore, the observation was semi-structured with a specific group of the sample and led by an observation grid in order to focus on behavior to be observed. The observation function was cognitive because the educational or training intervention was depended on the in-depth knowledge of the starting situation for the determination of the effectiveness level of the method proposed.

Furthermore, it was also a control tool to collect feedback on the proposed action in real-time, detecting any occurring problematic and/or uncomfortable actions, also foreseeing situations of improvement, or identifying any obstacles for effective action.[292]

There are three characteristics of observation tools to be taken into account in their preparation and implementation: validity, reliability, and relevance.[293]

Validity: in the sense that the categories foreseen for the observation must constitute indicators that are related to the factors to be detected during the investigation.

Reliability: different observers watch the same situation with the same instruments that must lead to the same results. In the case of research, the observers were the researcher and

[291] Braga, P., & Tosi, P. (1995). L'osservazione. S. Mantovani (a cura di), *La ricerca sul campo in educazione. I metodi qualitativi*, Milano, Bruno Mondadori.

[292] Trinchero, R. (2002). *Op.Cit.*p.254.

[293] Merriam, S. B. (1998). *Qualitative Research and Case Study Applications in Education. Revised and Expanded from" Case Study Research in Education."*. Jossey-Bass Publishers, 350 Sansome St, San Francisco, CA 94104.

the teachers involved who observed and followed the same training situation using the same data collection tools.

Relevance: the data collected must be consistent with the predetermined cognitive objectives and the categories and items of the grid must be those that we need to detect the factors under examination.[294]

In addition, at the end of the experimentation, one discussion group was organized with the students to let them express freely their opinions and feedback about the lived experience and their suggestions regarding possible improvement for the future implementation of the method.

[294] Denzin, K. N., and Lincoln, Y., S.,(editors)(2000); Handbook of Qualitative Reserach.

4.4 Coding and data processing

This phase aimed to classify and manage the collected data to analyze them also on the base of relations among variables through the data matrix construction.

Coding is an operation through which the collected data are organized into classes and/or categories to be identified by a specific number or symbol.[295]

At first, a different numerical code was transcribed near the answers provided for all the questions included in the whole questionnaire.

For the codification of the open questions (*content analysis*), the determination of which is left to the respondent, it was deemed necessary to read the answers given by all the respondents. Each answer was transcribed into an Excel file.

Similar responses were identified and placed in a class, to which was assigned a numeric code. The non-answers were assigned the *code 999*, which indicates that the respondent refused to answer the question, or the *code zero* if the question was not relevant to the case.

Also, a code was assigned to each response method within each question.

In the case of questions with numerical answers, a pre-assigned code was used. The same code was used as a symbol for the different answer modes.

In fact, the code, that identifies the answer of a determined question, was recorded in a *record*[296] and every question, including all its answer modalities, was registered in different positions to avoid mistakes and misunderstandings. This was useful to draw up a coding table, divided into rows and columns, where all codes, referring to all answer modalities, were registered.

Each row represented the questionnaire and each column, to which a title was assigned (V1, V2, V3, etc.), replaced the questions in the questionnaire.

All information achieved was recorded on electronic support in which *fields* corresponded to columns and *records* to rows.

Therefore, a field for each variable and a record for each unit of analysis were foreseen.

The *record* was constructed using the Microsoft Excel spreadsheet.

Finished the recording of all data, a data coding file/ data matrix was obtained allowing further statistical processing.

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[295] Goode, W. J. (2017). &Hatt. PK (1952).'Methods in Social Research'.

[296] IT support is used for recording the data collected by the investigation. The computer stores the information in a structure very similar to the data matrix, organizing it into files, which contains one or more records. Therefore the file coincides with the data matrix and the records with the rows of the matrix. The columns of the matrix are represented by the fields or bytes that make up a record (Bailey K. D., *Op.* cit., p. 383).

In this research, the statistical package, SPSS / PC (*Statistical Package for Social science*s) was used for data processing. The *Statistical Package for Social Sciences* is a set of programs created for the elaboration and statistical analysis of data concerning social sciences.

Due to the research objectives, the software was useful to construct the simple distributions, such as *frequencies* and double distributions (*crosstabs*) according to the plan of crossed variables, previously established, to highlight the possible relationships among them.

The data statistical analysis was the last step of the research phases, carried out according to the objectives fixed at the beginning and it was both descriptive and explanatory. In the first case the sample was described on the base of the identified features; in the second one more variables, as the cause of another variable, were analysed. Even though, there is no real distinction between these two levels because they are a part of the same continuum in the research phases.

CHAPTER 5. PRESENTATION OF RESEARCH RESULTS: DATA ANALYSIS

The piloting phase was implemented with a sample of about 130 secondary school students of different ages, 11-13 years-old for the first cycle (44,62%) and 14-16 years-old the second one (55,38%) with the 43,1% male and the 56,9% female.

They show an everyday use of the Internet (60,8%) independently of the age and the gender of the students. If the results are analysed considering the difference of the age and the gender of our samples in the frequent use of the internet, the results show a clearer situation. In fact, there is no difference between females and males of 11-13 years-old students, but the data demonstrate that the female sample of 14-16 years-old is more inclined to use the Internet every day: 44,44% against 20,83% of male students.

However, if we cross the gender and the two-sample of age, the 11-13 years-old male students used Internet very often against a percentage of 20,83% of 14-16 years-old male students, as showed in the table below.

Table 8 - How often do you use the Internet?

	Student Age 11-13 (*)		**Student Age 14-16 (*)**		**Final Trend**	
	Gender N-%		**Gender N-%**		**Gender pt**	
	F	**M**	**F**	**M**	**F**	**M**
Several times during the week	11 18,97%	11 18,97%	11 15,28%	12 16,67%	-3.69	-2,3
Every day	16 27,59%	16 27,59%	32 44,44%	15 20,83%	+16,89	- 6.76

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-samples. The first sub-sample (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

The prevalent tool used for Internet connection is the Smartphone for 86,9% of the research sample combining with 40,8% for the use of PC.

These data confirm that the new generations are *digital natives*[297] because they, from childhood, have been starting to use the technology. Actually, the data show that, except for female kids aged 14-16, the new generations use frequently Internet and more modern devices.

[297] Prensky, M. (2010). H. Sapiens Digitale: dagli Immigrati digitali e nativi digitali alla saggezza digitale. *Italian Journal of Educational Technology*, 18(2), 17-17.

Moreover, the data register an increasing tendency among the girls in the frequent use of Internet in the age group of 14-16 years-old (+16,89%) against a negative one in the boys' group from the same age (-6,76%).

These data are supported by another research conductedin 2017 by *Jean M. Twenge*, Professor of San Diego State University on this new generation, called *iGen* (iPhone generation). In fact, the girls use social networks more often than boys, in particular, by trying and hit "enemies" using the words up to commit real cyber-bullying attacks. While the boys, growing up, prefer to solve their conflicts physically.[298]

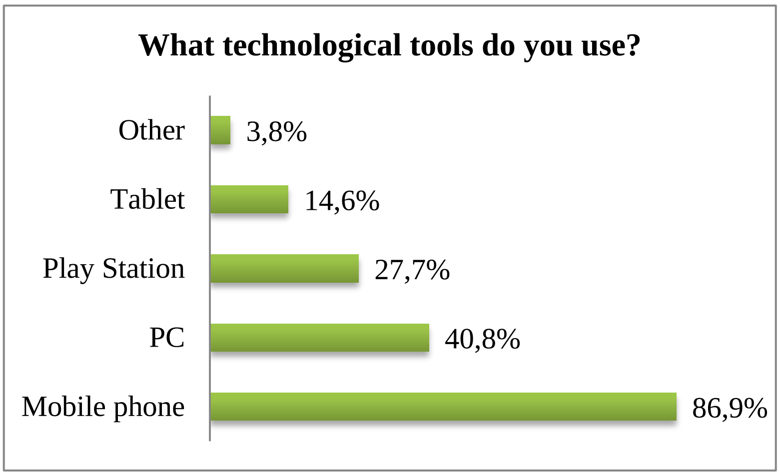

Figure 82: The technological tools used.

Other: Xbox One S, Nintendo.

Mobile phones such as Smartphones have already gained a dominant importance and presence within the everyday life of young people (Figure 82).

Mobile technology, currently, allows us to fully redefine the space-time dimensions in which the most common actions are carried out, and above all, it allows the construction of identity and action practices of the individual. Compared to the younger part of the population, these technologies are spreading quickly and effectively allowing portability and integrated multimedia in an effective and fast way. An example is the latest smartphones generation that offers a superior service to the simple mobile phone, ensuring a high standard of personalization, globalization, and localization of content and communication relationships.

[298] Twenge, J. M. (2017). iGen: *Why today's super-connected kids are growing up less rebellious, more tolerant, less happy--and completely unprepared for adulthood--and what that means for the rest of us. Simon and Schuster.*

Today many children from the age of 11-12 years-old have a mobile around which every moment of the individual's day revolves.[299]

Actually, children, aged between 8 and 14 years old, have already a mobile phone by using it on average 12 hours a day, with 10.3% for more than 12 hours on the base of the research report presented during *V Convegno Nazionale La prevenzione nella scuola e nella comunità" Dal cambiamento individuale al cambiamento sociale*" in Padova.[300]

The mobile phone remains the most important technological tool whether the subject is a regular user (*Addicted use* daily),[301] considering it as a fundamental object for social relations (often checking for the presence of messages, calls, and other notifications for no reason).

