Institute of Mathematics and Informatics

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MATHEMATICS EDUCATION, WESTERN AND EASTERN TEACHING APPROACHES COMBINED WITH ARTS

Executive Summary

by

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A dissertation submitted for the award of educational and scientific degree "Doctor"
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The PhD thesis was discussed and proposed for defense at the meeting of the Pre-Defence Unit, which took place on 20.11.2020 from 10 am in a hybrid mode in the Hall No. 115 of the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences (IMI-BAS) and online through a conference call.

The defense of the PhD thesis will take place on …………………...... from ………… in IMI-BAS, Acad. G. Bonchev St., block 8 at an open meeting of the Scientific Jury.

The PhD thesis is presented in 211 pages. It includes an introduction, 5 chapters, conclusion, 3 annexes, list of scientific and applied contributions, list of tables, list of figures, declaration of originality, list of references used from 223 literature sources and list of 8 author's publications related to the presented PhD thesis, list of citations.

The defense materials are available to those interested in the Library of IMI- BAS, Acad. G. Bonchev " St. block 8, Sofia.

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ACKNOWLEDGMENTS

Starting from my very first visit to the Institute of Mathematics and Informatics, Bulgarian Academy of Science, Sofia, Bulgaria, I have found myself immersed in an atmosphere full of a high professional level. These factors were determinative to start the first passes in a new research field.

First of all, I would like to express the deepest gratitude to my principal supervisor Professor Desislava Paneva-Marinova, whose experience, knowledge and wisdom guided me through all these years. Her presence has favored my formation as a researcher, keeping the sense of the taste towards the scientific work.

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Massive thanks also to the Scientific Council of the Institute of Mathematics and Informatics.

CHAPTER 1. MOTIVATION, GOAL, OBJECTIVES AND TASKS OF THE PHD THESIS

1.1 Motivation

Nowadays, when Europe is coping with a period of change and with the economic crisis, in particular, which is slowing down the development and the social progress highlighting the structural weaknesses within the European Countries, the role of R&D is difficult to overestimate.

On the other hand, there are other international challenges, such as globalization, resource exploitation, and human ageing. As stated in the European Communication EUROPE *2020 - A strategy for smart, sustainable and inclusive growth* of 2010, the Commission focuses on a stronger strategy that can transform the European Union into a smart, sustainable and inclusive growth for a high level of employment, productivity and social cohesion.

According to EUROSTAT, published in Science, technology, and innovation in Europe, 1.68% of people in Europe work in the Research & Development (R&D) field, but the situation is changing completely at a National level for some countries as Italy where the percentage of people working as "scientists and engineers" is very low, around 0.61%.

Although research & development are the keywords for Europe of the 21st century, the obvious national and institutional barriers limit the strengthening of the European Research.

These data are still confirmed by the following figure representing the intensity of R&D activities in 2016 in Europe where Italy is placed in a low ranking of the average of 2.03% reached by some European countries.

Schreiner and *Sjøberg* suggest that *it might be that we have now passed the era in which the work of physicists, technicians, and engineers is seen as crucial to people's lives and well-being.* Today's youths are more interested in who they will be rather than what they will do. Negative stereotypes of scientists, engineers, researchers, and other STEM (Science, Technology, Engineering and Mathematics) experts' careers can be found amongst youth in most of the western world, even the United States. The only exception is Japan, where wide investments are made in the educational field by the Government.

School, television, and newspapers feed the people's imagination with the idea that scientist's work is a hard and demanding activity as well as unknown and mysterious. It is enough to think about emblematic figures of scientists like Frankenstein or Doc, the character of the movie *Back to the Future*, or cartoons, which continue to represent *science* in a caricatured and imaginative way. As a result, the scientist's figure appears isolated, immersed in a special work – strange and incomprehensible to everyone at the same time.

Further studies show that women remain under-represented in R&D in every region worldwide. Some of the latest UNESCO data shows that, in North America and Western Europe, the average representation for the women in R&D is 32% (is the lowest average found in Luxemburg with only a 21% and in the Netherlands with a 24%). A different situation is found in Central and Eastern Europe, where the overall average rises to 40%. In fact, in Latvia, Lithuania and Northern Macedonia, the woman represents more than 50%, and in Albania, Bulgaria, Croatia, Estonia, Moldova and Romania the percentage exceeds 40%. Despite the positive trend occurring in Eastern Europe, a wide array of studies indicates a disparity between the number of women studying science and those who go working as scientists professionally. Overall, women account for a minority of the world's researchers *(UNESCO Institute for Statistics, June 2018).*

In this context, the European Commission's High-Level Expert Group on Science Education Renewal has made the points that *Teachers are key players…being part of a network allowing them to improve the quality of their teaching and support their motivation* and that *the articulation between national activities and those funded at the European level must be improved*.

In this framework, it is necessary to develop a high-quality teaching and learning environment where students can approach the study of scientific subjects with more interest and motivation adequately supported by teachers in their learning, which will be more effective if it is *meaningful*, i.e. active, constructive, intentional, authentic and collaborative.

1.2 Goal, objectives and tasks of the thesis

The research intended to exploit the potentiality emerging from a creative integration between Western and Eastern learning approaches combined with the use of the arts. The potential effect would allow students to develop and improve their creativity-based learning skills, and solving problem skills.

Moreover, it aimed at approaching the students to the study of STEM (*Science, Technology, Engineering and Mathematics*) subjects through the combination of both meaningful and mastery learning. This allowed the development of a more effective Educational and Training environment for teachers and their students who have benefited from the use of more attractive and fun pedagogical tools in the study of mathematics.

In detail, the research objectives can be synthesized as follows:

- Exploiting the possibility to find a combination between Western and Eastern approaches (mainly Singapore's method) in mathematics teaching.
- Finding the result of this combination in the arts.

Demonstrating the objectives above encourage meaningful learning in the development of the research skills to improve the school performance (short-term objective) and an increased interest towards a future perspective in the mathematics field (long-term objective).

The goal and objectives of the thesis were reached through the following tasks:

Task 1: Study of the problem

This task aimed to investigate how math topics are defined in PISA, TIMSS, and national surveys and to compare the school curriculum on math in diverse learning environments. It was accomplished through the following steps:

- Analysis of the existing practices of the school approaches for mathematics topics in lower and upper schools by comparing the learning and teaching approaches applied to the mathematical subject between Eastern (Singapore's method) and Western methodologies.

- Study of the relevant factors such as the current learning approaches and methods on mathematics education through an in-depth study of the relevant bibliography available in a public database and specialized libraries.
- Analysis of the students and teachers attitude towards the introduction of innovation into school and in learning and teaching approaches.

Task 2: Development of a research teaching/learning approach combined with the arts

This task aimed to define the research model approach for mathematics education as a result of the combination of Western and Eastern teaching approaches with the arts.

It was carried out through the following steps:

- Analysis of the exercises and mathematics topics from the school curricula to define the "mathematization" concept in terms of students' knowledge, skills and attitude (Ref. PISA, TIMSS and National school curriculum).
- Selection of the mathematics exercises and concepts and the artworks together with teachers for the experimental activities with the students.
- Preparation of the questionnaires to be submitted during the experimentation phase.
- Preparation of the worksheets and guidelines for the experimentation activity management into the class.
- Definition of the main elements to be carried out during the experimentation phase, in terms of reference sample (teachers and students) and data collection tools (both qualitative and quantitative).

Task 3: Experimentation phase

The aim was to carry out the experimentation activities to collect data and to start the evaluation of the research results.

This required the following steps:

- Organisation of the initial meetings with teachers and students to show different types of real connections between the artistic expression/creation and math patterns.
- Submission of the first questionnaire and the mathematics exercises previously selected.
- Realization of the experimentation activities in Italian secondary schools by involving both teachers and students.
- Organisation of the final meetings to let the students show their works, to collect qualitative and quantitative data on their feedback and to submit a set of the mathematics exercises related to the topic studied.

Task 4: Data processing and evaluation of the achieved results

The aim was to start the processing and evaluation of the research results through the coding and analysis phases, data interpretation (both qualitative and quantitative) and drawing up the final report with a discussion of the results.

CHAPTER 2. STUDY OF THE PROBLEM

The globalized labor world requires knowledge based on solid basic skills such as mathematics literacy. The development of math skills, teaching strategies, and practices become primary for educational growth during all school years. An alarming picture of the current situation is provided by PISA and TIMSS international surveys. The primary aim of these assessments is to assess different features of student learning and to assist policymakers to better understand to what extent their educational systems are measuring up with developments taking place in other countries.

2.1 The current situation provided by the international surveys

The PISA survey is promoted by the Organization for Economic Cooperation and Development (OECD). It aims at assessing the performances of 15 years-old students in reading, mathematics, and science every three years. It focuses, especially, on how far students, near the end of compulsory education, have acquired knowledge and skills essential for full participation in society. This survey contributes to the assessment of mathematical and science literacy through several proposed questions always set in a real context.

Concerning the PISA last survey realized in 2018, the mean scores achieved for the mathematics have registered negative tendency in countries considered to be at the forefront of education, like Finland (where the score has decreased of -12 in last three years), on one hand. While, on the other hand, the results have confirmed the negative ranking (below the OECD average) in other countries, like Greece.

