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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

Directional Continuity and Upper Semi-Continuity in Differential Inclusions

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Directional Continuity and Upper Semi-Continuity in Differential Inclusions *

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Abstract

An existence theorem is proved for solutions of autonomous differential inclusions with an upper semicontinuous and nonconvex right-hand side. The proof is based on an inner and directional continuous parameterization. The solution is obtained as a limit of disturbed systems solutions. An example of differential equation with discontinuous right-hand side is considered.

Key words: multi-function, differential inclusion, directional continuity, upper and low semi-continuity.

AMS (MOS) subject classification. 34A60.

1 Introduction

This work considers autonomous differential inclusions. The first section presents a set of conditions, named the **Z condition**, is presented in the first section. These conditions guarantee the existence of solutions for the differential inclusions with an upper semi-continuous (u.s.c.) right-hand side. This part is closely related to the ideas of many authors, see f.e. [1] [2], [5 - 10], [12 - 14], [16 - 17]. The first part of the **Z condition** is equivalent to the existence of the fixed point of some map. This fixed point is is equivalent to the solution of the Euler's implicit scheme for numerical solving of differential equations, as well as the Yosida approximation of maximal monotone operators [2].

An example of autonomous differential equation with discontinuous right-hand side is considered in the second section. In abstract way, a low semicontinuous and left-hand side continuous scalar function is constructed by two continuous functions. Applying the existence theorem, the respective differential equation (or differential inclusion) admits a solution.

Consider the following differential inclusion:

$$\dot{x} \in F(x), \quad x(0) = x_0, \quad t \in [0,1], \tag{1}$$

where $x \in D \subset \mathbb{R}^n$, $F(x)$ is a upper semicontinuous (u.s.c.) multi-function with compact values, D is a bounded domain, $x_0 \in intD$.

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Notice that the existence of solutions of (1) is problematic when the right-hand side $F(x)$ is an upper semicontinuous multi-function with nonconvex values. There are simple examples of differential inclusions with above mentioned right-hand sides which have no solutions.

Solutions of differential inclusion (1) exist in the convex case (f.e. [1], [2], [4], [9–10], [14], [16], [17]) as well as in the continuous or low semicontinuous case (f.e. [5–7], [13]). There are few papers (f.e. [8]) with sufficient conditions for the existence of the solutions in nonconvex and u.s.c. case.

2 Main Result.

Definition 1 *Every absolutely continuous function $x(t)$ which almost everywhere in $[0,1]$ satisfies the differential inclusion (1) is said to be a solution.*

Definition 2 (see [6]) *Let Γ be a cone in $\mathbb{R}^m$ and le Y be a metric space. A map $f: \mathbb{R}^m \to Y$ is Γ-continuous at a point $\bar{x} \in \mathbb{R}^m$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(\bar{x})) < \varepsilon$ for all $x \in B(\bar{x}, \delta) \cap (\bar{x} + \Gamma)$. We say that f is Γ-continuous on a set A if f is Γ-continuous at every point $\bar{x} \in A$.*

Z condition:

1. For every $(s, x) \in [0, \bar{s}] \times intD$ there exists a vector $z(s, x)$ for which the following inclusion

$$z(s,x) \in F(x + sz(s,x)) + \omega(s,x)B \tag{2}$$

holds, where $\bar{s} > 0$ is any constant, B is the unit ball centered at the origin, $\lim_{s\to 0} \omega(s,x) = 0$, and $\omega(s,x)$ is a nonnegative scalar function which is measurable in s and continuous in x.

2. The multi-function $\{z| \quad z \in F(x + sz) + \omega(s,x)B\}$ has a selection $z(s,x)$ which is jointly measurable in (s,x) (or only in x) and directionally continuous at $(0,x)$ with a constant M (see [4], [6], [7]), i.e. for every $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon, x) > 0$ for which

$$\| z(0,x) - z(s,y) \| < \varepsilon \quad \text{if} \quad \| x - y \| < Ms, \quad 0 \le s < \delta(\varepsilon, x), \tag{3}$$

where M is some constant.

3. $\| F(x) \| + \omega(\cdot, x) \le L, \quad x \in D, \quad L < M.$

The first part of the **Z condition** holds for the u.s.c. multi-functions $F(\cdot)$ with compact and convex values. The prove is easy, applying the Kukutani fix point theorem. Continuous multi-functions $F(x)$ with compact values satisfy the first and the second parts of the **Z condition**.