In addition, it is a *Trendy use*, considering it a gadget of the latest technology and equipped above all with Internet access.[302]

The ubiquitous penetration of technologies into our everyday life may be treated as something inescapable and natural.

However, the ways the students interact, get updated and organized are essentially influenced and facilitated by Smartphones, social networks and other ways of mobile communication.

Our sample underlines that it is used, mainly, for entertainment (93,10%) and to support homework (45,42%) as shown in the graphics below:

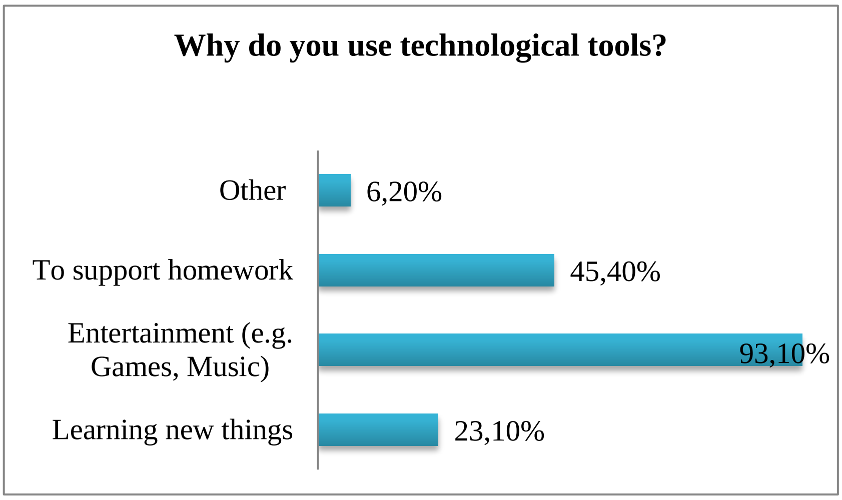

Figure 83: The reasons to use technological tools.

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[299] Rossato, M. (2014). A aprendizagem dos nativos digitais. *O sujeito que aprende: diálogo entre a psicanálise e o enfoque histórico-cultural*. Brasília: Liber Livro, 151-178.

[300] Petralia, V. (2005). Una comunità tra i sordi e gli udenti: alla ricerca di un senso di comunità condiviso. In *V Convegno Nazionale La prevenzione nella scuola e nella comunità" Dal cambiamento individuale al cambiamento sociale"*.

[301] Wilska, T. A. (2003). Mobile phone use as part of young people's consumption styles. *Journal of consumer policy*, *26*(4), 441-463.

[302] Ibid.

This is probably due to the easy accessibility of Smartphone games and the growing number of hours dedicated to these activities by young people.

This is confirmed from the *Q6 – Where do you mostly use technological tool?*, where the data show that students use mostly the technological tools *at home* (99,20%) and *at friend's home* (29,20%). Only a percentage of 21,50% of students states to use technological tool "at school".

Although Italian Ministry of Education (MIUR) has launched at the end of 2015 an important reform, named National Digital School Plan (PNSD), there are still schools which find difficult to use technology to support teaching and learning processes.[303]

The plan is divided into *35 Actions* grouped into four fundamental areas (instruments, skills, contents, training and support) and the implemented measures concerning both school administrative and infrastructural aspects, such as the Internet accessibility in schools, the development of digital teaching environments, as well as staff training. Moreover, a new school role has been identified such as *Digital Animator* who has the task to promote and organize learning activities for the digital skills development in students.[304]

Despite these facts, the survey shows that the teaching process is still bounded to manage the classes with traditional approaches.

If the results are analysed in-depth, they present a different tendency by comparing the data collected among 11-13 years-old students and 14-16 years old students. Actually, from one side, as shown in the following table, they reveal a negative tendency from the youngest male people (46,55%) to the oldest ones (36,11%), equal to -10,44 points in the use of the smartphone for entertainment reasons. From the other side, they show an increasing tendency (+16,62 points) comparing the youngest female people (43,10%) and the oldest ones (59,72%).

[303] Consoli, D., & Aureli, S. (2018). Un framework integrato per la misura dell'innovazione del Piano Nazionale Scuola Digitale (PNSD). *Management Control*.

[304] Mangione, G. R., Mosa, E., & Pettenati, M. C. (2015). Dalla Gelmini alla Giannini. Il Piano Nazionale Scuola Digitale, i PON disciplinari e il ruolo dell'INDIRE nella formazione continua degli insegnanti. *FORMAZIONE & INSEGNAMENTO. Rivista internazionale di Scienze dell'educazione e della formazione*, 13(3), 139-166.

Table 9 - What should technology be used for?

	Student Age (*) 11-13		Student Age (*) 14-16		Final Trend	
	Gender N-%		Gender N-%		Gender pt	
	F	**M**	**F**	**M**	**F**	**M**
Entertainment (e.g. Games, Music, etc.)	25 43,10%	27 46,55%	43 59,72%	26 36,11%	**+16,62**	**- 10,44**
Learning new things/to support homework	7 12,07%	10 17,24%	30 41,67%	17 23,61%	**+29,6**	**+6,36**

130 valid cases - Multiple answers were possible

(*) The percentage calculation is made on the number of units in the single sub-samples. The first sub-sample (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

Concerning the use of the technology for *learning new things/to support homework*, the final evolution trend of the two different student groups is positive in both the cases: +29,6 points for girls and +6,36 for boys.

Therefore, this underlines that the use of technology changes accordingly the student's age: from a simply entertaining tool it becomes the learning instrument for both learning new things and supporting homework.

5.1 Student initial attitude towards Mathematics.

Usually, students define mathematics as a big obstacle for their study path, because the difficulties often revealed are related to its being considered more abstract than the other disciplines.[305]

In particular, 57,70% of students stated that their major difficulty in mathematics study is that this discipline is too abstract than the other subjects as following:

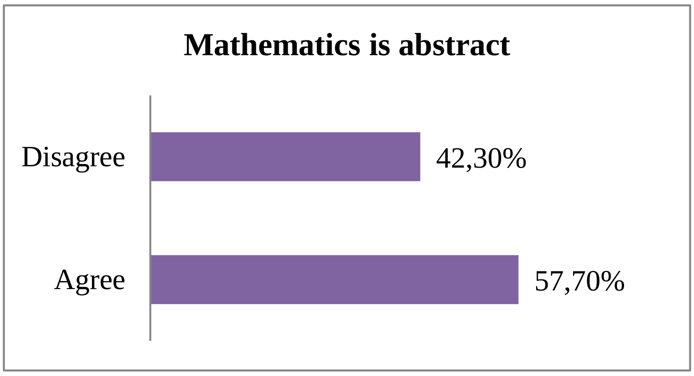


Figure 84: Attitude towards mathematics study.

If the two samples (11-13 and 14-16 years-old students) are compared, the table below shows the final tendency. It demonstrates that the students, growing up, change their attitude towards the mathematics study: most of them consider this discipline abstract and less connected to reality. The data underline a relevant increase from the age 11-13 years old to 14-16 years old, respectively +38,93 points for females and +20 for males.

Table 10 - Mathematics is abstract

	Student Age (*) 11-13		Student Age (*) 14-16		Final Trend	
	Gender N-%		Gender N-%		Gender pt	
	F	M	F	M	F	M
Disagree (including strongly disagree)	21 36,21%	24 41,38%	8 11,11%	4 5,56%	**-13**	**-20**
Agree (including strongly agree)	4 6,90%	3 5,17%	33 45,83%	23 31,94%	**+38,93**	**+20**

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-simples. The first sub-simple (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

[305] OECD. Organisation for Economic Co-operation and Development. (2016). PISA 2015 Results in Focus. and OECD, (2019); OECD, *Combined Executive Summary - PISA 2018.* Retrieved from: https://www.oecd.org/pisa/Combined_Executive_Summaries_PISA_2018.pdf [accessed: 16, February 2020].

This factor derives from the different nature of the topics to be studied in the mathematics curricula and/or from the teaching method applied in the classroom. Usually, in secondary school – first-grade teachers often use visual tools to explain and introduce scientific connections to their students.[306] However, these methods not always work effectively if we consider the diversity of subjects that students should study at school. The students are expected to use a different kind of ability and capability to learn the lesson of histories, algebraic inequalities, or other literature subjects.

The visual memory is not always a useful tool able to support students to cope with scientific subjects.[307]

Students, growing up, start to perceive mathematics as an abstract subject and for this reason, most of them see this discipline as something far from reality and not easily applicable. Therefore, it becomes boring and useful just to do homework.

Table 11 - What do you think of Mathematics?

	Strongly agree N-%	**Agree N-%**	**Neutral N-%**	**Disagree N-%**	**Strongly disagree N-%**
Mathematics is boring.	31 23,85%	26 20,00%	17 13,07%	31 23,85%	25 19,23%
I only use math at school or to do my homework.	58 44,60%	21 16,20%	14 10,80%	18 13,80%	19 14,60%
When I finish school, I won't need math.	22 16,90%	20 15,40%	13 10,00%	20 15,40%	55 42,30%

130 valid cases

Nevertheless, the students (57, 7%) understand that mathematics is a part of their life and then they will need it after school leaving.

However, if the results are analysed on the base of the two sub-samples, the situation presented in the Table n.10 confirms that growing students don’t see easily the applicability of this subject to their everyday life by considering it as an abstract topic.