In Italy, if the survey realized in 2015 have shown that the pupils' performance reached in mathematics a score on the OECD average (490), the last results collected in 2018 present again a negative situation (487, -2) under the OECD average (489) (Table 1). Probably this inverse tendency is still due to the use of traditional learning-teaching methodologies in the mathematics study.

Table 1- Extracted from the Snapshot of performance in mathematics, reading and science – PISA 2018 results

	Mean score in PISA 2018			Long-term trend: Average rate of change in performance, per three-year-period			Short-term change in performance (PISA 2015 to PISA 2018)			Top-performing and low-achieving students	
	Reading	Mathematics	Science	Reading	Mathematics	Science	Reading	Mathematics	Science	Share of top performers in at least one subject (Level 5 or 6)	Share of low achievers in all three subjects (below Level 2)
	Mean	Mean	Mean	Score dif.	Score dif.	Score dif.	Score dif.	Score dif.	Score dif.	%	%
OECD average	487	489	489	0	-1	-2	-3	2	-2	15,7	13,4
Italy	476	487	468	0	**5**	-2	-8	-3	**-13**	12,1	13,8

Source: OECD, PISA 2018 Database

If the total score achieved in Italy (487) is examined taking into account the gender difference, the final result represents a different scenario: the boys have achieved the main score of 494 corresponding to +5 with the respect of the OECD Average 489, but the girls - only a score of 479 with a decreasing of -10 below the OECD average. This factor underlines the complexity of the Italian situation where gender differences have a strong effect on the students' performance in mathematics.

Another important international survey is the TIMSS, developed by the International Association for the Evaluation of Educational Achievement (IEA) to allow participating nations to compare students' educational achievement. It is an international assessment of the mathematics and science knowledge of fourth-grade and eighth-grade students. This includes achievement in each of the content and cognitive domains as well as overall mathematics and science achievement.

The data, collected by last TIMSS survey (2015) on mathematics, show that the East Asian countries, in particular, Singapore, are top achievers at fourth and eighth grades in Mathematics. Italy has achieved an invariable average (507 and 494) in both at fourth and eighth grades in mathematics considering trends from 2011 to 2015. While countries like Finland and Germany show a lower average achievement at fourth grade in the same period.

However, if only the last collected data (2015) are considered, the results show two different situations for students at fourth grade and at eighth grade in Italy as represented in the two Figures1 and 2 below:

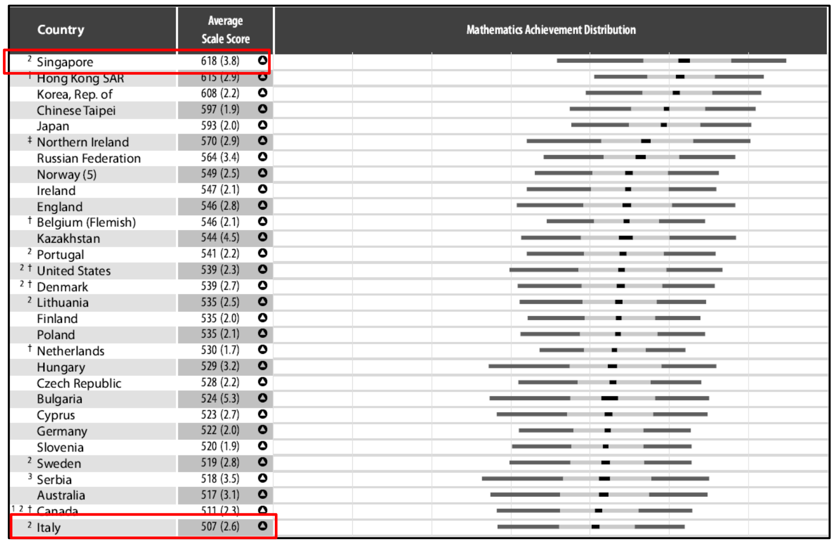

Figure 1: Extracted from the distribution of Mathematics Achievement at fourth grade
Source: IEA's Trends in International Mathematics and Science Study – TIMSS 2015

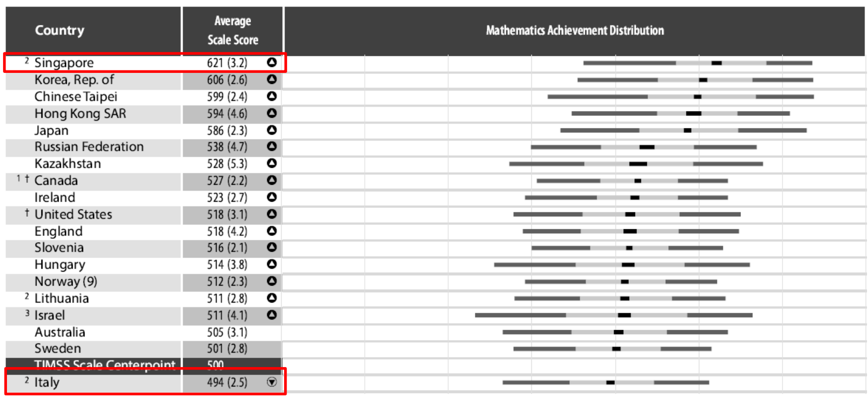

Figure 2: Extracted from the distribution of Mathematics Achievement at eighth grade
Source: IEA's Trends in International Mathematics and Science Study – TIMSS 2015

The Figures demonstrate that for fourth-grade students Italy has achieved +2.6 from the TIMSS Scale Centerpoint (500), while for eighth-grade students the results show still a negative situation -2.5 from TIMSS Scale Centerpoint (500). At the same levels, Singapore is placed at the top, reaching respectively 618 (+3.8) for fourth grade and 621 (+3.2) for eighth grade.

The gap between Asian countries and the next highest countries was 23 points in 2015, unchanged from 2011 at fourth grade, while at eighth grade the gap between Asian countries and the next highest countries was 48 points in 2015, increasing from 31 in 2011. However, with a few exceptions, like Norway, Belgium (at fourth grade) and Slovenia, Hungary (at eighth grade), European students often lack key basic competences in mathematics, science and technology.

In this context, most Ministries of Education share and agree on the educational objectives and results that are the base of these surveys by underlining the importance to encourage the development of the mathematics skills and their assessment through the national exams at the end of the compulsory education. In fact, on the base of these last results reached by the performances of the students, the European Ministers aim to reduce the average of students with difficulties in literacy, mathematics and science reaching a percentage lower than 15% within 2020 (from the section *News* in EU Commission web site).

In this direction, the Italian Ministry of Education has implemented different strategies as a result of a joint effort, carried out by government institutions with higher education partners or outside the education sector with the goals linked to broader educational aims than just improving student performance, such as:

- Promoting a positive feeling towards STEM subjects;
- Increasing public knowledge of STEM;
- Improving the teaching and learning of STEM in the school environment;
- Increase students' interest in STEM subjects and, consequently, incentivizing the choice of scientific careers at the upper secondary level and higher education level;
- Reinforcing gender equity in studies and professions related to mathematics, science and technology.

Therefore, the main objective of both the politics held and the current work is not only students' increasing scores but improving their performance starting from the implementation of strategies and methods which can improve the process of teaching and learning science by making it more effective through pointing on both intrinsic and extrinsic factors.

2.1.1 Substantial differences in mathematics performances between Western and Eastern countries

In 2018, students participated in the OECD-PISA international survey on mathematics, science and reading by providing an opportunity to make comparisons in student performance over time from 2009. On the base of the results achieved, the students from the following Asian countries were top-ranked: Singapore reached a score of 569 above the OECD average

489 followed by Hong Kong – China (551), Macau-China (558), Beijing-Shanghai-Jiangsu-Zhejiang (Regions of China participating in testing) (591).

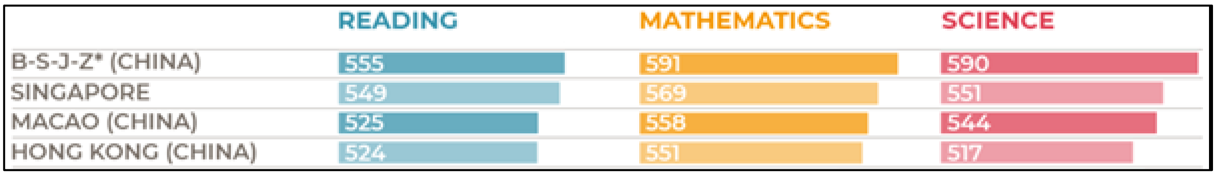

Figure 3: Extracted from the snapshot of students' performance in reading, mathematics and science

Source: OECD Education Statistics: PISA Results -2018

With the comparison of the previous PISA results, these countries confirmed their higher positions with reporting no differences between boys and girls. While Italian students' last performances in mathematics demonstrate a negative trend not overcoming the OECD average score by ranking at 31° place. Also in other countries, like Finland, even if still placed over the OECD average, the final ranking list underlines a negative trend concerning the previous results. In particular, in Finland, PISA data in 2006 indicates a negative relation between the performance mean score and the students' interest score: highest in science score, lowest in interest. This shows that the negative score in the students' interest has, over the years, led to a lowering of the performance score, losing in PISA survey of 2018 points.

