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Commutants of the Euler Operator and Corresponding Mean-Periodic Functions

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Abstract

The Euler operator $\delta = t\dfrac{d}{dt}$ is considered in the space $C = C(\mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$, and the operators $M : C \to C$ such that $M\delta = \delta M$ in $C^1(\mathbb{R}_+)$ are characterized. Next, for a non-zero linear functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ the continuous linear operators M with the invariant hyperplane $\Phi\{f\} = 0$ and commuting with δ in it are also characterized. Further, mean-periodic functions for δ with respect to the functional Φ are introduced and it is proved that they form an ideal in a corresponding convolutional algebra $(C(\mathbb{R}_+), *)$. As an application the mean-periodic solutions of Euler differential equations are characterized.

Key words and phrases: commutant, Riesz-Markov theorem, invariant hyperplane, convolutional algebra, multiplier, cyclic element, mean-periodic function.

1 Introduction

Compared with the case of the differentiation operator $D = \dfrac{d}{dt}$ in a space C of continuous functions, the problem of characterizing the continuous linear operators $M : C \to C$ commuting with the Euler operator $\delta = t\dfrac{d}{dt}$, i.e. such that

$$M\delta = \delta M$$

in C^1, had not been so intensively treated as the corresponding problem for D. Here we can mention only the classical book of B. Ya. Levin [12], Ch. 8 and 9, Theorem 20, pp. 379-380, where δ is considered in spaces of entire functions.

In the operational calculus developed in Elizarraraz and Verde-Star [9] in fact some operators commuting with the Euler operator are found.

Due to the analogy of the considerations for δ and D, a short survey of the results for the differentiation operator will be made.

N. Bourbaki [1], Chapter 6, seems to be the first to characterize the linear continuous operators $M : C(\mathbb{R}) \to C(\mathbb{R})$ with $MD = DM$ in $C^1(\mathbb{R})$: These are the operators of the form

$$Mf(t) = \Phi_\tau\{f(t+\tau)\},$$

where Φ is a linear functional on $C(\mathbb{R})$. According to F. Riesz - A. Markov theorem ([8], Theorem 4.10.1) Φ has the form

$$\Phi(f) = \int_\alpha^\beta f(\tau)d\sigma(\tau)$$

where $-\infty < \alpha < \beta < \infty$ and $\sigma(\tau)$ is a Radon measure.

J. Delsarte [2] introduced the space of the mean-periodic functions determined by the functional Φ as the kernel space of M. For details see also L. Schwartz [13].

One of the authors (I. Dimovski [3]) had found the linear continuous operators $M : C(\mathbb{R}) \to C(\mathbb{R})$, such that the subspace $C_\Phi = \{f \in C(\mathbb{R}), \Phi(f) = 0\}$ is an invariant subspace of M and M commutes with D in C^1_Φ. It happened that these are the operators of the form

$$Mf = \mu f(t) + m * f,$$

where μ =const, $m \in C(\mathbb{R})$ and $*$ is the operation

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t+\tau-\sigma)d(\sigma)d\sigma \right\}.$$

Quite natural is the question about the relationship between the two types of commutants. A partial answer is given by the following theorem (Dimovski and Skórnik [6],[7]):

The space of the mean-periodic functions determined by the functional Φ forms an ideal in the convolutional algebra $(C(\mathbb{R}), *)$.

Similar results for the Pommiez operator $\Delta f(z) = [(f(z) - f(0)]/z$ are presented by the authors in [5].

An interesting historical survey about commutants of the differentiation operator and related operators like the Euler one can be found in the book [11] of Yu. F. Korobeinik.

2 General commutant

Theorem 1. *A linear continuous operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ commutes with $\delta = t\frac{d}{dt}$ in $C^1(\mathbb{R}_+)$ iff it admits a representation of the form*

$$(Mf)(t) = \Phi_\tau\{f(t\tau)\} \tag{1}$$

with a continuous linear functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$.

Proof: Consider the one-parameter family $T^\tau, 0 < \tau < \infty$, of the shift operators defined by

$$(T^\tau f)(t) := f(t\tau), \qquad 0 < \tau < \infty. \tag{2}$$

Each of them commutes with $\delta = t\frac{d}{dt}$ in $C^1(\mathbb{R}_+)$. Indeed,

$$(\delta T^\tau f)(t) = tf'(t\tau)\tau = t\tau f'(t\tau) = (\delta f)(t\tau) = (T^\tau \delta f)(t).$$

Lemma 1. *A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ commutes with $\delta = t\frac{d}{dt}$ in $C^1(\mathbb{R}_+)$ iff M commutes with T^τ for all $\tau, 0 < \tau < \infty$.*

Proof: First a "multiplicative" version of the Taylor formula is needed. Let f be a polynomial and g be the function defined by

$$g(x) = f(e^x).$$

Then

$$f(t\tau) = g(\ln(t\tau)) = g(\ln t + \ln \tau).$$

Denote $x = \ln t$ and $\xi = \ln \tau$, i.e. $t = e^x$ and $\tau = e^\xi$, and apply the usual Taylor formula for g:

$$f(t\tau) = g(x + \xi) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x)}{n!}\xi^n. \tag{3}$$

Then,

$$g'(x) = \frac{dg(x)}{dx} = \frac{dg(\ln t)}{dt}\cdot\frac{dt}{dx} = \frac{df(t)}{dt}\cdot\frac{de^x}{dx} = f'(t)e^x = tf'(t) = (\delta f)(t). \tag{4}$$

Further,

$$g''(x) = (\delta^2 f)(t), \ldots, g^{(n)}(x) = (\delta^n f)(t), \ldots \tag{5}$$

Substituting (4) and (5) in (3) gives the desired "multiplicative" Taylor formula:

$$(T^{\tau}f)(t) = f(t\tau) = \sum_{n=0}^{\infty}(\delta^n f)(t)\frac{(\ln\tau)^n}{n!}. \tag{6}$$

It is true for arbitrary polynomial $f(t)$.