[306] OECD, (2016).Op.Cit.

[307] Raghubar, K. P., Barnes, M. A., & Hecht, S. A. (2010). Working memory and mathematics: A review of developmental, individual difference, and cognitive approaches. *Learning and individual differences*, *20*(2), 110-122.

Table 12 - When I finish school, I won't need math

	Student Age (*) 11-13 Gender N-%		Student Age (*) 14-16 Gender N-%		Final Trend Gender pt	
	F	M	F	M	F	M
Disagree (including strongly disagree)	23 39,66%	27 46,55%	17 23,61%	8 11,11%	**-6**	**-19**
Agree (including strongly agree)	2 3,45%	2 3,45%	23 31,94%	15 20,83%	**+21**	**+13**

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-simples. The first sub-simple (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls 27 boys.

Students often do not perceive a practical utility in mathematics study unless through specific subjects, such as economics where the utility is directly evident (Table 12). The reason is that they often don't have a stimulating overview of being interested in mathematics.

Despite these difficulties and resistance towards the mathematics study, students (for a total of 60,8%) try to solve the mathematics problems by reasoning and applying different methods from the ones taught by the teacher as shown in the following figure:

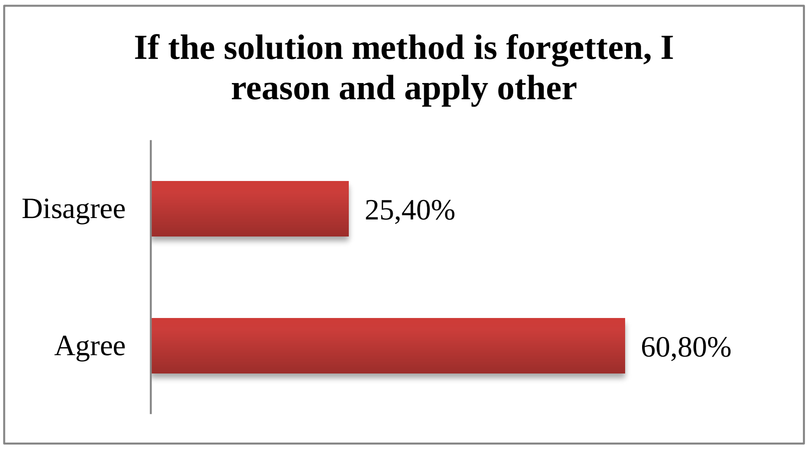


Figure 85: Problem-solving approach from student attitude.

5.2 Student and technology support: initial phase

Meaningful learning environments, where knowledge is built, co-constructed and shared, can be supported by digital and e-learning technologies. Students learn in a meaningful way if they can master the use of technologies creatively by organizing and representing what they know and learn, by creating products and solving problems anchored to real life, reflecting on contents and processes. The technologies or *collaboration tools*, as defined by *Jonassen*, can promote collaboration, cooperation, and distribution of knowledge in knowledge-building communities; make possible and support dialogic processes, then discussions, productive confrontations, meanings negotiation, consensus-building through a critical reflection on a *progressive*, improving knowledge.[308]

However, the survey shows that 75,40% of students (against the 24,60%) don't know any applications/programs to be used to improve mathematics skills (Figure 86).

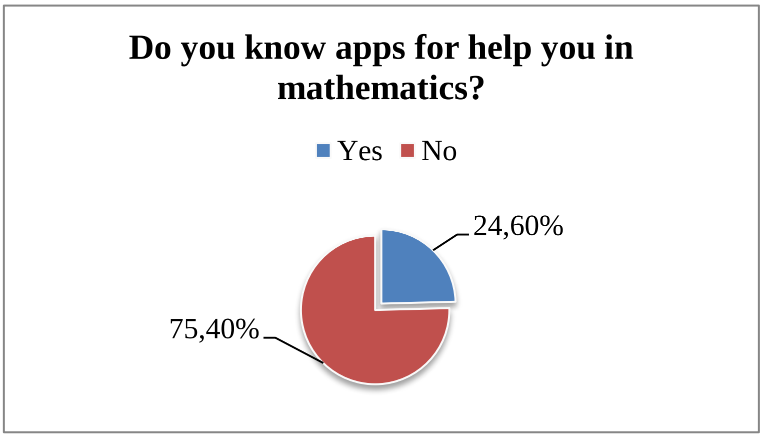


Figure 86: Student knowledge of apps for studying mathematics.

Among the 24,60% of students who know apps for supporting mathematics study the following were nominated: *GeoGebra*, Math games, *Photomath*, calculator (even hand-made with *Arduino*) and INVALSI Test for the exam simulation. More general use of the applications/programs, as *Duolingo* for language improvement and *Gego* for tracking everything with no limit on distance, were indicated by students.

The high percentage of 75,40% in the negative answer is confirmed by the data achieved in Table 9 where a stronger use of technology for entertainment, such as music and games with respect to study support emerges..

[308] Jonassen, D., Howland, J., Marra, R. M., & Crismond, D. (2008). How does technology facilitate learning. *Meaningful learning with technology*, 5-10.

Although students understand that these applications are useful (58,16%), they usually don't like studying online, preferring a more traditional way (67,35%). However, another section of the sample states that they are not necessary because the school teaching materials are enough (71,14%). While 72% of students involved states that they never tried to use them.

Given a deep look at the two-sample, the results underline an important aspect, as follows:

Table 13 - Do you know apps to help you in mathematics?

	Student Age (*) 11-13		**Student Age (*) 14-16**		**Final Trend**	
	Gender N-%		**Gender N-%**		**Gender pt**	
	F	**M**	**F**	**M**	**F**	**M**
SI	6 10,34%	7 12,07%	11 15,28%	19 26,39%	**+5**	**+12**
NO	23 36,66%	22 37,93%	34 47,22%	8 11,11%	**+11**	**-14**

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-simples. The first sub-simple (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

The collected data reveals an increasing trend in elder students in discovering different kinds of learning applications or programs that can be used to support their study. Probably, this depends on the different use of the smartphone among the two target groups: a frequent use in the 14-16 years-old student group against a less frequent use in the 11-13 years-old group due to the minor accessibility or the parental control of this tool for the youngest.

On the other hand, the results show that, in the second target group, girls are not so familiar with the existing applications and programs that can be used to improve mathematics knowledge or any other deepening in the scientific subjects. An increase of 11 points for the girls in the first target group was gathered. This is easily understandable from the fact that 14-16 years-old girls are more inclined to use a Smartphone for entertainment and, in particular, social networks.

In addition, the picture, emerged by the survey, highlights that students are not so familiar with the use of technology in the learning process and, consequently, in the teaching process. This once again underlines the resistance of some teachers to consider technology as an additional resource in the classroom besides their everyday job to motivate and to help students in their learning process.[309, 310]

[309] Gardner, H. (1994). *Intelligenze multiple, traduzione dall'ingles*e di I. Blum, Edizioni Anabasi, Milano.
[310] Gardner, H. (2005). *Educazione e sviluppo della mente. Intelligenze multiple e apprendimento*. Edizioni Erickson.

5.3 Students' initial attitude towards the combination of Mathematics and Art

Another relevant element investigated with the first part of the questionnaire is the initial capability to perceive the connections between mathematics concepts and art, seen as art-works but also as real objects to be found in nature.

The data show that most of the students (82,23%) never thought that mathematics can be moved by this unusual aspect, the participation in the research learning activity was considered by students as a way to try an opportunity aimed to increase the interest in scientific subjects and, in particular, in mathematics (the 60,8% against a percentage of 15,4% of students who has denied any possibility of interest improvement). Between these two extreme positions (Agree-Disagree), a quite high percentage of students (23,80%) held a neutral viewpoint by underlining the difficulty to be able to see and comprehend the relation between mathematics and art.

This last percentage includes an increase of +4 points in the girls of the second target group (the eldest) and an insignificant +1 point in the boys of the same group, as follows:

Table 14 - Art could be a way to increase my interest in math

	Student Age (*) 11-13		**Student Age (*) 14-16**		**Final Trend**	
	Gender N-%		**Gender N-%**		**Gender pt**	
	F	**M**	**F**	**M**	**F**	**M**
Agree	16 27,59%	17 29,31%	27 37,50%	19 26,39%	**+11**	**+2**
Disagree	5 8,62%	7 12,07%	6 7,69%	2 2,78%	**+1**	**-5**
Neutral position	8 13,79%	5 8,62%	12 16,67%	6 8,33%	**+4**	**+1**

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-simples. The first sub-simple (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

It seems that 14-16 years-old girls have more difficulty to perceive this possible combination or inclusion of art with mathematics and, generally, with scientific topics.

If on one hand, students consider more motivating to learn through having fun (87,69%), on the other hand, they are not convinced that studying mathematics using the arts can facilitate and support the learning process.