Theorem 1 *Let the* **Z condition** *hold for the differential inclusion (1) and let the following equality*

$$z(s + \tau, x - \tau\, z(s,x)) = z(s,x), \quad s, \tau \ge 0 \tag{4}$$

be fulfilled for the selection $z(s,x)$ of (2).

Then there exists a solution of (1) which can be continued up to the bound of the domain $[0,1] \times D$.

If the differential inclusion (1) locally has a solution then, by traditional methods, it can be continued up to the bound of the domain $[0,1] \times D$. As long as $z(0,x) \in F(x)$, the above theorem 1 immediately follows from:

Theorem 2 *Let the bounded function $z(s,x)$ ($\| z(s,x) \| \leq L$) be jointly measurable in (s,x) (or only in x) and directionally continuous at $(0,x)$ with a constant M (see (3)). Let $L < M$ and the following equality be fulfilled*

$$z(s+\tau, x - \tau\, z(s,x)) = z(s,x), \quad s,\tau \geq 0. \tag{4}$$

Then the Cauchy problem

$$\dot{x} = z(0,x), \quad x(0) = x_0 \tag{5}$$

locally has a solution.

Proof. Let us denote

$$Z(s,x) = \operatorname*{ess\,lim}_{y \to x} z(s,y), \tag{6}$$

where $u \in \operatorname*{ess\,lim}_{y\to x} z(s,y)$ if for every set $N \subset \mathbb{R}^n$ with Lebesgue's measure $\mu(N) = 0$ there exists a sequence $\{y_k\}_{k=1}^{\infty} \notin N$ for which $\lim_{k\to\infty} y_k = x$ and $\lim_{k\to\infty} z(s,y_k) = u$. If the function $z(s,x)$ is jointly measurable in (s,x) then the multi-function $coZ(s,x)$ is the Filippov's extension of the right-hand side for the ordinary differential equations with a measurable right-hand side ([10], [11]). This function is u.s.c. in x, jointly measurable in (s,x), measurable in s and for any continuous function $x(s)$ the multi-function $Z(s,x(s))$ is measurable (see [10], [11]).

Consider the following differential inclusion:

$$\dot{x}(s,t) \in coZ(s,x(s,t)), \quad x(s,0) = x_0, \quad s > 0, \tag{7}$$

where $s > 0$ is a constant, $Z(s,x)$ is defined by (6) and co means the convex hull. It is wellknown that the differential inclusion (7) has a solution (see f.e. [10], [11]) which can be extended closely to the bound of the domain $[0,1] \times D$ (see [10]). As long as $x_0 \in D$ there exists $T > 0$ for which the solutions $x(s,t)$, $s \geq 0$ of (7) are well defined on the interval $[0,T]$.

The derivative $\dot{x}(s,t)$ of the solutions (7) a.e. can be represented as follows (see [11]):

$$\dot{x}(s,t) = \sum_{k=1}^{n+1} \alpha_k(s,t) z_k(s,t), \quad s > 0, \tag{8}$$

where $z_k(s,t)$ and $\alpha_k(s,t)$ are measurable functions on $[0,T]$, and a.e. in t

$$z_k(s,t) \in Z(s,x(s,t)), \quad \alpha_k(s,t) \geq 0, \quad \sum_{k=1}^{n+1} \alpha_k(s,t) = 1, \quad k = 1,2,\ldots,(n+1). \tag{9}$$

For fixed t which satisfies (8) we are going to estimate $\| \dot{x}(s,t) - z(0,x(0,t)) \|$, where $x(s,\cdot)$ uniformly converges to $x(0,\cdot)$ if $s \to +0$. As far as $z(s,x)$ is a bounded function with a constant L, the set of solutions of (7) is conditionally compact in the space of continuous functions $C[0,T]$. Thus, we can choose a subsequence which uniformly converges to some function $x(0,t)$.

