SEVERAL MATHEMATICAL PAPERS

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The present book contains seven unpublished yet mathematical papers. Some of them resolve old problems. For instance, the problem of the geometrical description of the polynomial ideals has been actual since 1873, when Max Noether geometrically described the zero-dimensional polynomial ideals. The first article here gives the geometrical description of the polynomial ideals by differential operators and their corresponding sets of common zeros on the described ideal in the sense of Max Noether's descrption of the sero-dimensional polynomial ideals. The given here description is for the polynomial ideals of the polynomial ring $K[\xi]$, where K is a field $\xi \in \Omega^n$, Ω being a universal extension of the field K. This description supplies with new properties Max Noether's description of the zero-dimensional polynomial ideals. The problem of the geometrical description of the polynomial ideals has been extensively elaborated, because it is connected with the resolution of the systems of linear differential equations.

The second article here introduces and studies functional spaces with strong generalized derivatives, which are a generalization of Sobolev's spaces, a generalization of the spaces of Schwartz, Bessov, Lizorkin, Triebel. The introduced here spaces do not always coincide with Sobolev's spaces. For instance, the space $\mathcal{W}_p^{A,I}$, $1 < p < \infty$, with A - a linear constant-coefficient differential operator of order N, coincides with Sobolev's space W_p^N if and only if the operator A is elliptic (I is the identity operator). Moreover, the introduced speces $\mathcal{W}_*^{A_1,\ldots,A_m}$ are such that each solution $f \in \mathcal{L}_*$ of the system $\{A_k f = u_k\}$, $u_k \in \mathcal{L}_{*k}$, $k=1,\ldots,m$, belongs to the space $\mathcal{W}_*^{A_1,\ldots,A_m}$. Fundamental law of Physics are expressed by differential equations. Their resolve has imposed extensions of the studied functional spaces. In this way Sobolev's spaces, Theory of distributions, the spaces of Bessov, Lizorkin and Triebel have appeared. Although the solutions of systems of linear constant-coefficient differential equations are expressed by distributions, i.e. by continuous linear operators, solutions of more classical kind are important for Applications of Mathematics.

That is why Sobolev's spaces are actual still and are an intensively developed branch of Analysis. Therefore the introduced here functional spaces are also important. Moreover, these spaces are genuinely connected with the functional algebras of type C and their connection is established in this study. Necessary and sufficient requirements for a closeness of these spaces relatively the multiplication of their functions are also formulated.

The article "A note on the automorphisms of the tori" gives a new more constructive form of the necessary and sufficient conditions for automorphisms of the elliptic algebraical curves (i.e. of tori), and as a corollary proves some results on integers.

The proposed here article "On a model of the real numbers" exposes a new model of the real numbers, constructed by the rational numbers, analogously with the well known models of Cantor-Méray, Dedekind, Bachmann and others. The model, present here, is more natural from the gnosiological (i.e. epistemological) and the ontological points of views. The problem to construct the real numbers by the intervals of the line in the sense of the method, built here, has been proposed by Whitehead and Russell in the beginning of this century. The idea of Whitehead and Russell of such constructions by psychologically more primary objects (for instance by the events for the instants, or by intervals for the real numbers) does not concern the instants only. This idea also includes the real numbers. This idea has found a general recognition since the events, or the intervals are psychologically more primary conceptions, while the instants or the real numbers are intuitive-mental constructions. That is why it is worth-while the conceptions of the instants or the real numbers to be built by logico-mathematical ways from the more primary psychologically objects, as the events or the intervals correspondingly.

The article "Two models of Time with Walker's definition of instants by events" constructs two models of Time, using Walker's definition of the instants by events. It follows from either of the proposed two systems of axioms on the events, that the instants, constructed by events after Walker's definition of the instants, compose an open-ended linear continuum with a " dense" sequence of instants. I.e. Time continuum has the properties, characterizing the real line. The exposition of the paper is based on Walker's definition of instants (without mixing up Russell's definition of instants). The used here systems of axioms are simplier than the previous in the literature and treat only events. The Walker's definition can be interpreted as defining the instants, belonging to the Present and borders, dividing the Past from the Future.

The sixth article is on the measurement of Time in the axiomatic theories of Time, which development has been begun by Bertrand Russell, A.N. Whitehead, Norbert Wiener,Gerald J. Whitrow. Here are formulated only exact mathematical results, almost without commentaries, following Newton's motto "Hypotheses non fingo". The result of this article is very small in comparison with the immensity and grandiosity of the problem of Time. This article further develops the axiomatic models of Time, proposing a measurement of Time in all of them together. Any measurement consists in an establishment of a correspondence between the measured object and a number, or a vector, or some other mathematical quantity. Here we establish a one-to-one correspondence between all moments of Time and the real numbers for each of these Time models. Moreover, this correspondence preserves the order, i.e. it maps to the later moments larger real numbers.

The article "On the derivatives of a composite function" gives the formula of the n-th derivative of composite functions. (Professor V.N.Vragov denoted on Conference of Mathematics and its Applications, Varna, 1989, that the coefficients for such a formula did not determined yet.)

CONTENTS

INTRODUCTION

This book presents seven unpublished yet mathematical papers. Some of them resolve old problems. For instance, the problem of the geometrical description of the polynomial ideals has been actual since 1873, when Max Noether geometrically described the zero-dimensional polynomial ideals. The first article here gives the geometrical description of the polynomial ideals by differential operators and their corresponding sets of common zeros on the described ideal in the sense of Max Noether's description of the zero-dimensional polynomial ideals. The given here description is for the polynomial ideals of the polynomial ring $K[\xi]$, where K is a field, $\xi \in \Omega^n$, Ω being a universal extension of the field K. This description supplies with new properties Max Noether's description of the zero-dimensional polynomial ideals. The problem of the geometrical description of the polynomial ideals has been extensively elaborated, because it is connected with the resolution of the systems of linear differential equations.

The second article here introduces and studies functional spaces with strong generalized derivatives, which are a generalization of Sobolev's spaces, a generalization of the spaces of Schwartz, Bessov, Lizorkin, Triebel. The introduced here spaces do not always coincide with Sobolev's spaces. For instance, the space $\mathcal{W}_p^{A,I}$, $1 < p < \infty$, with A - a linear constant-coefficient differential operator of order N, coincides with Sobolev's space W_p^N iff the operator A is elliptic (I is the identity operato[r]. Moreover, the introduced spaces $\mathcal{W}_*^{A_1,\dots,A_m}$ are such that each solution $f \in \mathcal{L}_*$ of the system $\{A_k f = u_k\}$, $u_k \in \mathcal{L}_{*k}$, $k=1,\dots,m$, belongs to the space $\mathcal{W}_*^{A_1,\dots,A_m}$. Fundamental laws of Physics are expressed by differential equations Their resolution has imposed extensions of the studied functional spaces. In this way have appeared Sobolev's spaces, Theory of distributions, the spaces of Bessov, Lizorkin, Triebel. Although the solutions of the systems of linear constant-coefficient differential equations are expressed by distributions, i.e. by continuous linear operators, solutions of more classical type are important for Applications of Mathematics. That is why Sobolev's spaces are still actual and are an intensively developed.

loped branch of Analysis. Therefore the introduced here functional spaces are also important. Moreover, these spaces are genuinely connected with functional algebras of type C and their connections are established in this study. Necessary and sufficient conditions for a closeness of these spaces relatively the multiplication of their functions are also formulated. Inclusions among the spaces $A_1,\dots,A_m$ and the Sobolev's spaces are investigated here.

The article also gives a generalization of Petrovsky's parabolicity together with characterizing inclusions among corresponding functional spaces These inclusions also demonstrated the necessity of a generalization of Petrovsky's parabolicity, although such investigations are new and for the classical parabolicity.

The article "A note on the automorphisms of the tori" gives a new more constructive form of the necessary and sufficient conditions for automorphisms of the elliptic algebraical curves (i.e. of tori), and as a corollary proves some results on integers.

The investigations of the automorphisms of the Riemann surfaces of the first genre are old and traditional for Complex Analysis. Such investigatior are included in basic courses as Hurwitz A., R.Courant. Allgemeine Funktionentheorie und Elliptische Funktionen. Geometrische Funktionentheorie, Serre J.-P. Seminar on Complex Multiplication, Lecture Notes in Mathematic:

The proposed here article "On a model of the real numbers" exposes a new model of the real numbers, constructed by the rational numbers, analogously with the well known models of Cantor-Méray, Dedekind, Bahmann and other: The model, present here, is more natural from gnosiological (i.e. epistemological) and ontological points of views. The problem to construct the real numbers by the intervals of the line in the sense of the method, built here has been proposed by Whitehead and Russell in the beginning of this century The idea of Whitehead and Russell of such constructions by psychologically more primary objects (for instance by the events for the instants, or by intervals for the real numbers) does not concern the instants only. This idea also includes the real numbers. This idea has found a general recognition since the events, or the intervals are psychologically more primary conceptions, while the instants or the real numbers are intuitive-mental constructions. That is why it is worth-while the conceptions of the instants or the real numbers to be built by logico-mathematical ways from the more primary psychologically objects as the events for the instants or intervals for the real numbers.

The exposed here model of the real numbers has a resemblance with the other models of the real numbers, as well as the other models of the real numbers have resemblances, although each of these models has its importance The model of Cantor-Méray and Dedekind's model of the real numbers ~~are~~ considered as the most different. But each of these constructions evidently can be reduced to the other. Although each of them is independently and thoroughly exposed in the literature. Here we shall show how easy and clearly the construction of Cantor-Méray of the real numbers by fundamental sequences can be reduced to Dedekind's construction of the real numbers by sections. Let us remind the basic definitions.

Definition. A sequence $\{r_n\}$ is called fundamental if for every natural number m there exists a natural number n_m, such that

$$|r_n - r_{n'}| < 1/m \qquad \text{for } \forall\, n,\ n \geqslant n_m,\ \forall\, n',\ n' \geqslant n_m$$

Definition. The nonempty disjoint subsets A and B of the set of the rational numbers Q form a section $A\,I\,B$ of Q if they have the following properties: I. If $p \in A$ and $p' < p$, then $p' \in A$; $p, p' \in Q$;

II. If $q \in B$ and $q' > q$, then $q' \in B$, $(q, q' \in Q)$;

III. The set $Q - (A \cup B)$ does not contain more than one rational number.

We shall easily reduce the construction of Cantor-Méray of the real numbers by fundamental sequences to the construction of the real numbers by sections. I.e. we must show how does any fixed fundamental sequence $\{r_n\}$ of rational numbers determine a corresponding Dedekind's section $A\,I\,B$ of Q. Let $A = \{p : p \in Q,\ p \leq r_n - 1/m \text{ for } \forall\, m, \forall\, n \text{ with } n \geqslant n_m,\ m, n, n_m \in Z_+\}$

and $B = \{q : q \in Q,\ q \geqslant r_n + 1/m \text{ for } \forall\, m, \forall\, n \text{ with } n \geqslant n_m,\ m, n, n_m \in Z_+\}$

Evidently, AIB is a Dedekind's section by the construction: Clearly, $A\,I\,B$ has the properties I and II of a Dedekind' section. III. If $a \notin A \cup B$, $a \in Q$, then we have

$r_n - 1/m < a < r_n + 1/m$ for $\forall m$, $\forall n$, $n \geqslant n_m$,

But this cannot be satisfied from more than one rational, since the sequence $\{r_n\}$ is fundamental. ∎

Although each of these two constructions of the real numbers is independently exposed in the literature and has its advantages.

Therefore the proposed in this article model of the real numbers has its rights to be exposed also.

The article "Two models of Time with Walker's definition of instants by events" constructs two models of Time, using Walker's definition of the instants by events. It follows from either of the proposed two different systems of axioms on the events, that the instants, constructed by events after Walker's definition of the instants, compose an open-ended linear continuum with a "dense" sequence of instants. I.e. Time continuum has the properties, characterizing the real line. The exposition of the paper is based on Walker's definition of instants (without mixing up Russell's definition of the instants.) The used here systems of axioms are simplier than the previous in the literature and treat only events. The Walker's definition can be interpreted as defining the instants, belonging to the Present and as borders, dividing the Past from the Future.

The attempts of mathematical constructions of the instants of Time by the events, derived from Russell and Whitehead. Such constructions of Time are also elaborated by Robbs, N.Wiener, Walker, G.J.Whitrow, Thomason. A.S. Madguerova has constructed in other articles two models of Time, based on Russell's definition of the instants by events. The natural sciences are in accord that the conception of the events is more primary and fundamental, whereas the instants are intuitive-mental constructions. Russell and Whitehead have posed the problem to obtain the construction of the instants from the events by a logico-mathematical way. The present article proposes two different models of Time, based on Walker's definition, and having more simple requirements about the events. For instance, here only the relations $\prec$ ("before") and $\odot$ ("simultaneously") are required among the events, whereas the literature usually needs the relations $\prec_1, \prec_0, \prec, \odot$ among the events. Here the constructions and proofs use only Walker's definition of the instants (without a mixing Russell's definition of instants.

The constructed two models of Time with Walker's definition of instants are based on two different systems of axioms on the events. It follows from either of these systems, that the instants, constructed by the events after Walker's definition, have the discussed in the literature properties of the Time continuum of Mathematical Physics. This is the instants compose an open-ended linear continuum with a "dense" sequence of instants, which are characterising properties of the real line. The second model of Time in this article is introduced not only to show a new possibility of a construction of Time. The second model of Time avoids the conceptual imperfection of the first more simple model of Time here. All events are finite in the first model, whereas the second model admits unbounded events also. The first system of axioms on the events is satisfied for instance by all nonempty compact segments of the real line. The second system of axioms on the events is satisfied for instance by all nonempty open intervals of the real line. The axiom 𝔄5 belongs to Russell, Wiener, Walker.

The sixth article is on the measurement of Time in the axiomatic theories of Time, which development has been begun by Bertrand Russell, A.N. Whitehead, Norbert Wiener, Gerald J.Whitrow. Here are formulated only exact mathematical results, almost without commentaries, following Newton's motto "Hypotheses non fingo". The result of this article is very small in comparison with the immensity and grandiosity of the problem of Time. This article further develops the axiomatic models of Time, proposing a measurement of Time in all of them together. Any measurement consists in an establishment of a correspondence between the measured object and a number, or a vector, or some other mathematical quantity. Here we establish an one-to-one correspondence between all moments of Time and the real numbers.for each of these Time models. Moreover, this correspondence preserves the order, i.e. it maps to the later moments larger real numbers.

The constructed correspondence can evidently be changed in many aspects The possibility of many kind of mesurements of Time reflects the real relativ tivity of the measurement of Time, depending on the choice of the "periodical" processes, i.e. depending on the choice of the "clocks" and their comparisons and confrontations. The choice of the clock would reflect on the choice of the basic dense sequence $\mathcal{K}$ of instants of Time. The sequence $\mathcal{K}$ is constructed by a given sequence K of events. That is why it is not difficult and it is almost evident to substitute the proposed here construction for a measurement of Time by such one, based on the sequence K of e events, avoiding the aidding sequence $\mathcal{K}$ of instants.

The proposed measurement of Time is for any arbitrarily chosen co-ordinate system of account and is based on the events of K in this system. This assures the compatibility of the measurement of Time with Theory of Relativity.

Coarsely, we can choose a suitable sequence of "periodical" events for K . As an example, K can consists of the motions of an"eternal" clock pendulum, whose motions are reduced to fragments. We can choose for $\mathcal{K}$ the instants of the fixed positions of the pendulum. Then the proposed in this article construction of a measurement will coincide with the usual measurement of Time.

The construction of a measurement in any co-ordinate system of account is necessary for the comparison of different co-ordinate systems of accoun The different measurement of Time in different co-ordinate systems of account can as usual be assured and obtained, postulating the Lorentz's formulas or Newton's formulas.

The article "On the derivatives of a composite function" gives the formula of the n-th derivative of composite functions. (Professor V. N.Vragov denoted on Conference of Mathematics and its Application, Varna, 1989, tha the coefficients for such a formula did not determined yet.)

ON THE GLOBAL DESCRIPTION OF THE POLYNOMIAL IDEALS

Andreana Stefanova Madguerova

The present paper gives the description of the polynomial ideals by linear differential operators, i.e. determines (roughly speaking) each polynomial ideal $\mathfrak{J}$ by sets of common zeros of differential operators over the polynomial of $\mathfrak{J}$. The results proposed here include new properties of Max Noether's description of the zero-dimensional polynomial ideals, 1873.

The problem of the geometrical description of polynomial ideals has been explored at least since 1873, when Max Noether has given the geometrical description of the zero-dimensional polynomial ideals [1,2]. Van der Waerden writes in all editions of his Algebra [2] that : "The main problem in the theory of the polynomial ideals consists in the establishing... a method,which would simultaneously elucidate the construction of the ideal and elicit the geometrical relation between its roots and its elements." That is, the main problem is the geometrical description of the polynomial ideals in the sense of the Noether's description of the zero-dimensional polynomial ideals. This problem is largely investigated because of its connections with the resolutions of the systems of linear constant-coefficient partial differential equations. That is why the geometrical description of the polynomial ideals and of the polynomial modules is important not only in the Algebra. Max Noether's description of the zero-dimensional polynomial ideals is only considered in the literature as geometrically satisfying [3]. Lasker has proved [2,4] in 1905 that every polynomial ideal is an intersection of algebraically primary polynomial ideals, where we have:

Definition. A polynomial ideal $\mathfrak{J}$ is algebraically primary iff the conditions $ab = 0(\mathfrak{J})$, $a \neq 0(\mathfrak{J})$ imply that there exists some integer ϱ such that $b^{\varrho} = 0(\mathfrak{J})$.