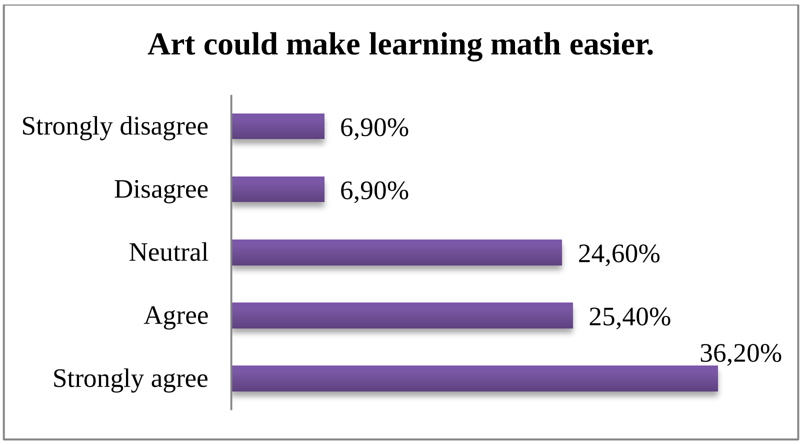


Figure 87: Art and its power to facilitate mathematics learning.

Therefore, from the disadvantage of not seeing the existing connection between mathematics and art, students take a fairly neutral position (a percentage of 26,60%) in considering art as a means of facilitating the understanding of mathematical concepts.

Nevertheless, 61,60% of students, intrigued by this new experience, consider this means, *art*, a way to support and make easier the mathematics learning (Figure 87).

5.4 Students' attitude and perception of the experience

After the piloting phase, a second part of the questionnaire was submitted to the students. It was focused to understand the impact of the carried out experience being on students in terms of initial motivation, technology application in mathematics study and art as a supporting tool in mathematics study.

First of all, students were expected to self-evaluate three initial variables: motivation, participation and, more in general, the experience.

The 43,85% of the students were very motivated and interested in the participation in this new experience because, according to the qualitative feedback, they discovered something of *unexpected*, or, in other words, something to never have thought about.

For this reason, their curiosity pressed them during all the experience collaborating with other classmates. Students have worked mainly in small groups.

This has allowed exploiting the potentialities of the collaborative learning approach where they are at the center of their learning process as underlined from a socio-constructivist point of view. As confirmed by students during the discussion groups, this approach has supported the development of the new relationships among them by promoting a positive interdependence, more individual responsibility, and relationship skills to carry out from the expected activities.

By collaborating, students learned to be more active and efficient, by developing their ability to learn as much as possible. This was favoured also by the teacher who acted as a facilitator for learning and not just as an instructor. The path realized has created and constructed meaningful learning experiences and has encouraged the reasoning in students through real-world problems for the achievement of specific objectives.

Therefore, their participation was self-evaluated as *very good* at 55,38%. This means that they were actively involved in all the three phases, concrete-pictorial-abstract, by guaranteeing the finalization of the whole tasks without raising any relevant critical situation.

This is confirmed by the Q3.3 which provides an overall judgment of their final experience with the 64,62% *Excellent*, 29,23% *Very good* and for 6,15 *Good* without receiving any negative feedback.

5.4.1 Students' perception and attitude towards the three phases

During the first (pictorial) phase, students were expected to build and manipulate some objects (e.g. pyramids to study geometrical figures, friezes, ornaments to study the symmetry) with instructions (provided by the teacher or the researcher) and equipment provided.

The evaluation of the technological impact on this experience, such as the use of the GeoGebra, was positive. Thanks to the object manipulation in the virtual mathematics lab, students reported that the mathematics concepts became clearer (63,08% - Strongly agree and 28,46% - Agree – Figure 88).

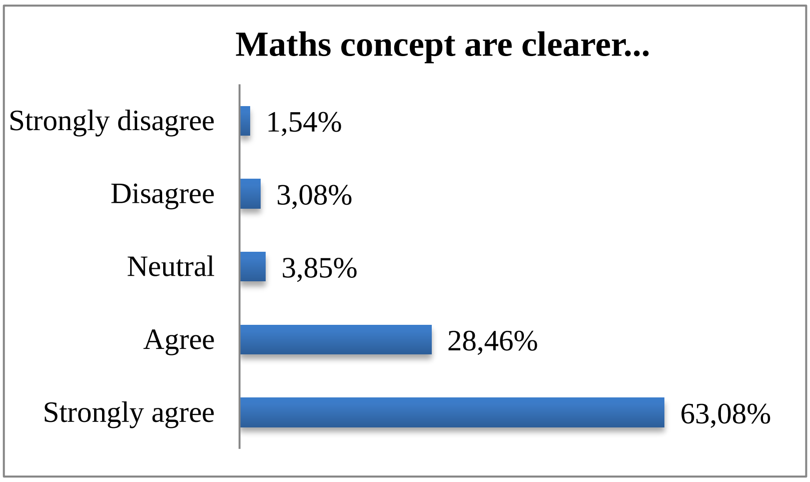


Figure 88: Clarity of mathematical concepts.

The proposed approach made the subject easier thanks to the fact that students could test and operate directly on the object or concept to be studied (78,46% - Strongly agree and 13,85% - Agree).

Besides, the new experience during the pictorial phase supported by *GeoGebra* made the mathematics subject, usually considered boring, more fun and attractive.

During the pictorial phase, students could make different kinds of virtual animations by exploiting all the technological options of this tool.

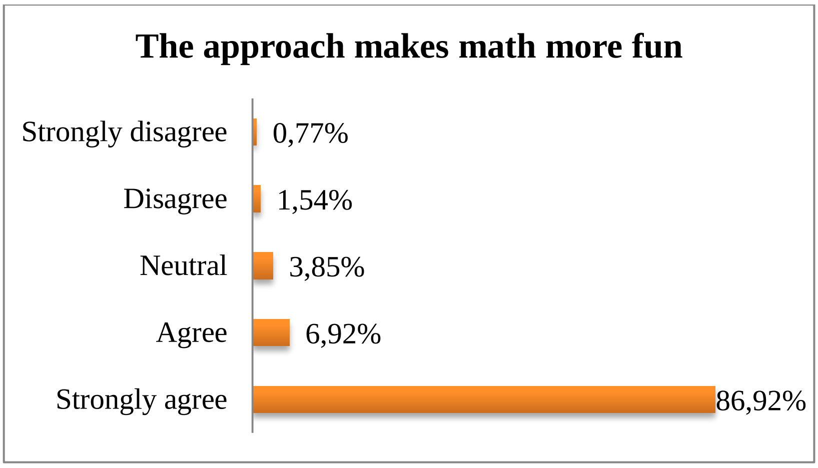


Figure 89: The attractiveness of mathematics study.

The Figure above shows that mathematics study became more attractive through this method because they could use their creativity, be original and work individually or in a group.

However, some of them, in particular, 4.1% (2.7% - Disagree and 1.4% - strongly Disagree) of girls have had some difficulties, mainly during the third phase, when students were expected to create their art-works. On the base of the results achieved boys seem to be more comfortable to use their creativity and to be original in the creation of their art-works. This low percentage is verifiable also in the question related to the comfortability of the method (Table 15).

Table 15 - Discovering mathematics in art ...

	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**
I don't feel comfortable using this method.	1 0,77%	1 0,77%	10 7,69%	31 23,85%	87 66,92%
It was easy to create my artwork using the mathematical concept studied.	61 46,92%	42 32,31%	15 11,54%	8 6,15%	4 3,08%

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-simples. The first sub-simple (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

However, the data described above show also that most of the students felt free to use their creativity and be original in their art-works production during the third phase (abstract). As particularly relevant is the increasing positive trend from the first target group to the second one. For the girls, +13 points were achieved after the experimentation and +4 - for boys (Table 16).

Table 16 - I can use my creativity and be original.

	Student Age (*) 11-13		Student Age (*) 14-16		Final Trend	
	Gender N-%		Gender N-%		Gender pt	
	F	M	F	M	F	M
Agree	28 48,28%	23 39,66%	41 56,94%	27 37,50%	+13	+4
Disagree	0	0	1 1,39%	0	+1	0
Neutral position	2 3,45%	5 8,62%	2 2,78%	0	0	-5

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-simples. The first sub-simple (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

Another interesting factor to be underlined is the different attitude of students towards the misconception that young people have towards mathematics, considered as something abstract.

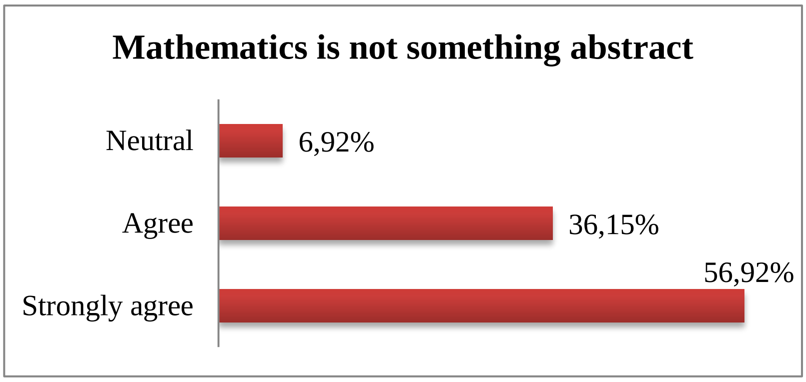


Figure 90: Student attitude towards mathematics after the method application.

As shown in Figure 90, at the end of the experimentation, students discovered that mathematics is not something abstract but it is related to a real-life application.

Also, there is no relevant difference between girls and boys.