Let $\varepsilon > 0$ be sufficiently small, for example $\varepsilon < \dfrac{M-L}{4}$. Let $\tau > 0$ be chosen under the **Z condition**, i.e.

$$\| z(\tau, y) - z(0, x(0,t)) \| \leq \varepsilon \quad \text{if} \quad \| y - x(0,t) \| < \tau M$$

As well as $z_k(s,t) \in Z(s, x(s,t))$, one can choose $y_k(s,t)$ which are sufficiently close to $x(s,t)$ such that $(i = k, j,\ i = 1, 2, \ldots, (n+1))$

$$\| z_k(s,t) - z(s, y_k(s,t)) \| \leq \varepsilon. \tag{10}$$

If $s > 0$ is sufficiently small we have

$$\| y_k(s,t) - \tau\, z(s, y_k(s,t)) - y_j(0,t) \| < \tau\,(L + \varepsilon) < \tau\, M.$$

Under the directional continuity of $z(s,x)$ and (4) we obtain

$$\| z(s, y_k(s,t)) - z(0, x(0,t)) \| = \| z(s+\tau, y_k(s,t) - \tau\, z(s, y_k(s,t))) - z(0, x(0,t)) \| < \varepsilon \tag{11}$$

We can write

$$\dot{x}(s,t) - z(0, x(0,t)) = \sum_{k=1}^{n+1} \alpha_k(s,t)\Big(z_k(s,t) - z(0, x(0,t))\Big) =$$

$$\sum_{k=1}^{n+1} \alpha_k(s,t)\Big(z_k(s,t) - z(s, y_k(s,t))\Big) + \sum_{k=1}^{n+1} \alpha_k(s,t)\,\Big(z(s, y_k(s,t)) - z(0, x(0,t))\Big)$$

By (9), (10) and (11) we have:

$$\| \dot{x}(s,t) - z(0, x(0,t)) \| < 2\varepsilon. \tag{12}$$

By (12), on the contrary, we obtain that a.e. in $t \in [0,T]$

$$\lim_{s \to +0} \dot{x}(s,t) = z(0, x(0,t)).$$

According Lebesgue's theorem, limiting s to $+0$, we obtain

$$x(0,t) = \lim_{s \to +0} x(s,t) = x_0 + \int_0^t \lim_{s \to +0} \dot{x}(s,\xi)\, d\xi = x_0 + \int_0^t z(0, x(0,\xi))\, d\xi$$

which is equivalent to

$$\dot{x}(0,t) = z(0, x(0,t)), \quad x(0) = x_0, \quad t \in [0,T].$$

Q.E.D.

3 Example

Consider the following illustration of theorem 1:

Let $f_1(x)$ and $f_2(x)$ be two continuous scalar functions which are defined on the real line $(-\infty, +\infty)$. Using these two functions one can construct some lower semicontinuous single-valued and left-hand side continuous function $f(x)$ for which

$$f(x) \in f_1(x) \cup f_2(x).$$

Suppose

$$|f_i(x)| \leq L < 1, \quad i = 1, 2. \tag{13}$$

Finally, denote

$$F(x) = \operatorname*{Lim\,sup}_{y \to x} f(y),$$

where Lim sup is the Kuratowski upper limit (see f.e. [3]). Applying theorem 1, we shall show that the differential inclusion

$$\dot{x} \in F(x), \quad x(0) = x_0 \tag{14}$$

has a solution. Obviously, it is sufficient to show that **Condition Z** is fulfilled. By (13) the third part of **Condition Z** holds. The first part of the **Z condition** follows from the following lemma, the proof of which is the same as the proof of lemma 5 [15].

Lemma 1 *If $g(z)$ is a lower semi-continuous and left-hand side continuous real function defined on the interval $[a, b]$ for which $g(a) > 0$ and $g(b) < 0$ then there exists $\xi \in (a, b)$ such that $g(\xi) = 0$.*

If $g(z) = f(x + sz) - z$ then for every sufficiently small s, $s \geq 0$, we have

$$\lim_{z \to -\infty} g(z) = \infty \text{ and } \lim_{z \to \infty} g(z) = -\infty.$$

Thus for every x and every sufficiently small s there exists a solution of the equation $z = f(x + sz)$ (respectively of the inclusion $z \in F(x + sz)$).

Note that the case $0 \in F(x)$ is trivial because the differential inclusion (14) has a solution $x(t) \equiv \text{const}$.