Now suppose that M commutes with the Euler operator δ, i.e. $M\delta = \delta M$. Then, for every $\tau, 0 < \tau < \infty$, (6) implies

$$(MT^{\tau}f)(t) = M\sum_{n=0}^{\infty}(\delta^n f)(t)\frac{(\ln\tau)^n}{n!} = \sum_{n=0}^{\infty}(M(\delta^n f))(t)\frac{(\ln\tau)^n}{n!} =$$
$$= \sum_{n=0}^{\infty}(\delta^n(Mf))(t)\frac{(\ln\tau)^n}{n!} = (T^{\tau}Mf)(t).$$

In order to prove the opposite implication, suppose $MT^{\tau} = T^{\tau}M$ for every $\tau, 0 < \tau < \infty$, and for arbitrary polynomial $f(t)$, and reverse the order in the last chain of equalities as follows:

$$\sum_{n=0}^{\infty}(M(\delta^n f))(t)\frac{(\ln\tau)^n}{n!} = (M(T^{\tau}f))(t) =$$
$$= (T^{\tau}(Mf))(t) = \sum_{n=0}^{\infty}(\delta^n(Mf))(t)\frac{(\ln\tau)^n}{n!}.$$

The sums have to coincide for every τ and hence the coefficients of $(\ln\tau)^n$ are equal for arbitrary n . For $n = 1$ it follows that

$$(M(\delta f))(t) = (\delta(Mf))(t). \tag{7}$$

Assuming that (7) is true for polynomials, it follows that it is true for arbitrary $f \in C^1(\mathbb{R}_+)$ since f could be approximated by polynomials. The proof of the lemma is completed.

Proof of Theorem 1:

It is a matter of a direct check to show that the operators of the form (1) commute with δ and here only the proof of the necessity is needed.

If M commutes with δ, then by the lemma

$$MT^{\tau}f(t) = T^{\tau}Mf(t), \qquad 0 < \tau < \infty. \tag{8}$$

Applying the symmetry property

$$(T^{\tau}f)(t) = f(t\tau) = f(\tau t) = (T^{t}f)(\tau) \tag{9}$$

to the right hand side of (8) gives

$$(M(T^\tau f))(t) = (T^t(Mf))(\tau). \tag{10}$$

Define the linear functional Φ as

$$\Phi\{f\} := (Mf)(1).$$

Then, substituting 1 for t in (10) and taking into account that T^1 is the identity operator, one has

$$(M(T^\tau f))(1) = (T^1(Mf))(\tau) = (Mf)(\tau).$$

The left hand side is the value of the functional Φ for the function $g(t) = (T^\tau f)(t)$, and hence

$$(Mf)(\tau) = \Phi_\sigma\{(T^\tau f)(\sigma)\} = \Phi_\sigma\{(T^\sigma f)(\tau)\}$$

Using (2) and (9), this is in fact the desired representation (1) of the commutant of δ with τ for t, and with the dumb variable σ instead of τ. This completes the proof.

The abundance of the operators, commuting with δ in $C(\mathbb{R}_+)$ given by Theorem 1 is in sharp contrast to the set of linear operators commuting with δ in $C(\Delta)$, where Δ is a segment $[a,b] \subset \mathbb{R}_+$. Then the only such operators are the trivial ones:

$$Mf(t) = cf(t), \qquad c = \text{const.}$$

Such a result for the differentiation operator $\dfrac{d}{dx}$ is shown by C. Kahane [10]. The corresponding result for the Euler operator δ will be stated in the following theorem:

Theorem 2. *Let $[a,b] \subset \mathbb{R}_+$. Then a continuous linear operator $M : C[a,b] \to C[a,b]$, such that $M : C^1[a,b] \to C^1[a,b]$, commutes with the Euler operator δ in $C^1[a,b]$ if and only if it is an operator of the form*

$$Mf(t) = cf(t),$$

with a constant c.