Gentzelt has reduced the case of the algebraically primary polynomial ideals to Max Noether's theorem about zero-dimensional polynomial ideals [2,3]. Gentzelt's method is not considered in the literature as sufficiently geometrical (see [3]: "This method has been proposed by Gentzelt. However, the described method, for the sake of its insufficient geometricity, does not give in full measure a solution of the problem of the description of the polynomial ideals"). Let $\mathfrak{P}$ be an arbitrarily fixed algebraically primary polynomial ideal. Let N be the algebraical manifold of all its zeros. Gentzelt's method consists in the following: The algebraical manifold N of the algebraically primary polynomial ideal $\mathfrak{P}$ can be parametrized as follows $\xi = (\xi_{d+1}, \dots, \xi_n) = \xi(\eta)$, $\eta = (\eta_1, \dots, \eta_d)$. Let us fix arbitrarily a value of the parameter η, $\eta = \eta_0$. Gentzelt has proved that the set $\mathfrak{P}_{\eta_0}$ of all restrictions of the polynomials of the algebraically primary ideal $\mathfrak{P}$ for $\eta = \eta_0$ is a zero-dimensional polynomial ideal in the polynomial ring $K[\xi]$. That is why the belonging of a polynomial p to the algebraically primary polynomial ideal $\mathfrak{P}$ is determined by Max Noether's conditions for any fixed value η_0 of the parameter η.

Max Noether's theorem asserts that a polynomial f belongs to the zero-dimensional polynomial ideal $\mathfrak{J}$ iff some finite number of linear relations among the coefficients of the expansion of the polynomial f at the point x° are satisfied at any fixed point x° of the manifold N of the ideal $\mathfrak{J}$ i.e. the ideal $\mathfrak{J}$ is determined by the manifold N and by a finite number of linear differential operators $A_1, \dots, A_q$ with $A_j f(x^\circ) = 0$ for $\forall f \in \mathfrak{J}$ $\forall x^\circ \in N$, $j = 1, \dots, q$. Further on, the sets

$$N_m = \left\{x : D^p f(x) = 0 \text{ for } \forall p \in Z_+^n, |p| \leq m, \forall f \in \mathfrak{J}\right\}, \quad m = 0, 1, 2, \dots$$

have been considered in the literature for the description of the polynomial ideal $\mathfrak{J}$ of $C[x]$, $x \in R^n$. In this way important results have been obtained. But these sets N_m do not uniquely determine and describe any ideal $\mathfrak{J}$ of $C[x]$, $x \in R^n$, in the case $n \geq 2$. For instance, the following ideals of $C[x]$, $x \in R^2$, $\mathfrak{J}^* = \{x_1^2 + x_2^2\}$ and $\mathfrak{J}^{**} = \{x_1^2, x_2^2\}$ have the same complex of sets $N_0^* = \{x : f(x) = 0, \forall f \in \mathfrak{J}^*\}$, $N_0^{**} = \{x : f(x) = 0, \forall f \in \mathfrak{J}^{**}\}$ and $N_1^* = \{x : f^{(p)}(x) = 0, \forall p \in Z_+^2, |p| \leq 1, \forall f \in \mathfrak{J}^*\}$, $N_1^{**} = \{x : f^{(p)}(x) = 0, \forall p \in Z_+^2, |p| \leq 1, \forall f \in \mathfrak{J}^{**}\}$. Thus we have $N_0^* = N_0^{**}$, $N_1^* = N_1^{**}$, but it holds simultaneously $\mathfrak{J}^* \neq \mathfrak{J}^{**}$.

The present paper gives a global description of the polynomial ideals in $K[\zeta_1, \dots, \zeta_n]$ by linear differential operators and by the sets of common zeros of these operators over the polynomials of the described ideal. Here K is an arbitrarily fixed field. Let Ω be a universal extension of the field

K , i.e. Ω is algebraically closed and with infinite degree of transcendentality over the field K . Let $\zeta = (\zeta_1,\ldots,\zeta_n)$ with $\zeta_j \in \Omega$, j = 1,...,n . It is convenient to examine $K[\zeta_1,\ldots,\zeta_n]$ with $\zeta_j \in \Omega$,where Ω is a universal extension of K as have been noticed by A. Weyl .

Definition. We shall denote by D^k , $k \in Z^n_+$, the operator

$$D^k : K[\zeta] \longrightarrow K[\zeta]$$

such that if

$$p = \sum_{|r|\le m} c_r \zeta^r ,\ c_r \in K \quad \text{we have} \quad D^k p = \sum_{|r|\le m,\ k\le r} c_r \binom{r}{k} k!\, \zeta^{r-k};$$

The operator $A : K[\zeta] \longrightarrow K[\zeta]$, $A = \sum_{|k|\le m} a_k D^k$,$a_k \in K$,

will be called a linear differential operator of order not larger than m . If at least one of the coefficients a_k , $|k| = m$, is not annulling at some point ζ° , we shall say that the operator A is of order m at ζ° ; If the coefficients a_k , $|k|\le m$, do not depend on ζ , then the operator A will be called constant-coefficient linear differential operator;

The operator

$$A^{(q)} = \sum_{|k|\le m,\ q\le k} a_k \binom{k}{q} q!\, D^{k-q},\quad q \in Z^n_+,$$

will be called an operator derivative of the operator A ;

A K-linear space α of linear differential operators will be called differential-invariant if the condition $A \in \alpha$ involves that $A^{(q)} \in \alpha$,$\forall q \in Z^n_+$.

Theorem 1. Let $\mathfrak{J}$ be an arbitrarily fixed polynomial ideal of $K[\zeta]$. The ideal $\mathfrak{J}$ determines a K-linear finite-dimensional differential-invariant space α of linear differential operators, such that the space α and its corresponding sets

$$N_{\alpha^{(q)}} = \{\zeta : Af(\zeta) = 0 \text{ for } \forall A \in \alpha^{(q)},\ \forall f \in \mathfrak{J}\} ,\ \forall\, q \in Z^n_+$$

where $\alpha^{(q)} = \{B^{(q)} ,\ \forall B \in \alpha\}$, completely describe the ideal $\mathfrak{J}$. This is, the complex of the space α and the sets $N_{\alpha^{(q)}}$, $\forall q \in Z^n_+$, determines the ideal $\mathfrak{J}$ (this complex is different for different ideals of $K[\zeta]$). I.

e. in the traditional terminology of the problem, the space α and its corresponding sets $N_{\alpha(q)}$, $\forall q \in Z^n_+$, geometrically describe the ideal $\mathfrak{I}$.

Moreover, we can choose the space α for the fixed ideal $\mathfrak{I}$ such that the coefficients of its operators are rational functions on the sets

$$\left\{ N_{\alpha(p^\circ)} - \bigcup_{p' < p^\circ} N_{\alpha(p')} \right\}.$$

(Such a space α can be constructed by an arbitrarily fixed basis $\mathcal{B}(\mathfrak{I})$ $(f_1, \ldots, f_m)$ of the ideal $\mathfrak{I}$. Let ζ° be an arbitrarily fixed point of the algebraical manifold F of the ideal $\mathfrak{I}$ in the case $F \neq \emptyset$. (If $F = \emptyset$, then then $\mathfrak{I} = K[\zeta]$ and the corresponding space $\alpha = \{0\}$.) Let ν be the largest degree of the polynomials $f_1, \ldots, f_m$ and $M(\zeta^\circ)$ be the maximal ideal of $K[\zeta]$ at the point ζ°. Let I_{ζ° be the image of the ideal $\mathfrak{I}$ in the canonical homomorphism

$$K[\zeta] \longrightarrow K^\nu = K[\zeta] / \mathcal{M}^{\nu+1}(\zeta^\circ).$$

The space of the linear functionals on K^ν which are annulling over the ideal I_{ζ°, generates the space α_{ζ° of linear differential operators, annulling over the ideal $\mathfrak{I}$ at the point ζ°. A space α with the properties of Theorem 1 can be constructed by the spaces $\{\alpha_{\zeta^\circ}\}_{\zeta^\circ \in F}$).

Remark 1. In the case of $C[x]$, $x \in R^n$, this is a result of A.S.Madguerova [5-7]. In the case of $C[\zeta]$, $\zeta \in C^n$, the result can be precised. A similar descriptionof the polynomial ideals of $C[\zeta]$, $\zeta \in C^n$, follows from the results [5-7], noting only that we have $z = x + iy$, $x, y \in R^1$, for any complex number z. That is why the polynomials in complex variables are polynomials in real variables with complex coefficients. Moreover, any ideal $\mathfrak{I}$ of $C[z_1, \ldots, z_n]$ uniquely generates an ideal $\mathfrak{I}^*$ of $C[x_1, \ldots, x_{2n}]$, where $(z_1, \ldots, z_n) \in C^n$, $(x, \ldots, x_{2n}) \in R^{2n}$. After the results [5-7], the ideal $\mathfrak{I}^*$ is uniquely determined by a C-linear finite-dimensional differential-invariant space α of linear differential operators on R^{2n} and by the corresponding sets $N_{\alpha(q)}$, $q \in Z^{2n}_+$. The coefficients of the operators of the space α have upper semicontinuous modules.

Definition. A polynomial ideal $\mathfrak{I}$ of $K[\zeta]$ will be called a primary polynomial ideal $\mathfrak{I}$ in the sense of G.E.Shilov (or Shilov's primary ideal) if the ideal $\mathfrak{I}$ is contained in a unique maximal ideal of K$[\zeta]$.

Remark 2. The maximal ideals of K$[\zeta]$ are determined by the points ζ (see [2]). The maximal ideal M of K$[\zeta]$, determined by the point ζ° will be denoted by $M = M(\zeta^{\circ})$ and will be called the maximal ideal of K$[\zeta]$ at the point ζ°. A Shilov's primary ideal $\mathfrak{I}$ of K$[\zeta]$, contained in the maximal ideal $M(\zeta^{\circ})$, will be denoted by $\mathfrak{I} = \mathfrak{I}(\zeta^{\circ})$ and will be called a Shilov's primary polynomial ideal of K$[\zeta]$ at the point ζ°.

Theorem 2. Any Shilov's primary polynomial ideal $\mathfrak{I}$ of K$[\zeta]$ has the kind

(1) $$\mathfrak{I} = \mathfrak{I}(\zeta^{\circ}) = \{f : f \in K[\zeta],\ (Af)(\zeta^{\circ}) = 0 \text{ for } \forall A \in \alpha\},$$

where α is a K-linear finite-dimensional differential-invariant space of linear constant-coefficient differential operators on K$[\zeta]$ and the identity operator $I \in \alpha$ ($If = f$).

Inversely, the set (1) is a Shilov's primary polynomial ideal of K$[\zeta]$ at the point ζ° for each fixed K-linear finite-dimensional differential-invariant space α of linear constant-coefficient differential operators with $I \in \alpha$, and for each fixed point ζ°.

Corollary. Each polynomial ideal of K$[\zeta]$ is an intersection of Shilov's primary polynomial ideals.

Proof of Theorem 2. Let the Shilov's primary ideal $\mathfrak{I}$ be contained in the maximal ideal $\mathfrak{M} = \mathfrak{M}(\zeta^*)$ (see Remark 2). Thus the ideal $\mathfrak{I}$ is of the kind $\mathfrak{I} = \mathfrak{I}(\zeta^*)$. The ideal $\mathfrak{I}$ is finitely generated after Hilbert's Theorem on the basis. Let $\mathfrak{I}$ be generated by the polynomials

$$f_1, \ldots, f_m .$$

Let N be the largest degree of the polynomials $f_1, \ldots, f_m$.

Let us scrutinize the natural homomorphism

$$K[\zeta] \longrightarrow K[\zeta]/\mathfrak{M}^{N+1}$$

and let us denote the quotion ring $K[\zeta]/M^{N+1}$ by Q^N. Let I^* be the image in Q^N of the ideal $\mathfrak{I}$ in the latter homomorphism. I^* is an ideal of Q^N. Let $\Psi^0_{\zeta^*}$ be the unique K-linear finite-dimensional space of all linear functionals on Q^N, annulling on I^*. This space Ψ_{ξ^*} uniquely generates the K-linear finite-dimensional space $\alpha = \alpha(\mathfrak{J})$ of all linear constant-coefficient differential operators, annulling on $\mathfrak{J}$ at the point ξ^*. The space α is the prototype of Ψ_{ξ^*} where D^k corresponds formally to the functionals on X^k. Thus the order of the operators of the space α is not larger than N.

This space α is also differential-invariant which is involved by the following : Let $A \in \alpha$. It is sufficient to prove that if

$$A = \sum a_k D^k$$

with some $k_1^* > 0$ for a corresponding $a_{k^*} \neq 0$, $k^* = (k_1^*, \ldots, k_n^*)$, then the operator

$$A^{(1,0,0,\ldots,0)} \in \alpha.$$

We have that

$$A(fg) = \sum \frac{1}{q!} g^{(q)} A^{(q)} f$$

if f, $g \in K[\xi]$ and $f \in \mathfrak{J}$. For the polynomial

$$g = (\xi_1 - \xi_1^*)$$

we receive that

$$0 = A(fg)(\xi^*) = A^{(1,0,\ldots,0)} f(\xi^*) = 0, \text{ i.e., } A^{(1,0,\ldots,0)} \in \alpha.$$

Therefore we have inductively that all operator derivatives $A^{(q)} \in \alpha$.

Moreover, the identity operator $I \in \alpha$, since we have $\mathfrak{J} = \mathfrak{J}(\xi^*)$.

Inversely, each fixed point ξ^* and each K-linear finite-dimensional differential-invariant space α of linear constant-coefficient differential operators with $I \in \alpha$, determine a primary polynomial ideal in the sense of G.E.Shilov in $K[\xi]$,

$$\mathfrak{J} = \mathfrak{J}(\xi^*) = \{ f : f \in K[\xi], Af(\xi^*) = 0 \text{ for } \forall A \in \alpha \}$$

as it is involved by the following: Let N be the largest order of the operators of α. The space α uniquely determines the corresponding space Ψ_{ξ^*} of linear functionals on Q^N. Let I^* be the ideal in Q^N on which ideal I^* all functionals of Ψ_{ξ^*} are annulling.

Then the ideal $\mathfrak{J}(\xi^*)$ is the prototype of $\mathbb{I}^*$ in the canonical homomorphism

$$K[\xi] \longrightarrow Q^N = K[\xi] / \mathfrak{M}^{N+1}(\xi^*).$$

The set $\mathfrak{J}(\xi^*)$ is an ideal in $K[\xi]$ since the space α is differential-invariant. ∎

Proof of Theorem 1. Let F be the analytic variety of all zeros of the ideal $\mathfrak{J}$. If $F = \emptyset$ then we have $\mathfrak{J} = K[\xi]$ and the space $\alpha = \{0\}$. Let now $F \neq \emptyset$ and let $\xi^* \in F$. For the arbitrarily fixed $\xi^* \in F$ let the ideal $\mathfrak{J}_{\xi^*}$ be the image of the ideal $\mathfrak{J}$ in the canonical homomorphism

$$K[\xi] \longrightarrow K[\xi] / \mathfrak{M}^{N+1}(\xi^*) = Q^N_{\xi^*},$$

where N is the largest order of the polynomials of a fixed finite basis of the ideal $\mathfrak{J}$. At first we receive the K-linear finite-dimensional space Ψ_{ξ^*} of all linear functionals, annulling on the ideal $\mathfrak{J}_{\xi^*}$ of $Q^N_{\xi^*}$. The space Ψ_{ξ^*} uniquely determines the K-linear finite-dimensional space α_{ξ^*} of all linear constant-coefficient differential-operators on $K[\xi]$, annulling on $\mathfrak{J}$ at the point ξ^*. Further on, since $\mathfrak{J}$ is an ideal, hence the space α_{ξ^*} is also differential-invariant (confer the proof of Theorem 2). Moreover, let $\alpha_{\xi^*} = \{0\}$ if we have $\xi^* \notin F$. By the set of all spaces

$$\{\alpha_{\xi^*}\}_{\xi^*}$$

we can construct a K-linear finite-dimensional differential-invariant space α of linear differential operators on $K[\xi]$ in a way that if $A \in \alpha$ then at each arbitrarily fixed point ξ^* there exists an operator $A_{\xi^*} \in \alpha_{\xi^*}$ such that A_{ξ^*} is equal to A at the point ξ^* in the sense

$$[A_{\xi^*} f(\xi)]_{\xi=\xi^*} = [Af(\xi)]_{\xi=\xi^*}, \quad \forall f \in K[\xi].$$

The space α can be chosen such that the coefficients of its operators to be rational functions on the indicated in Theorem 1 subsets. We shall construct α more effectively to prove this : Let Φ_{ξ^*} be the canonical homomorphism

$$\Phi_{\xi^*} : K[\xi] \longrightarrow K[\xi] / \mathfrak{M}^{N+1}(\xi^*),$$

where ξ^* is an arbitrarily fixed point of F. Let

$$g=\sum_{r}\frac{(\zeta-\zeta^*)^r}{r!}D^r g(\zeta^*).$$

Then we get

$$\Phi_{\zeta^*}(g)=\sum_{|r|\le N}\frac{Z^r}{r!}D^r g(\zeta^*),$$

with $\Phi_{\zeta^*}(\zeta-\zeta^*)=Z$.