An interesting factor, underlining the difference between Western and Asian students, emerging from these administered surveys, is the quality of relationships between teacher and student analyzed in the framework of survey *Do teacher-student relation affect student's well-being at school*. As a result, students in Belgium, Hong Kong China, Japan, Liechtenstein, Shanghai-China, Singapore, Switzerland, and Chinese Taipei have achieved a high score (above-average OECD) as well as above-average performances in mathematics.

The collected data show a strict relation between student well-being and their school performances. Indeed, the countries, such as China, at the top of the ranking, are still those where young students get the best scores in mathematics and language understanding tests. The results of the PISA survey show that positive and constructive teacher-student relations are a good ground for better performance in mathematics. Therefore this could be a key factor for fostering the social and emotional development of students at school.

Comparing countries and economies with two indicators, *teacher-student relation and mathematics performance*, shows that in some countries, where the development in a competitive, knowledge-based global economy is crucial, like Shangai-China, Singapore, Hong Kong-China and Chinese Taipei, students show high performance. On the contrary, countries, like Italy, France, Hungary, Bulgaria, Greece, are still below the OECD average.

Looking through the PISA results in 2015, another relevant factor comes out. In particular, the results related to students' science beliefs, engagement and motivation are interesting if the achievements in Asian countries are compared with some Western countries, such as Italy. High performances in Asian countries, e.g. Singapore, express a more positive and inclusive image of science in students. This encourages a positive attitude towards scientific careers and learning motivation enhanced.

While low performances, as in Italy, are strictly related to the creation of a mental negative image, stereotyped on who are the scientists and the engineers and on who are the people choosing this kind of career. Since science knowledge and understanding are useful even beyond the scientists' work and are necessary for full participation in a science-based technology world, school scientific topics should be promoted more positively by enhancing new resources and methodologies to increase student interest and enjoyment.

2.2 The existing approaches in mathematics study

The national and international investigations report that the learning process in mathematics becomes thwarting, exhausting and non-immediate for many students. This sense of discomfort gets worse by the fact that, at the social level, people think being good at math is synonymous to be smart. As a result, the failures in mathematics can affect student’s self-esteem. If success in mathematics requires special skills, then already first failures wouldn't foster in younger students the certainty of being able to improve. The vicious circle made by failures, low self-esteem, decreased motivation and reduced care can’t be solved without the support of a teacher, as an expert who can facilitate the reconstruction of a positive relationship with mathematics at school.

2.3 Student motivation in mathematics study

About 79% of students define mathematics as a big obstacle for their learning process. The difficulties revealed are often related to its being considered more abstract than the others. About 83% of students work using visual memory. This means that if we can imagine a history lesson as a film or a cartoon, It is hardly possible to do the same with a mathematics lesson. For instance, a lesson on inequalities hardly can activate visual memory as a literature subject does. At most, it can stimulate the photographic memory which is only 7% of the visual one.

Besides, the theory should be learned through doing exercises to understand the solution process by activating the procedural memory. This allows students to get skills through learning by doing.

Students often do not perceive a practical utility in mathematics study unless they study specific subjects, such as economics where the utility is directly evident. In short, most students don't have enough stimuli of being interested in mathematics.

Mathematics is not complicated by itself but must be studied differently from other disciplines. The suggested learning approaches usually are the following: full immersion, an association between image and concept or using memory techniques.

In case of the memory techniques, for example, the use of mnemonics can help students remember formulas or demonstrations by looking at them once because these stimulate the innate abilities of the student’s mind.

However, these learning methods can help students in a short-term study but they don’t seem effective if we think in long-term learning.

Another issue, which can impact the student motivation in mathematics study, is directly related to the teaching-learning process.

The traditional one is focused more on content mastery than on the learning abilities development and enhancement of research aptitude. The mainstream education system is oriented towards the teacher: the teacher gives and students receive. Therefore the student evaluation is related to *a correct or incorrect answer*. This education system is focused on school performance and results and it doesn't develop their lifelong learning capacity.

The didactic activities in the classroom can be realized in different ways by teachers. For example, by presenting, in the beginning, general principles on the topic to be studied, and then letting students deduce particular behaviours or on the contrary, by starting from phenomena, partial information, well-defined known behaviours up to building laws, processes, considerations, and general behaviour. This way to proceed is a part of the traditional teaching methodology, focused on the discipline and knowledge and the completeness of the message communicated¸ exploiting the learning deductive mode.

On the other hand, in a context of an active school, deductive teaching is replaced by the inductive modality, centered on the student's educational process involving his/her bio-psychological, socio-cultural and value structure.

Consequently, the factors, which can make inductive teaching interesting, regard the increased attention given to students, in terms of effective learning aiming to exploit and develop their motivation.

According to the motivational theories, motivation is a mainspring for the individual to act. If the individuals are driven by a strong reason, they will be able to: reach own objectives; develop a positive vision on their work/study; produce new energies needed to change; increase the self-esteem and their abilities; realize their personal and professional development helping others in this process. One of the factors that increase the motivation is the interest: motivating students means to arouse their interest in the study through a process of search and discovery of information that will promote student learning.

The meaningful learning aims at connecting a learning process to relevant concepts already owned, existing in the cognitive structure of the subject. This concept comes from the constructivist theory of knowledge, in other words, there is no knowledge without a process of meaning construction by the learner.

The learners construct their knowledge in meaningful contexts through the object's manipulation, tools and through the observation and interpretation of their actions' results. In these terms, meaningful learning becomes contextualized and complex. The students learn more and better if they cope with authentic tasks strictly connected to the real world, where they meet their *everyday life's real problems*.

The same thing occurs when a student approaches math studies. People often forget that reality, as well as all disciplines, as known by man, is based on mathematical concepts: math could be found in human beings, in architecture, plants, and animals, and as a part of them it regulates also their characteristics.

Despite this strong presence of mathematics concepts in the reality, their connection doesn’t appear so evident during the learning process and often teachers offer to students a too theoretical approach causing the perception that mathematics is abstract and far away from everyday life. This influences the way to learn mathematics privileging more storage capacity than problem-solving skills.

CHAPTER 3. LEARNING AND TEACHING APPROACHES: THE STATE-OF-THE-ART

During recent years, researchers have been identifying different approaches for mathematics learning and teaching to improve student performance. In a multiplicity of theories and methods, we considered only some of them, especially, the ones functional to the research objective, such as:

(i) *Cooperative Learning* - a method involving students divided into small groups working on an assignment or project under certain conditions. It refers to work done by students delivering a product, such as a set of problem solutions, a laboratory or project report, or the design of a product or a process.

(ii) *Problem-based Learning* - an instructional methodology aiming at learning to solve problems. The focus is on the problem which leads the student learning. Knowledge becomes a means not the final achievement by favouring the development of general skills such as critical thinking and abstract reasoning.

(iii) *Inquiry-based Learning in mathematics education* (IBME) - it refers to student-centred teaching, in which students have to observe phenomena, ask questions, look for mathematical and scientific ways how to answer these questions, such as carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalizations, interpreting and evaluating their solutions, and communicate and discuss with their school mates their solutions effectively. A fundamental part of the IBME method is the laboratory (e.g. GEOMLAND — a mathematical laboratory in Logo style).

(iv) *Technology-enhanced learning (TEL)* - refers to the support of teaching and learning through the use of technology. The technologies can become meaningful learning tools if students master the use of technologies; learn how to use them creatively, organize and represent what they know and learn; create products, solve problems connected real life, to reflect on contents and processes.

There are several examples of technology integration into the classroom, such as a *serious game, virtual reality, augmented reality, educational robotics* and, especially for mathematical topics, *virtual laboratories*.

3.1 Western and Eastern (Chinese) learning processes

The work of *Di Paola B.* and *Spagnolo F.,* on learning styles of mathematics comparing Italian and Chinese students, underlines how mathematics teaching/learning can't exclude a multicultural analysis regarding social, cultural, anthropological and geographical aspects of students.

This research, through an epistemological path of Chinese culture comparing with Western one, investigated the logical argumentation schemes used by lower and upper secondary school students in algebraic and geometry problem-solving tasks.

According to the collected data on the analysis of cognitive processes in a specific mathematics context (e.g arithmetic, algebra) with both Chinese and Italian students come from primary and secondary schools, Chinese students result to have highly pragmatic and concrete behaviour highlighted by procedural algorithmic reasoning to holistic thinking about coding and decoding of information in various situations/problems. Actually, during the mathematical argumentation and conjecturing, they often show a heuristic approach by trial and error, aimed at finding a *fundamental algorithm* as a proof tool. The same type of arithmetical reasoning is used by Italian students, but this tends more towards exploration.

For Chinese students, the argument and organization of reasoning take place through the hierarchization of reasoning models (models and sub-models assets and sub-sets) that don’t seem to refer to bivalent logic, but, by extensive use of the variability concept.