Proof: Let $[a,b]$ be an arbitrary segment of $\mathbb{R}_+$ and let $M\delta = \delta M$ in $C^1[a,b]$. Consider the substitution $t = e^x$ as the transformation

$$Sf(t) = f(e^x) =: \widetilde{f}(x). \tag{11}$$

Obviously $S : C[a,b] \to C[\ln a, \ln b]$ and $S : C^1[a,b] \to C^1[\ln a, \ln b]$. Then, denoting $D := \frac{d}{dt}$, one has as in (4)

$$S\delta f(t) = f'(e^x) = DSf(t). \tag{12}$$

It is supposed that

$$M\delta f(t) = \delta M f(t).$$

Applying S on the left and using (12) yields

$$SM\delta f(t) = S\delta M f(t) = DSMf(t). \tag{13}$$

Denoting by $\widetilde{M}$ the operator

$$\widetilde{M} = SMS^{-1}. \tag{14}$$

it is easily seen from (13) and (12) that

$$\widetilde{M} D\widetilde{f}(x) = D\widetilde{M}\widetilde{f}(x). \tag{15}$$

This means that the conditions of Kahane's theorem ([10]) are fulfilled for the operator $\widetilde{M}$ in $C[\ln a, \ln b]$ and the result is that

$$\widetilde{M}\widetilde{f}(x) = c\widetilde{f}(x), \qquad c = \text{const},$$

which in view of (11) and (14) gives also the desired

$$Mf(t) = cf(t), \qquad c = \text{const}.$$

3 A general convolution related to the Euler operator

Basic for the theory of the differentiation operator $\frac{d}{dt}$ considered in a space $C(\Delta)$ of continuous functions on an interval Δ is the operation

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t + \tau - \sigma) g(\sigma) d\sigma \right\}, \tag{16}$$

where Φ is a linear functional on $C(\Delta)$. Its properties are studied in details in [4]. The operation (16) is bilinear, commutative, and associative in $C(\Delta)$. It generalizes the classical Duhamel convolution

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau \tag{17}$$

when the functional Φ in (16) is $\Phi(f) = f(0)$.

In [3] it is shown that any operator of the commutant of $\dfrac{d}{dt}$ in $C(\Delta)$ with an invariant hyperplane $C_\Phi(\Delta) = \{f \in C(\Delta), \Phi(f) = 0\}$ has the form $Mf(t) = \mu f(t) + (m * f)(t)$ with $\mu = \text{const}$ and $m \in C(\Delta)$.

In order to extend this result to the Euler operator an analogue of the operation (16) is needed. In the literature only the analogue

$$(f * g)(t) = \int_1^t f\left(\frac{t}{\tau}\right) g(\tau)\frac{d\tau}{\tau}$$

of the Duhamel convolution (17) is known (see [9]).

Definition 1. *The analytic function*

$$E(\lambda) = \Phi_\tau(\tau^\lambda) \tag{18}$$

is said to be the Euler indicatrix of the functional Φ.

It is also convenient to denote for the rest of the paper

$$\varphi_\lambda(t) = \frac{t^\lambda}{E(\lambda)} = \frac{t^\lambda}{\Phi_\tau(\tau^\lambda)}. \tag{19}$$

Here a "multiplicative variant" of (16) is proposed:

Theorem 3. *Let* Φ *be a continuous non-zero linear functional on* $C(\mathbb{R}_+)$. *Then the operation*

$$(f * g)(t) = \Phi_\tau\left\{\int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma)\frac{d\sigma}{\sigma}\right\} \tag{20}$$

is a separately continuous, bilinear, commutative, and associative operation in $C(\mathbb{R}_+)$ *such that*

$$\Phi(f * g) = 0. \tag{21}$$

Proof: According to Riesz-Markov theorem ([8], Theorem 4.10.1)

$$\Phi\{f\} = \int_\alpha^\beta f(\tau)d\sigma(\tau)$$

with $\Delta = [\alpha, \beta] \subset \mathbb{R}_+$ and a Radon measure $\sigma(t)$. Hence (20) is a separately continuous operation in $C(\Delta)$.

The bilinearity and the commutativity of the operation (20) are almost evident, while the associativity of (20) is by no means obvious and needs a proof.

Let $f(t) = t^\mu$ and $g(t) = t^\nu$. Then

$$\begin{aligned} \{t^\mu\} * \{t^\nu\} &= \Phi_\tau\left\{\int_\tau^t \frac{(t\tau)^\mu}{\sigma^\mu}\sigma^\nu \frac{d\sigma}{\sigma}\right\} = t^\mu \Phi_\tau\left\{\tau^\mu \int_\tau^t \sigma^{\nu-\mu-1}d\sigma\right\} = \\ &= t^\mu \Phi_\tau\left\{\tau^\mu \frac{t^{\nu-\mu} - \tau^{\nu-\mu}}{\nu - \mu}\right\} = \frac{E(\mu)t^\nu - E(\nu)t^\mu}{\nu - \mu}. \end{aligned}$$

Using this expression, it follows that

$$(\{t^\mu\} * \{t^\nu\}) * \{t^\varkappa\} = \{t^\mu\} * (\{t^\nu\} * \{t^\varkappa\}) \tag{22}$$

because both sides of (22) have one and the same symmetric form

$$t^\mu \frac{E(\nu)E(\varkappa)}{(\mu-\nu)(\mu-\varkappa)} + t^\nu \frac{E(\varkappa)E(\mu)}{(\nu-\varkappa)(\nu-\mu)} + t^\varkappa \frac{E(\mu)E(\nu)}{(\varkappa-\mu)(\varkappa-\nu)}.$$

with respect to μ, ν, and $\varkappa$. Then, (22) differentiated m, n, and k times with respect to μ, ν, and $\varkappa$ correspondingly, gives

$$(\{t^\mu(\ln t)^m\} * \{t^\nu(\ln t)^n\}) * \{t^\varkappa(\ln t)^k\} = \{t^\mu(\ln t)^m\} * (\{t^\nu(\ln t)^n\} * \{t^\varkappa(\ln t)^k\}).$$