The ideal J_{ζ^*} of $Q^N_{\zeta^*}$ has a basis

$$B[J_{\zeta^*}]=\left\{\sum_{Q<|r|\le N}B^j_r Z^r,\ j=0,1,\dots,t\right\},$$

where the elements

$$\sum_{Q<|r|\le N}B^j_r Z^r,\ j=0,1,\dots,t$$

with $B^0_r=0$, $Q\geq 0$ if $\zeta^*\in F$, are linearly independent ; B^j_r are constants in Z but eventually depend on ζ^*. Therefore we may determine

$$Z^{r_j}=\sum_{Q<|r|\le N,\ r\ne r_\tau,\ |r|\le|r_j|}A^j_r Z^r \pmod{J_{\zeta^*}},\ \tau,j=0,1,\dots,t;$$

$A^0_r=0$; $A^j_r\in K$ and if $|k|>|k_j|$, then $A^j_r=0$. Thus we receive

$$Q_*=Q^N_{\zeta^*}/J_{\zeta^*}=\left\{\sum_{|r|\le N}a_r Z^r,\ Z^{r_j}=\sum_{Q<|r|\le N;\ r\ne r_\tau;\ |r|\le|r_j|}A^j_r Z^r,\ j,\tau=0,1,\dots,t\right\}.$$

Let

$$Q_*\ni\sum_{|r|\le N}a_r Z^r=\sum_{|r|\le N,\ r\ne r_j}a_r Z^r+\sum_{j=0}^{t}a_{r_j}\sum_{\substack{Q<|r|\le N\\ r\ne r_\tau;\ |r|\le|r_j|}}A^j_r Z^r=$$

$$\sum_{|r|\le Q}a_r Z^r+\sum_{Q<|r|\le N;\ r\ne r_\tau,\ |r|\le|r_j|}\left[a_r+\sum_{j=0}^{t}a_{r_j}A^j_r\right]Z^r,\ j,\tau=0,1,\dots,t.$$

Let

$$P(\zeta)=\sum_{r}\left[(\zeta-\zeta^*)^r(D^rP)(\zeta^*)\right]/r!.$$

Then we have

$$\Phi_{\zeta^*}(P)=\sum_{|\varkappa|\le Q} D^{\varkappa}P_*(\zeta^*)Z^{\varkappa}+\sum_{Q<|\varkappa|\le N;\ \varkappa\ne\varkappa_\tau;\ |\varkappa|\le|\varkappa_j|} d_{\varkappa}(\zeta^*).$$

$$\Big[D^{\varkappa}P_*(\zeta^*)/\varkappa!+\sum_{j=0}^{t}A^j_{\varkappa}(\zeta^*)D^{\varkappa_j}P_*(\zeta^*)/\varkappa_j!\Big]Z^{\varkappa}\ (\mathrm{mod}\ J_{\zeta^*}).$$

That is why the space α_{ζ^*} has a basis

$$\mathcal{B}(\alpha_{\zeta^*})=\Big\{D^{\varkappa},\ \forall\varkappa \text{ with } |\varkappa|\le Q;\ M_{\varkappa}=\Big(D^{\varkappa}/\varkappa!+\sum_{j=0}^{t}A^j_{\varkappa}(\zeta^*)D^{\varkappa_j}/\varkappa_j!\Big),$$

$$Q<|\varkappa|\le N,\ \varkappa\ne\varkappa_\tau,\ |\varkappa|\le|\varkappa_j|,\ \tau,j=0,1,\dots,t\Big\}$$

The coefficients $A^j_{\varkappa}$ are rationals relatively $p_1(\zeta),\dots,p_\ell(\zeta)$, and their derivatives where $\{p_1(\zeta),\dots,p_\ell(\zeta)\}$ is a fixed basis of the ideal J. If the matrix $(B^j_{\varkappa}(\zeta^*))_{j,\varkappa}$ at the point ζ^* has the same character relatively its adjunctive elements as the matrix $(B^j_{\varkappa}(\zeta^{**}))_{j,\varkappa}$ then we can choose at the point ζ^{**} the same $Z^{\varkappa_1},\dots,Z^{\varkappa_t}$ as at the point ζ^*, (here $\zeta^*,\zeta^{**}\in F$). Let $\mathcal{P}$ be the set of all zeros of at least one of the polynomials $p_1(\zeta),\dots,p_\ell(\zeta)$ and their derivatives. We can choose the same $Z^{\varkappa_1},\dots,Z^{\varkappa_t}$ out of the set $\mathcal{P}$. Therefore we can construct the space α such that the coefficients of the operators of α to be rational functions out of the set $\mathcal{P}$. ∎

The assertion of Corollary immediately follows from Theorems 1 and 2.

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THE GENERALIZED PETROVSKY'S PARABOLICITY AND FUNCTIONAL SPACES WITH GENERALIZED DERIVATIVES

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A generalization of Petrovsky's parabolicity is given here. Some apriori estimations for the generalized Petrovsky's parabolic differential operators are investigated. The results are new even in the classical case. Some inclusions among functional spaces, characterizing the parabolicity, are studied. The strong generalized derivatives (introduced by the author [1-4]) can be extended in several directions. Fundamental physical laws are expressed by differential equations. In this way Sobolev's spaces, Theory of distributions, the spaces of Bessov, Lizorkin, Triebel are introduced. Although the solutions of systems of linear constant-coefficient differential equations are expressed by distributions, i.e. by linear continuous operators, solutions with more classical properties are important in the mathematical applications. That is why Sobolev's spaces are actual even today and ar an intensively developed part of Analysis. The spaces $\mathcal{W}_{*k}^{A_1,\ldots,A_m}$ of functions with strong generalized derivatives are generalizations of the spaces of Sobolev, Schwartz, Bessov, Lizorkin, Triebel. They do not always coincide with Sobolev spaces W_p^N. For instance, the space $\mathcal{W}_p^{A,I}$, $1 < p < \infty$, where A is a linear constant-coefficient differential operator of order N, coincides with the Sobolev space W_p^N if and if the operator A is elliptic. Furthermore, the spaces

$$\mathcal{W}_{*k}^{A_1,\ldots,A_m}$$

are such that each solution $f \in \mathcal{L}_*$ of the system $\{A_k f = u_k,\ k=1,\ldots,m,\}$ $u_k \in \mathcal{L}_{*k}$, belongs to the space $\mathcal{W}_{*k}^{A_1,\ldots,A_m}$. Moreover, it is given the necessary and sufficient condition for to be these spaces algebras.

The generalization of I.G.Petrovsky [5] of the classical parabolicity is a recognized stage in the development of the theory of the differential operators. Petrovsky's parabolicity has been studied by I.G.Petrovsky and its scool, by O.A.Oleinic, M.S.Agranovich, S.D.Eidel'man, M.I.Vishic, M.I. Ventzel, S.D.Ivasishen. I.M.Gel'fand and G.E.Shilov [6] made a generalization of Petrovsky's parabolicity in some cases. G.E.Shilov denoted in [7] that a future description of some algebras of type C"may serve as a basis of a specific classification of the second order differential operators". Indeed, elaborating such a description, the author observed the necessity of a supplement to the classical definition of a parabolic operator of second order with complex coeffitients, so that the same operator not to be simultaneously elliptic and parabolic, which is a natural requirement to any classification (see Definition 3 and [8,9]). Moreover, the description of the algebras of type C, their inclusions and their comparisons with the classical Sobolev spaces W_p^N open and impose the necessity to extend the definition of Petrovsky's parabolicity. For instance, if A is a constant-coefficient elliptic operator of order $N \geqslant 2$ then the space W_∞^{N-1} with the corresponding supremum norm contains the space $\mathfrak{W}_\infty^{A,I}$,

$$W_\infty^{N-1} \supset \mathfrak{W}_\infty^{A,I}$$

(Here I is the identity operator.) But it is not so if A is a parabolic second-order constant-coefficient differential operator, or a parabolic after I.G.Petrovsky differential operator, or a generalized parabolic differential operator, introduced here. Then we have

$$W_\infty^{N-1} \not\supset \mathfrak{W}_\infty^{A,I}$$

Thus there exists a function $f \in \mathfrak{W}_\infty^{A,I}$, which does not have all continuous partial derivatives till order N-1 inclusive, $f \notin W_\infty^{N-1}$. This is a new result even in the case

$$A = \partial^2/\partial x^2 + a\partial/\partial t, \ a \neq 0.$$

The same is also true for the spaces $\mathfrak{W}_\infty^\alpha$, where α is the linear differential-invariant space, generated by a generalized Petrovsky's parabolic differential operator A of order $N \geq 2$,

$$\mathfrak{W}_\infty^\alpha \not\subset W_\infty^{N-1}$$

Similar results are true for the spaces $\mathfrak{W}_p^{A,I}$, $1 \leq p < \infty$.

Definition 1. The system $(A_q)_q$, q=1,...,m, of linear constant-coefficient differential operators in n+1 variables, $n \geqslant 1$, is called parabolic of order $N \geq 2$ if each operator A_q is of order not larger than N ;

at least one of the operators A_q is of order N ; each operator $\mathcal{A}$ belonging to the linear huul of the system $(A_q)_q$, $q=1,\dots,m$, which operator is of order N, can be transformed by some real linear nondegenerated transform L of R^{n+1} into a differential operator A of the following kind:

(1)
$$Au = \sum_{j=1}^{m} \sum_{|s|+l_{sj}=N} a_{sj} D^s D_t^{l_{sj}} u_j + \sum_{j=1}^{m} D_t^{N-k} \sum_{|s|<k} b_{sj} D^s u_j +$$
$$+ \sum_{j=1}^{m} P_j(D,D_t)\, u_j \quad \text{with} \quad u=(u_1,\dots,u_m), \quad \sum_{j=1}^{m} \sum_{|s|<k} |b_{sj}| \neq 0,$$

where $0 \leq l_{sj} < N-k$ if $N-k \neq 0$, and $l_{sj}=0$ if $N=k$; $0<k\leq N$; P_j, $j=1,\dots,m$, are polynomials of D and D_t of degree less than N and their degrees relatively D_t are less than N-k if $N>k$ and 0 if $N=k$; u_1, $\dots$, u_m are complex-valued functions in (x,t) ; $L\mathcal{U} = u$, $L\mathcal{A}\mathcal{U} = AL\mathcal{U}$ $= Au$; $s=(s_1,\dots,s_n) \in Z_+^n$, $x=(x_1,\dots,x_n)$, $D=\partial/\partial x$.

Remark. The definition of the generalized parabolicity of a system for the nonconstant-coefficient case can be received exiging the requirements of Definition 1 to hold for any point (x,t) of some domain $\mathcal{S}$. The transformation L can depend on $(x,t)\in\mathcal{S}$.

Theorem 1. The Definition 1 is a generalization of the Petrovsky's parabolicity 5 in the constant-coefficient case. (The proof in the nonconstant-coefficient case is almost the same.)

(In the particular case of $m=1$ we have the following simpler

Definition 2. The linear constant-coefficient differential operator $\mathcal{A}$ in n+1 variables, $n\geq 1$, of order $N \geq 2$, is called parabolic if there exists a real nondegenerated linear transformation L of R^{n+1} such that the operator $\mathcal{A}$ is transformed in an operator A of the kind:

(2)
$$A = \sum_{|s|+l_s=N} a_s D^s D_t^{l_s} + D_t^{N-k} \sum_{|s|<k} b_s D^s + P(D,D_t), \quad \sum_{|s|<k} |b_s| \neq 0$$

where $0 \leq l_s < N-k$ if $N-k\neq 0$, and $l_s=0$ if $N=k$; $0<k\leq N$; P is a polynomial of degree less than N and its degree relatively D_t is less than N-k if $N>k$, and 0 if $N=k$; $D=\partial/\partial x$; $x=(x_1,\dots,x_n)$; $D_t = \partial/\partial t$;

$s=(s_1,\dots,s_n) \in Z_+^n$, $|s| = s_1+\dots+s_n$.

Remark. Definition 2 is equivalent for the second-order operators in two variables to the following Definition 3.

Definition 3. The constant-coefficient linear differential operator of second order

$$\mathcal{A} = a_{20}\partial^2/\partial x^2 + 2a_{11}\partial^2/\partial x\partial y + a_{02}\partial^2/\partial y^2 + \sum_{j+r\leq 1} a_{jr}\partial^{j+r}/\partial x^j \partial y^r$$

is called parabolic if $a_{20}a_{02} - a_{11}^2 = 0$ and if $a_{20} \neq 0$ then a_{11}/a_{20} is real, if $a_{02} \neq 0$ then a_{11}/a_{02} is real. Here the constants $a_{jk} \in C$.

Theorem 2. The Definition 2 and Definition 3 are equivalent in the case of linear constant-coefficient differential operators of second order in two variables.

Theorem 3. Definition 2 is a generalization of the Petrovsky's parabolicity in the case of one linear constant-coefficient differential operator.

Theorem 4. Let $A, B_1,\dots,B_m$ be linear constant-coefficient differential operators in variables (x,t), $x=(x_1,\dots,x_n)$, $n \geqslant 1$; Let A be parabolic (after Definition 1) of order $N \geqslant 2$ and respectively D_t let its order be $N-k$, $0 < k < N$. Let

$$A = \sum_{|s|+l_s=N} a_s D^s D_t^{l_s} + D_t^{N-k} \sum_{|s|<k} b_s D^s + P(D,D_t),$$ where $0 \leqslant l_s < N-k$; P is a polynomial in D, D_t of degree less than N and P is of degree less than $N-k$ respectively D_t (see (2)). Let all operators $B_1,\dots,B_m$ be of orders less than N and respectively D_t be of orders less than $N-k$.

Then the operators $A, B_1,\dots,B_m$ do not jointly dominate the operator D^{N-k_*}, $k \geqslant k_* > 0$ on $\mathcal{D}_\rho$, $0 < \rho \leq \infty$. I.E; does not exist such a constant $\varkappa$ that

$$\|D_t^{N-k_*} g\| \leq \varkappa \{\|Ag\| + \|B_1 g\| + \dots + \|B_m g\|\}$$

for $\forall g \in \mathcal{D}_\rho$, where $\mathcal{D}_\rho$ is the set of all real-valued infinitely differentiable functions with compact supports in $\{(x,t) : |(x,t)| < \rho\}$;
$\|f\| = \sup_{(x,t)} |f(x,t)|$.

Remark. The additional condition in Definition 3 for the parabolicity of a linear second order constant-coefficient differential operator assures that the operator $\mathcal{A}$ cannot be simultaneously parabolic and elliptic, which permits the usual definition for the parabolicity of linear second-order constant-coefficient differential operators. For instance, if $B = (\partial/\partial x + i\partial/\partial y)^2$ then we have for the operator B that $a_{20}a_{02} - a_{11}^2 = 0$. But simultaneously the second order homogeneous part of the characteristic polynomial of the operator B is zero on R^2 only at the origin of the plane R^2. Thus the operator B is also simultaneously and elliptic. ~~But~~ fundamental exigement to any classification is different classes not to intersect.

Lemma 5. Let A be a linear constant-coefficient differential operator in n variables. Let K be a compact in R^n with $\overset{\circ}{\bar{K}} = K$, $K \neq \emptyset$. Let the space L_* of complex-valued functions, $L_* \subseteq L_1$, be the completion of $C^\infty | K$ in a norm τ, which is stronger than the convergence in the distribution space $\mathcal{D}'$. Let there exist for the function $f \in L_*$ such a sequence $\{\varphi_m\}$, $\forall \varphi_m \in C^\infty | K$, that

$$\{\varphi_m\} \xrightarrow{*} f, \text{ i.e. } \tau(\varphi_m - f) \longrightarrow 0 \text{ and } \{A\varphi_m\} \xrightarrow{*} F.$$

If for another sequence $\{\psi_m\}$, $\forall \psi_m \in C^\infty | K$, we have

$$\{\psi_m\} \xrightarrow{*} f \quad \text{and} \quad \{A\psi_m\} \xrightarrow{*} G, \text{ then we have}$$

$F = G$ in L_*. (Here C^∞ is the set of all infinitely differentiable functions on R^n.)

Definition 4. Let the function $f \in L_*$. Let exist such a sequence $\{\varphi_m\}$ $\forall \varphi_m \in C^\infty | K$, that

$$\{\varphi_m\} \xrightarrow{*} f \quad \text{and} \quad \{A\varphi_m\} \xrightarrow{*} H \text{ in } L_*.$$

Then the function H will be called a strong generalized A_* derivative of the function f and will be denoted by $A_* f$.

Theorem 5'. If a function $f \in L_*$ has an A_* strong generalized derivative, then this derivative is uniquely determined by the function f and by the operator A_*. (This derivative does not depend on the choice of the auxiliary sequence $\{\varphi_m\}$, used in Definition 4).

Let $A_1, \ldots, A_m$ be linear constant-coefficient differential operators. The subspace of L_*, consisting of all functions $f \in L_*$ with $A_1, \ldots, A_m$ strong generalized derivatives $A_{1*}f, \ldots, A_{m*}f$ in L , we shall denote by $\mathcal{W}_*^{A_1, \ldots, A_m}$.

The strong generalized derivatives are extended further in the following direction (cf. [4]): 1. It is constructed an extension of the strong generalized derivatives in the case when the base spaces $\mathcal{L}_*$ have a more complicated structure than the spaces $L_* \subseteq L_1$;

2. It is constructed an extension of the strong generalized derivatives in the case when the used operators $A_1, \ldots, A_m$ are not linear constant-coefficient differential operators;

3. It is constructed an extension of the strong generalized derivatives in the case when the function f belongs to the space $\mathcal{L}_*$ and its strong generalized derivatives $A_{k*}f$ belong to the spaces $\mathcal{L}_{*k}$. The corresponding space of all such functions f is denoted by $\mathcal{W}_k^{A_1, \ldots, A_m}$;

4. It is construct an extension of the strong generalized derivatives in the case of functions, defined on manifolds.

Now let W_*^N be the Sobolev space of the restrictions on K of all complex-valued functions of L_* with partial derivatives of order till N inclusive in L_* . Further, let $L_* = L_p = L_p(K)$, $p=1,2,\ldots;\infty$. Here L_∞ is examined with the supremum norm for the importance of the continuity and for concisement.

Theorem 6. Let A be a linear constant-coefficient differential operator of order $N \geq 2$ in $n \geq 2$ variables. We have

$$\mathcal{W}_p^{A,I} = W_p^N \quad \text{for } 1 < p < \infty$$

if and only if the operator A is elliptic.

We have

$$\mathcal{W}_\infty^{A,I} \subset W_\infty^{N-1} \quad \text{and} \quad \mathcal{W}_1^{A,I} \subset W_1^{N-1}$$

if A is elliptic (and only if A is elliptic for $n \geq 3$).

Theorem 7. Let A be a linear generalized Petrovsky's parabolic (after

Definition 2) constant-coefficient differential operator of order $N \geqslant 2$ in n $n \geqslant 2$ variables. Then we have

$$\mathcal{W}^{A}_{\infty} \not\subset W^{N-1}_{\infty}$$

and moreover,

$$\mathcal{W}^{A,\ldots,A^{(s)},\ldots,I}_{\infty} \not\subset W^{N-1}_{\infty},$$

where $A^{(s)}$ is an operator derivative of the operator A, $s = (s_1,\ldots,s_n) \in Z^n_+$. This is, if

$$A = \sum_{\ell} c_{\ell} D^{\ell} \text{, then } A^{(s)} = \sum_{s \leq \ell} c_{\ell}\, s! \binom{\ell}{s} D^{\ell - s}.$$

Definition . A set α of linear constant-coefficient differential operators is called differential-invariant set of linear constant coefficient differential operators if α is such that $A \in \alpha$ implies that all operator derivatives $A^{(s)}$, $s \in Z^n_+$, of the operator A belong also to the set α.