On the other side, the typical reasoning of Italian students is Aristotelian-Euclidean, which is hypothetical-deductive, expressed through a finite chain of conjunctions included in a bivalent logic process.

The arguments in a generalization process refer to algorithmic-procedural reasoning. Their good manipulative skills on the algebraic symbolism could result from the features of their ideographic written language that helps them in algebraic syntax control formalized.

This shows that the difference in behaviour between the two student styles come out, especially, from their language differences. Indeed, in Chinese students there is a constant balance between a serial, local thinking and global, holistic one, favoring cognitive categorizations and generalizations.

As a result, the constructive elements of the Chinese ideographic script may favor easier access to Algebra.

3.1.1 Singapore’s method to study mathematics

In 1980, the Institute for Curriculum Development (CDI) created the mathematics curriculum for primary school, later recognized as the "Singapore method". This system is based on learning studies of *Richard Skemp*, *Zoltan Dienes* and *Jerome Bruner*.

Singapore mathematics is not different mathematics, it is simply a method created by collecting the teacher experience and good practices realized during their professional development path.

Singapore mathematics supports the students to perceive this subject, not as a set of arbitrary rules or procedures but emphasizes the relational understanding among the parts. This concept derives from the study of Richard Skemp on the instrumental and relational learning where the author distinguishes between the ability to perform a procedure (*instrumental*) and the ability to explain the procedures (*relational*) which favor the development of two corresponding learning methods – *instrumental* and *relational*.

In detail, the main features of Singapore’s method are as follows:

(i) The concept of *variation*: teacher varies the lesson through a series of examples that deal with the same problem or topic. *Variation* can take the form of mathematical variability and perceptual variability. The first implies that students need to experience many variations linked to the concept structure, in order to single out the general mathematical concept which is constant to all manipulations. The second suggests that conceptual learning is maximized when children are exposed to a concept through a variety of physical contexts or embodiment. The final result of the problem's variations inclusion in a systematic way in the mathematics study is that the students become more aware of what they are learning.

(ii) The structure into three phases of the learning process - the first moves from the awareness of real objects (concrete or action-based - enactive representation) to proceed, then, towards the pictorial representations – second phase (image-based - iconic representation), and finally to the symbols (abstraction) – third phase.

(iii) The spiral approach of the curriculum which is built on the idea that when teaching new topics a constant revisiting of the basic ideas that were previously taught allows students to activate previously formed neural pathways. This, in turn, facilitates more effective understanding. Therefore, it is not important to teach the same topic again and again in the same manner but to bring the students to a higher level as a spiral. The adoption of Singapore's method invites the teacher to slow down the transition towards symbols during the teaching process, above all to avoid that the learner develops a gap among the knowledge stages, and then finds difficulty in the assimilation process.

(iv) Pursuing the best possible result without excessively extending the number of notions to be learned mechanically by the students. In fact, Singapore's national guidelines of 2006 invite teachers to develop the capacity of the learner in a relatively limited number of skills. This ensures more attention to the qualitative aspect over the quantitative in the teaching process.

(v) The student’s skills are developed and improved through a problem-based approach and not only with the exercises' mechanical repetitions. The problem-based approach can affect positively the student's cognitive and emotional abilities, arouses attention, and involves them in solving the problem by encouraging them to use their resources and to develop new skills.

CHAPTER 4. THE RESEARCH MODEL/APPROACH DEVELOPED

The reference theoretical framework of the proposal had a heuristic function and practice orientation by allowing guiding the tasks to be carried out up to the achievement of the results.

The main research model was adapted from the *Theory of Didactical Situations in Mathematics*, elaborated by *Guy Brousseau,* aiming to define the conditions under which "a *learner* is led to *do* mathematics, how to use it and how to invent it".

On the base of the constructivist approach, Brousseau defines three situations in the teaching process:

1. *A-didactical* situation is a context containing all the conditions which allow students to establish a relationship with a specific knowledge regardless of their teacher. This means that students' actions and answers depend on such relationship with the problem or the difficulty to be solved or overcome, even if this relation is no so explicit.
2. *Non-didactical* situation is the context that is not organized to allow students to learn specific knowledge. For example, all operations with natural numbers may be considered as a non-didactical situation.
3. *Didactical* situation is the one in which the situation is designed and well-organized to favor knowledge. It is enough to consider as didactical all the activities done in the classrooms where teacher intentionally "teaches" and students are forced to learn.

The three situations identified can be imagined like interaction systems between one or more subjects with a *milieu* that is a context or means. According to the theory of didactical situations, teachers have to encourage in their students a *behaviour,* which they should take independently to demonstrate their knowledge. For this reason, an important element is a *milieu* that teachers know well or have prepared by themselves for students.

To synthesize the different situations occurring, Brousseau refers to the didactical quadrangle where the four apices are constituted by Knowledge (in this case Mathematics), Teacher and Learner/Student and the *milieu* (Figure 4).

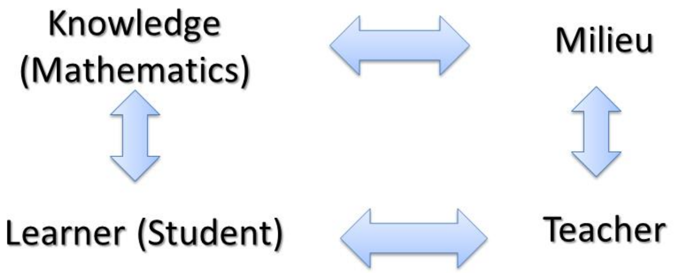

Figure 4: The didactical quadrangle of *Guy Brousseau*

In this constructivist theoretical framework, problem-solving is a fundamental tool to learn mathematics. This means that student's need to solve a problematic situation gives rise to a reflection action which, accordingly, becomes knowledge. Therefore, the encouragement, in mathematics teaching, of student's problem-solving ability can be a good means to favor learner's reflection and motivation.

The model proposed in this research work is the result of the integration of the three types of didactical situations identified by Brousseau. Therefore, the learning/teaching situation developed according to the three phases of the Eastern learning/teaching approach (concrete, pictorial and abstract), was *A-didactical*, *Non-didactical*, *Didactical* as follows:

1. *A-didactical*, because the students learned mathematical topics by discovering that different relationships exist among things or mathematical concepts (even if they cannot be so explicit) and by developing, accordingly, problem-solving skills avoiding the memorization of the solution procedure only.
2. *Didactical*, because several worksheets were prepared for the students before starting the experimentation phase. These contained instructions to lead the student from the concrete phase to pictorial up to the abstract one.
3. *Non-didactical*, because the teachers had the function to mediate and support the learning process through the creativity and the imagination of their students. The use of creativity from students was let free, especially when they produced their art-works on the base of mathematics concepts studied.

Therefore, re-considering the figure of *Didactical Quadriangle* of Guy Brousseau (Figure 4) from the point view of the proposed model, the art was identified as the context or better the *milieu* to be used to reach knowledge.

4.1 Arts as learning means in mathematics study

The introduction of arts, in the proposed approach, helps students connect better the mathematics to reality understanding its actual application in their everyday life.

As *Steiner and Schwarz* claim in education, it is necessary to have tools and methodologies that can guarantee an active involvement and creative inclusion of learners to let them test the interconnection of different languages, such as visual, sensory, verbal and nonverbal. This favors the development of both cognitive and emotional dimensions.

The artists of all ages have tried to create works that represent human beings and dimensions. But to do it in the best way, they had also to devote themselves to the studies of geometry and architecture. In arts, we find some mathematics concepts such as perspective and proportion. One of the well-known artists, who used mathematics as one of the most important components in his artworks, was the artist *Mauritius Cornelius Escher* (1898-1972). (Figure 5-6)

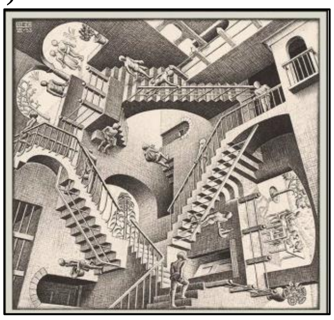

Figure 5: Relativity.

Figure 6: Gravity.

Source: http://www.mcescher.com/

In this context, art is a way for students to express their creativity and to find harmony in the development of both cognitive and emotional dimensions, but, at the same time, it obeys to mathematics rules and propositions. Similarly, mathematics reconstructs reality according to its laws getting an idealized replica of the subject. Consequently, art and mathematics are closely related.

Exploiting the potentialities of the interconnections between mathematics and art, students can discover how mathematics and scientific rules have an impact on various aspects of reality.

4.2 An Eastern learning approach: the Singapore method

The research approach was focused on the combination of the mathematics approaches set in West and East, between Western analytical methods and Eastern holistic approaches, namely Singapore's method.

The theoretical reference system under consideration could be schematized by referring to *Aristoteles* with the organization of bivalent logic characterizing the way of arguing in Western culture and therefore of our students, and to Confucius with Confucian method transmission of the Tao, and the book *I Ching*.