These data are confirmed by the control question as shown in the figure below:

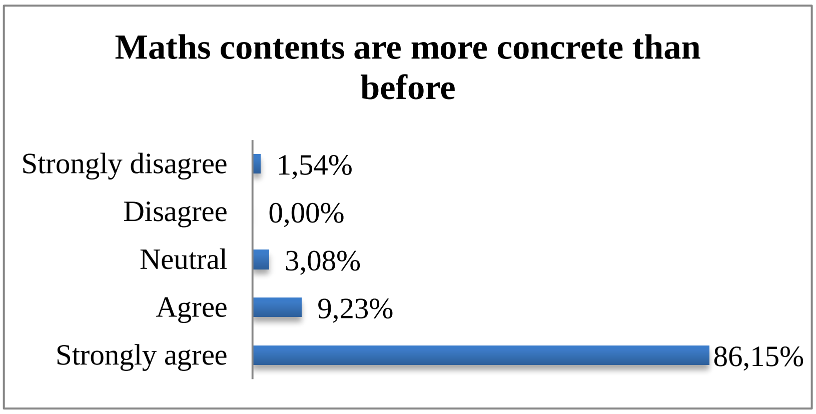
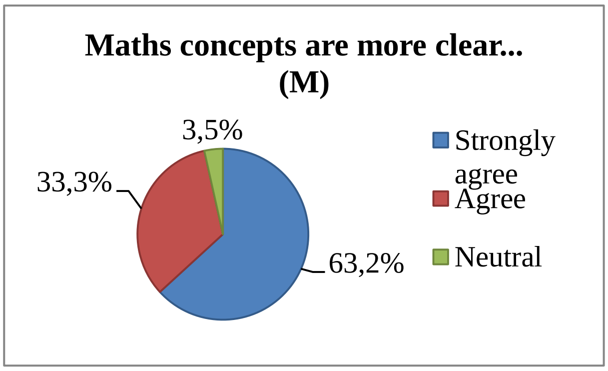


Figure 91: Student attitude towards mathematics after the method application – Control Question.

Although the relationship between art and mathematics does not appear evident at first sight, the students understood the intertwining and convergence between these two spheres of human culture while remaining fascinated. Some of the students defined, during the discussion groups, this experience as "spiritual", because the path started from the analysis of beauty (contained in the art-works) led them to rediscover previously to be useless and far from reality (mathematics) in a remarkable way.



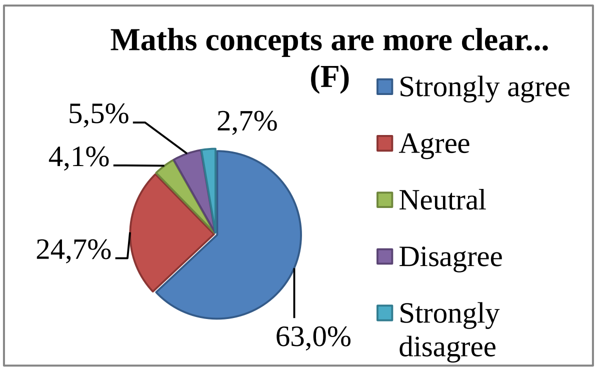


Figure 92-93: Student's Attitude towards mathematics after the method implementation related to two sub-samples. The first sub-simple (female students) includes 74 units and the second sub-sample (male students) 56.

Comparing the answers gathered, despite an insignificant percentage of the *strongly disagree* option in girls' sample, the majority of respondents demonstrate a strong opinion that the method has enabled students to consider the mathematics concepts as more concrete and applicable to the reality (Figure 92-93).

Moreover, the main perception revealed is that the use of the art-works in the mathematics study favors the development of an enhanced learning setting, enabling students to enjoy the learning process more compared to the traditional frontal lessons thanks to the exploitation of different languages, such as visual, graphical, verbal, non-verbal, representational and pictorial.

Concerning the contents learned, the survey and the exercise submission have demonstrated that students developed not only their knowledge in mathematics but also the reasoning process based on the applicability, imagination, creativity, and problem-solving skills. They learned to deal with the mathematical problem from different point views thanks to the use of the problem variations. In fact, the use of the different art-works in the study of the same mathematical concept allowed them to analyse it from a different perspective and learn logical reasoning bounded to the problem studied. This means that even if the art-works can change the context and the background, the mathematical concept, behind these works, is always recognized and become applicable more easily in the everyday life as shown in the table below.

Table 17 – Content learned

As for the contents learned ...						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
The content was easily understood.	85,38% (111)	7,69% (10)	4,62% (6)	0,00% (0)	2,31% (3)	0,00% (0)
The contents learned seem to me more concrete and practical than before.	86,15% (112)	9,23% (12)	3,08% (4)	0,00% (0)	1,54% (2)	0,00% (0)
The content learned can support my study outside the classroom.	60,77% (79)	21,54% (28)	15,38% (20)	0,00% (0)	2,31% (3)	0,00% (0)

130 valid cases

With in-depth analysis, although the 27,69 % of students were comfortable with the whole method proposed, the results show that some students had a different attitude towards the single phases tested.

Actually, most of them (31,54%) preferred the third phase where students were required to create their art-works and to use their creativity to produce something original.

On the other hand, 26,92% of students liked the second phase, where they could work in small groups and do research to find the combination between the math concept to be studied and the art-works related.

The qualitative survey showed that students are usually not used to working together, but in most cases, they study individually.

Therefore, this experience demonstrated that teamwork among students can be more motivating and interesting by making the performances more positive in terms of achieved results.

However, a small percentage of 13,85% preferred the first phase where they could use applications, like *GeoGebra*, to manipulate math concepts and objects. The reason was that they could use technological tools differently from the simple entertainment by making animation and playing around (Figure 94).

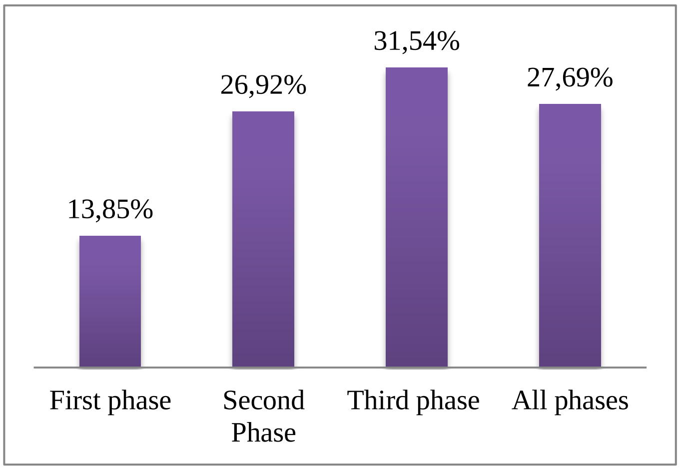


Figure 94: The phase that liked most.

On the other hand, looking at the results achieved to the question *Which phase of the activities did you like least?*, the 15,50% (against 3,10% for the second phase and the 7,75% for the third one) of students indicated the first phase (concrete) addressed to the object and concept manipulation related to mathematics study by using the technological applications. The reason was that in the first step of the method they didn't feel free to use their creativity and their work like they did in the third phase.

Besides, it was revealed through the participatory observation that usually students are not used to involving the technology into the classroom to reinforce the learning process. Instead, they use the technology generally for entertainment as confirmed in the profile description of the sample and in the control questions.

Consequently, the survey reveals that the introduction of technology into the classroom by the teachers is not so popular.

This remains an experience of a few schools while most of them are linked to the traditional frontal teaching method: the teacher explains and students receive the information.

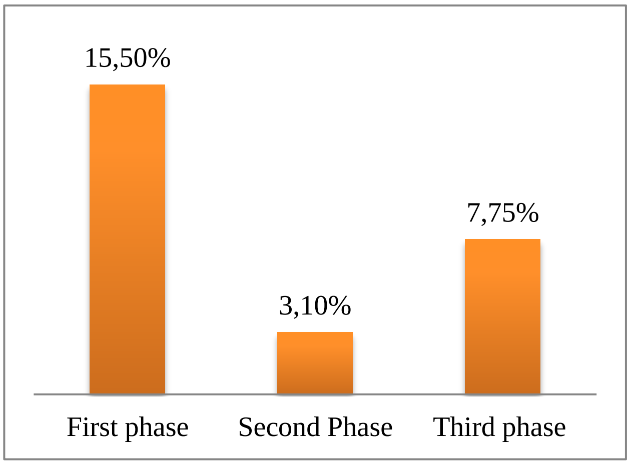


Figure 95: The phase that liked least.

This is demonstrated, also, by the fact that not all the students (7,75%) feel comfortable to create something and to use their creativity, because, often they are not used to on their own without being told exactly what to do (Figure 95).

Comparing the results of pre and after the piloting experiences, the student attitude (82,31%) towards mathematics has changed. Students discovered that different things can be connected with art as well as the beauty of the art can be derived from the mathematical concepts. Although a percentage of 12,31% remains "neutral" towards this discipline.

However, the 91,54% of the sample agree with the desire to go on using the arts in the study of mathematics, mainly with the proposed method (91,5%) because it could be a good incentive to improve and promote the study of mathematics.

In fact, most of the students (79,23% against the 16,92% with a neutral position) agree that the teaching method, generally used for the scientific topics, should be changed and diversified by using both the technology and arts to underline the real applicability of the scientific concepts.

CONCLUSION

This thesis aimed to analyse and exploit the possibility to find a combination between Western and Eastern approaches, in mathematics teaching using the arts. Starting from a study of the strengths and weaknesses of both Eastern and Western learning and teaching approaches, we designed a model exploiting the potentialities of both without leaving cultural differences out.