Theorem 8. Let the set $\alpha \neq \emptyset$ of linear constant-coefficient differential operators $A_1,\ldots,A_m$ in n variables is differential-invariant. Let K be a compact connected set with $\overline{\overset{\circ}{K}} = K$, $K \subset R^n$. Then the space $\mathcal{W}^{\alpha}_{\infty}(K) = \mathcal{W}^{A_1,\ldots,A_m}_{\infty}(K)$ is an algebra of complex-valued functions of type C on K respectively the pointwise multiplication and the norm

$$\|f\| = \sum_{j=1}^{m} \sup_{x \in K} \left| A_j f(x) \right| .$$

Theorem 9. Let the system $\alpha = (A_q)_q$, $q = 1,\ldots,m$, of linear constant-coefficient differential operators be differential-invariant and parabolic (in the generalized Petrovsky's sense of Definition 1) system of order $N \geqslant 2$ in $n \geqslant 2$ variables. Then we have

$$\mathcal{W}^{\alpha}_{\infty} \not\subset W^{N-1}_{\infty}$$

Theorem 10. If the system $\alpha = (A_q)_q$, $q=1,\ldots,m$, of linear constant-coefficient differential operators is differential-invariant system of second order in $n \geqslant 2$ variables and if

$$\mathcal{W}^{\alpha}_{\infty} \not\subset W^{1}_{\infty}$$

then α is a generalized parabolic system (in the sense of Definition 1).

Remark. Theorem 8 shows the sufficiency of the differential-incariance of α for the algebraicity of the space $\mathcal{W}^{\alpha}_{\infty}$. It was proved [8-10] and the necessity of this condition.I.e. it is proved that if $\mathcal{W}^{\alpha}_{\infty}$ is an algebra of type C then the space α is differential-invariant.

We use un the proof the following generalization of a theorem of K.de Leeuw,H.Mirkil [11]:

Theorem 11. Let $A, A_1, \ldots, A_m$ be linear differential operators for which we have: 1. The coefficients of the operators $A_1, \ldots, A_m$ are defined and are continuous on some neighbourhood $U \subset R^n$ of the origin $0 = (0,\ldots,0)$; 2. The orders of the operators $A_1, \ldots, A_m$ are not larger than N on U ; 3. The operator A is defined on U and there exists a constant $\varkappa$, for which

$$|Ag(0)| \leq \varkappa \left(\|A_1 g\|_C + \cdots + \|A_m g\|_C\right)$$

for $\forall\, g \in \mathcal{D}^{\infty}$ with a support - $\operatorname{supp} g \subset \{x : |x| < \varepsilon\} \subset U$ for some $\varepsilon > 0$. Here $\|f\|_C = \sup\limits_{x \in R^n} |f(x)|$, $f \in \mathcal{D}^{\infty}$. Then it is true the following:

1. The order of the operator A at the point $0 = (0,\ldots,0)$ is not larger than N ; 2. If we denote the homogeneous parts of the operators A , $A_1, \ldots, A_m$ of order N by A^N ,A_1^N ,..., A_m^N correspondingly, then we have

$$A^N(0) = \sum_{j=1}^{m} c_j A_j^N(0)$$ at the origin , where $c_1, \ldots, c_m$ are constants.

Corollary 12. Let $A, A_1, \ldots, A_m$ be linear differential operators with coefficients in the open set $U \subset R^n$,

$$A = \sum_{|k| \leq M} a_k D^k \;, \quad A_j = \sum_{|k| \leq N} a_{jk} D^k \;, \quad j = 1,\ldots, m \;,$$ for which

1. The complex-valued functions a_{jk} , j , $k = 1,\ldots,m$, are continuous ;
2. The orders of the operators $A_1, \ldots, A_m$ are not larger than N on U ;
3. There exists such a constant $\varkappa$, that

$$\|Ag\|_C \leq \varkappa(\|A_1 g\|_C + \cdots + \|A_m g\|_C)$$ for $\forall\, g \in \mathcal{D}^{\infty}$ with sup

ports , supp $g \subset U$ and diam supp $g < \varepsilon$ for some $\varepsilon > 0$, then :

1. The order of the operator A on U is not larger than N ;
2. We have on U that

$$A^N = c_1(x)\, A_1^N + \dots + c_m(x)\, A_m^N ,$$ where $c_1(x), \dots, c_m(x)$ are complex-valued functions on U ;

3. If the functions $a_k(x)$, $a_{jk}(x)$, $|k| = N$, $j = 1,\dots,m$, are continuous on U and if

$$\operatorname{rang}\ (\, a_{jk}(x)\,)_{(|k| = N\ ,\ j=1,\dots,m)} = m \ ,$$ then

the functions $c_j(x)$, $j=1,\dots,m$, are also continuous on U .

The generalized here Theorem of K.de Leeuw, H.Mirkil [11] is about linear constant-coefficient differential operators.

We have constructed the following extensions of the spaces $\mathcal{L}_p^{A_1,\dots,A_m}$ (see [3,4,10]):

I. Let the sets K_α , G be subsets of R^n , $n \geqslant 1$, with the closures $\overline{K_\alpha}$, $\overline{G}$, coinciding with the closures of their corresponding open sets,i.e.

$$\overline{\overset{\circ}{K}_\alpha} = \overline{K}_\alpha \ , \quad \text{and} \quad \overline{\overset{\circ}{G}} = \overline{G} ,$$ where $\alpha \in \mathcal{S}$, and $\mathcal{S}$ is a family of indexes.

Let $\mathcal{P}(K_\alpha)$ be a linear space of complex-valued functions on K_α with a locally convex topology, generated by the set of seminorms

$$\{ \gamma_{\alpha,s} \}_s \quad , \quad s = 0 \ , 1, 2 \ , \dots \ .$$

The corresponding locally convex topology of $\mathcal{P}(K_\alpha)$ is as usual with the base of neighbourhoods U_ψ of any function $\psi \in \mathcal{P}(K_\alpha)$ of the kind

$$U_\psi = \{ \varphi \in \mathcal{P}(K_\alpha),\ \gamma_{\alpha,s_r}(\varphi - \psi) < \varepsilon_r,\ r = 1,\dots,z \},\ \varepsilon_r > 0 .$$

Let $\mathcal{J}$ be a up-ordered Shatunovsky system, i.e., $\mathcal{J}$ is a partially ordered family of indexes such that if $\alpha, \beta \in \mathcal{J}$ then there exists $\delta \in \mathcal{J}$ with

$$\alpha \leqq \delta , \ \beta \leqq \delta .$$

(In the exposed case it suffices that $\mathcal{J} = \{ 0, 1 , 2 , \dots \}$, but there exist interesting generalizations.)

Let the space $\mathcal{D}$ (G) be the inductive limit of the spaces $\mathcal{D}(K_\alpha)$ with

$$\bigcup_{\alpha\in\mathcal{S}} K_\alpha = G, \quad \text{if } \alpha \leq \beta \text{ then } K_\alpha \subseteq K_\beta,\ \mathcal{D}(K_\alpha) \subseteq \mathcal{D}(K_\beta)$$

and the maps i_α^β be inclusions, where i_α^β are the maps from the definition of the inductive limit.

Definition 5. Let $\mathcal{X}$ be a system of locally convex topological spaces. X_α and continuous maps $i_\alpha^\beta : X_\alpha \longrightarrow X_\beta,\ \alpha, \beta \in \mathcal{S}$, $\alpha \leq \beta$,

$$\mathcal{X} = \{X_\alpha,\ i_\alpha^\beta,\ \alpha, \beta \in \mathcal{S}\},$$

where $\mathcal{S}$ be a up-ordered Shatunovsky system of indexes, such that

1. i_α^α is the identity map for $\forall \alpha \in \mathcal{S}$;
2. $i_\beta^\delta\, i_\alpha^\beta = i_\alpha^\delta$ if $\alpha \leq \beta \leq \delta$.

Let us introduce the relation of equivalence in the space $X = \bigcup_{\alpha\in\mathcal{S}} X_\alpha$ so that the element $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$ are equivalent iff there exists index $\delta \in \mathcal{S}$, $\alpha \leq \delta$, $\beta \leq \delta$ such that

$$i_\alpha^\delta x_\alpha = i_\beta^\delta x_\beta$$

Let $\overset{In}{X}$ be the space of all classes of equivalence in the space $X = \bigcup_{\alpha\in\mathcal{S}} X_\alpha$.

Let i_α be the mapping $i_\alpha : X_\alpha \longrightarrow \overset{In}{X}$ which corresponds to any element $x_\alpha \in X_\alpha$ its class of equivalence. The space $\overset{In}{X}$, examined with the most strong locally convex topology in which all maps i_α, $\alpha \in \mathcal{S}$, are continuous, will be called the inductive limit of the system of the locally convex vector spaces X_α and their inclusions i_α^β , $\alpha, \beta \in \mathcal{S}$.

Further on, let the space $\mathcal{D}$ (G) be with the strongest locally convex topology in which all maps $i_\alpha : \mathcal{D}(K_\alpha) \longrightarrow \mathcal{D}$ (G) for $\forall \alpha \in \mathcal{S}$ are continuous, where i_α , $\alpha \in \mathcal{S}$, are the maps from Definition 5 of the inductive limits.

Let the space $\mathcal{D}'$(S) be the dual space of the space $\mathcal{D}$ (S) , where S may be some of the sets K_α, $\alpha \in \mathcal{S}$ or G . I.e. $\mathcal{D}'$(S) is the space of all linear continuous functionals f defined on the space $\mathcal{D}$ (S) .

The functional f(φ) for $\varphi \in \mathcal{D}$ (S) let us denote by (f ,φ) .

Let us fix a method by which every $\psi \in \mathcal{D}$ (S) defines a linear continuous functional on the space $\mathcal{D}(S)$, denoted by (ψ, φ) and such that the functional (ψ, φ) on the space $\mathcal{D}(G)$ is the corresponding projective limit of the functionals $(\psi_\alpha, \varphi_\alpha)$ on the spaces $\mathcal{D}(K_\alpha)$, $\alpha \in \mathcal{S}$, requiring in addition that the family $\{\mathcal{D}(K_\alpha)\}_\alpha$, $\alpha \in \mathcal{S}$ is regular (see Definition 7).

Definition 6. Projective limits. Let $\mathcal{X}$ be a system of linear locally convex topological spaces X_α with projective maps $i_\alpha^\beta : X_\alpha \longrightarrow X_\beta$, $\alpha \succeq \beta$, $\alpha, \beta \in \mathcal{S}$, where $\mathcal{S}$ is a up-ordered Shatunovsky system of indexes and the projective maps i_α^β are such that 1. i_α^α is the identity map for $\forall \alpha$; 2. $i_\beta^\delta\, i_\alpha^\beta = i_\alpha^\delta$, $\delta \preceq \beta \preceq \alpha$, $\alpha, \beta, \delta \in \mathcal{S}$.

Threads will be called any construction of the kind

$$x = \left\{ x_\alpha \in X_\alpha,\ i_\alpha^\beta x_\alpha = x_\beta,\ \alpha \succeq \beta,\ \alpha, \beta \in \mathcal{S} \right\}.$$

The linear space of all treads $\overset{Pr}{X} = \lim_{proj} \left\{ X_\alpha, i_\alpha^\beta, \alpha, \beta \in \mathcal{S} \right\}$ will be called a projective limit of the system $\mathcal{X}$ of the linear locally convex topological spaces X_α and the projective maps i_α^β, $\alpha, \beta \in \mathcal{S}$, if the space $\overset{Pr}{X}$ is examined with the most weak locally convex topology, in which all maps $i^\alpha : \overset{Pr}{X} \longrightarrow X_\alpha$ are continuous, $\forall \alpha \in \mathcal{S}$, where $i^\alpha x = x_\alpha$.

Definition 7. A family $\mathcal{X} = \left\{ X_\alpha, i_\alpha^\beta, \mathcal{S}, \alpha \preceq \beta, \alpha, \beta \in \mathcal{S} \right\}$ is called a regular family if any bound subset B of the inductive limit $\overset{In}{X}$ is equal to $i_\alpha(B_\alpha)$ for some $\alpha \in \mathcal{S}$ and for some bound subset B_α of the space X_α, where the spaces X_α, the maps i_α^β and $\mathcal{S}$ satisfy the conditions of Definition 5.

Theorem (Godement [12], V.P.Palamodov [13]). Let $\mathcal{X}$ be a regular family $\left\{ X_\alpha, i_\alpha^\beta, \alpha \preceq \beta, \alpha, \beta \in \mathcal{S} \right\}$ satisfying the conditions of Definition 5. Let X'_α, $\alpha \in \mathcal{S}$, $(\overset{In}{X})'$ be the dual spaces of the spaces X_α, $\overset{In}{X} = \lim_{ind} \left\{ X_\alpha, i_\alpha^\beta, \alpha \preceq \beta, \alpha, \beta \in \mathcal{S} \right\}$ correspondingly and the mappin j_β^α be the dual mappings of i_α^β, $\alpha, \beta \in \mathcal{S}$. Then it holds the natural isomorphism

$$(\overset{In}{X})' \cong \lim_{proj} \left\{ X'_\alpha, j_\beta^\alpha, \alpha \preceq \beta, \alpha, \beta \in \mathcal{S} \right\}.$$

That is why the convergence in the dual space $\mathcal{D}'(S)$ is the usual weal convergence, i.e. $(f_m) \longrightarrow f$, where f_m, $f \in \mathcal{D}'(S)$ if

$\{(f_m,\varphi)\} \longrightarrow (f,\varphi)$ for $\forall \varphi \in \mathcal{D}$ (S).

Definition 8. We shall say that the space $\mathcal{D}$(S) is a space of fundamental functions and that its dual space $\mathcal{D}$'(S) of all linear continuous functionals on $\mathcal{D}$ (S) is a distribution space.

Let the operators A and B with A : $\mathcal{D}(K_\alpha) \longrightarrow \mathcal{D}(K_\alpha), \forall \alpha \in \mathcal{J}$, B : $\mathcal{D}(K_\alpha) \longrightarrow \mathcal{D}(K_\alpha)$, $\forall \alpha \in \mathcal{J}$, be linear continuous operators for which

$(A\psi,\varphi) = (\psi, B\varphi)$ for $\forall \varphi, \forall \psi \in \mathcal{D}(K_\alpha)$, $\forall \alpha \in \mathcal{J}$ and let

$i^{\beta}_{\alpha} \omega\varphi = \omega\, i^{\beta}_{\alpha} \varphi$, $\forall \varphi \in \mathcal{D}(K_\alpha)$, $\forall \alpha$, $\forall \beta \in \mathcal{J}$, $\alpha \leq \beta$, $\omega =$ A , B .

Then the projective limits of the operators A and B exist on the space $\mathcal{D}$ (G) , and will be denoted also by A and B correspondingly. We shall define the operator A on the distribution space $\mathcal{D}$'(G) by the equation

$$(Af,\varphi) \overset{\text{def.}}{=\!=\!=} (f, B\varphi) \;.$$

Let the space of complex-valued functions $\mathcal{L}_*(K_\alpha)$ be the completion of the space $\mathcal{D}(K_\alpha)$ in the complete family of seminorms $\{\varrho^{\alpha}_{\delta}\}_{\delta}$, which induces a locally convex topology in the space $\mathcal{L}_*(K_\alpha)$, and $\mathcal{D}(K_\alpha) \subset \mathcal{L}_*(K_\alpha)$. If the sequence $\{\varphi_m\}$, $\varphi_m \in \mathcal{D}(K_\alpha)$, is a fundamental Cauchy sequence relatively the family of seminorms $\{\varrho^{\alpha}_{\delta}\}_{\delta}$, determing the function $h \in \mathcal{L}_*(K_\alpha)$, let this imply that the sequence $\{i^{\beta}_{\alpha}\varphi_m\}$ is a fundamental Cauchy sequence relatively the family of seminorms $\{\varrho^{\alpha}_{\delta}\}_{\delta}$, determing the function $g \in \mathcal{D}(K_\beta)$. Then we can extend the mappings i^{β}_{α} ; $\alpha,\beta \in \mathcal{J}$, on the spaces $\mathcal{L}_*(K_\nu)$ in the following way:

$i^{*\beta}_{\alpha} : \mathcal{L}_*(K_\alpha) \longrightarrow \mathcal{L}_*(K_\beta)$, $i^{*\beta}_{\alpha}$ is with $i^{*\beta}_{\alpha}$ h = g , where h and g are from the latter requirement.

Now we can construct the inductive limit $\mathcal{L}_*(G)$ of the system $\{\mathcal{L}_*(K_\alpha)$, $i^{*\beta}_{\alpha}$, $\alpha \leq \beta$, $\alpha,\beta \in \mathcal{J}\}$; $\mathcal{L}_*(G) = \lim_{\text{ind}}\{\mathcal{L}_*(K_\alpha), i^{*\beta}_{\alpha}, \alpha\leq\beta, \mathcal{J}$ since the requirements from Definition 5 of the inductive limit are satisfied by this system. Let the method, by which each $\psi \in \mathcal{D}$ (S) defines a linear continuous functional on $\mathcal{D}$ (S), be extended on $\mathcal{L}_*$(S) : any function $\psi \in \mathcal{L}_*$(S) defines a linear continuous functional on $\mathcal{D}$(S), denoted by (ψ,φ) and such that (ψ,φ) on $\mathcal{D}$(G) is the corresponding projective limit of the functionals on $\mathcal{D}(K_\alpha)$, $\alpha \in \mathcal{J}$. Moreover, let the topology in the space $\mathcal{L}_*$ (S) be stronger than the convergence in the distribution space $\mathcal{D}'$(S) in the mention sense. We shall use the notation $\{\mathcal{X}_m\} \xrightarrow{*} \mathcal{X}$ in the space $\mathcal{L}_*$ (S) iff $\rho_\delta^\alpha(\mathcal{X}_m - \mathcal{X}) \longrightarrow 0$ for $\forall\delta$ when $S = K_\alpha$, and $\rho_\delta^\alpha(\mathcal{X}_{\alpha m} - \mathcal{X}_\alpha) \longrightarrow 0$ for $\forall\delta$ for $\forall\, \alpha \in \mathcal{J}$, $\mathcal{X}_{m\alpha} \in \mathcal{X}_m$, $\mathcal{X}_\alpha \in \mathcal{X}$, when S = G .

Let $\{\mathcal{X}_m\} \xrightarrow{*} 0$ in the space $\mathcal{L}_*$ (S) implies its weak convergence in the distribution space $\mathcal{D}'$(S) ,i.e. implies that $\{(\mathcal{X}_m, \varphi)\} \longrightarrow 0$ for $\forall\, \varphi \in \mathcal{D}$(S) , $S = K_\alpha$, G , $\alpha \in \mathcal{J}$.