As *Leung* stated: *Aristoteles, more than any other thinker, determined the orientation and the content of Western intellectual history. He was the author of a philosophical and scientific system that through the centuries became the support and vehicle for both medieval Christian and Islamic scholastic thought: until the end of the 17th century, western culture was Aristotelian. And, even after the intellectual revolutions of centuries to follow, Aristotelian concepts and ideas remained embedded in Western thinking*.[1] For example, Euclidean geometry, the first structured language in mathematics history, is a model of Aristotelian bivalent logic.

In fact, Eastern researchers prefer, in general, to determine the relationships among things, such as among phenomena or objects following a different approach from Aristotelian hypothetical deductive one.

In other words, the aim is to favor a relational thought connected to a concrete arithmetic concept and to develop skills for the recognition of relationships among variables, to be able to work on them dynamically. This approach, mediated by the teacher, can promote in the student a stronger recognition of the relationship between syntax and semantics of arithmetic writing. For example, the theory of the elements is defined on the base of their function in a situation. This is analogous to the method mostly used in mathematics, according to which an object is determined from the relationships established with other objects.

Therefore, the mathematical reasoning results can be assimilated to a set of analysis and synthesis processes that evolve in the development of both abstraction and generalization

[1] Leung, F. K. S., Mathematics Education in East Asia and the West: Does Culture Matter? in Leung, F. K. S., Graf, K. D., & Lopez-Real, F. J. (Eds.). (2006). Mathematics education in different cultural traditions-A comparative study of East Asia and the West: The 13th ICMI study (Vol. 9) Springer Science &Business Media.

phases through so-called *variations* as well as the observation and recognition of invariant elements can give rise to generalizations through the search of algorithmic procedures unifying and contextualized in more fields. This process has been called by some mathematics teachers the *trilogy of problem-solving*.

In this meta-cognition *independent* process, the definitions, patterns of hypothetical-deductive reasoning demonstrations are not used. However, this is focused on the ability to review, rationally, the information and data provided by the problematic situation met; to learn how to categorize them concerning knowledge previously achieved and to choose the best ways for their representation and use to draw up concrete conclusions.

Generally, the Singapore approach is defined as the *Concrete-Pictorial-Abstract approach* (CPA) applying the so-called *Concrete-Representational-Abstract* (CRA) *strategies*. The *concrete* phase refers to a manipulative experience with real objects to understand how they work. During the *pictorial* phase, students learn how to transfer their knowledge acquired through the real objects into a mental image, into a diagram or drawing. Finally, the *abstract*, they learn to use mathematics symbols, such as *X* for multiplication.

Thanks to this transition, starting from concrete objects to pictures arriving then to symbols, the Singapore approach offers many and various opportunities to learn mathematical concepts, especially for those students who have difficulties. In this way, students have time to face mathematics concepts by testing many and different approaches to solve similar problems.

The main feature of this approach is the use of visual and model-drawing strategies which underline mathematics and word problems and leave out the memorization occurring through repetitive exercises. In effect, Singaporean textbooks use very often the visual aids like *strip models*, which are pictorial representations of mathematical problems requiring a basic arithmetic operation to be solved. These tools are used to help the young student learn to choose more appropriate arithmetic operations to solve a certain problem.

The approach aims to teach the student how to represent mathematics concepts visually before applying the abstract symbols like numbers and equations. This helps students make their connections and draw generalizations about the concepts learnt and not simply memorization of disconnected and isolated facts.

In this context, students experience different types of representations of the same concept by finding correspondences and relationships among them. This step conducts students to consolidate the conceptual understanding.

Inspired by the Singapore method, we combined the three phases (concrete, pictorial and abstract) with the Aristotelian method of logic deductive by using a particular tool: the art (Figure 7).

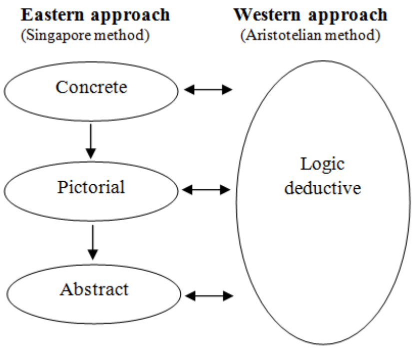

Figure 7: Combination between Singapore method and the Aristotelian method.

The introduction of the art to combine and join Eastern and Western learning/teaching approaches turns Figure 7 into the following diagram:

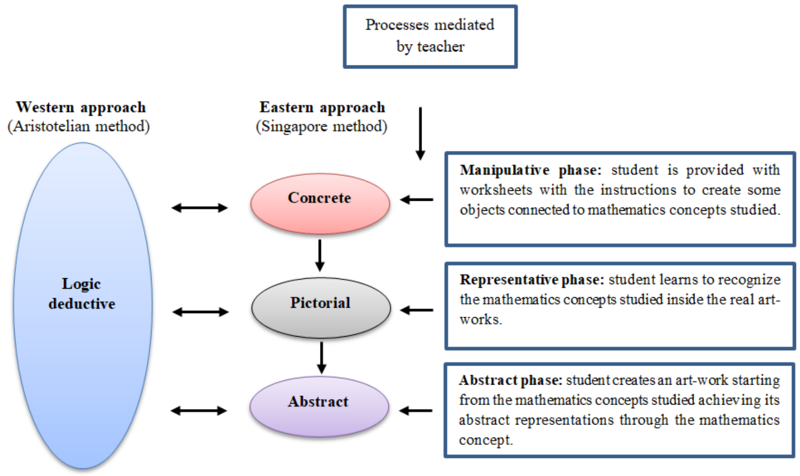

Figure 8: Representation of Western and Eastern approaches (Singapore's method) combined with art.

This procedure lets students face mathematics problems with variations, a fundamental element used systematically in Chinese mathematics teaching. In these types of problems, we have a constant element that can be represented by Arabic numbers as usually they are specified in the textbook or by the general mathematics concept to be studied while the context or other surrounded elements change. For instance, a student can learn to recognize the different symmetries working on several friezes and plan ornaments. In this example, the symmetry is the constant element and the ornament is the context changing.

This allows students to learn, analyze and to see the problem from different point views. Moreover, identifying the relationships among the different mathematics problems, they don't only memorize the resolution procedure, but also develop and reinforce problem-solving skills.

CHAPTER 5. EXPERIMENTATION AND EVALUATION

The preliminary phase of the experimentation included the following phases:

(i) Definition of the mathematics concepts, on the base of the school curriculum, and the selection of the art-works to be investigated by the students. Teachers and researcher worked together during this task.

(ii) Construction of the teaching/learning materials to be used during the experimentation activity. In particular, three worksheets, as guidelines, were constructed to provide the teachers with the instructions to proceed through the three phases concrete, pictorial and abstract. The instructions described the materials (such as paper with specific shape, pencils, and colours) and/or software applications (e.g. GeoGebra) to be used, the objective, the vocabulary, activity sequence structured on the base of the three phases (concrete, pictorial and abstract).

The experimentation phase divided into concrete – pictorial and abstract were mediated by teachers and/or the researcher who supported the students' study by introducing every phase and by stimulating them with questions and observations.

In detail, these three phases were implemented as follows:

(i) *Concrete phase* (Figure 9) - students were invited, for example, to build some objects (e.g. pentagram to study geometrical figures), figures (e.g. friezes, ornaments to study the symmetry) with instructions (provided by the teacher or the researcher) and the equipment available. The aim was to let students make the experience and become familiar with the mathematical concepts studied through object manipulation.

As agreed with teachers, GeoGebra application was used for this purpose for several reasons.

First of all, Italian teachers consider it (GeoGebra) user-friendly when used with low ICT skilled students on PC, Mobile, and Tablet.

Secondly, due to students' previous experiences with the software, they were able to concentrate directly on the subject to be explored without first being worried about a new tool to be learnt.

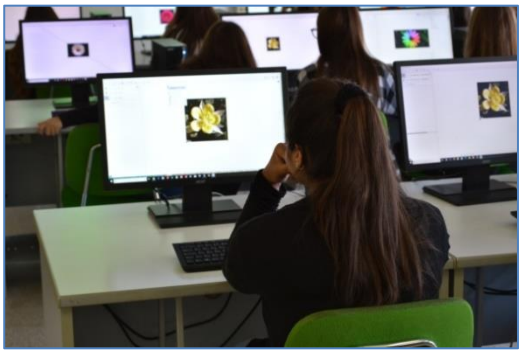

Figure 9: Experimentation time - Concrete phase with students at *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

(ii) *Pictorial (or also representative) phase*, students started their research through the finding of the art-works or the real-life application containing the mathematical concept or formula studied of the previous phase both individually and in team working. This helped students recognize the same concept in different contexts (and then in different art-works – Figure 10). This put their attention on the existing different representations of mathematics concepts.

During this phase, they used not only what is strictly known as the art-works, but also have found the mathematical concept in other contexts as well, e.g. in nature (Figure 10).