Next, passing to the limits $\mu \to +0, \nu \to +0$ and $\varkappa \to +0$, one gets

$$(\{(\ln t)^m\} * \{(\ln t)^n\}) * \{(\ln t)^k\} = \{(\ln t)^m\} * (\{(\ln t)^n\} * \{(\ln t)^k\}).$$

But the bilinearity of (20) implies for arbitrary polynomials P, Q and R

$$(\{P(\ln t)\} * \{Q(\ln t)\}) * \{R(\ln t)\} = \{P(\ln t)\} * (\{Q(\ln t)\} * \{R(\ln t)\}).$$

To finish the proof, note that if $t \in \mathbb{R}_+$, then $\ln t$ covers the whole real line $\mathbb{R}$. Then Weierstrass' theorem allows any function in $C(\mathbb{R}_+)$ to be approximated almost uniformly by polynomials of $\ln t, t > 0$, i.e. by a sequence uniformly convergent to the function on each segment $[a, b] \subset \mathbb{R}_+$. Due to the continuity of the functional Φ the desired equality holds for every $f, g, h \in C(\mathbb{R}_+)$

$$(f * g) * h = f * (g * h).$$

The second statement (21) of the theorem can fe checked as follows: The function

$$h(t,\tau) = \int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma)\frac{d\sigma}{\sigma}$$

is antisymmetric with respect to t and τ, i.e. $h(t,\tau) = -h(\tau,t)$, and, hence

$$\begin{aligned} \Phi\{f*g\} &= \Phi_t\{(f*g)(t)\} = \Phi_t\Phi_\tau\{h(t,\tau)\} = \\ &= \Phi_t\Phi_\tau\{-h(\tau,t)\} = -\Phi_t\Phi_\tau\{h(\tau,t)\} = \\ &= -\Phi_\tau\Phi_t\{h(\tau,t)\} = -\Phi_t\Phi_\tau\{h(t,\tau)\} = -\Phi\{f*g\}. \end{aligned} \tag{23}$$

Here the Fubini property of the functional Φ is used , i.e. the possibility of interchanging of Φ_t and Φ_τ. At the end, t and τ are also interchanged, since they are "dumb" variables in the expression. Thus the last chain of equalities gives $2\Phi\{f*g\} = 0$ and $\Phi\{f*g\} = 0$ holds.

4 The commutant of δ in an invariant hyperplane

In this section another commutant of δ will be described. Here it is supposed that the operators $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ preserve $C^1(\mathbb{R}_+)$, i.e. $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$, and additionally they preserve invariant also a hyperplane

$$C_\Phi := \{f \in C(\mathbb{R}_+) : \Phi\{f\} = 0\}, \tag{24}$$

i.e. $M : C_\Phi \to C_\Phi$, where $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ is an arbitrary non-zero linear functional.

The main result of this section is the explicit representation $Mf = \mu f + m * f$ of any linear continuous operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C_\Phi \to C_\Phi$ and commutting with $\delta = t\dfrac{d}{dt}$ in $C^1_\Phi := C_\Phi \cap C^1(\mathbb{R}_+)$.

To this end some auxilliary results will be considered.

Lemma 2. *A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ and $M : C_\Phi(\mathbb{R}_+) \to C_\Phi(\mathbb{R}_+)$, commutes with the Euler operator δ in $C^1_\Phi(\mathbb{R}_+)$ iff M commutes with L_λ in $C(\mathbb{R}_+)$, where L_λ is the right inverse in $C(\mathbb{R}_+)$ of the perturbed Euler operator $\delta_\lambda = \delta - \lambda$, satisfying the boundary condition $\Phi(L_\lambda f) = 0$.*

Proof: First an explicit expression for L_λ will be found. Let λ be such that $E(\lambda) \neq 0$. Then

$$L_\lambda f(t) = \int_1^t \left(\frac{t}{\tau}\right)^\lambda f(\tau)\frac{d\tau}{\tau} - \frac{t^\lambda}{E(\lambda)}\Phi_\tau\left\{\int_1^\tau \left(\frac{\tau}{\sigma}\right)^\lambda f(\sigma)\frac{d\sigma}{\sigma}\right\}. \tag{25}$$

Indeed, the general solution of the linear differential equation $t\dfrac{dy}{dt} - \lambda y = f(t)$ is $y = t^\lambda\left(c + \displaystyle\int_1^t \frac{f(\tau)}{\tau^{\lambda+1}}d\tau\right)$ with an arbitrary constant c. Then, using the condition $\Phi\{y\} = 0$, one obtains the value

$$c = -\frac{t^\lambda}{E(\lambda)}\Phi_\tau\left\{\int_1^\tau \left(\frac{\tau}{\sigma}\right)^\lambda f(\sigma)\frac{d\sigma}{\sigma}\right\}.$$

Now suppose that $ML_\lambda = L_\lambda M$ in $C(\mathbb{R}_+)$ and $f \in C^1_\Phi(\mathbb{R}_+)$. To prove that

$$h = (M\delta_\lambda - \delta_\lambda M)f = 0,$$

consider

$$L_\lambda h = L_\lambda M\delta_\lambda f - L_\lambda\delta_\lambda Mf = M(L_\lambda\delta_\lambda)f - (L_\lambda\delta_\lambda)Mf) = Mf - Mf = 0.$$

But $L_\lambda h = 0$ implies $\delta_\lambda L_\lambda h = 0$, i.e. $h = 0$. Hence $M\delta_\lambda f = \delta_\lambda Mf$.