Let H = M in $\mathcal{D}'$(S) and H , M $\in \mathcal{L}_*$ (S) involve that H = M in $\mathcal{L}_*$(S) (Some of these requirements are consequences, but since the main subject here is not this extension, we shall not scrutinize the spaces $\mathcal{L}_*$ (S) minutely in the general case.)

Theorem 12. Let the function $h \in \mathcal{L}_*$ (S). Let there exist a sequence $\{\varphi_m\}$, $\varphi_m \in \mathcal{D}$ (S), such that $\{\varphi_m\} \xrightarrow{*} h$ in $\mathcal{L}_*$ (S) and $\{A\varphi_m\} \xrightarrow{*} H$ in $\mathcal{L}_*$ (S) . If there exists another sequence $\{\psi_m\}$, $\psi_m \in \mathcal{D}$ (S) , with

$$\{\psi_m\} \xrightarrow{*} h \ , \qquad \{A\psi_m\} \xrightarrow{*} M ,$$

then we have H = M in $\mathcal{L}_*$ (S) .

Proof. Since $(\varphi_m - \psi_m) \longrightarrow 0$ in the space $\mathcal{L}_*$(S) , then $(\varphi_m - \psi_m) \longrightarrow 0$ in the distribution space $\mathcal{D}'$(S) . This involves that H = M in $\mathcal{D}'$(S). Since we have also H , M $\in \mathcal{L}_*$ (S) (as functions, uniquely determing functionals on $\mathcal{D}$ (S)), hence H = M as elements of the space $\mathcal{L}_*$(S) . ∎ Therefore we can give the following definition:

Definition 9. Let the function $h \in \mathcal{L}_*$ (S). Let there exist a sequence $\{\varphi_m\}$, $\varphi_m \in \mathcal{D}$ (S) , such that $\{\varphi_m\} \xrightarrow{*} h$ in $\mathcal{L}_*$ (S) and $\{A\varphi_m\} \xrightarrow{*} H$ in $\mathcal{L}_*$ (S) . Then the function H will be called a generalized strong A_* derivative of the function H and will be denoted by A_*h . Theorem 12 assures that if such derivative exists for the function $h \in \mathcal{L}_*$(S) it would be unique. (Here S may be equal to K_α, $\alpha \in \mathcal{J}$, or to G).

Let the space $W_*^{A_1,\ldots,A_m}(K_\alpha)$ be the completion of the space $\mathcal{D}(K_\alpha)$ in the complete family of seminorms $\{\pi_\delta\}_\delta$, $\pi_\delta\varphi = \rho_\delta\varphi + \sum_{k=1}^{m} \rho_\delta A_k\varphi$,

where $A_1,\dots A_m$ are linear continuous operators with the properties of the operators A and B above. Hence $\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(K_\alpha)$ is the space of all complex-valued functions from the space $\mathcal{L}_*(K_\alpha)$ which also have the strong generalized derivatives $A_1,\dots, A_m$ in the space $\mathcal{L}_*(K_\alpha)$.

Furthermore, each solution $h \in \mathcal{L}_*(K_\alpha)$ of the system $\{A_k h = u_k\}$, $k=1,\dots,m$, $u_k \in \mathcal{L}_*(K_\alpha)$, belongs to the space $\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(K_\alpha)$.

Remark. Analogous nontrivial constructions by projective limits of functional spaces give analogous results. Constructions by projective limits of spaces are not without interest, but the inductive limits of spaces are traditional in Distribution Theory (see Britchkov , Prudnikov [14]).

It is almost evident now that the system

$$\left\{\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(K_\alpha),\ i^{\beta}_{\alpha}\Big|\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(K_\alpha),\ \alpha \leq \beta,\ \alpha, \beta \in \mathcal{S}\right\}$$

has the properties, exiged by Definition 5.That is why we can construct the space

$$\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(G) = \lim_{\text{ind}}\left\{\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(K_\alpha),\ i^{\beta}_{\alpha}\Big|\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(K_\alpha)\ \ \alpha \leq \beta, \alpha, \beta \in \mathcal{S}\right\}.$$

The space $\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(G)$ also has the mentioned properties of the spaces $\mathcal{W}^{\ell\, A_1,\dots,A_m}_{*}(K_\alpha)$.

II. Let us fix the ideas and the results for the following case. Let the space of complex-valued functions $\mathcal{L}_*(K) \subseteq L_1(K)$ be the completion of $\mathcal{D}^\infty|K$ in the norm ρ , which is stronger than the weak convergence in the distribution space $\mathcal{D}'|K$. Here $\mathcal{D}^\infty$ is the space of all complex-valued function on R^n with compact supports and infinitely differentiable; the compact K is a subset of R^n with $\overline{\mathring{K}} = K$; $n \geq 2$. Let $H, M \in \mathcal{L}_*(K)$ and $H = M$ in $\mathcal{D}'|$ imply that $H = M$ in $\mathcal{L}_*(K)$ also. Let A be a linear differential operator on the open subset Ω of R^n , $\Omega \supseteq K$, A be with infinitely differentiable coefficients on Ω. It is well known that the operator A has a conjugate operator B on Ω with infinitely differentiable coefficients linear differential operator. The assertion that the operators A and B are conjugate signifies that

$\int (A\Phi)\Psi\, dx = \int \Phi\, B\Psi\, dx$ for $\forall \Phi, \forall \Psi \in \mathcal{D}^\infty(\Omega)$, where $\mathcal{D}^\infty(\Omega)$ is the space of all complex-valued infinitely differentiable functions with compact supports in Ω.

Let the function $h \in \mathcal{L}_*(K)$ and let there exist a sequence $\{\varphi_m\}$, $\varphi_m \in \mathcal{D}^\infty | K$, such that $\{\varphi_m\} \xrightarrow{*} h$, $\{A\varphi_m\} \xrightarrow{*} H$ (i.e., $\rho(\varphi_m - h) \longrightarrow 0$ and $\rho(A\varphi_m - H) \longrightarrow 0$).

Let $\{\psi_m\}$, $\psi_m \in \mathcal{D}^\infty | K$ be another sequence for which $\{\psi_m\} \xrightarrow{*} h$, $\{A\psi_m\} \xrightarrow{*} M$.

Then we have

$$\int_K [A(\varphi_m - \psi_m)]\,\Phi\, dx = \int_K (\varphi_m - \psi_m) B\Phi\, dx \longrightarrow 0 \text{ as } m \longrightarrow \infty,$$

for $\forall \Phi \in \mathcal{D}^\infty | K$. Therefore we receive $H = M$ in $\mathcal{D}' | K$. After the requirements on the space $\mathcal{L}_*(K)$, this involves that $H = M$ in the space $\mathcal{L}_*(K)$ also. Hence we can define uniquely the function H as a generalized strong A derivative of the function h and we shall denote it Ah, since this derivative depends only on the function h and the operator A and does not depend on the choice of the sequence $\{\varphi_m\}$.

Remark. The spaces $L_p(K)$, $1 \le p \le \infty$, satisfy all requirements on the space $\mathcal{L}_*(K)$ here.

Let the space $W_*^{A_1,\dots,A_m}(K)$ be the completion of the space $\mathcal{D}^\infty | K$ in the norm π,

$$\pi\varphi = \rho\varphi + \sum_{k=1}^{m} \rho A_k \varphi ,$$

where $A_1,\dots,A_m$ are linear differential operators on Ω with infinitely differentiable coefficients. Hence $W_*^{A_1,\dots,A_m}(K)$ is the space of all complex-valued functions of the space $\mathcal{L}_*(K)$ which have strong generalized A_k, $k=1,\dots,m$, derivatives in the space $\mathcal{L}_*(K)$.

Further on, let the function $h \in \mathcal{L}_*(K)$ be a solution of the system $\{A_k h = u_k\}$, $k=1,\dots,m$, with $u_k \in \mathcal{L}_*(K)$. Then the function h belongs to the space $W_*^{A_1,\dots,A_m}(K)$.

It is proved in [10] stronger result than Theorem 8, i.e. it is proved that the spaces $\mathcal{W}_{\infty}^{A_1,\dots,A_m}(K)$ are algebras of complex-valued functions of type C on K iff the system $\{A_k, k=1,\dots,m\}$ of linear differential operators on Ω with infinitely differentiable coefficients is a differential-invariant system, i.e. the linear hull of the system $\{A_k, k=1,\dots,m\}$ is a linear differential-invariant space on K.

III. Let the linear continuous operators $A_1,\dots,A_m$, $A_k : \mathcal{D}(K_\alpha) \longrightarrow \mathcal{D}(K_\alpha)$, $\forall\alpha$, $k=1,\dots,m$, be with the properties of the operator A from page 21, with the corresponding conjugate linear continuous operators $B_1,\dots,B_m$. Let the spaces $\mathcal{L}_{1*}(S),\dots,\mathcal{L}_{m*}(S)$, $S=K_\alpha$, G, be analogous of the spaces $\mathcal{L}_*(S)$, received by completions with the complete systems of seminorms $\{{}_k\rho_\delta^\alpha\}_\delta$ correspondingly, $k=1,\dots,m$. Let the spaces $\mathcal{L}_{k*}(S)$, $k=1,\dots,m$, have the properties of the space $\mathcal{L}_*(S)$, so that the result of Theorem 12 to be true for any of the spaces $\mathcal{L}_{k*}(S)$ when the function $h\in\mathcal{L}_*(S)$ and its strong generalized derivatives $A_{k*}h$ belong to $\mathcal{L}_{k*}(S)$, $k=1,\dots,m$. We have the following:

Theorem 13. Let the space $\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha)$ be the completion of the space $\mathcal{D}(K_\alpha)$ in the complete family of seminorms $\{\pi_\delta^\alpha\}_\delta$,

$$\pi_\delta^\alpha\varphi = \rho_\delta^\alpha\varphi + \sum_{k=1}^{m} {}_k\rho_\delta^\alpha\, A_k\varphi .$$

Hence $\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha)$ is the space of all complex-valued functions f from the space $\mathcal{L}_*(K_\alpha)$, which have the strong generalized derivatives $A_{k*}f$ in the space $\mathcal{L}_k(K_\alpha)$, $k=1,\dots,m$.

Furthermore, each solution $h\in\mathcal{L}_*(K)$ of the system $\{A_kh=u_k,\ k=1,\dots,m\}$ with $u_k\in\mathcal{L}_{k*}(K_\alpha)$, belongs to the space $\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha)$.

It is almost evident that the family

$$\left\{\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha),\ i_\alpha^{*\beta}\big|\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha),\ \alpha\preceq\beta,\ \alpha,\beta\in\mathcal{S}\right\}$$

has the properties, exiged by Definition 5 of the inductive limits. That is why we can construct the space $\mathcal{W}_*^{A_1,\dots,A_m}(G)=\lim_{\text{ind}}\{\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha),\ i_\alpha^{*\beta}|\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha),\ \alpha\preceq\beta,\ \alpha,\beta\in\mathcal{S}\}$.

The space $\mathcal{W}_*^{A_1,\dots,A_m}(G)$ also has the properties of the spaces $\mathcal{W}_*^{A_1,\dots,A_m}(K_\alpha)$ of the previous points of Theorem 13.

IV. Let G be an n-dimensional C^∞ differentiable manifold. Let C_G^ν be the algebra of all complex-valued functions f on G , for which $f\circ\varphi^{-1}\in C^{\nu}_{\varphi(U)}$ for each local chart (U,φ) of the manifold G , where $C^{\nu}_{\varphi(U)}$ is the algebra of all continuously differentiable complex-valued functions till order ν inclusive on $\varphi(U)$, $\nu = 0,1,2,\dots;\infty$.

Definition . A linear continuous map $A: C^{\infty}_G \longrightarrow C^{0}_G$ is called a linear differential operator on G if for any fixed local chart (U,φ) of G, the transfer $A\circ\varphi^{-1}$ is a linear differential operator on $C^{\infty}_{\varphi(U)}$ (see Treves [15]). Moreover, if the coefficients of the linear differential operators $A\circ\varphi^{-1}$ are in $C^{\nu}_{\varphi(U)}$ for any local chart (U,φ) of the manifold G , then the linear differential operator A is called a linear differential operator with coefficients in the algebra C^{ν}_G .

Theorem 14. Let A be a linear differential operator on the n -dimensional C^{∞} differentiable manifold G with coefficient in the algebra C^{∞}_G . Let the function $h\in C^{0}_G$. Let the sequence $\{\varphi_m\}$, $\varphi_m\in C^{\infty}_G$, be such that

$$\{\varphi_m\}\xrightarrow{\text{uniformly on } G} h\ , \quad \{A\varphi_m\}\xrightarrow{\text{uniformly on } G} H .$$

Then if we have for another sequence $\{\psi_m\}$, $\psi_m\in C^{\infty}_G$, that

$$\{\psi_m\}\xrightarrow{\text{uniformly on } G} h \quad\text{and}\quad \{A\psi_m\}\xrightarrow{\text{uniformly on } G} M$$

it follows that $H \equiv M$ on G .

Proof. Let the linear differential operator B on G be the congugate operator of the operator A^{*}, i.e. the linear differential operator B on G $B: C^{\infty}_G \longrightarrow C^{0}_G$, is such that its transfer $B\circ\varphi^{-1}$ is the conjugate operator of the transfer $A\circ\varphi^{-1}$ of the operator A for any fixed local chart (U,φ) of the manifold G . This is, we have

$$\int\{[A\circ\varphi^{-1}]\,\Phi\}\,\overline{\Psi} = \int\Phi\,[B\circ\varphi^{-1}]\,\overline{\Psi} \quad \text{for } \forall\Phi, \forall\Psi\in\mathcal{D}^{\infty}_{\varphi(U)},$$

where $\mathcal{D}^{\infty}_{\varphi(U)}$ is the algebra of all complex-valued functions on $\varphi(U)$, which are infinitely differentiable and with compact supports.

Such conjugate operator B exists. Moreover, the coefficients of the conjugate operator B are also in the algebra C^{∞}_G .

Since it holds $\int\{(A\circ\varphi^{-1})[(\varphi_m-\psi_m)\circ\varphi^{-1}]\}\,\overline{\Phi} =$

$$\int [(\varphi_m - \psi_m)\circ \varphi^{-1}](B\circ\varphi^{-1})\Phi \longrightarrow 0$$

as $m \longrightarrow \infty$ for $\forall \Phi \in \mathcal{D}^{\infty}_{\varphi(U)}$, hence

$$\lim_m (A\circ\varphi^{-1})(\varphi_m \circ \varphi_m^{-1})(x) \equiv \lim_m (A\circ\varphi^{-1})(\psi_m \circ \varphi^{-1})(x)$$

at any point $x \in \varphi(U)$ and on each local chart (U, φ) of the manifold G . Therefore we obtain $H \equiv M$ on G . ∎

The uniquely determined function H will be denoted by $A_\infty h$ and will be called a generalized strong A_∞ derivative of the function h .

Theorem 15. Let G be a compact n-dimensional C^∞ differentiable manifold. Let $A_1,\ldots,A_m$ be linear differential operator on G with coefficients in C^∞_G . Let $W_\infty^{A_1,\ldots,A_m}(G)$ be the completion of the algebra C^∞_G in the norm π ,

$$\pi f = \sup_{s\in G} |f(s)| + \sum_{k=1}^{m} \sup_{s\in G} |A_k f(s)| .$$

The space $W_\infty^{A_1,\ldots,A_m}(G)$ is a space of all continuous complex-valued functions on the manifold G with continuous $A_{k\infty}$, $k=1,\ldots,m$, strong generalized derivative on G (up to the natural isomorphism), according to Theorem 14. Furthermore, each solution $h \in D^0_G$ of the system $\{A_k h = u_k ,\ k=1,\ldots,m\}$ with $u_k \in D^0_G$, belongs to the space $W_\infty^{A_1,\ldots,A_m}(G)$.

Moreover, the space $W_\infty^{A_1,\ldots,A_m}(G)$ is an algebra of type C of complex-valued functions on G iff the linear hull of the system $\{A_1,\ldots,A_m\}$ is differential-invariant.

Remark. The expression "up to the natural isomorphism" here signifies the following: any element of the completion $W_\infty^{A_1,\ldots,A_m}(G)$ formally consists of several limits, i.e. for any fixed element $\mathcal{F} \in W_\infty^{A_1,\ldots,A_m}(G)$ there exists a sequence $\{f_p\}$, $f_p \in C^\infty_G$, such that

$$\{f_p\} \xrightarrow{\text{uniformly on } G} f \quad , \quad \{A_k f_p\} \xrightarrow{\text{uniformly on } G} g_{A_k} ,$$

$k=1,\ldots,m$. This is, the element $\mathcal{F}$ formally consists of f , g_{A_k} , $k=1,\ldots,m$. But according to Theorem 14 , the functions g_{A_k} , $k=1,\ldots,m$, do not depend on the choice of the sequence $\{f_p\}$. They depend only on the function f . Moreover, the function g_{A_k} is $A_{k\infty}$ strong generalized derivative of the function f , $A_k f$, which is uhiquely determined by f , $k=1,\dots,m$,. That is why if we map to the element $\overline{f}$ the function f this will be an isomorphism betwen the completion $^{q}W^{l}_{\infty}{}^{A_1,\dots,A_m}(G)$ and the set of all functions $f \in D^{o}_{G}$ with strong generalized derivatives $A_{k\infty}f$, on G , $k=1,\dots,m$. The differential-invariance of the system $\{A_k, k=1,\dots,m\}$, of linear differential operators on G signifies, that the linear hull of the system of the transfers $\{A_k \circ \varphi^{-1}, k=1,\dots,m,\}$ is differential-invariant for any fixed local chart (U, φ) of G . A proof of Theorem 15 is contained in [10]. Let us remind the following :

Definition 9. I.G.Petrovsky, see I.G.Petrovsky [5, 16], S.D.Eidel'man [17], M.S.Agranovich, M.I.Vishik [18], S.D.Ivasishen [19]). Let us study the system

(3)
$$D_t^{n_i} u_i = \sum_{j=1}^{m} \sum_{k_0 p + |s| \le n_j p} A_{ij}^{(k_0,s)}(t,x) D_t^{k_0} D^{s} u_j , \quad s=(s_1,\dots,s_n)\in Z_+^n$$

with $k_0 < n_j$; $A_{ij}^{(k_0,s)}(t,x)$, $u_i(t,x)$ are complex-valued functions in the variables t and $x=(x_1,\dots,x_n)$; p is an even integer, $p \geq 2$, $n_j > 0$, $k_0 \geq 0$.