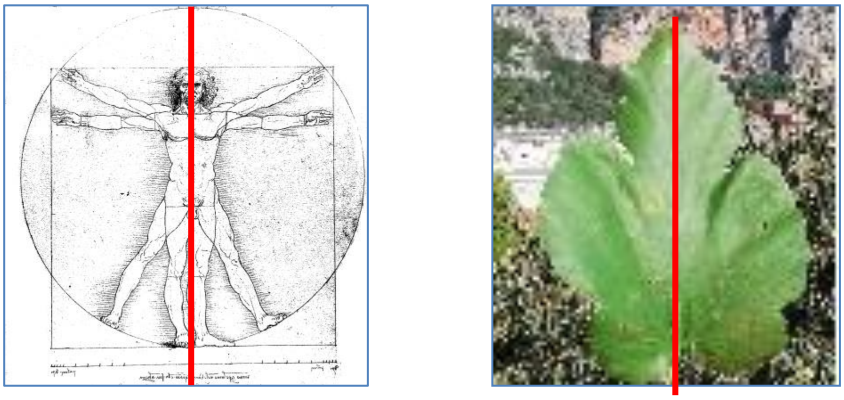

Figure 10: Experimentation time - Pictorial phase – artwork and object from nature selected for the symmetry study by students at Istituto Tecnico Superiore Bianchini in Terracina (Italy).

(iii) *Abstract phase* - students were invited to create their art-work starting from the mathematics concept studied in the two previous phases by achieving the abstraction representation of the mathematical concept (Figure 11).

Figure 11: Experimentation time - Abstract phase – artwork realized by a group of students from *Istituto Tecnico Superiore Bianchini* in Terracina (Italy).

(iv) Finally, a part of these works was uploaded in a 3D virtual museum, developed by *Institute for Computer Science and Control, Hungarian Academy of Science* in collaboration with *Institute of Mathematics and Informatics – Bulgarian Academy of Sciences.*

The virtual museum, *Mathematics and Arts*[2], was realized to show some of the final outputs of the research work, in particular, the objectives and model proposed by showing the structure and the tasks followed during the three-phases (concrete-pictorial-abstract).

Moreover, by clicking on the art-works exposed, users can get some information about the author(s), the object created and its relation to the real world, as well as the mathematics concept represented (Figure 12) and they can launch a video showing the key moments of the experimentation phase realized.

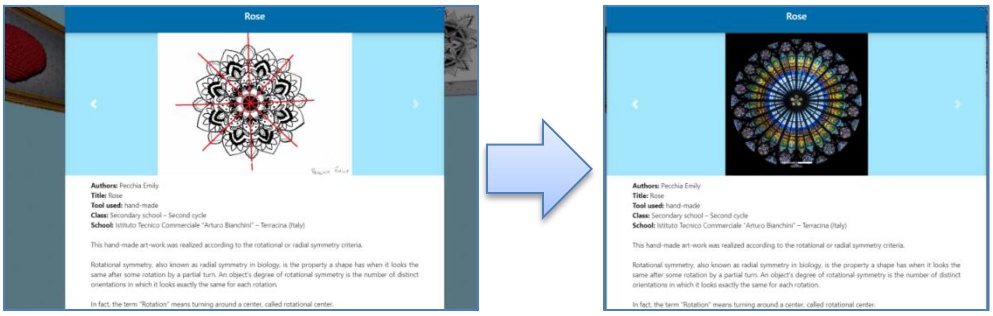

Figure 12: An example of the information card prepared per each art-work created by students and published in the virtual 3D Museum.

The <u>evaluation phase</u> included different stages:

(i) Research sample formation - 3 mathematics teachers of lower and upper secondary school and 3 art teachers and 130 secondary school students of different ages, 11-13 for the first cycle and 14-16 - the second one.

(ii) Qualitative and quantitative data collection - different tools were prepared and used to verify the effectiveness and the usability of the method – semi-structured questionnaires and grid for the observation participatory method and group discussion.

(iii) Coding and data processing – classification and management of the quantitative and qualitative data collected to analyze them on the base of relations among variables found.

[2] The virtual museum “Mathematics and Arts” is available at the link: http://files.elearning.sztaki.hu/Escape3D/Mathematics_and_Arts/Intro_HTML/index.html

CHAPTER 6. PRESENTATION OF RESEARCH RESULTS

The piloting phase was implemented with a sample of about 130 secondary school students of different ages, 11-13 years-old for the first cycle (44,62%) and 14-16 years-old the second one (55,38%) with the 43,1% male and the 56,9% female.

6.1 Student initial attitude towards Mathematics.

Usually, students define mathematics as a big obstacle for their study path, because the difficulties often revealed are related to its being considered more abstract than the other disciplines.

In particular, 57,70% of students stated that their major difficulty in mathematics study is that this discipline is too abstract than the other subjects.

This factor derives from the different nature of the topics to be studied in the mathematics curricula and/or from the teaching method applied in the classroom. Usually, in secondary school – first-grade teachers often use visual tools to explain and introduce scientific connections to their students. However, these methods not always work effectively if we consider the diversity of subjects that students should study at school. The students are expected to use a different kind of ability and capability to learn the lesson of histories, mathematics or other subjects.

Students, growing up, start to perceive mathematics as an abstract subject and for this reason most of them see this discipline as something far from reality and not easily applicable. Students often do not perceive a practical future utility in mathematics study unless they study specific subjects, such as economics where the utility is directly evident. The reason is that they often don’t have a stimulating overview of being interested in mathematics.

6.2 Students’ initial attitude towards the combination of Mathematics and Art

Another relevant element investigated with the first part of the questionnaire is the initial capability to perceive the connections between mathematics concepts and art, seen as art-works but also as real objects to be found in nature.

The data show that most of the students (82,23%) never thought that mathematics can be included in this unusual aspect, "art". Besides, a quite high percentage of students (23,80%) held a neutral position by underlining the difficulty to be able to see and comprehend the relation between mathematics and art.

However, 61,60% of students, intrigued by this new experience, consider this means, art, a way to support and make easier the mathematics learning (Figure 13).

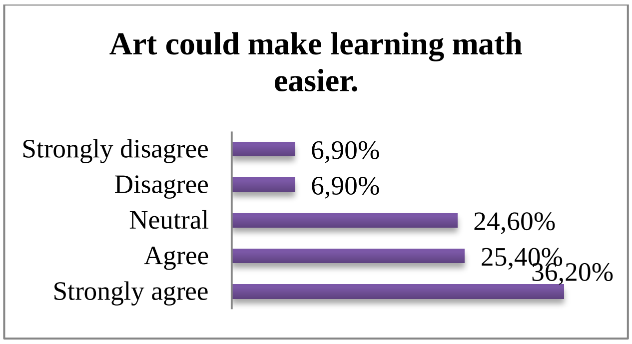

Figure 13: Art and its power to facilitate mathematics learning.

6.3 Students' attitude and perception of the experience

After the piloting phase, a second part of the questionnaire was submitted to the students. It was focused on understanding the impact of the carried out experience on students in terms of initial motivation, technology application in mathematics study and art as a supporting tool in mathematics study.

The 43,85% of the students were very motivated and interested in the participation in this new experience because, according to the qualitative feedback, they discovered something of *unexpected*, or, in other words, something of never have thought about.

For this reason, their curiosity pressed them during all the experience collaborating with other classmates by working in small groups.

Their participation was self-evaluated as very good at 55,38%, because they were actively involved in all the three phases, concrete-pictorial- abstract, by guaranteeing the finalization of the whole tasks without raising any relevant critical situation. This is confirmed by the Q3.3 which provides an overall judgment of their final experience with the 64,62% *Excellent*, 29,23% *Very good* and for 6,15 *Good* without receiving any negative feedback.

However, some of them, in particular, 4.1% (2.7% - Disagree and 1.4% - strongly Disagree) of girls have had some difficulties, mainly during the third phase, when students were expected to create their art-works. On the base of the results achieved boys seem to be more comfortable to use their creativity and to be original in the creation of their art-works (Table 3).

Table 2 - Discovering mathematics in art ...

	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**
I don't feel comfortable using this method.	1 0,77%	1 0,77%	10 7,69%	31 23,85%	87 66,92%
It was easy to create my artwork using the mathematical concept studied.	61 46,92%	42 32,31%	15 11,54%	8 6,15%	4 3,08%

130 valid cases

(*) The percentage calculation is made on the number of units in the single sub-simples. The first sub-sample (11-13 years old students) includes 29 girls and 29 boys. The second sub-sample (14-16 years old students) comprehends 45 girls and 27 boys.

The data described above show also that most of the students felt free to use their creativity and be original in their art-works production during the third phase (abstract). The characteristic trend revealed is in the positive increase from the first target group to the second one.

Another interesting factor to be underlined is the different attitude of students towards the misconception that young people have towards mathematics, considered as something abstract. At the end of the experimentation, students discovered that mathematics is not something abstract but it is related to a real-life application. Also, there is no relevant difference between girls and boys.

Comparing the answers gathered, despite an insignificant percentage of the *strongly disagree* option in girls' sample, the majority of respondents demonstrate a strong opinion that the method has enabled them to consider the mathematics concepts as more concrete and applicable to the reality.

Moreover, the main perception revealed is that the use of the art-works in the mathematics study favors the development of an enhanced learning setting enabling students to enjoy the learning process more to the traditional frontal lessons thanks to the exploitation of different languages, such as visual, graphical, verbal, non-verbal, representational and pictorial.