Conversely, let $M\delta_\lambda f = \delta_\lambda Mf$ for every $f \in C^1_\Phi(\mathbb{R}_+)$. If $g \in C(\mathbb{R}_+)$, then there is a function $f \in C^1_\Phi(\mathbb{R}_+)$, for which $f = L_\lambda g$. After the substitution $f = L_\lambda g$ in $\delta_\lambda Mf = M\delta_\lambda f$, one gets

$$\delta_\lambda(ML_\lambda g) = M\delta_\lambda L_\lambda g = Mg.$$

Since $\Phi\{L_\lambda g\} = 0$, then $\Phi\{ML_\lambda g\} = 0$. But the solution of the equation $\delta_\lambda y = Mg$ with the condition $\Phi\{y\} = 0$ by definition is $y = L_\lambda(Mg)$, which implies

$$ML_\lambda g = L_\lambda Mg$$

in $C(\mathbb{R}_+)$, which completes the proof.

Lemma 3. *The operator L_λ given by (25) is a convolution operator of the form*

$$L_\lambda f = \varphi_\lambda * f = \left\{\frac{t^\lambda}{E(\lambda)}\right\} * f. \tag{26}$$

Proof: The equality (26) can be checked directly using (21) and the representation $\int_\tau^t = \int_1^t - \int_1^\tau$.

Theorem 4. *The commutant of δ in the invariant hyperplane C_Φ coincides with the commutant of any of the operators L_λ in $C(\mathbb{R}_+)$.*

Proof: Let $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be a linear operator commuting with L_λ for some $\lambda \in \mathbb{C}$, i.e. $ML_\lambda = L_\lambda M$. First, it will be proved that C_Φ is an invariant hyperplane for M. Indeed, let g be a function from $C(\mathbb{R}_+)$ and f be the solution of the problem

$$\delta f - \lambda f = g, \qquad \Phi\{f\} = 0. \tag{27}$$

Then

$$L_\lambda M g = M L_\lambda g = M f \tag{28}$$

and hence

$$Mg = (\delta - \lambda) M f.$$

Using (27) this can be written as

$$M(\delta - \lambda) f = (\delta - \lambda) M f$$

or, equivalently,

$$(M\delta) f = (\delta M) f.$$

It remains to show that $\Phi\{Mf\} = 0$. This follows using (28) and the representation (26) of L_λ as a convolutional operator, along with the property $\Phi\{p * q\} = 0$ for arbitrary $p, q \in C(\mathbb{R}_+)$ of the convolution (20).

Conversely, let $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ have the hyperplane C_Φ as an invariant subspace and let $M\delta = \delta M$ in C^1_Φ. One has to prove that $ML_\lambda = L_\lambda M$ for $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$.

Let $f \in C(\mathbb{R}_+)$ be arbitrary and denote $h = (ML_\lambda - L_\lambda M) f$. Then

$$(\delta - \lambda) h = (\delta - \lambda) M L_\lambda f - M f = M(\delta - \lambda) L_\lambda f - M f = 0$$

and also

$$\Phi\{h\} = \Phi\{M L_\lambda f\} - \Phi\{L_\lambda M f\} = 0,$$

according to our assumptions. Since λ is not an eigenvalue, i.e. $E(\lambda) \neq 0$, then $h = 0$, or

$$M L_\lambda f = L_\lambda M f.$$

The proof is completed.

Definition 2. *A linear operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is said to be a multiplier of the convolutional algebra $C(\mathbb{R}_+), *)$ when for arbitrary $f, g \in C(\mathbb{R}_+)$ it holds*

$$M(f * g) = (Mf) * g.$$

Theorem 5. *A linear operator* $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ *with* $M : C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ *is a multiplier of the convolution algebra* $(C(\mathbb{R}_+), *)$ *iff it has a representation of the form*

$$Mf(t) = \mu f(t) + (m * f)(t), \tag{29}$$

where $\mu = \mathrm{const}$ *and* $m \in C(\mathbb{R}_+)$.

Proof: The sufficiency is obvious. In order to prove the necessity, the notations from (18) and (19) will be used for convenience.

Let $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be an arbitrary multiplier of $(C(\mathbb{R}_+), *)$. Applying (26), one has

$$ML_\lambda f = M(\varphi_\lambda * f) = (M\varphi_\lambda) * f = \varphi_\lambda * Mf = L_\lambda Mf, \tag{30}$$

i.e. $ML_\lambda f = L_\lambda Mf$. Also, denoting $e_\lambda = M\varphi_\lambda$, one has $e_\lambda \in C^1(\mathbb{R}_+)$, and (30) gives

$$L_\lambda Mf = e_\lambda * f.$$

It remains to apply the operator $\delta_\lambda = \delta - \lambda$ and the definition of L_λ as the right inverse operator of δ_λ to obtain

$$Mf = \delta_\lambda(e_\lambda * f).$$

The right hand side can be represented in a different way using the identity

$$\delta_\lambda(u * v) = (\delta_\lambda u) * v + \Phi(u)v \tag{31}$$

which can be checked directly. Then

$$(Mf)(t) = [(\delta_\lambda e_\lambda) * f](t) + \Phi(e_\lambda)f(t),$$

which is the representation (29) with $\mu = \Phi(e_\lambda) = \Phi\{M\varphi_\lambda\}$ and $m(t) = (\delta_\lambda e_\lambda)(t) = [\delta_\lambda M\varphi_\lambda](t)$. Thus, the necessity is proved.