The system (3) is called Petrovsky's parabolic in the domain $\mathcal{G}_0$ $R^{n+1}(x,t)$ if for any point $(x,t) \in \mathcal{G}$ the real parts of the equation

(4)
$$\det\left\{ \left\| \sum_{k_0 p + |s| = n_j p} A_{ij}^{(k_0,s)}(t,x)(i\sigma_1)^{s_1}\dots(i\sigma_n)^{s_n} \right\|_{i,j=1}^{m} - \left\| \begin{matrix} \lambda^{n_1} & & \\ & \ddots & \\ & & \lambda^{n_m} \end{matrix} \right\| \lambda^{-m k_0} \right\} = 0$$

satisfy the inequality

(5)
$$\mathrm{Re}\, \lambda(t,x,\sigma_1,\dots,\sigma_n) \le -\delta(t,x)$$

for any real $\sigma_1, \sigma_2, \dots, \sigma_n$, for which

$$\sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2 = 1,$$

$$\delta(t, x) > 0 .$$

Proof of Theorem 1. We must prove that Definition 1 is a generalization of any Petrovsky's parabolic system. Let us prove that an arbitrary fixed Petrovsky's parabolic system $\mathcal{L} = (\mathcal{P}_i)_i$, $i=1,\dots,m$, (3), parabolic after Definition 9, is also a generalized parabolic system after Definition 1 in the constant-coefficient case, where after Definition 9 we have

$$\mathcal{P}_i u = D_t^{n_i} u_i - \sum_{\varkappa_0 p+|s| \le n_j p} A_{ij}^{(\varkappa_0, s)} D_t^{\varkappa_0} D^s u_j .$$

(Remark. The proof in the nonconstant-coefficient case is almost the same).

Since (4) and (5) hold for $\mathcal{L}$, then all

(6)
$$\sum_{j=1}^{m} \sum_{p\varkappa_0+|s|=pn_j} |A_{ij}^{(\varkappa_0,s)}| \ne 0 \quad i=1,\dots,m.$$

(If we assume that for some i_o we have

$$\sum_{j=1}^{m} \sum_{p\varkappa_0+|s|=pn_j} |A_{i_0 j}^{(\varkappa_0,s)}| = 0 ,$$

then the first determinant in (4) is zero and (5) cannot hold).

Therefore the order of $\mathcal{L}$ is $N = p(\max_j n_j - k_o) + k_o$. Let us denote $\max_j n_j$ by $n^\wedge$. Let $n_j = n^\wedge$ iff $j = j_1, \dots, \tau$, $1 \le \tau \le m$.

Now let

$$\mathcal{P}_u = \sum_{i=1}^{m} \alpha_i \mathcal{P}_i u$$

be of order N, where α_i, $i = 1,\dots,m$, are complex constants. We shall prove that $\mathcal{P}_u$ has the kind (1). We have

$$\mathcal{P}u = -\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{\varkappa_0 p+|s|=N} \alpha_i A_{ij}^{(\varkappa_0,s)} D_t^{\varkappa_0} D^s u_j +$$

$$\sum_{i=1}^{c} \alpha_{j_i} D_t^{n^\wedge} u_{j_i} + \sum_{\substack{i=1 \\ (i \neq j_1,\dots,j_c)}}^{m} \alpha_i D_t^{n_i} u_i -$$

$$- \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k_0 p + |s| < N} \alpha_i A_{ij}^{(k_0, s)} D_t^{k_0} D^s u_j .$$

The order N of $\mathcal{P}$ is larger than n_j, $N > n_j$, $j=1,\dots,m$, since $N - n_j = p(n^\wedge - k_0) + k_0 - n_j = p(n^\wedge - k_0) - (n_j - k_0) > 0$, ($n_j \leq n^\wedge$, $p \geq 2$).

Since $k_0 < n_j$, $\forall j$, then N-k from Definition 1 here is not larger than $n^\wedge$, and $N-k > k_0$. That is why $\mathcal{P}u$ has the kind (1). Therefore any Petrovsky's parabolic system after Definition 9 is also a generalized parabolic system after Definition 1 in the constant-coefficient case . ■

Remark. The most important characteristics of the linear constant-coefficient differential operators on R^n remain the same after linear real nondegenerated transformations of R^n. The Definitions 1 and 2 satisfy the natural requirement that the property of parabolicity to remain after real linear nondegenerated transformations of R^n. (Definition 9 does not satisfy this requirement.)

Proof of Theorem 2. I. Let us set N = 2 in (2) . Since $0 < k \leq N$, hence k might only be either k=1 or k=2 . Since $0 \leq l_s < N-k$ in Definition 2 when $k \neq N$, then in the case k=1 we have $l_s = 0$. We receive also $l_s = 0$ in Definition 2 in the case k = N = 2 . Thus the operator A from (2) here might only have the following forms - either:

(2')

$A' = a\partial^2/\partial x^2 + b\partial/\partial t + c\partial/\partial x + d$ with $a \neq 0$, $b \neq 0$, or

(2")

$A'' = a\partial^2/\partial x^2 + c\partial/\partial x + d$ with $a \neq 0$, (c might be 0 although $\sum_s |b_s| \neq 0$ in Definition 2 , since the corresponding term $b\,\partial/\partial x$ in the case N = k = 2 might be annulled by $P(\partial/\partial x, \partial/\partial t)$ after Definition 2 .

Let $\mathcal{H} = a_{20}\partial^2/\partial u^2 + 2a_{11}\partial^2/\partial u\partial v + a_{02}\partial^2/\partial v^2 + P^*(\partial/\partial u, \partial/\partial v)$, (where P^* be a polynomial of degree not larger than 1) , be an arbitrary fixed constant-coefficient linear differential operator, parabolic after Definition 2 , and of second order in two variables. Then there exists a linear real nondegenerated transformation $T : (u,v) \longrightarrow (x,t)$ such that the transfer $T\mathcal{H}$ of $\mathcal{H}$ has either the kind (2') or the kind (2") after Definition 2. As it is well known, this signifies that $a_{20}a_{02} - a_{11}^2 = 0$ Since the transformation T must also be real, then if $a_{20} \neq 0$ it follows that a_{11}/a_{20} is real in this case; if $a_{02} \neq 0$ then a_{11}/a_{02} must be real. Therefore the operator $\mathcal{H}$ is parabolic after Definition 3 also. Thus roughly speaking, we have Definition 2 $\subseteq$ Definition 3 in the case of second-order linear constant-coefficient differential operators in two variables.

II. Now let $\mathcal{H} = a_{20}\partial^2/\partial u^2 + 2a_{11}\partial^2/\partial u\partial v + a_{02}\partial^2/\partial v^2 + P^*(\partial/\partial u, \partial/\partial v)$,

(where P^* be a polynomial of degree not larger than 1,) be a linear constant-coefficient differential operator of second order in two variables and $\mathcal{H}$ be parabolic after Definition 3. Since $\mathcal{H}$ is of second order and $a_{20}a_{02} - a_{11}^2 = 0$, then we have $|a_{20}| + |a_{02}| \neq 0$. Without loss of generality let us suppose $a_{20} \neq 0$. Then we receive $a_{02} = a_{11}^2/a_{20}$ and we have

$$\mathcal{H} = a_{20}\left(\partial/\partial u + \frac{a_{11}}{a_{20}}\,\partial/\partial v\right)^2 + P^*(\partial/\partial u; \partial/\partial v) .$$

Hence the sought real nondegenerated linear transformation is $T = x = u + (a_{11}/a_{20})v$, $t = v$. Then $T\mathcal{H} = a_{20}\partial^2/\partial x^2 + TP^*$ has either the form (2') or the form (2") . Therefore the operator $\mathcal{H}$ is parabolic after Definition 2 also. Thus roughly speaking, we receive Definition 3 $\subseteq$ Definition 2 . ∎

The assertion of Theorem 3 follows from Theorem 1, but since this particular case is important we shall prove Theorem 3 independently:

Proof of Theorem 3. We must prove that the operator

$$\mathcal{P} = D_t^{n_{i_0}} - \sum_{r_0 p + |s| \leq n_{i_0} p} A^{(r_0, s)} D_t^{r_0} D^s , \quad 0 \leq r_0 < n_{i_0}, \quad p = 2b,\ b = 1, 2, \ldots$$

for which the requirements (4) and (5) holds, is parabolic after Definition 2. We have for the operator $\mathfrak{P}$ that

$$\sum_{pr_0+|s|=pn_{i_0}} |A^{(r_0,s)}| + C ,$$

since (4) and (5) hold. Then the order of $\mathfrak{P}$ is $N = p(n_{i_0} - k_0) + k_0 > n_{i_0}$. As $k_0 < n_{i_0}$, then N-k from Definition 2 is equal to n_{i_0}. That is why the operator $\mathfrak{P}$ has the kind

$$\mathfrak{P} = -\sum_{r_0 p+|s|=N} A^{(r_0,s)} D_t^{r_0} D^s + D_t^{n_{i_0}} + P(D_t, D),$$

where P is a polynomial in D_t, D. The order of $\mathfrak{P}$ relatively D_t is n_{i_0} = N-k and the sum $\sum_{|s|<k} |b_s| = 1$ has only one member, corresponding to s = (0,...,0) ; moreover, the order of $\mathfrak{P}$ is N . The rest of the addends of $\mathfrak{P}$ are with orders less than N , and their orders relatively D_t are equal to $k_0 < n_{i_0}$ = N-k . Thus these addends can be included in the corresponding from Definition 2 member $P(D_t, D)$. That is why the operator $\mathfrak{P}$ is of the kind (2). Therefore any Petrovsky's parabolic after Definition 9 linear differential operator is also a generalized parabolic operator after Definition 2. ■

Theorem 16. Let $0 < \varrho \leq \infty$ and $\lambda \in C$ be a constant. It does not exist such a constant K , that

$$\|g'_y\| \leq K \{\|g''_{xx} + \lambda g'_y\| + \|g'_x\| + \|g\|\}$$

for $\forall g \in \mathcal{D}_\varrho$, where $\mathcal{D}_\varrho$ is the set of all real-valued infinitely differentiable functions with compact supports in

$$\{(x,y) : |(x,y)| \leq \varrho\} \quad ; \quad \|f\| = \sup |f(x,y)| .$$

Let us remind that if the linear differential operator B is equal to

$$B = \sum_{r+\tau \leq N} b_{r\tau} \partial^{r+\tau}/\partial x^r \partial y^\tau$$

then its full characteristic polynomial $\sigma(B)$ is equal to

$$\sigma(B) = \sum_{r+\tau \leq N} b_{r\tau} (iX)^{r} (iY)^{\tau} .$$

We shall use the following theorem:

Theorem 17. (K. de Leeuw , H. Mirkil [11]). Let $A, A_1, A_2, \ldots, A_m$ be constant-coefficient linear differential operators on R^n . There exists a constant $\varkappa$ such that

(7)
$$\| A g \| \leq \varkappa \left\{ \| A_1 g \| + \ldots + \| A_m g \| \right\} \text{ for } \forall g \in \mathcal{D}_\infty = \mathcal{D} \text{ iff}$$
$$\sigma(A) = \sum_{k=1}^{m} M_k \sigma(A_k)$$

for suitable Fourier-Stieltjes transforms $M_1, \ldots, M_m$ of some integrable (i.e. with final total mass) measures $\mu_1, \ldots, \mu_m$.

If the condition (7) holds for the operators $A_1, A_2, \ldots, A_m$ of order $\leq N$, then the order of A is also $\leq N$, and the homogeneous part of order N of A , A^N , is a linear combination of A_k^N , $k=1,\ldots,m$, where A_k^N is the corresponding homogeneous part of A_k of order N , $k=1,\ldots,m$,

(8)
$$A^N = \sum_{k=1}^{m} c_k A_k^N , \text{ where } c_k \text{ is the mass assi-}$$

gned at the origin by μ_k , $k=1,\ldots,m$.

Theorem 18 (W F Eberlein [20]). Let μ be an integrable measure; let c be the assigned mass at the origin by μ and let M be the Fourier-Stieltjes transform of μ . Then the constant function c can be approximated uniformly by $\pi \star M$ with π - a probably measure of finite support.

Proof of Theorem 16. If $\lambda = 0$ then the assertion of Theorem 16 is evident. So further let $\lambda \neq 0$. I. Let us scrutinize the case $\varrho = \infty$ at first : If we assume the contrary of Theorem 16 in the case $\varrho = \infty$, then it follows from Theorem 17 that

(9)
$$i Y = M_1 (- X^2 + i \lambda Y) + i M_2 X + M_3 \quad \text{for suitable Fou-}$$

rier-Stieltjes transforms M_1 , M_2 , M_3 of some integrable measures μ_1 , μ_2 , μ_3 . In (9) let us fix $X = C$, where C is a constant ; Let us divide it by $Y \neq 0$ and let $|Y| \longrightarrow \infty$. M_1, M_2 , M_3 are Fourier-Stielties transforms of integrable measures and therefore are bounded as $|Y| \longrightarrow \infty$. That is why we receive that the limit

$$\lim_{|Y| \to \infty} M_1 (0 , Y) = 1/\lambda \neq 0 .$$ We also get from Theorem 17 that $0 = \alpha \partial^2/\partial x^2$, where α is the mass at the origin of the measure μ_1. Hence $\alpha = 0$. The Eberlein's Theorem 18 interprets α in term of the Fourier-Stieltjes transform M_1. So we obtain a contradiction, since $\alpha = 0$, but $\lim_{|Y| \to \infty} M_1 (0 , Y) \neq 0$. The constructed contradiction proves that Theorem 16 is true for $\varrho = \infty$.

II. Let now $\varrho < \infty$. Again let us assume the contrary, i.e. that there exists a constant $\varkappa$ for $\mathcal{D}_\varrho$ with

$$\| g'_y \| \leq \varkappa \left\{ \| g''_{xx} + \lambda g'_y \| + \| g'_x \| + \| g \| \right\} \quad \text{for } \forall g \in \mathcal{D}_\varrho .$$

(Obviously if this assumption is true for some fixed $\varrho_0 < \infty$, then this will be true for every fixed $\varrho < \infty$). Let $\varkappa_\varrho$ denote

$$\varkappa_\varrho = \min \left\{ \varkappa : \| g'_y \| \leq \varkappa \left[\| g''_{xx} + \lambda g'_y \| + \| g'_x \| + \| g \| \text{ for } \forall g \in \mathcal{D} \right\} \right. .$$ Evidently we have $\varkappa_\varrho > 0$. Then for $\varkappa_\varrho - \varepsilon$, $0 < \varepsilon < \varkappa_\varrho$, there exists a function $f_\varrho \in \mathcal{D}_\varrho$, such that

(10)
$$(\varkappa_\varrho - \varepsilon) \left\{ \| (\partial^2/\partial x^2 + \lambda \partial/\partial y) f_\varrho \| + \| \tfrac{\partial}{\partial x} f_\varrho \| + \| f_\varrho \| \right\} < \| \tfrac{\partial}{\partial y} f_\varrho \|$$

Let the function $g_\varrho(u,v)$ be defined by

$$g_\varrho (ax , a^2 y) = f_\varrho (x , y)$$

and let $a = \varrho_0 / \varrho$. As $|ax| \leq \varrho_0$, $|a^2 y| \leq \varrho_0^2/\varrho$ then if $0 < \varrho_0 < \varrho$, we have $g_\varrho \in \mathcal{D}_{\varrho_0}$. The inequality (10) can be transformed in

(11)
$$(\varkappa_\varrho - \varepsilon) \left\{ a^2 \| (\partial^2/\partial u^2 + \lambda \partial/\partial v) g_\varrho \| + a \| \tfrac{\partial}{\partial u} g_\varrho \| + \| g_\varrho \| \right\} \leq a^2 \| \tfrac{\partial}{\partial v} g_\varrho \| .$$

Since $g_\varrho \in \mathcal{D}_{\varrho_0}$, hence we also have

(12)
$$\| \tfrac{\partial}{\partial v} g_\varrho \| \leq \varkappa_{\varrho_0} \left\{ \| (\partial^2/\partial u^2 + \lambda \partial/\partial v) g_\varrho \| + \| \tfrac{\partial}{\partial u} g_\varrho \| + \| g_\varrho \| \right\} .$$

Putting (12) in (11) we get

$$(\varkappa_\varrho-\varepsilon)\{a^2\|(\partial^2/\partial u^2+s\partial/\partial v)g_\varrho\|+a\|\tfrac{\partial}{\partial u}g_\varrho\|+\|g_\varrho\|\}<$$

$$<a^2\varkappa_{\varrho_0}\{\|(\partial^2/\partial u^2+s\partial/\partial v)g_\varrho\|+\|\tfrac{\partial}{\partial u}g_\varrho\|+\|g_\varrho\|\}.$$

Therefore we receive

(13)

$$a^2\|(\partial^2/\partial u^2+s\partial/\partial v)g_\varrho\|[\varkappa_\varrho-\varepsilon-\varkappa_{\varrho_0}]+a\|\tfrac{\partial}{\partial u}g_\varrho\|\cdot[\varkappa_\varrho-\varepsilon-a\varkappa_{\varrho_0}]+\|g_\varrho\|[\varkappa_\varrho-\varepsilon-a^2\varkappa_{\varrho_0}]<0.$$

The case $\varrho=\infty$ of Theorem 16 is already proved, hence $\sup_\varrho\varkappa_\varrho=\infty$. That is why we have

$$[\varkappa_\varrho-\varepsilon-\varkappa_{\varrho_0}]>0,\ [\varkappa_\varrho-\varepsilon-a\varkappa_{\varrho_0}]>0,\ [\varkappa_\varrho-\varepsilon-a^2\varkappa_{\varrho_0}]>0$$

for any sufficiently large ϱ. This contradicts with the inequality (13) The obtained contradiction finishes the proof of Theorem 16 . ■

Further $\mathcal{D}_\varrho$, $0<\varrho\le\infty$, $(\mathcal{D}_\infty=\mathcal{D})$, be the set of all infinitely differentiable complex-valued functions compact supports in the closed ball with a radius ϱ and a center - the point 0 .