We need the following lemma:

Lemma 2 *Every real-valued function $g(x) : (-\infty, +\infty) \to R$ which is continuous on the some interval $[a, b]$ satisfies the second part of* **Condition Z** *at the point $(0, a)$ (respectively $(0, b)$) if $g(a) > 0$ (respectively $g(b) < 0$).*

Proof. We are going to prove only one part of the lemma (the case $g(a) > 0$) because the other part can be proved symmetrically. As well as $g(x)$ is continuous, for every $\varepsilon = g(a)/4$ there exists $\delta > 0$ for which $|g(x) - g(a)| < \varepsilon$ if $0 \leq x - a < \delta$. Choosing $M = g(a)/2$ and $0 \leq s < 4\delta/7g(a)$ consider the following equation:

$$z = g(y + sz), \quad |y - a| < sM, \quad \frac{3}{4}g(a) \leq z \leq \frac{5}{4}g(a). \tag{15}$$

As long as $-sg(a)/2 < s - a < sg(a)/2$ and $-g(a)/4 < g(y+sz) - g(a) < g(a)/4$ we have

$$0 < \frac{1}{4}sg(a) < y + sz - a < \frac{7}{4}sg(a) < \delta.$$

and

$$\frac{3}{4}g(a) < g(y+sz) < \frac{5}{4}g(a).$$

For instance from lemma 1, follows the existence of a solution $z(s,y)$ of the equation (15). Now, for every arbitrarily chosen $0 < \varepsilon \leq g(a)/4$ and its respective $\delta > 0$ we have

$$|z(s,y) - z(0,a)| = |g(y + sz(s,y)) - g(a)| < \varepsilon$$

if $|y - a| < sM$, where $0 \leq s < 4\delta/7g(a)$ because

$$0 \leq s\frac{g(a)}{4} = -sM + \frac{3}{4}sg(a) < y - a + sz(s,y) < sM + \frac{5}{4}sg(a) < \delta.$$

Q.E.D.

Note that $z(\lambda s, y)$ is directionally continuous with a constant λM. Thus we can suppose that $M > L$ at every point. As far as $g(\cdot)$ is single valued the condition (4) obviously holds. By the above lemma 2, $F(x)$ satisfies the second part of **Condition Z** if $f(\cdot)$ is continuous at x or $f(x) < 0$.

Let x be a point of discontinuity and $f(x) > 0$. In this case we have $F(x) = f_1(x) \cup f_2(x)$. As far as $f(\cdot)$ is lower semi-continuous we obtain that $0 < f(x) = \min\{f_1(x), f_2(x)\}$. For example let $f_1(x) < f_2(x)$. We shall show that $f(y) = f_2(y)$ on some interval $(x, x + \delta x]$. Choosing $\varepsilon < (f_2(x) - f_1(x))/2$ there exists $\delta > 0$ for which $|f_i(y) - f_i(x)| < \varepsilon$ $(i = 1,2)$ if $|y - x| < \delta$. There exists a sequence $\{x_k\}_{k=1}^{\infty}$, $\lim_{k\to\infty} x_k = x$, for which $x_k > x$, $f(x_k) = f_2(x_k)$ and $|x_k - x| < \delta$, $k = 1, 2, \ldots$. Consider the following numbers:

$$y_k = \sup\{y \in [x_k, x + \delta) \,|\, f(z) = f_2(z) \quad \forall\, z \in [x_k, y)\}.$$

As long as $f(\cdot)$ is a left-hand side continuous function, we have $f(y_k) = f_2(y_k)$. If $x_k \leq y_k < x + \delta$ then there exist $z_i > y_k$, $i = 1, 2, \ldots$, for which $\lim_{i\to\infty} z_i = y_k$. As well as $f(\cdot)$ is a lower semi-continuous function we obtain

$$f_2(x) - \varepsilon < f_2(y_k) = f(y_k) \leq \lim_{i\to\infty} f_1(z_i) = f_1(y_k) < f_1(x) + \varepsilon$$

or $f_2(x) - f_1(x) < 2\varepsilon$. This contradiction implies that $y_k = x + \delta$ for all $k = 1, 2, \ldots$. Thus $f_2(y) \in F(y)$ on the interval $[x, x + \delta]$. According to lemma 2, the multi-function $F(x)$ satisfies the second part of **Condition Z** as well as the equality (4) holds with a constant $M\,L < M$.

Applying theorem 1 we obtain that the differential inclusion (14) has a solution.

The existence of the function $z(s,x)$ (see (2)) without a directional continuity is not sufficient for the admission of solutions. In the paper [12] is presented a respective example.

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