We adopted Singapore's method defined through three phases (concrete, pictorial, abstract) combined with the Western mathematics teaching implemented mainly in Italian schools.

The complex combination was created by using art.

The major result obtained is that students improved not only their understanding of the subject but they develop their creativity through their art-works. This increased the motivation and interest in the study of mathematics, on one hand, and meaningful improvement of student final performances, on the other hand.

Besides, the use of the artworks in the proposed learning and teaching approach offered students a way to go beyond the pure theory and to apply their knowledge in the surrounding world.

Moreover, the research work, through the collaboration of teachers from different subjects like art and mathematics, showed the importance of a novel interdisciplinary and multidisciplinary approach in the school curriculum needed to improve and develop, in this case, mathematics skills.

To achieve the research objectives described in this thesis, the foreseen tasks led to the following concrete results (Ref. Chapter 4, Chapter 5):

- Starting from the reference theoretical framework described in the “theory of Didactical Situations in mathematics” of Guy Brousseau, we designed an innovative model by integrating the three phases of the Singapore’s method (concrete, pictorial and abstract) and art. (Ref. Chapter 3, Paragraph 3.1 – 3.2)
- Furthermore, we developed a model which was turned into a new didactic approach, including didactic materials, worksheets, and guidelines for the experimentation activity management. This aimed at valorizing mainly the problem-solving skills, the creativity and the imagination of the students without losing knowledge value related to the mathematics envisaged for the corresponding age. The documents prepared for the experimentation phase described and suggested also (i) the materials (such as

paper with a specific shape, pencils and colors) and software applications (e.g. GeoGebra) to be used, (ii) the objectives, (iii) the vocabulary, (iv) the activity sequence structured on the base of the three phases (concrete, pictorial and abstract) to be carried out. (Ref. Chapter 4)

- We implemented the proposed model and approach in secondary schools in Italy. Therefore, we defined, furthermore, the reference sample and we designed the evaluation and validation tools to verify its effectiveness and efficiency. This allowed us to reach the results discussed. (Ref. Chapter 4, Paragraph 4.2 – 4.3 – 4.4)
- We developed a 3D virtual museum "Mathematics and Arts" in collaboration between the Institute for Computer Science and Control, Hungarian Academy of Science and the Institute of Mathematics and Informatics – Bulgarian Academy of Sciences, showing the art-works realized by some students during the experimentation phase; the real objects and the mathematics concepts they referred; the video describing the main phases of the proposed approach. (Ref. Chapter 4, Paragraph 4.1)

In the implementation of Task 1 (Study of the problem) we:

- analyzed the existing practices of the school approaches for mathematics topics in lower and upper schools by comparing the learning and teaching approaches applied to the mathematical subject between Eastern (Singapore's method) and Western methodologies. (Ref. Chapter 2, Paragraph 2.3 - 2.4)
- studied the relevant factors such as the current pedagogies on mathematics education through an in-depth study of the relevant bibliography available in the public database and specialized libraries. (Ref. Chapter 1, Chapter 2, Paragraph 2.1 - 2.3 - 2.4)
- analyzed the students and teachers attitude towards the introduction of innovation into school and in learning and teaching approaches. (Ref. Chapter 5).

In the implementation of Task 2 (Development of a research teaching/learning approach combined with the arts) we (involving directly the teachers):

- analyzed the exercises and mathematics topics from the school curricula to define the "mathematization" concept in terms of students' knowledge, skills and attitude. (Ref. Chapter 1, Chapter 3, Paragraph 3.2.1)
- selected the mathematics exercises and concepts and the artworks for the experimental activities with the students. (Ref. Chapter 4)

- prepared the worksheets and guidelines for the experimentation activity management into the class. (Ref. Chapter 3, Paragraph 3.2.1 - 3.2.2, Chapter 4)
- defined the main elements to be carried out during the experimentation phase, in terms of the reference sample (teachers and students) and data collection tools (both qualitative and quantitative). (Ref. Chapter 4, Paragraph 4.2 - 4.3 - 4.4)

In the implementation of Task 3 (Experimentation phase) we (Ref. Chapter 4):

- organized the initial meetings with teachers and students to show different types of real connections between the artistic expression/creation and math patterns.
- submitted the first questionnaire and the mathematics exercises previously selected.
- carried out the experimentation activities in the Italian secondary schools by involving both teachers and students.
- organized final meetings, in the end, to let the students show their works (in every phase), to collect qualitative and quantitative data on their feedback and to submit a set of the mathematics exercises related to the topic studied.

In the implementation of Task 4 (Data processing and evaluation of the achieved results) we performed the processing data and the evaluation of the proposed model. (Ref. Chapter 5).

CONTRIBUTIONS

The most important contributions of the thesis are:

- A model and framework combining Western and Eastern (Singapore's method) teaching and learning approaches using the art have been developed: (i) the main Western learning and teaching approaches (with a special focus on the ones used in Italian schools) such as cooperative, problem-solving, inquiry-based, technology-enhanced learning (e.g. virtual laboratories in mathematics education, the use of the serious game, virtual reality and augmented reality, educational robotics) have been investigated; (ii) the main features of Eastern learning processes with special attention to Chinese methods and culture have been analyzed; (iii) the main features of Singapore's method have been explored and harnessed in mathematics classes (learning/teaching); (iv) the existing connections between mathematics and arts are explored.
- The reference theoretical framework (described in the "theory of Didactical Situations in mathematics" by Guy Brousseau) has been adapted for an innovative didactic model through the integration of the three phases of the Singapore's method (concrete, pictorial and abstract) and art.
- The didactic materials have been developed such as worksheets and guidelines to manage the three phases in the classes during the experimentation activity. The aim was to provide both teachers and students with clear information about (i) the materials (such as paper with a specific shape, pencils and colors) and software applications (e.g. GeoGebra) to be used, (ii) the objectives, (iii) the vocabulary, (iv) the activity sequence structured on the base of the three phases (concrete, pictorial and abstract) to be carried out.
- The proposed model and approach have been implemented in several secondary schools in Italy by involving teachers of mathematics and arts: 130 secondary school students of different ages, 11-13 years-old for the first cycle and 14-16 years-old for the second one. The effectiveness and efficiency of the proposed approach have been analysed well as the relevance of the results achieved.
- A 3D virtual museum "Mathematics and Arts" has been developed in collaboration between the Institute for Computer Science and Control, Hungarian Academy of Science and the Institute of Mathematics and Informatics – Bulgarian Academy of Sciences, exhibiting the final outputs of the research work. Also, a video summarizing

the activities carried out was produced and made available inside the 3D virtual museum.

DECLARATION OF ORIGINALITY OF RESULTS

I declare that the present dissertation contains original results obtained in my research with the support and assistance of my scientific advisers. The results obtained, described and / or published by other scholars are duly and in detail cited in the bibliography.

This dissertation is not applied for obtaining a scientific degree in another higher school, university or scientific institute.

Signature: Michela Tramonti……………………………..

(Michela Tramonti)

DISSEMINATION OF THE RESULTS AND FUTURE WORK

The framework presented in this thesis has been exploited within Erasmus+ project n. 2018-1-FI01-KA201-047215 – *G.A. STEM - Enhancing STEM skills through arts and mini-games*, aiming to develop STEM skills in 13-16 years old students reinforced by the use of technology. In particular, it has inspired the project idea by supporting the development of the state-of-the-art as regards to the relation of mathematics education and arts.

Moreover, parts of the work done for the scope of the thesis have been published in several journals and conference proceedings (Ref. LIST OF THE AUTHOR'S PUBLICATIONS RELATED WITH PHD THESIS): International Education Conference Proceedings - EDULEARN 2017 (Barcelona, Spain, 2017); Central Bohemia University International Conference Proceedings – CBU 2017 (Prague, Czech Republic, 2017); Digital Presentation and Preservation of Cultural and Scientific Heritage—DiPP2017 (Burgas, Bulgaria, 2017); 12th International Technology, Education and Development Conference Proceedings, INTED2018, Valencia, Spain, 2018); Digital Presentation and Preservation of Cultural and Scientific Heritage—DiPP2018 (Burgas, Bulgaria, 2018); Technology, Education, Management Journal (TEM 2019); Digital Presentation and Preservation of Cultural and Scientific Heritage—DiPP2019 (Burgas, Bulgaria, 2019); 12th annual International Conference of Education, Research and Innovation – ICERI2019 (Seville, Spain, 2019).

The results obtained during the work on the thesis suggest that the study can be extended and developed further in the following areas:

- Theoretical direction: 1) Extension/adaptation of the developed approach to other scientific subjects, such as physics. 2) Extension/adaptation of the model through the integration of arts and scientific subjects with the designing of a mini-game conceptual idea (Ref. G.A. STEM Project).
- Applied/practical direction: Implementation and experimentation with other software (apart from GeoGebra) in the first phase (pictorial); Extension of the 3D Virtual Museum "Mathematics and Arts" by adding more topics and exercises.