Definition. Let A , A_1 ,..., A_m be linear constant-coefficient differential operators. If there exists a constant $\varkappa$ such that

$\|Ag\|\le\varkappa\{\|A_1 g\|+\ldots+\|A_m g\|\}$ for $\forall g\in\mathcal{D}_\varrho$, where $\|f\|=\sup_w|f(w)|$, then we shall say that A_1 ,..., A_m jointly dominate the operator A on $\mathcal{D}_\varrho$ (in the supremum norm).

Proof of Theorem 11. Let us scrutinize the set S ,

$S=\{(A_1 g,\ldots,A_m g)$, $\forall g\in\mathcal{D}_\infty$ with $\operatorname{supp} g\subset\{x:|x|<\varepsilon\}$.

S is a subspace of the space $\bigoplus^m C^0$, where C^0 is the space of all continuous functions, vanishing at infinity, with the usual topology.

The correspondence $(A_1 g,\ldots,A_m g)\longmapsto(Ag)(0)$ determines a continuous functional on S . Let us extend this functional on the spa- $\bigoplus^m C^o$ after Hahn-Banach' Theorem . According to Riez' Theorem there exist regular and bounded measures $\mu_1, \dots, \mu_m$ (see for instance Danford, Schwarz [21], p. 284) such that

(14)
$$Ag(0) = \int (A_1 g)\, d\mu_1 + \dots + \int (A_m g)\, d\mu_m .$$

Let $g \in \mathcal{D}_\varepsilon$ and $r > 0$. Let us study (14) for $g_r(x) = g(rx) = g(rx_1, \dots, rx_n)$; $A = \sum_{|k| \leq M} a_k(x) D^k$; $A_j = \sum_{|k| \leq N} a_{jk}(x) D^k$, $j=1,\dots,m$. We have

(15)
$$\sum_{|\kappa| \leq M} a_\kappa(0) r^{|\kappa|} (D^\kappa g)(0) = \sum_{j=1}^{m} \sum_{|\kappa| \leq N} r^{|\kappa|} \int a_{j\kappa}(x) (D^\kappa g)(rx)\, d\mu_j$$

We divide the equality (15) by r^N and let $r \longrightarrow \infty$. Since the measures $\mu_1, \dots, \mu_m$ are bounded and regular, and the coefficients $a_{jk}(x)$ are continuous on U , moreover without loss of generality U can be changed by another neighbourhood of the origin in which the coefficients a_{jk} are not only continuous but also and bounded, hence the right part of (15) has the limit

$$\sum_{j=1}^{m} \sum_{|\kappa| = N} c_j a_{j\kappa}(0) (D^\kappa g)(0) = \sum_{j=1}^{m} c_j (A_j^N g)(0),$$

where $c_1, \dots, c_m$ are constants. But then the left part of (15) must also have a limit as $r \longrightarrow \infty$, which is possible only if $M \leq N$.

Thus we obtain from (15) that

$$\sum_{|\kappa| = N} a_\kappa(0) (D^\kappa g)(0) = \sum_{j=1}^{m} c_j (A_j^N g)(0)$$

i.e.

$$A^N g(0) = \sum_{j=1}^{m} c_j (A_j^N g)(0) .$$

Since the functions $g \in \mathcal{D}_\varepsilon$ are sufficiently much, hence we get

$$A^N(0) = c_1 A_1^N(0) + \dots + c_m A_m^N(0) \quad . \blacksquare$$

Proof of Corollary 12. The first two points of Corollary 12 immediately follows from Theorem 11. Let us prove the point 3 . Let choose an arbitrary x_0 U . Let the functions $g_1, \dots, g_m$ $D^N(U)$. Let us study the system

$$A^N g_1(x) = \sum_{j=1}^{m} c_j(x)\left(A_j^N g_1\right)(x)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$A^N g_m(x) = \sum_{j=1} c_j(x)\left(A_j^N g_m\right)(x).$$

If we can choose the functions $g_1, \dots, g_m \in D^N(U)$ such that the determinant

$\det((A_j^N g_q)(x_0)) \neq 0$, then since $A_j g_q$ are continuous hence $\det((A_j^N g_q)(x))$ is also continuous and therefore nonannulling on some neighbourhood $V_{x_0} \subset U$. The functions $c_1(x), \dots, c_m(x)$ can be determined in the neighbourhood V_{x_0} by the Kramer's formulas. It is clear by these formulas that $c_1(x), \dots, c_m(x)$ would be continuous on V_{x_0} . Since the point x_0 was arbitrarily chosen, this would prove the continuity of $c_1(x), \dots, c_m(x)$ on U .

But we have

$$\det\left((A_j^N g_q)(x)\right) = \sum_{(i_1 \dots i_m)} (-1)^{[i_1 \dots i_m]} A_{i_1}^N g_1(x) \dots A_{i_m}^N g_m(x) =$$

$$= \sum_{(i_1 \dots i_m)} (-1)^{[i_1 \dots i_m]} \left(\sum_{|p_1|=N} a_{i_1 p_1} D^{p_1} g_1(x)\right) \dots \left(\sum_{|p_m|=N} a_{i_m p_m}\right.$$

$$\left. \cdot D^{p_m} g_m(x)\right) = \sum_{\substack{p,\,|p_j|=N \\ j=1,\dots,m}} \Delta_p D^{p_1} g_1(x) \dots D^{p_m} g_m(x),$$

with Δ_p ,

$$\Delta_p = \sum_{(i_1 \dots i_m)} (-1)^{[i_1 \dots i_m]} a_{i_1 p_1}(x) \dots a_{i_m p_m}(x).$$

Since $\operatorname{rang}_{|k|=N} \left(a_{jk}(x) \right) = m$, hence not all Δ_p are annulling . According to Borel's Theorem of Taylor's series, there exist such functions $G_1 , \dots , G_m \in D^N(U)$, for which

$$\det \left(\left(A_j^N G_q \right)(x_0) \right) \neq 0 .$$

This finishes the proof of Corollary 12. ∎

Proof of Theorem 4. I. Let us prove the theorem in the case $\varrho = \infty$ at first. Let us assume the contrary of the assertion of Theorem 4 in this case. Then from Leeuw-Mirkil's Theorem 17 as well from Theorem 11 it follows that

$$\sigma\left(D_t^{N-k_*}\right) = M_0\, \sigma(A) + M_1\, \sigma(B_1) + \dots + M_m\, \sigma(B_m) \tag{16}$$

for the Fourier-Stieltjes transforms $M_0 , M_1 , \dots , M_m$ of suitable integrable measures $\mu_1 , \dots , \mu_m , \mu_0$. In (16) let us fix $X = C = (C_1 , \dots , C_m)$, where $C_1 , \dots , C_n$ are constants. Let us divide the obtained equation by T^{N-k} and let $|T| \longrightarrow \infty$ for a fixed $X = C$. Thus we receive that $\lim_{|T|\to\infty} \left| M_0(C,T) \right|$ exists (it may be equal to ∞ if $k > k_*$ and the contradiction, so obtained, proves our assertion in this case of $k > k_*$). Moreover, $\lim_{|T|\to\infty} \left| M_0(C,T) \right|$ is strictly positive for each C with eventual exeptions of the zeros of a polynomial in variables $C_1, \dots,$ C_n , which polynomial is not equivalent to 0 . The Leeuw-Mirkil's Theorem 17(Theorem 11) and the equation (16) yield that $0 = \alpha A^N$, where A^N is the homogeneous part of A of order N and α is the mass , assigned at the origin by the measure μ_0. Thus $\alpha = 0$. The Eberlein's Theorem 18 interprets the constant function α in terms of the Fourier-Stieltjes transform M_0 . The function $\alpha \equiv 0$ can be approximate uniformly by $\overset{a}{\pi} * M_0$, with π - a probably measure with finite support. Hence we obtain a contradiction since $\alpha \equiv 0$, but $\lim_{|T|\to\infty} \left| M_0(C,T) \right| > 0$ almost everywhere .

Therefore the assertion of Theorem 4 is true in the case $\varrho = \infty$.

II. Let now $0 < \varrho < \infty$. Let us assume the contrary of the assertion of Theorem 4 in these cases, i.e. let us assume that for some ϱ_0, $0 < \varrho_0 < \infty$, the operators A, $B_1,\dots,B_m$ do jointly dominate $D_t^{N-k_*}$ on $\mathcal{D}_{\varrho_0}$. But it follows from this assumption that $A, B_1,\dots,B_m$ jointly dominate $D_t^{N-k_*}$ on each $\mathcal{D}_\varrho$, $0 < \varrho < \infty$. Thus let us assume that A, $B_1,\dots,B_m$ jointly dominate $D_t^{N-k_*}$ on each $\mathcal{D}_\varrho$ with $0 < \varrho < \infty$.

If $k > k_*$, then the proof of Theorem 4 might be simpler by choosing any $\varphi(u,v) \in \mathcal{D}_1$, $u = (u_1,\dots,u_n)$, $\varphi \not\equiv 0$, and examing the assumed inequality for the functions $g(x,t) = \varphi(\alpha x, \beta t)$, $\alpha \geq 1$, $\beta \geq 1$, $\alpha x = (\alpha x_1, \dots, \alpha x_n)$, i.e. examing the inequality

(17)
$$\| D_t^{N-k_*} g(x,t) \| \leq \varkappa \left\{ \| A\, g(x,t) \| + \sum_{j=1}^{m} \| B_j\, g(x,t) \| \right\},$$

where $\varkappa$ is a corresponding constant for $\mathcal{D}_1$. The inequality (17) is respectively α and β for a fixed φ a "polynomial" inequality with coefficients
$$D_u^k\, D_v^l\, \varphi(u,v).$$

α and β can increase to infinity, remaining g in $\mathcal{D}_1$. Then a necessary requirement for (17) is that the order of $D_t^{N-k_*}$, $N-k_*$, to be not larger than the order of the operator A relatively D_t. Thus a contradiction of the assumption is obtained in the case $k > k_*$, i.e. Theorem 4 is true in this case.

Furthermore, as $B_1,\dots,B_m$ are of orders less than N and their orders relatively D_t are less than $N-k$, then a similar argument proves that it is sufficient to carry out the proof for the case $P \equiv 0$, $B_1 \equiv 0$, $\dots$, $B_m \equiv 0$. Thus let $k = k_*$, $B_q \equiv 0$, $q=1,\dots,m$,

$$A = \sum_{|s| + l_s = N} a_s\, D^s\, D_t^{l_s} + D_t^{N-k} \sum_{|s| < k} b_s\, D^s \quad , \ 0 \leq l_s < N-k \ .$$

Moreover, we have $\sum_s |a_s| \neq 0$, since it is given that the order of the operator A is N. We also have $\sum_s |b_s| \neq 0$, since the order of the operator A respectively D_t is $N-k$.

We have assumed the contrary of Theorem 4 on each $\mathcal{D}_\varrho$, $0 < \varrho < \infty$.

Let

$$\varkappa_\varrho = \min \left\{ \varkappa : \| D_t^{N-k} f \| \leq \varkappa \| A f \|, \forall f \in \mathcal{D}_\varrho \right\} .$$

Evidently $\varkappa_\varrho > 0$ for $\forall \varrho$, $0 < \varrho < \infty$. Hence for $\varkappa_\varrho - \varepsilon > 0$ and for fixed $\varepsilon > 0$, there exists such a function $f_\varrho \in \mathcal{D}_\varrho$ that we have

(18)
$$\| D_t^{N-k} f_\varrho \| \leq \varkappa_\varrho \| A f_\varrho \| \quad \text{and} \quad (\varkappa_\varrho - \varepsilon) \| A f_\varrho \| < \| D_t^{N-k} f_\varrho \| .$$

Let the function $g_\varrho(u_1, \dots, u_n, v)$ be determined by

$g_\varrho(\alpha_1 x_1, \dots, \alpha_n x_n, \beta t) = f_\varrho(x_1, \dots, x_n, t)$, where α_j, $j = 1, \dots, n$, β are constants. If $\alpha_j \leq \varrho_0 / \varrho$, $j = 1, \dots, n$, $\beta \leq \varrho_0 / \varrho$, then we have $g_\varrho \in \mathcal{D}_{\varrho_0}$ for $\forall \varrho_0$, $\forall \varrho$, $0 \leq \varrho_0 < \infty$, $0 < \varrho < \infty$. In these cases the last of the inequalities (18) might be transformed in

(19)
$$(\varkappa_\varrho - \varepsilon) \| A_{x,t} f_\varrho \| < \| D_t^{N-\varkappa} f_\varrho \| = \beta^{N-\varkappa} \| D_v^{N-\varkappa} g_\varrho \| \leq$$

$$\leq \varkappa_{\varrho_0} \beta^{N-\varkappa} \| A_{u,v} g_\varrho \| .$$

If some function $\varphi_\varrho \in \mathcal{D}_\varrho$ satisfies the inequalities (18), then each function $\psi_\varrho = C_\varrho \varphi_\varrho$ (where $C_\varrho \neq 0$ is a constant), also satisfies the inequalities (18). So we can supose that $\| A_{u,v} g_\varrho \| = 1$ for $\forall \varrho$, $0 < \varrho < \infty$. Then it follows from (19) that

(20)
$$(\varkappa_\varrho - \varepsilon) \left\| \sum_{|s| + l_s = N} a_s \alpha^s \beta^{l_s - N + \varkappa} D_u^s D_v^{l_s} g_\varrho + \right.$$

$$\left. + D_v^{N-\varkappa} \sum_{|s| < \varkappa} b_s \alpha^s D_u^s g_\varrho \right\| < \varkappa_{\varrho_0} ,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$.

Theorem 4 is already proved on $\mathcal{D} = \mathcal{D}_\infty$, therefore $\varkappa_\varrho \longrightarrow \infty$ as $\varrho \longrightarrow \infty$.

Since Theorem 4 requires $0 < k < N$, and since the operator A is parabolique after Definition 2, then $0 \leq l_s < N-k$. Thus we have $N-k-l_s > 0$.

Let now $\alpha_j = \rho_0/\rho^{\delta_j}$, $\beta = \rho_0/\rho^{\gamma}$ with $\delta_j \geq 1$, $\gamma \geq 1$, $j=1,\dots,n$, and eventually $\delta_j = \delta_j(\rho)$, $\gamma = \gamma(\rho)$; $0 < \rho_0 < \infty$, ρ_0 is fixed. Let us scrutinize the expression

$$\alpha^s \beta^{l_s - N + k} = \rho_0^{|s| + l_s - N + k} / \rho^{|s\delta| + \gamma(l_s - N + k)} = M \rho^{\gamma(N - l_s - k) - |s\delta|}, \tag{21}$$

where $\delta = (\delta_1, \dots, \delta_n)$, $|s\delta| = \sum_j s_j \delta_j$; M is a positive constant. Then if we choose $\gamma > |s\delta|/(N-k-l_s)$, thus the degree of ρ in (21) is strictly positive. That is why

$$\alpha^s \beta^{l_s-N+k} \longrightarrow \infty \quad \text{as } \rho \longrightarrow \infty \quad \text{for } \gamma > |s\delta|/(N-k-l_s)$$

if ρ_0 is fixed , $0 < \rho_0 < \infty$.

Let now $R(\rho, \delta, \gamma) A_{x,t} f_\rho =$

$$\sum_{|s|+l_s=N} a_s \alpha^s \beta^{l_s-N+k} D_u^s D_v^{l_s} g_\rho + D_v^{N-k} \sum_{|s|<k} b_s \alpha^s D_u^s g_\rho =$$

$$\sum_{|s|+l_s=N} \rho^{\gamma(N-l_s-k)-|s\delta|} \underbrace{a_s \rho_0^{|s|+l_s-N+k}}_{a'_s} D_u^s D_v^{l_s} g_\rho +$$

$$+ \sum_{|s|<k} \underbrace{b_s \rho_0^{|s|}}_{b'_s} \rho^{-|\delta s|} D_v^{N-k} D_u^s g_\rho = \sum_{|s|+l_s=N} \rho^{\gamma(N-l_s-k)-|s\delta|} \cdot$$

$$\cdot a'_s D_u^s D_v^{l_s} g_\rho + \sum_{|s|<k} \rho^{-|\delta s|} b'_s D_v^{N-k} D_u^s g_\rho$$

with $g_\rho(u,v) = f_\rho(\frac{u}{\alpha}, v/\beta)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j = \rho_0/\rho^{\delta_j}$,

$\beta = \rho_0/\rho^{\gamma}$, $1 \leq \rho_0 < \rho$, $1 \leq \delta_j$, $j=1,\dots,n$, $1 \leq \gamma$.

The derivatives $D_u^s g_\varrho(u,v)$, $D_v^l g_\varrho(u,v)$ are bounded when is fixed as $\delta_j \longrightarrow \infty$, $\gamma \longrightarrow \infty$, where l and s are such that the corresponding coefficients of the operator A , $a_s \neq 0$, $b_s \neq 0$. This follows from from $\| A_{u,v} g_\varrho(u,v) \| = 1$, since

$$A_{u,v} g_\varrho(a^* u, b^* v) = \sum_{|s|+l_s=N} a_s a^{*s} b^{*l_s} D_u^s D_v^{l_s} g_\varrho + \sum_{|s|<r} b_s \cdot a^{*s} b^{*N-r} D_v^{N-r} D_u^s g_\varrho,$$

for any $a^* = (a_1^*, \ldots, a_n^*)$, b^*.