Theorem 6. *The function* $\varphi_\lambda(t) = \dfrac{t^\lambda}{E(\lambda)}$ *is a cyclic element of the operator* L_λ.

Proof: Let $f \in C(\mathbb{R}_+)$ be arbitrarily chosen. It is needed to prove that there is a sequence of functions of the form

$$f_n(t) = \sum_{k=0}^{n} c_{nk} L_\lambda^k \varphi_\lambda(t), \quad n = 1, 2, \ldots$$

converging to $f(t)$ uniformly on any segment $[a,b]$ of $\mathbb{R}_+$.

First, it is easy to show by induction that

$$L_\lambda^k \varphi_\lambda(t) = t^\lambda p_k(\ln t), \tag{32}$$

where p_k is a polynomial of degree k, i.e. $p_k(\ln t) = \sum_{s=0}^{k} a_{ks}(\ln t)^s$.

Indeed, if $k=1$, then by (26) and (20)

$$\begin{aligned} L_\lambda \varphi_\lambda(t) &= \left\{\frac{t^\lambda}{E(\lambda)}\right\} * \left\{\frac{t^\lambda}{E(\lambda)}\right\} = \frac{1}{E^2(\lambda)}\Phi_\tau \left\{\int_\tau^t \left(\frac{t\tau}{\sigma}\right)^\lambda \sigma^\lambda \frac{d\sigma}{\sigma}\right\} = \\ &= \frac{1}{E^2(\lambda)} t^\lambda \Phi_\tau \left\{\tau^\lambda \int_\tau^t \frac{d\sigma}{\sigma}\right\} = t^\lambda \left[\frac{\Phi_\tau\{\tau^\lambda\}}{E^2(\lambda)}\ln t - \frac{\Phi_\tau\{\tau^\lambda \ln \tau\}}{E^2(\lambda)}\right]. \end{aligned}$$

Next, the inductive step will be made. Suppose that

$$L_\lambda^{k-1}\varphi_\lambda(t) = t^\lambda p_{k-1}(\ln t).$$

Then

$$\begin{aligned} L_\lambda^k \varphi_\lambda(t) &= L_\lambda(L_\lambda^{k-1}\varphi_\lambda(t)) = \left\{\frac{t^\lambda}{E(\lambda)}\right\} * L_\lambda^{k-1}\varphi_\lambda(t) = \\ &= \frac{1}{E(\lambda)}\Phi_\tau\left\{\int_\tau^t \left(\frac{t\tau}{\sigma}\right)^\lambda \sigma^\lambda p_{k-1}(\ln\sigma)\frac{d\sigma}{\sigma}\right\} = \\ &= \frac{1}{E(\lambda)} t^\lambda \Phi_\tau \left\{\tau^\lambda \int_\tau^t p_{k-1}(\ln\sigma) d\ln\sigma\right\}. \end{aligned}$$

The integration of p_{k-1} gives a polynomial q_k of $\ln t$ of degree k and the above chain of equalities can be continued as

$$\begin{aligned} L_\lambda^k \varphi_\lambda(t) &= \frac{1}{E(\lambda)} t^\lambda \Phi_\tau \left\{\tau^\lambda [q_k(\ln t) - q_k(\ln\tau)]\right\} = \\ &= t^\lambda \left[\frac{\Phi_\tau\{\tau^\lambda\}}{E(\lambda)} q_k(\ln t) - \frac{\Phi_\tau\{\tau^\lambda q_k(\ln\tau)\}}{E(\lambda)}\right], \end{aligned}$$

where the expression in the square brackets is obviously a polynomial p_k of $\ln t$ of degree k, as desired.

Now let $f \in C(\mathbb{R}_+)$ be arbitrarily chosen. Consider the function $\widetilde{f}(t) = \dfrac{f(t)}{t^\lambda}$, which is again in $C(\mathbb{R}_+)$. Making the substitution $t = e^x, x = \ln t$, the new function $g(x) = \widetilde{f}(t)$ is in $C(-\infty, \infty)$. By Weierstrass' theorem, g

can be approximated almost uniformly on $(-\infty,\infty)$ by a sequence of polynomials $\{r_n(x)\}_{n=1}^{\infty}, r_n(x)=\sum_{k=0}^{n} b_{nk}x^k$, i.e. the convergence is uniform on any segment $[a,b]\subset(\mathbb{R}_+)$. Returning to the old variable, $\widetilde{f}(t)$ can be approximated by the sequence of polynomials $\{r_n(\ln t)=\sum_{k=0}^{n} b_{nk}(\ln t)^k\}_{n=1}^{\infty}$. Finally, multiplying by t^λ and using (32), the desired approximation of $f(t)$ on $(\mathbb{R}_+)$ follows from the representation

$$f_n(t)=t^\lambda r_n(\ln t)=\sum_{k=0}^{n} b_{nk}t^\lambda(\ln t)^k=\sum_{k=0}^{n} c_{nk}t^\lambda p_k(\ln t)=\sum_{k=0}^{n} c_{nk}L_\lambda^k\varphi_\lambda(t).$$

The new coefficients c_{nk} can be calculated from the old ones b_{nk}. Thus, φ_λ is a cyclic element of L_λ in $C(\mathbb{R}_+)$.