Concerning the contents learned, the survey and the exercise submission have demonstrated that students developed not only their knowledge in mathematics but also the reasoning process based on the applicability, imagination, creativity, and problem-solving skills. They learned to deal with the mathematical problem from different point views thanks to the use of the problem variations. In fact, the use of the different art-works in the study of the same mathematical concept allowed them to analyse it from a different perspective and learn logical reasoning bounded to the problem studied. This means that even if the art-works can change the context and the background, the mathematical concept, behind these works, is always recognized and become applicable more easily in the everyday life as shown in the table below.

Table 3 – Content learned

As for the content learned ...						
	Strongly agree	**Agree**	**Neutral**	**Disagree**	**Strongly disagree**	**I don't know**
The content was easily understood.	85,38% (111)	7,69% (10)	4,62% (6)	0,00% (0)	2,31% (3)	0,00% (0)
The contents learned seem to me more concrete and practical than before.	86,15% (112)	9,23% (12)	3,08% (4)	0,00% (0)	1,54% (2)	0,00% (0)
The content learned can support my study outside the classroom.	60,77% (79)	21,54% (28)	15,38% (20)	0,00% (0)	2,31% (3)	0,00% (0)

130 valid cases

The qualitative survey showed that students are usually not used to working together, but in most cases, they study individually.

Therefore, this experience showed that teamwork among students can be more motivating and interesting by making the performances more positive in terms of achieved results.

However, a small percentage of 13,85% preferred the first phase where they could use applications, like *GeoGebra*, to manipulate math concepts and objects. The reason was that they could use technological tools differently from the simple entertainment by making animation and playing around.

Looking at the results achieved to the question *Which phase of the activities did you like least?*, the 15,50% (against 3,10% for the second phase and the 7,75% for the third one) of students indicated the first phase (concrete) addressed to the object and concept manipulation related to mathematics study by using the technological applications. The reason was that in the first step of the method they didn't feel free to use their creativity and their work the way they did in the third phase.

Finally, not all the students (7,75%) feel comfortable to create something and to use their creativity, because, often they are not used to proceed on their own without being told exactly what to do.

6.4 Comparison of the results of pre and after the experimentation

Comparing the results of pre and after the piloting experience, the students' attitude (82,31%) towards mathematics has changed. Students discovered that different things can be connected with art as well as the beauty of the art can be derived from the mathematical concepts. Although a percentage of 12,31% remains "neutral" towards this discipline.

However, the 91,54% of the sample agree with the desire to go on using the arts in the study of mathematics, mainly with the proposed method (91,5%) because it could be a good incentive to improve and promote the study of mathematics.

In fact, most of the students (79,23% against the 16,92% with a neutral position) agree that the teaching method, generally used for the scientific topics, should be changed and diversified by using both the technology and arts to underline the real applicability of the scientific concepts.

CONCLUSION

This thesis aimed to analyse and exploit the possibility to find a combination between Western and Eastern approaches, in mathematics teaching using the arts. Starting from a study of the strengths and weaknesses of both Eastern and Western learning and teaching approaches, we designed a model exploiting the potentialities of both without leaving cultural differences out.

We adopted Singapore's method defined through three phases (concrete, pictorial, abstract) combined with the Western mathematics teaching implemented mainly in Italian schools.

The complex combination was created by using art.

The major result obtained is that students improved not only their understanding of the subject but they develop their creativity through their art-works. This increased the motivation and interest in the study of mathematics, on one hand, and meaningful improvement of student final performances, on the other hand.

Besides, the use of the artworks in the proposed learning and teaching approach offered students a way to go beyond the pure theory and to apply their knowledge in the surrounding world.

Moreover, the research work, through the collaboration of teachers from different subjects like art and mathematics, showed the importance of a novel interdisciplinary and multidisciplinary approach in the school curriculum needed to improve and develop, in this case, mathematics skills.

To achieve the research objectives described in this thesis, the foreseen tasks led to the following concrete results (Ref. Chapter 4, Chapter 5):

- Starting from the reference theoretical framework described in the "theory of Didactical Situations in mathematics" of Guy Brousseau, we designed an innovative model by integrating the three phases of the Singapore's method (concrete, pictorial and abstract) and art. (Ref. Chapter 3, Paragraph 3.1 – 3.2)
- Furthermore, we developed a model which was turned into a new didactic approach, including didactic materials, worksheets, and guidelines for the experimentation activity management. This aimed at valorizing mainly the problem-solving skills, the creativity and the imagination of the students without losing knowledge value related to the mathematics envisaged for the corresponding age. The documents prepared for the experimentation phase described and suggested also (i) the materials (such as paper with a specific shape, pencils and colours) and software applications (e.g. GeoGebra) to be used, (ii) the objectives, (iii) the vocabulary, (iv) the activity sequence structured on the base of the three phases (concrete, pictorial and abstract) to be carried out. (Ref. Chapter 4)
- We implemented the proposed model and approach in secondary schools in Italy. Therefore, we defined, furthermore, the reference sample and we designed the

evaluation and validation tools to verify its effectiveness and efficiency. This allowed us to reach the results discussed. (Ref. Chapter 4, Paragraph 4.2 – 4.3 – 4.4)

- We developed a 3D virtual museum "Mathematics and Arts" in collaboration of the Institute for Computer Science and Control, Hungarian Academy of Science and the Institute of Mathematics and Informatics – Bulgarian Academy of Sciences from Bulgaria, showing the art-works realized by some students during the experimentation phase; the real objects and the mathematics concepts they referred; the video describing the main phases of the proposed approach. (Ref. Chapter 4, Paragraph 4.1)

In the implementation of Task 1 (Study of the problem) we:

- analyzed the existing practices of the school approaches for mathematics topics in lower and upper schools by comparing the learning and teaching approaches applied to the mathematical subject between Eastern (Singapore's method) and Western methodologies. (Ref. Chapter 2, Paragraph 2.3 - 2.4)
- studied the relevant factors such as the current pedagogies on mathematics education through an in-depth study of the relevant bibliography available in the public database and specialized libraries. (Ref. Chapter 1, Chapter 2, Paragraph 2.1 - 2.3 - 2.4)
- analyzed the students and teachers attitude towards the introduction of innovation into school and in learning and teaching approaches. (Ref. Chapter 5).

In the implementation of Task 2 (Development of a research teaching/learning approach combined with the arts) we (involving directly the teachers):

- analyzed the exercises and mathematics topics from the school curricula to define the "mathematization" concept in terms of students' knowledge, skills and attitude. (Ref. Chapter 1, Chapter 3, Paragraph 3.2.1)
- selected the mathematics exercises and concepts and the artworks for the experimental activities with the students. (Ref. Chapter 4)
- prepared the worksheets and guidelines for the experimentation activity management into the class. (Ref. Chapter 3, Paragraph 3.2.1 - 3.2.2, Chapter 4)
- defined the main elements to be carried out the experimentation phase, in terms of the reference sample (teachers and students) and data collection tools (both qualitative and quantitative). (Ref. Chapter 4, Paragraph 4.2 - 4.3 - 4.4)

In the implementation of Task 3 (Experimentation phase) we (Ref. Chapter 4):

- organized the initial meetings with teachers and students to show different types of real connections between the artistic expression/creation and math patterns.
- submitted the first questionnaire and the mathematics exercises previously selected.
- carried out the experimentation activities in the Italian secondary schools by involving both teachers and students.

- organized final meetings, in the end, to let the students show their works (in every phase), to collect qualitative and quantitative data on their feedback and to submit a set of the mathematics exercises related to the topic studied.

In the implementation of Task 4 (Data processing and evaluation of the achieved results) we performed the processing data and the evaluation of the proposed model. (Ref. Chapter 5).

CONTRIBUTIONS

The most important contributions of the thesis are:

- A model and framework combining Western and Eastern (Singapore's method) teaching and learning approaches using the art have been developed: (i) the main Western learning and teaching approaches (with a special focus on the ones used in Italian schools) such as cooperative, problem-solving, inquiry-based, technology-enhanced learning (e.g. virtual laboratories in mathematics education, the use of the serious game, virtual reality and augmented reality, educational robotics) have been investigated; (ii) the main features of Eastern learning processes with special attention to Chinese methods and culture have been analyzed; (iii) the main features of Singapore's method have been explored and harnessed in mathematics classes (learning/teaching); (iv) the existing connections between mathematics and arts are explored.
- The reference theoretical framework (described in the *Theory of Didactical Situations in mathematics* by Guy Brousseau) has been adapted for an innovative didactic model through the integration of the three phases of the Singapore's method (concrete, pictorial and abstract) and art.
- The didactic materials have been developed such as worksheets and guidelines to manage the three phases in the classes during the experimentation activity. The aim was to provide both teachers and students with clear information about (i) the materials (such as paper with a specific shape, pencils and colours) and software applications (e.g. GeoGebra) to be used, (ii) the objectives, (iii) the vocabulary, (iv) the activity sequence structured on the base of the three phases (concrete, pictorial and abstract) to be carried out.
- The proposed model and approach have been implemented in several secondary schools in Italy by involving teachers of mathematics and arts: 130 secondary school students of different ages, 11-13 years-old for the first cycle and 14-16 years-old for the second one. The effectiveness and efficiency of the proposed approach have been analysed well as the relevance of the results achieved.
- A 3D virtual museum "Mathematics and Arts" has been developed in collaboration of the Institute for Computer Science and Control, Hungarian Academy of Science and the Institute of Mathematics and Informatics – Bulgarian Academy of Sciences, exhibiting the final outputs of the research work. Also, a video summarizing the activities carried out was produced and made available inside the 3D virtual museum.