LIST OF THE TABLES

LIST OF THE FIGURES

ABBREVIATIONS

R&D: Research and Development

OECD: Organisation for Economic Co-Operation and Development

EUROSTAT: European Statistical Office

STEM: Science, Technology, Engineering and Mathematics

UNESCO: United Nations Educational, Scientific and Cultural Organization

MST: Mathematics, Science and Technology

PISA: Programme for International Student Assessment

TIMSS: International Mathematics and Science Study

IEA: International Association for the Evolution of Educational Achievement

PON: Programma Operativo Nazionale (National Operative Programme)

FSE: Fondo Sociale Europeo - European Social Fund (ESF)

PIRLS: Progress in International Reading Literacy Study

TEL: Technology-Enhanced Learning

TEAL: Technology Enabled Active Learning

MIT: Massachusetts Institute of Technology

VR: Virtual Reality

AR: Augmented Reality

APPENDIX 1 - DATA COLLECTION TOOLS

QUESTIONNAIRE – FIRST PART

GENERAL INFORMATION

1	**Age**	....	

2	**Gender**	M	F

3	**How often do you use the Internet?** Please, indicate with an "X" the selected answer. (A maximum of 2 answers can be selected)	
a	Once a month or less	
b	Once a week	
c	Several times during the week	
d	Every day	
e	Never	

4	**What technological tools do you use?** Please, indicate with an "X" the selected answer. (A maximum of 2 answers can be selected)	
a	Smartphone	
b	PC	
c	Play Station	
d	Tablet	
e	None	
f	Other, please, specify...	

5	**What should technology be used for?** Please, indicate with an "X" the selected answer. (A maximum of 2 answers can be selected)	
a	Learning new things	
b	Entertainment (e.g. Games, Music, etc.)	
c	To support homework	
d	Other, please, specify...	

6	**Where do you mostly use technological tools?** Please, indicate with an "X" the selected answer. (A maximum of 2 answers can be selected)	
a	At school	
b	At home	
c	At your friend' home	
d	Other, please, specify...	

Please, indicate your answer by selecting a number from 1 (strongly agree) to 5 (strongly disagree)

7	**What do you think of Mathematics?**	**1** *Strongly agree*	**2**	**3**	**4**	**5** *Strongly disagree*	*I don't know*
7.1	Mathematics is boring.						
7.2	I only use math at school or to do my homework.						
7.3	I think math is something abstract.						
7.4	When I finish school, I won't need math.						
7.5	I like to attend math class.						
7.6	When I don't remember exactly how I was taught to solve a math problem, I try to reason and apply other methods.						

Please, indicate with "X" the selected answer

8	**Today there are many applications/programs that can help you improve your math skills. Do you know any of them?**	
a	Yes	
b	NO	

8.1 If your answer is "yes", please, specify which are the applications or programs that you use:

____

____

____

Please, indicate your answer by selecting a number from 1 (strongly agree) to 5 (strongly disagree)

9	**If your answer is "No" to the previous questions, please try to indicate a possible motivation.**	**1** *Strongly agree*	**2**	**3**	**4**	**5** *Strongly disagree*	*I don't know*
9.1	I think they are not useful.						
9.2	I don't like studying online.						
9.3	They are not necessary. The school teaching materials are enough.						
9.4	I never tried.						

10	**What do you think of the relationship between mathematics and art?**	**1** *Strongly agree*	**2**	**3**	**4**	**5** *Strongly disagree*	*I don't know*
10.1	I never thought I could study math using art.						
10.2	It could be a way to increase my interest in math.						
10.3	Learning while having fun is more motivating.						
10.4	It might make learning math easier.						

QUESTIONNAIRE – SECOND PART

GENERAL INFORMATION

1	**Age**	....	

2	**Gender**	M	F

Please, indicate your answer by selecting a number from 1 (Excellent) to 5 (Weak)

3	**Please, provide your opinion related to participation in this new experience.**	**1** *Excellent*	**2**	**3**	**4**	**5** *Weak*
3.1	My motivation was...					
3.2	My participation was...					
3.3	The experience was…					

Please, indicate your answer by selecting a number from 1 (strongly agree) to 5 (strongly disagree)

4	**Please, describe your experience in using applications like GeoGebra in mathematics**	**1** *Strongly agree*	**2**	**3**	**4**	**5** *Strongly disagree*	*I don't know*
4.1	The mathematics concepts are clearer to me.						
4.2	The way to approach mathematics makes the subject easier.						
4.3	The way we approach math makes the subject more fun.						

5	**Discovering mathematics in art ...**	**1** *Strongly agree*	**2**	**3**	**4**	**5** *Strongly disagree*	*I don't know*
5.1	It was fun ...						
5.2	It helped me understand that math is not something abstract.						
5.3	I can use my creativity and be original.						
5.4	I don't feel comfortable using this method.						
5.5	It was easy to create my artwork using the mathematical concept studied.						

6	**As for the contents learned ...**	**1** *Strongly agree*	**2**	**3**	**4**	**5** *Strongly disagree*	*I don't know*
6.1	The content was easily understood.						
6.2	The contents learned seem to be more concrete and practical than before.						
6.3	The content learned can support my study outside the classroom.						

Please, express your overall judgment on the activities carried out.

7	**Which phase of the activities did you like most?**	

8	**Which phase of the activities did you like least?**	

Please, indicate your answer by selecting a number from 1 (strongly agree) to 5 (strongly disagree)

9	**In brief, at the end of this new experience, you can say ...**	**1** *Strongly agree*	**2**	**3**	**4**	**5** *Strongly disagree*	*I don't know*
9.1	My attitude towards mathematics has changed.						
9.2	I would like to continue using art in the study of mathematics.						
9.3	The method could be a good incentive to study math.						
9.4	I would not change anything in the method used so far to study mathematics.						
9.5	Using technology in the study of mathematics has increased my interest.						

OBSERVATION GRID

Write down if the students:

1. actively participate in the activities:

2. interact during the lesson:

3. contribute with their ideas to teamwork:

4. find difficult in the use of the ICT applications:

5. can work independently:

6. can explore the available material:

7. know how to look for new materials related to the topic studied:

____

____

8. understand better the studied concepts:

____

____

9. feel comfortable using the proposed method:

____

____

10. Other observation notes:

____

____

APPENDIX 2 - CONTENT ANALYSIS

QUESTIONNAIRE – FIRST PART: OPEN QUESTION RECORDING

Q4 - What technological tools do you use?

Other, specify:

- Xbox One S
- I use at least one time per day mainly my mobile and my Nintendo.
- TV
- I use my mobile for sending messages and to listen to music and my Play Station to play and have a good time.

Q6 - Where do you mostly use technological tools?

Other, specify:

- Walking around
- Away from home
- When I am outside
- Only at home, sometimes I use the mobile outside to call
- In general, outside the home
- I use technological devices only at home because I think it is not polite to use them at home of others.
- At my grandma’s home or my aunt’s home
- At restaurant, pizzeria

8.1 If your answer is “yes”, please, specify which are the applications or programs that you use:

- GeoGebra
- Math games
- Photomath
- Calculator

- Calculator realized by me with Arduino
- I don’t remember
- Duolingo
- INVALSI test per level for primary and secondary schools.
- I don’t use them but I know some of them but I don’t remember the name.
- Mobile applications
- Gego
- I don’t remember the name.

QUESTIONNAIRE – SECOND PART: OPEN QUESTION RECORDING

Q7 - Which phase of the activities did you like most?

- First phase
- Second phase
- Third phase
- All phases
- The second phase, because we had to show our works to the classes.
- The second phase where we have searched images in nature
- The activity that I liked most was the last one when we had to design and create something. I enjoyed it a lot.
- When we have created the art-works
- My favourite activity was to draw the works.
- The one addressed to realize a drawing with axial symmetry.
- The phase where we have created hands-on works.
- Creating drawings and taking pictures.
- Creating works by hand.
- The first with GeoGebra
- The second for the group experience.
- I liked the second activity.

Q8 - Which phase of the activities did you like least?

- First phase
- Second phase
- Third phase
- All phases
- None of them
- Beautiful all of them
- I liked everything a lot
- I liked everything
- The activity with GeoGebra
- The activity with a straight line
- I liked all the activities. Each of them had something special.
- The third phase because it was not easy to create.
- The third was the most difficult.

APPENDIX 3 - TABLES

QUESTIONNAIRE- FIRST PART: RESULTS

Q1. Student Age		
	N	**%**
11-13	58	44,62
14-16	72	55,38
Total	130	100

130 valid cases

Q2. Genre		
	N	**%**
M	56	43,1
F	74	56,9
Total	130	100

130 valid cases

Q3. How often do you use the Internet?		
	N	**%**
Once a month or less	0	0
Once a week	6	4,6
Several times during the week	45	34,6
Every day	79	60,8
Never	0	0
Total	130	100

130 valid cases

Q4. What technological tools do you use?		
	N	**%**
Smartphone	113	86,9
PC	53	40,8
Play Station	36	27,7
Tablet	41	14,6
Other *	5	3,8
Total	248	173,8

130 valid cases

The total is not equal to 100 because multiple answers were possible.
(*) The "other" item includes: Xbox One S, Nintendo

Q5. What should technology be used for?		
	N	**%**
Learning new things	30	23,10
Entertainment (e.g. Games, Music, etc.)	121	93,10
To support homework	59	45,40
Other *	8	6,20
Total	218	167,80

130 valid cases

The total is not equal to 100 because multiple answers were possible.

Q6. Where do you mostly use technological tools?		
	N	**%**
At school	27	21,50
At home	129	99,20
At your friend' home	38	29,20
Other *	18	13,80
Total	212	163,70

130 valid cases

The total is not equal to 100 because multiple answers were possible.