Theorem 7. *A linear operator $M : C(\mathbb{R}_+)\to C(\mathbb{R}_+)$, such that $M : C^1(\mathbb{R}_+)\to C^1(\mathbb{R}_+)$, and with an invariant hyperplane $C_\Phi=\{f\in C(\mathbb{R}_+) : \Phi\{f\}=0\}$ commutes with δ in C_Φ^1 if and only if it has a representation of the form*

$$(Mf)(t)=\mu f(t)+(m*f)(t) \tag{33}$$

with a constant $\mu\in\mathbb{C}$ and $m\in C(\mathbb{R}_+)$.

Proof: Since $\Phi\{f*g\}=0$ for $f,g\in C(\mathbb{R}_+)$ (see (10)), then each operator of the form (33) has C_Φ as an invariant subspace. It commutes with δ in C_Φ^1. Indeed, if $f\in C_\Phi^1$, then (31) gives

$$\delta(m*f)=m*\delta f+\Phi\{f\}m$$

and, using (33),

$$\delta Mf=\mu\delta f+m*(\delta f)+\Phi\{f\}m=\mu\delta f+m*(\delta f)=M\delta f.$$

The sufficiency is proved.

In order to prove the necessity of (33), according to Lemma 2, $ML_\lambda=L_\lambda M$ for $\lambda\in\mathbb{C}$ with $E(\lambda)\neq 0$. As it is shown in the book [4] (Theorem 1.3.11, p.33) , the commutant of L_λ coincides with the ring of the multipliers of the convolution algebra $(C(\mathbb{R}_+),*)$ since L_λ has a cyclic element. By Theorem 6 such a cyclic element is the function $\varphi_\lambda(t)=\dfrac{t^\lambda}{E(\lambda)}$ for which $L_\lambda f=\varphi_\lambda * f$. The proof is completed.

Remark. The constant μ and the function $m\in C(\mathbb{R}_+)$ in (29) are uniquely determined. Indeed, assume that $\mu f+m*f=\mu_1 f+m_1*f$. Take f such that $\Phi(f)\neq 0$. Then, (23) implies $\mu\Phi(f)=\mu_1\Phi(f)$, and hence $\mu=\mu_1$. From $m*f=m_1*f$ for arbitrary $f\in C(\mathbb{R}_+)$ it follows that $(m-m_1)*f=0$, and hence $m=m_1$.

5 Mean-periodic functions for the Euler operator

Definition 3. *A function $f \in C(\mathbb{R}_+)$ is said to be mean-periodic for the Euler operator with respect to the linear functional Φ if*

$$\Phi_\tau\{f(t\tau)\} = 0$$

identically in $\mathbb{R}_+$.

It is clear that the mean-periodic functions with respect to Φ form the kernel space of the operator

$$Mf(t) = \Phi_\tau\{f(t\tau)\}$$

commuting with the Euler operator δ in $C(\mathbb{R}_+)$.

Now a connection between the mean-periodic functions and the convolutional algebra $(C(\mathbb{R}_+), *)$ will be shown.

Theorem 8. *The mean-periodic functions for the Euler operator δ with respect to any non-zero functional $\Phi : C(\mathbb{R}_+) \to \mathbb{C}$ form an ideal in the convolutional algebra$(C(\mathbb{R}_+), *)$.*

Proof: One need to prove only that the convolutional product $(f * g)(t)$ of a mean-periodic function f and an arbitrary function $g \in C(\mathbb{R}_+)$ is a mean-periodic function, too, i.e. it is given that $\Phi_\tau\{f(t\tau)\} = 0$ and then $\Phi_\tau\{(f * g)(t\tau)\} = 0$ is to be shown. By (20)

$$(f * g)(t\tau) = \Phi_\sigma\left\{\int_\sigma^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\}$$

and

$$\begin{aligned}\Phi_\tau\{(f * g)(t\tau)\} &= \Phi_\tau\Phi_\sigma\left\{\int_\sigma^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\} = \\ = \Phi_\tau\Phi_\sigma\left\{\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\} &+ \Phi_\tau\Phi_\sigma\left\{\int_\tau^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\}. \quad (34)\end{aligned}$$

Interchanging τ and σ in the first term of (34) and using the Fubini commutational property of the functionals yields

$$\Phi_\tau\Phi_\sigma\left\{\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\} = \Phi_\sigma\Phi_\tau\left\{\int_\tau^{\sigma} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\} =$$

$$= \Phi_\sigma\Phi_\tau\left\{-\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\} = -\Phi_\tau\Phi_\sigma\left\{\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{d\eta}{\eta}\right\},$$

thus obtaining

$$\Phi_\tau \Phi_\sigma \left\{ \int_\sigma^\tau f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} = 0. \tag{35}$$

The second term in (34) also vanishes

$$\Phi_\tau \Phi_\sigma \left\{ \int_\tau^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} = \Phi_\tau \left\{ \int_\tau^{t\tau} \Phi_\sigma \left\{ f\left(\frac{t\tau\sigma}{\eta}\right) \right\} g(\eta) \frac{d\eta}{\eta} \right\} = 0 \tag{36}$$

since f is mean-periodic and hence

$$\Phi_\sigma \left\{ f\left(\frac{t\tau\sigma}{\eta}\right) \right\} = 0.$$

Finally, (34), (35), and (36) give the desired result $\Phi_\tau\{(f * g)(t\tau)\} = 0$.