DISSEMINATION OF THE RESULTS AND FUTURE WORK

The framework presented in this thesis has been exploited within Erasmus+ project n. 2018-1-FI01-KA201-047215 – *G.A. STEM - Enhancing STEM skills through arts and mini-games*, aiming to develop STEM skills in 13-16 years old students reinforced by the use of technology. In particular, it has inspired the project idea by supporting the development of the state-of-the-art concerning the relation between math and art.

Moreover, parts of the work done for the scope of the thesis have been published in several journals and conference proceedings (Ref. LIST OF THE AUTHOR'S PUBLICATIONS RELATED WITH PHD THESIS): International Education Conference Proceedings - EDULEARN 2017 (Barcelona, Spain, 2017); Central Bohemia University International Conference Proceedings – CBU 2017 (Prague, Czech Republic, 2017); Digital Presentation and Preservation of Cultural and Scientific Heritage—DiPP2017 (Burgas, Bulgaria, 2017); 12th International Technology, Education and Development Conference Proceedings, INTED2018, Valencia, Spain, 2018); Digital Presentation and Preservation of Cultural and Scientific Heritage—DiPP2018 (Burgas, Bulgaria, 2018); Technology, Education, Management Journal (TEM 2019); Digital Presentation and Preservation of Cultural and Scientific Heritage—DiPP2019 (Burgas, Bulgaria, 2019); 12th annual International Conference of Education, Research and Innovation – ICERI2019 (Seville, Spain, 2019).

The results obtained during the work on the thesis suggest that the study can be extended and developed further in the following areas:

- Theoretical direction: 1) Extension/adaptation of the developed approach to other scientific subjects, such as physics. 2) Extension/adaptation of the model through the integration of arts and scientific subjects with the designing of a mini-game conceptual idea (Ref. G.A. STEM Project).
- Applied/practical direction: Implementation and experimentation with other software (apart from GeoGebra) in the first phase (pictorial); Extension of the 3D Virtual Museum "Mathematics and Arts" by adding more topics and exercises.

LIST OF THE AUTOR'S PUBLICATIONS RELATED WITH THE TOPIC OF THE PHD THESIS

YEAR 2017

1. **Tramonti, M.**, Mathematics Education Reinforced through Innovative Learning Processes. Proceedings of the 9th International Conference on Education and New Learning Technologies, Barcelona, Spain, 2017, ISBN:978-84-697-3777-4, ISSN:2340-1117, DOI:10.21125/edulearn.2017.0744, 9279-9284, Available at: https://library.iated.org/view/TRAMONTI2017MAM (indexed in Web of Science)
   Note: published

2. **Tramonti, M.**, Paneva-Marinova, D., Pavlov, R., Math and Art Convergence for Education. Proceedings of CBU International Conference on Innovation in Science and Education, Prague, Czech Republic, 2017, ISSN:1805-997X, Online ISSN 1805-996, DOI:10.12955/cbup.v5.1037, 851-854, Available at: http://ojs.journals.cz/index.php/CBUIC/article/view/1037/pdf (indexed in Web of Science)
   Note: published

3. **Tramonti, M.**, Reinforcing Learning Setting through the Use of Digital Tools. Digital Presentation and Preservation of Cultural and Scientific Heritage. DiPP2017 Conference Proceedings, 7, 2017, ISSN:1314-4006, 159-167, Available at: http://dipp.math.bas.bg/images/2017/159-168_9_707_fDiPP2017-2_Tramonti.pdf
   (indexed in Scopus)
   Note: published

YEAR 2018

4. **Tramonti, M.**, Paneva-Marinova, D., Towards Improving Math Understanding Using Digital Art Library as a Source of Knowledge. Proceedings of the 12th annual International Technology, Education and Development Conference (IATED2018), Valencia, Spain, 2018, ISBN:978-84-697-9480-7, ISSN:2340-1079, DOI:10.21125/inted.2018.0517, 2751-2756, Available at: https://library.iated.org/view/TRAMONTI2018TOW (indexed in Web of Science)
   Note: published

5. **Tramonti, M.**, Technology and Art to Improve Mathematics Learning. Proceedings of the 12th annual International Technology, Education and Development Conference (IATED2018), Valencia, Spain, 2018, ISBN:978-84-697-9480-7, ISSN:2340-1079,

DOI:10.21125/inted.2018.0254, 1492-1497, Available at: https://library.iated.org/view/TRAMONTI2018TEC (indexed in Web of Science)
Note: published

YEAR 2019

6. **Tramonti, M.**, Paneva-Marinova, D., Maths, Art and Technology: a Combination for an Effective Study. TEM Journal, 8, 1, 2019, ISSN:2217-8309, DOI: 10.18421/TEM81-11, 82-86. SJR (Scopus):0.167,
Available at: http://www.temjournal.com/content/81/TEMJournalFebruary2019_82_86.html (indexed in Scopus and Web of Science)
Note: published

7. **Tramonti, M.**, Art and Science: Combining Learning Tool. Digital Presentation and Preservation of Cultural and Scientific Heritage, DiPP2019 Conference Proceedings, 9, 2019, ISSN:1314-4006, 145-152, Available at: http://dipp.math.bas.bg/images/2019/145-152_8_2.7_fDiPP2019-09_f_v.1.F_20190908.pdf (indexed in Scopus and Web of Science)
Note: published

8. **Tramonti, M.,** Mathematics and Science Study through the Arts. Proceedings of the 12th annual International Conference of Education, Research and Innovation (ICERI2019), Seville, Spain, 2019, ISBN:978-84-09-14755-7, ISSN:2340-1095, DOI:10.21125/iceri.2019.0518 1837-1842
Note: published

LIST OF CITATIONS

Tramonti, M., Mathematics Education Reinforced through Innovative Learning Processes. Proceedings of the 9th International Conference on Education and New Learning Technologies, Barcelona, Spain, 2017, ISBN:978-84-697-3777-4, ISSN:2340-1117, DOI:10.21125/edulearn.2017.0744, 9279-9284,

Citation (1):

Bawa, S. K., Kaushal, R., & Dhillon, J. K. (2020). Unification of Multimedia with Techniques of Art and Vedic Aphorisms for Development of Mathematical Skills: A Study of Indian and UK School Students.

Tramonti, M., Paneva-Marinova, D., Pavlov, R., Math and Art Convergence for Education. Proceedings of CBU International Conference on Innovation in Science and Education, Prague, Czech Republic, 2017, ISSN:1805-997X, Online ISSN 1805-996, DOI:10.12955/cbup.v5.1037, 851-854.

Citation (2):

Sabev, N., Georgieva-Tsaneva, G., & Bogdanova, G. (2018, September). Creating a Software System with Functionality to Help Make it Accessible for People with a Visual Deficit. In *CBU International Conference Proceedings* (Vol. 6, pp. 734-738).

Sabev, N., Bogdanova, G., & Hristov, P. (2019, September). Interactive System For Digital Presentation Of Cultural Routes And Specialized Collections. In *CBU International Conference Proceedings* (Vol. 7, pp. 996-1000).

Tramonti, M., Technology and Art to Improve Mathematics Learning. Proceedings of the 12th annual International Technology, Education and Development Conference (IATED2018), Valencia, Spain, 2018, ISBN:978-84-697-9480-7, ISSN:2340-1079, DOI:10.21125/inted.2018.0254, 1492-1497,

Citation (1):

Bawa, S. K., Kaushal, R., & Dhillon, J. K. (2020). Unification of Multimedia with Techniques of Art and Vedic Aphorisms for Development of Mathematical Skills: A Study of Indian and UK School Students.

Tramonti, M., Paneva-Marinova, D., Maths, Art and Technology: a Combination for an Effective Study. TEM Journal, 8, 1, 2019, ISSN:2217-8309, DOI: 10.18421/TEM81-11, 82-86. SJR (Scopus):0.167.

Citation (1):

Georgieva-Tsaneva, G. N. (2019). Innovative Methods in Medical Education in Bulgaria: Video Materials and Serious Games. *International Journal of Emerging Technologies in Learning (iJET), 14*(16), 165-171.

Tramonti, M., Reinforcing Learning Setting through the Use of Digital Tools. Digital Presentation and Preservation of Cultural and Scientific Heritage. DiPP2017 Conference Proceedings, 7, 2017, ISSN:1314-4006, 159-167.

Citation (1):

Dochshanov A. M. (2018) Multidisciplinary Roadmap for STEM education: a case study. Conference: 12th Annual International Technology, Education and Development Conference, pages: 2278-2282 - INT