(*) The "other" item includes: outside home, at my grandmother's house or my aunt's house, a restaurant/pizzeria

Q7. What do you think of Mathematics?						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
7.1 Mathematics is boring.	31 23,85%	26 20,00%	17 13,07%	31 23,85%	25 19,23%	0 0,00%
7.2 I only use math at school or to do my homework.	58 44,60%	21 16,20%	14 10,80%	18 13,80%	19 14,60%	0 0,00%
7.3 I think math is something abstract.	40 30,80%	23 17,70%	12 9,20%	19 14,60%	36 27,70%	0 0,00%
7.4 When I finish school, I won't need math.	22 16,90%	20 15,40%	13 10,00%	20 15,40%	55 42,30%	0 0,00%
7.5 I like to attend math class.	24 18,50%	23 17,70%	22 16,90%	29 22,30%	32 24,60%	0 0,00%
7.6 When I don't remember exactly how I was taught to solve a math problem, I try to reason and apply other methods.	37 28,50%	42 32,30%	18 13,80%	20 15,40%	13 10,00%	0 0,00%

130 valid cases

Q8. Today there are many applications / programs that can help you improve your math skills. Do you know any of them?		
	N	**%**
YES	32	24,60
NO	98	75,40
Total	130	100

130 valid cases

Q9. If your answer is "No" to the previous questions, please try to indicate a possible motivation.						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
9.1 I think they are not useful.	0 0,00%	13 13,27%	28 28,57%	25 25,51%	32 32,65%	0 0,00%
9.2 I don't like studying online.	46 46,94%	20 20,41%	4 4,08%	16 16,33%	12 12,24%	0 0,00%
9.3 They are not necessary. The school teaching materials are enough. **(*)**	52 53,61%	17 17,53%	12 12,37%	7 7,22%	9 9,28%	0 0,00%
9.4 I never tried.	54 55,00%	17 17,00%	13 13,27%	10 10,20%	4 4,08%	0 0,00%

98 valid cases with n.1 no response.

Q10. What do you think of the relationship between mathematics and art?						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
10.1 I never thought I could study math using art.	81 62,31%	22 16,92%	12 9,23%	6 4,62%	9 6,92%	0 0,00%
10.2 It could be a way to increase my interest in math.	55 42,30%	24 18,50%	31 23,80%	8 6,20%	12 9,20%	0 0,00%
10.3 Learning while having fun is more motivating.	99 76,15%	15 11,54%	9 6,92%	2 1,54%	5 3,85%	0 0,00%
10.4 It might make learning math easier.	47 36,20%	33 25,40%	32 24,60%	9 6,90%	9 6,90%	0 0,00%

130 valid cases

QUESTIONNAIRE - SECOND PART: RESULTS

Q3. Please, provide your opinion related to participation in this new experience.					
	Excellent	Very good	Good	Fair	Weak
3.1 My motivation was...	21 16,15%	57 43,85%	41 31,54%	11 8,46%	0 0,00%
3.2 My participation was...	42 32,31%	72 55,38%	14 10,77%	2 1,54%	0 0,00%
3.3 The experience was…	84 64,62%	38 29,23%	8 6,15%	0 0,00%	0 0,00%

130 valid cases

Q4. Please, describe your experience in using applications like GeoGebra in mathematics.						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
4.1 The mathematics concepts are clearer to me.	82 63,08%	37 28,46%	5 3,85%	4 3,08%	2 1,54%	0 0,00%
4.2 The way to approach mathematics makes the subject easier.	102 78,46%	18 13,85%	5 3,85%	3 2,31%	2 1,54%	0 0,00%
4.3 The way we approach math makes the subject more fun.	113 86,92%	9 6,92%	5 3,85%	2 1,54%	1 0,77%	0 0,00%

130 valid cases

Q5. Discovering mathematics in art ...						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
5.1 It was fun ...	114 87,69%	12 9,23%	4 3,08%	0 0,00%	0 0,00%	0 0,00%
5.2 It helped me understand that math is not something abstract.	74 56,92%	47 36,15%	9 6,92%	0 0,00%	0 0,00%	0 0,00%
5.3 I can use my creativity and be original.	75 57,69%	44 33,85%	10 7,69%	1 0,77%	0 0,00%	0 0,00%
5.4 I don't feel comfortable using this method.	1 0,77%	1 0,77%	10 7,69%	31 23,85%	87 66,92%	0 0,00%
5.5 It was easy to create my art work using the mathematical concept studied.	61 46,92%	42 32,31%	15 11,54%	8 6,15%	4 3,08%	0 0,00%

130 valid cases

Q6. As for the contents learned ...						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
6.1 The content was easily understood.	111 85,38%	10 7,69%	6 4,62%	0 0,00%	3 2,31%	0 0,00%
6.2 The contents learned seem to me more concrete and practical than before.	112 86,15%	12 9,23%	4 3,08%	0 0,00%	2 1,54%	0 0,00%
6.3 The content learned can support my study outside the classroom.	79 60,77%	28 21,54%	20 15,38%	0 0,00%	3 2,31%	0 0,00%

130 valid cases

Q7 - Which phase of the activities did you like most?		
	N	**%**
First phase	18	13,85
Second Phase	35	26,92
Third phase	41	31,54
All phases	36	27,69
Total	130	100

130 valid cases

Q8 - Which phase of the activities did you like least?		
	N	**%**
First phase	20	15,50
Second Phase	4	3,10
Third phase	10	7,75
None	95	73,64
Total	129	100

129 valid cases with n.1 no response.

Q9. In brief, at the end of this new experience, you can say ...						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
9.1 My attitude towards mathematics has changed.	63 48,46%	44 33,85%	16 12,31%	4 3,08%	3 2,31%	0 0,00%
9.2 I would like to continue using art in the study of mathematics.	110 84,62%	9 6,92%	8 6,15%	3 2,31%	0 0,00%	0 0,00%
9.3 The method could be a good incentive to study math.	96 73,85%	23 17,69%	9 6,92%	2 1,54%	0 0,00%	0 0,00%
9.4 I would not change anything in the method used so far to study mathematics.	4 3,08%	1 0,77%	22 16,92%	35 26,92%	68 52,31%	0 0,00%
9.5 Using technology in the study of mathematics has increased my interest.	67 51,54%	43 33,08%	13 10,00%	5 3,85%	2 1,54%	0 0,00%

130 valid cases

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LIST OF THE AUTOR'S MAIN PUBLICATIONS RELATED WITH THE TOPIC OF THE PHD THESIS

YEAR 2017

1. **Tramonti, M.**, Mathematics Education Reinforced through Innovative Learning Processes. Proceedings of the 9th International Conference on Education and New Learning Technologies, Barcelona, Spain, 2017, ISBN:978-84-697-3777-4, ISSN:2340-1117, DOI:10.21125/edulearn.2017.0744, 9279-9284
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2. **Tramonti, M.**, Paneva-Marinova, D., Pavlov, R., Math and Art Convergence for Education. Proceedings of CBU International Conference on Innovation in Science and Education, Prague, Czech Republic, 2017, ISSN:1805-997X, Online ISSN 1805-996, DOI:10.12955/cbup.v5.1037, 851-854, Available at: http://ojs.journals.cz/index.php/CBUIC/article/view/1037/pdf (indexed in Web of Science).
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3. **Tramonti, M.**, Reinforcing Learning Setting through the Use of Digital Tools. Digital Presentation and Preservation of Cultural and Scientific Heritage. DiPP2017 Conference Proceedings, 7, 2017, ISSN:1314-4006, 159-167, Available at: http://dipp.math.bas.bg/images/2017/159-168_9_707_fDiPP2017-2_Tramonti.pdf (indexed in Scopus).
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YEAR 2018

4. **Tramonti, M.**, Paneva-Marinova, D., Towards Improving Math Understanding Using Digital Art Library as a Source of Knowledge. Proceedings of the 12th annual International Technology, Education and Development Conference (IATED2018), Valencia, Spain, 2018, ISBN:978-84-697-9480-7, ISSN:2340-1079, DOI:10.21125/inted.2018.0517, 2751-2756, Available at: https://library.iated.org/view/TRAMONTI2018TOW (indexed in Web of Science)
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5. **Tramonti, M.**, Technology and Art to Improve Mathematics Learning. Proceedings of the 12th annual International Technology, Education and Development Conference (IATED2018),

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6. **Tramonti, M.**, Paneva-Marinova, D., Maths, Art and Technology: a Combination for an Effective Study. TEM Journal, 8, 1, 2019, ISSN:2217-8309, DOI: 10.18421/TEM81-11, 82-86. SJR (Scopus):0.167,
Available at: http://www.temjournal.com/content/81/TEMJournalFebruary2019_82_86.html (indexed in Scopus and Web of Science)
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7. **Tramonti, M.**, Art and Science: Combining Learning Tool. Digital Presentation and Preservation of Cultural and Scientific Heritage, DiPP2019 Conference Proceedings, 9, 2019, ISSN:1314-4006, 145-152, Available at: http://dipp.math.bas.bg/images/2019/145-152_8_2.7_fDiPP2019-09_f_v.1.F_20190908.pdf (indexed in Scopus and Web of Science)
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8. **Tramonti, M.,** Mathematics and Science Study through the Arts. Proceedings of the 12th annual International Conference of Education, Research and Innovation (ICERI2019), Seville, Spain, 2019, ISBN:978-84-09-14755-7, ISSN:2340-1095, DOI:10.21125/iceri.2019.0518 1837-1842.
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