6 Application to the Euler differential equation

Now Theorem 8 will be applied to find necessary and sufficient conditions in order an Euler differential equation

$$P(\delta)y(t) = f(t), \qquad 0 < t < \infty, \tag{37}$$

to have a unique mean-periodic solution with respect to a non-zero linear functional Φ in $C(\mathbb{R}_+)$. Here $\delta = t\dfrac{d}{dt}$ is the Euler operator and $P(\mu) = a(\mu - \mu_1)(\mu - \mu_2)\ldots(\mu - \mu_k)$ is a polynomial.

Theorem 9. *In order the Euler differential equation (37) to have a unique mean-periodic solution with respect to a non-zero linear functional Φ in $C(\mathbb{R}_+)$, it is necessary and sufficient no roots of the equation $P(\lambda) = 0$ to be roots of the Euler indicatrix $E(\lambda) = \Phi_\tau(\tau^\lambda)$.*

Proof: Consider the Euler differential equation (37). It is clear that in order y to be a mean-periodic solution, the right hand side, i.e. the function $f(t)$, should be mean-periodic, too. Formally, let $Mf(t) = \Phi_\tau\{f(t\tau)\}$. Applying M to (37) and using the commutativity of $\delta = t\dfrac{d}{dt}$ and M yields

$$P(\delta)My(t) = Mf(t).$$

Then from $My = 0$ it follows that $Mf = 0$, i.e. the requirement f to be mean-periodic is a necessary condition for existing of a mean-periodic solution y. It can be shown that it is also a sufficient condition, but in general the solution may not be unique. Indeed, if a root μ of the equation $P(\lambda) = 0$ is a root of the Euler indicatrix $E(\lambda)$, then the function t^μ is a solution of the homogeneous equation $P(\delta)u = 0$, and hence the uniqueness of the solution holds no more.

Now it will be shown that if neither of the roots $\mu_1, \mu_2, \ldots, \mu_k$ of the equation $P(\lambda) = 0$ is a root of the Euler indicatrix $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$, then there exists a unique mean-periodic solution of the Euler equation $P(\delta)y = f$, provided f is a mean-periodic function with respect to Φ.

Assuming that y is a mean-periodic solution of (37), an explicit expression for y will be obtained. Let P be a polynomial of degree k

$$P(\mu) = a(\mu - \mu_1)(\mu - \mu_2)\ldots(\mu - \mu_k).$$

From the assumption that y is a mean-periodic solution it follows that

$$\Phi\{y\} = \Phi\{\delta y\} = \ldots = \Phi\{\delta^{k-1}y\} = 0. \tag{38}$$

Indeed, the mean-periodicity of y means that

$$\Phi_\tau\{y(t\tau)\} = 0.$$

Applying the operator δ to this identity with respect to t, Theorem 1 gives

$$\Phi_\tau\{(\delta^n y)(t\tau)\} = 0, \qquad n = 1, 2, \ldots, k-1.$$

It remains to put $t = 1$ in order to obtain the boundary conditions (38). Next, the unique solution of (37) is

$$y = \frac{1}{a} L_{\mu_k} L_{\mu_{k-1}} \ldots L_{\mu_1} f(t). \tag{39}$$

Indeed, the equation (37) can be represented as

$$(\delta - \mu_1)[(\delta - \mu_2)\ldots(\delta - \mu_k)y(t)] = \frac{1}{a} f(t).$$

Denoting the square brackets by $u_1(t)$ yields

$$\delta u_1 - \mu_1 u_1 = \frac{1}{a} f,$$

for u_1 with $\Phi\{u_1\} = 0$, as it follows from (38). This equation has the unique solution $u_1 = \frac{1}{a} L_{\mu_1} f$ with L_{μ_1} defined as in Lemma 2. Next solve

$$\delta u_2 - \mu_2 u_2 = u_1, \qquad \Phi\{u_2\} = 0,$$

for $u_2(t) = (\delta - \mu_3)\dots(\delta - \mu_k)y(t)$ with the unique solution $u_2 = L_{\mu_2}u_1$. Continuing in the same manner one gets the unique solution (39) of the initial equation (37). Now it is easy to verify that (39) is indeed a mean-periodic solution. It can be written in the form of convolutional product using Lemma 3:

$$\begin{aligned} y &= \frac{1}{a} L_{\mu_k} L_{\mu_{k-1}} \dots L_{\mu_1} f(t) = \\ &= \left(\frac{1}{a} \varphi_{\mu_k} * \varphi_{\mu_{k-1}} * \dots * \varphi_{\mu_1} \right) * f = \varphi * f \end{aligned} \tag{40}$$

with $\varphi := \frac{1}{a}\varphi_{\mu_k} * \varphi_{\mu_{k-1}} * \dots * \varphi_{\mu_1}$. It remains to use Theorem 8 to assert that the mean-periodicity of f implies the mean-periodicity of y.

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