Let now l_s^* be the least of l_s such that $\| a_{s^*} D_u^{s^*} D_v^{l_s} g_\varrho \| \neq 0$, for some fixed s^* with $|s^*| + l_s = N$. Since $\varrho > 1$, $\delta_j = \delta_j(\varrho) \geqslant 1$, it f follows that we can evidently choose $\delta_j(\varrho) > 1$ such that to hold simultaneously i). - iii). :

i). $\left\| D_v^{N-r} \sum_{|s|<r} \varrho^{-|\delta s|} b_s' D_u^s g_\varrho \right\| \leq C, \quad \forall s, \ |s|<r;$

ii). $\left\| \varrho^{-|\delta s|} a_s' D_u^s D_v^{l_s} g_\varrho \right\| \leq C$ for $\forall$ s with $|s| + l_s = N$, C is a constant

iii). $\left\| \varrho^{-|\delta s^*|} a_{s^*}' D_u^{s^*} D_v^{l_s^*} g_\varrho \right\| \geq 2 \left\| \sum_{|s|+l_s^*=N,\, s\neq s^*} \varrho^{-|\delta s|} a_s' D_u^s D_v^{l_s^*} g_\varrho \right\|$

Let $\gamma > |\delta s|/(N-k-l_s)$ and $\gamma \geqslant 1$ for $\forall$ s with $|s| + l_s = N$. Then it fpllows that $R(\varrho, \delta, \gamma) \longrightarrow \infty$ as $\varrho \longrightarrow \infty$ in the indicated choice of $\delta = \delta(\varrho)$ and $\gamma = \gamma(\varrho)$. Therefore we obtain the wanted contradiction from (20) as $\varrho \longrightarrow \infty$ for $\alpha_j = \varrho_0/\varrho^{\delta_j}$, $\beta = \varrho_0/\varrho^{\gamma}$, j=1,..., n, when ϱ_0 is fixed . Thus Theorem 4 is true . ∎

Theorem 4 permits to investigate some inclusions among functional spaces with strong generalized derivatives $W_\lambda^{\ell A_1,\ldots,A_m}$, which inclusions are characteristically different for elliptic and parabolic operators.

Let the space of complex-valued functions $\mathcal{L}_* = \mathcal{L}(K) \subseteq L_1 = L_1(K)$ be the completion of $C^\infty|K$ in a family of seminorms $\{\gamma_j\}_j$, $j=0,1,2,\ldots$ which induces a locally convex topology in $\mathcal{L}_*$. Let this topology be stronger than the weak convergence in the distribution space $\mathcal{D}'$ (and as $\mathcal{L}_* \subseteq L_1$, then the topology in $\mathcal{L}_*$ is stronger even than the L_1-topology.) Here again C^∞ is the space of all complex-valued infinitely differentiable functions on R^N; $K \subset R^n$ with $\overline{\mathring{K}} = \overline{K}$; $\mathcal{D}'$ is the dual of $\mathcal{D}$ ($\mathcal{D}$ is the space of all complex-valued infinitely differentiable functions on R^n with compact supports), i.e., $\mathcal{D}'$ is the space of all linear continuous functionals on $\mathcal{D}$; L_p, $p=1,2,\ldots$, is the space of all complex-valued measurable functions f on K, for which $|f|^p$ is integrable on K; $L_p = L_p(K)$ is examined with its usual norm. Let the space $L_\infty = L_\infty(K)$ be here with the norm $\sup_{x\in K} |f(x)|$ (see K.de Leeuw, H.Mirkil [11]). We have:

Lemma 5'''. For the function $h \in \mathcal{L}_*$ let there exist such a sequence $\{\varphi_m\}$, $\varphi_m \in C^\infty|K$, that $\{\varphi_m\} \xrightarrow{*} h$ and $\{A\varphi_m\} \xrightarrow{*} H$. Here $\{g_m\} \xrightarrow{*} g$ denotes that $\gamma_j(g_m - g) \longrightarrow 0$ as $m \longrightarrow \infty$ for $\forall$ j; A is a linear constant-coefficient differential operator. If for another sequence $\{\psi_m\}$, $\psi_m \in C^\infty|K$, we have $\{\psi_m\} \xrightarrow{*} h$ and $\{A\psi_m\} \xrightarrow{*} M$, then H = M in $\mathcal{L}_*$. (Lemma 5" is a little stronger than Lemma 5 and will be proved non using results on $\mathcal{D}'$.)

Proof of Lemma 5". The properties of the space $\mathcal{L}_*$ assure that $\{\varphi_m - \psi_m\} \longrightarrow 0$ in $\mathcal{D}'$. Therefore $\{A(\varphi_m - \psi_m)\} \longrightarrow 0$ in $\mathcal{D}'$. So we have

$$\int A(\varphi_m - \psi_m)(w)\,\varphi(w)\,dw = \int (\varphi_m - \psi_m)(w)\,A^*\varphi(w)\,dw.$$

for $\forall \varphi \in \mathcal{D}$, where $A^* = \sum_k (-1)^k a_k D^k$ if $A = \sum_k a_k D^k$

Thus H=M in L_1. However H, M $\in \mathcal{L}_* \subseteq L_1$. That is why H=M in $\mathcal{L}_*$. ∎

Definition 4'. For the function $h \in \mathcal{L}_*$ let there exist such a sequence $\{\varphi_m\}$, $\varphi_m \in C^\infty|K$ that $\{\varphi_m\} \xrightarrow{*} h$ and $\{A\varphi_m\} \xrightarrow{*} H$.

Then the function H will be called a generalized strong A_* derivative $A_* h$ of the function h and will be denoted by $A_* h$ (=Ah). According to Lemma 5" if there exists such a derivative $A_* h$ for the function $h \in \mathcal{L}_*$ it should be unique.

Definition. The completion of $C^\infty | K$ by the family of seminorms $\{\pi_j\}_j$, $j=0,1,2,\ldots$, where

$$\pi_j f = \gamma_j f + \sum_{q=1}^{m} \gamma_j A_q f$$

and $A_1,\ldots,A_m$ are linear constant-coefficient differential operators, will be denoted by $\mathcal{W}_*^{A_1,\ldots,A_m} = \mathcal{W}_*^{A_1,\ldots,A_m}(K)$ and will be called a space of functions with $A_1,\ldots,A_m$ strong generalized derivative. Now it is evident the following:

Theorem. The completion $\mathcal{W}_*^{A_1,\ldots,A_m}$ is the space of all complex-valued functions of $\mathcal{L}_*$ with strong generalized derivatives $A_q h$ in $\mathcal{L}_*$, $q=1,\ldots,m$.

Moreover, each solution $h \in \mathcal{L}_*$ of the system $\{A_q h = u_q,\ q=1,\ldots,m\}$ belongs to the space $\mathcal{W}_*^{A_1,\ldots,A_m}$ if $u_q \in \mathcal{L}_*$, $q=1,\ldots,m$.

Remark. In the case $\mathcal{L}_* = C_0(R^2)$ and $A_1,\ldots, A_q,\ldots,A_m$ with $A_1,\ldots,A_q$ - homogeneous constant-coefficient linear differential operators of order N in two variables and $A_{q+1},\ldots,A_m$ equal to $D^s = \partial^{|s|}/\partial x^s$, $s=(s_1,s_2)$, $|s| < N$, $x=(x_1,x_2) \in R^2$, such spaces $\mathcal{W}_*^{A_1,\ldots,A_q,\ldots A_m}$, a completion in the supnorm, are proposed by K.de Leeuw, H.Mirkil [22].

Proof of Theorem 6. It is well known (see K.de Leeuw, H.Mirkil [11]) that an elliptic linear constant-coefficient differential operator A of order N and the identity operator I jointly dominate in the L_p - norm for $\forall$ p with $1 < p < \infty$ any linear constant-coefficient differential operator B of order not larger than N. Inversely, if the latter is true then A is elliptic.

Any function $h \in W_p^N$ has a generalized strong A_p derivative in L_p, that is why

$$W_p^N \subseteq \mathcal{W}_p^{A,I}.$$

Now if $h \in \mathcal{W}_p^{A,I}$ then there exists a sequence $\{\varphi_m\}$, $\varphi_m \in C^\infty | K$ after Definition 4, for which $\{\varphi_m\} \xrightarrow{*} h$, $\{A\varphi_m\} \xrightarrow{*} A_p h$

It follows from the cited result that

$$\| D^s \varphi_m \|_p \leq \varkappa \left(\| A \varphi_m \|_p + \| \varphi_m \|_p \right), \; 1 < p < \infty,$$ where $\| \cdot \|_p$ is the norm in L_p, for $\forall s \in Z_+^N$ with $|s| \leq N$, $\varkappa$ is a constant, $\varkappa = \varkappa(p)$. This implies that $D^s h$ exists for $\forall s$ with $|s| \leq N$. Thus

$$\mathcal{W}_p^{A,I} \subseteq W_p^N, \text{ i.e., } \mathcal{W}_p^{A,I} = W_p^N, \; 1 < p < \infty.$$

Furthermore, let us have

$$\mathcal{W}_p^{A,I} = W_p^N \qquad \text{for some fixed } p, \; 1 < p < \infty.$$ We can imply the Closed Graph Theorem to the imbedding $\mathcal{W}_p^{A,I} \subseteq W_p^N$, where the spaces $\mathcal{W}_p^{A,I}$ and W_p^N are studied with their natural topologies. The continuity of the imbedding mapping signifies that

$$\| D^s f \|_p \leq \varkappa^* \left\{ \| f \|_p + \| A f \|_p \right\} \qquad \forall f \in \mathcal{D}_\infty | K, \; \varkappa^*$$ is a constant, for $\forall s, |s| \leq N$, $1 < p < \infty$, i.e. $\mathcal{W}_p^{A,I} = W_p^N$ implies that the operator A is elliptic.

We have for L_1 and for L_∞ with the supremum norm the following :

Theorem (K. de Leeuw, H.Mirkil [11], D.Ornstein [23]). Let A be a linear constant-coefficientdifferential operator in $n \geq 3$ variables of order $N \geq 2$. Then a necessary and sufficient condition that A be elliptic is that A and I jointly dominate all linear constant-coefficient differential operators of order not larger than N. If $n=2$, the condition is only necessary.

Now let $h \in \mathcal{W}_\infty^{A,I}$, A be elliptic, $n \geq 2$. Then the strong generalized derivative $A_\infty h$ exists. Let the sequence $\{\varphi_m\}$, $\varphi_m \in C^\infty | K$, be such that

$$\{\varphi_m\} \xrightarrow{L_\infty} h, \quad \{A \varphi_m\} \xrightarrow{L_\infty} A_\infty h.$$ (Such a sequence exists after Definition 4. We have according to the previous Theorem that

$$\| D^s \varphi_m \|_{L_\infty} \leq \varkappa_* \left\{ \| A \varphi_m \|_{L_\infty} + \| \varphi_m \|_{L_\infty} \right\}, \; \forall m,$$

for any $s \in Z_+^n$ with $|s| \leq (N-1)$ for some constant $\varkappa_*$. This is sufficient to affirm that the derivative $D^s h$ exists in L_∞, $\forall s, |s| \leq (N-1)$. So we obtain $\mathcal{W}_\infty^{A,I} \subset W_\infty^{N-1}$.

Further, let $n \geq 3$ and let we have $\mathcal{W}_{\infty}^{A,I} \subset W_{\infty}^{N-1}$, where the spaces $\mathcal{W}_{\infty}^{A,I}$ and W_{∞}^{N-1} are studied with their natural topologies. Implying the Closed Graph Theorem to the corresponding imbedding map, we receive that

$$\|D^s f\|_{L_\infty} \leq æ' \left\{ \|Af\|_{L_\infty} + \|f\|_{L_\infty} \right\} \text{for } \forall f \in C^\infty|K ,\ æ' \text{ is a}$$

constant. According to the previous cited Theorem , we obtain that the operator A is elliptic.

We prove with similar reasonning that the assertion about W_1^N and $\mathcal{W}_1^{A,I}$ also is true . ■

Proof of Theorem 7. The assertion is a consequence from Theorem 4 : As A is a parabolic constant-coefficient linear differential operator of order N in $n \geq 2$ variables, then there exists a linear real nondegenerated transform L of R^n , such that the operator A is transformed in LA in the form (2) in variables $(x_1,\ldots,x_{n-1},t)$. If we assume that

$\mathcal{W}_{\infty}^{LA} \subset W_{\infty}^{N-1}$ (where spaces $\mathcal{W}_{\infty}^{LA}$ and W_{∞}^{N-1} are studied with their natural topologies), then applying the Closed Graph Theorem to the corresponding imbedding map, we shall receive that the operator LA dominates the operator D_t^{N-k} ,$k > 0$, in the supremum norm. But this contradicts with the assertion of Theorem 4. That is why we have

$$\mathcal{W}_{\infty}^{LA} \not\subset W_{\infty}^{N-1} \text{. Hence we obtain}$$

$$\mathcal{W}_{\infty}^{A} \not\subset W_{\infty}^{N-1} \text{ also.}$$

Furthermore, it is evident that any operator derivative $(LA)^s$, $s \in Z_+^n$, of the operator LA in the transformation L cannot dominate the operator D_t^{N-1} . That is why we have

$$\mathcal{W}_{\infty}^{LA,\ldots,(LA)^{(s)},\ldots,I} \not\subset W_{\infty}^{N-1} \text{ .Hence we also get}$$

$$\mathcal{W}_{\infty}^{A,\ldots,A^{(s)},\ldots,I} \not\subset W_{\infty}^{N-1} \text{ . ■}$$

The proof of Theorem 8 is contained in [10], although Theorem 8 is connected with this article. That is why we shall only remind the subject of this Teorem: So we have the following:

Definition (G.E.Shilov[24]). An algebra R of type C of complex-valued functions on a compact K , $\overset{\circ}{\bar{K}} = K$, is a Banach algebra of complex-valued functions on K for which 1). The norm $\| f \|$ in R of $f \in R$ is equivalent to the norm

$$\sup_{w \in K} \inf_{g \in R} \left\{ g \text{ , with } g = f \text{ in some neighbourhood of } w \right\}.$$

2). R is without radical , i.e. the intersection of all maximal ideals of R consists of the zero element of R only.

Since $\alpha \neq \emptyset$ is differential-invariant set of linear differential operator, then the identity operator $I \in \alpha$. Hence the topology in ${}^{\S}W^{\alpha}_{\infty}$ is stronger than the pointwise convergence. This permits to apply the Closed Graph Theorem to the inclusions of the kind

$${}^{\S}W^{\alpha}_{\infty} \subseteq W^{\mu}_{\infty}.$$

Proof of Theorem 9. Now let $\alpha = (A_q)_q$, $q=1,\dots,m$, be a generalized parabolic system of order $N \geq 2$ of linear constant-coefficient differential operators in $n \geq 2$ variables. There exists such a variable x_{j_0} after Theorem 4 that all operators $A_1,\dots,A_m$ do not dominate jointly the operator

$$D^{N-1}_{x_{j_0}} = (\partial/\partial x_{j_0})^{N-1} .$$

Let us assume the contrary of Theorem 9, i.e., ${}^{\S}W^{\alpha}_{\infty} \subseteq W^{N-1}_{\infty}$, where the spaces ${}^{\S}W^{\alpha}_{\infty}$ and W^{N-1}_{∞} are studied with their natural topologies. The Closed Graph Theorem is applyable to the latter inclusion, since the natural topologies in ${}^{\S}W^{\alpha}_{\infty}$ and W^{N-1}_{∞} are stronger than the pointwise convergence. It follows the continuity of the corresponding imbedding map after the Closed Graph Theorem. The continuity of the imbedding map implies that the operators $A_1,\dots,A_m$ jointly dominate each operator $D^{N-1}_{x_{j_0}}$, i.e.,

$$\left\{ \| (\partial/\partial x_{j_0})^{N-1} g \|_{L_\infty} \leq \varkappa \left\{ \| A_1 g \|_{L_\infty} + \dots + \| A_m g \|_{L_\infty} \right\} \right.$$

for $\forall\, g \in C^{\infty}|_{K}$, where $\varkappa$ is a constant. So constructed contradiction involves that

$${}^{\S}W^{\alpha}_{\infty} \not\subseteq W^{N-1}_{\infty} . \blacksquare$$

Proof of Theorem 1C. Now, let us have $\mathcal{W}^{\alpha}_{\infty} \not\subseteq W^1_{\infty}$, where the differential-invariant space $\alpha = (A_1,\dots,A_m)$ in $n \geq 2$ variables is of second order. It follows from $\mathcal{W}^{\alpha}_{\infty} \not\subseteq W^1_{\infty}$ that we also have

$$\mathcal{W}^{L\alpha}_{\infty} \not\subseteq W^1_{\infty}$$

for each linear real nondegenerated transformation L of R^n, where $L\alpha = (LA_1,\dots,LA_m)$. Now, let the second-order operator A belong to the linear hull of the system α. We must prove that A is a generalized Petrovsky's parabolic operator: It follows from $\mathcal{W}^{LA}_{\infty} \not\subseteq W^1_{\infty}$ and from the closed Graph Theorem that the operator LA for each fixed L together with its operator derivatives jointly do not dominate all operators $\partial/\partial u_j$, $j=1,\dots,n$, where $(LR^n)(u_1,\dots,u_n)$.

Let $B^{[2]}$ be the homogeneous part of second order of the operator B, where B is a linear differential operator.

Let L^* be a real linear nondegenerated transformation of R^n, which transforms $A^{[2]}$ into

(22)
$$L^* A^{[2]} = (L^* A)^{[2]} = \sum_j \varepsilon_j \partial^2/\partial u_j^2 + i\sum_{j,e} \beta_{j,e}\, \partial^2/\partial u_j \partial u_e ,$$

where $\varepsilon_j = \pm 1, 0$, and the constants $\beta_{jl} \in R^1$. Such a transformation L^* exists since we have

$A^{[2]} = \sum a_{jl}\partial^2/\partial x_j \partial x_l$ with $a_{jl} = a'_{jl} + i a''_{jl}$, $a'_{jl}, a''_{jl} \in R^1$, $\sum |a'_{jl}| \neq 0$. Therefore L^* is a canonical Lagrange transformation for the real quadratic form

$$\sum a'_{jl}\, x_j\, x_l \; .$$

Since $\mathcal{W}^{L^*\alpha}_{\infty}(K^*) \not\subseteq W^1_{\infty}(K^*)$ and since K^* is a compact with $\overline{\mathring{K}^*} = K^*$, hence there exists a function $\varphi^* \in \mathcal{W}^{L^*\alpha}_{\infty}(K^*)$ and $\varphi^* \notin W^1_{\infty}(K^*)$, without all continuous partial derivatives of first order on $\mathring{K}^*$. It is evident, that we can suppose, that φ^* is a real function and that does not exist the continuous $(\partial/\partial u_n)\varphi^*$ on $\mathring{K}^*$, where $K^* = LK$.

Some of the coefficients $\mathcal{E}_j$ in (22) are zero, since we have that the functions of $\mathcal{W}_\infty^{L^*\alpha}$ have all strong generalized derivatives for the supremum norm - L^*A_p, $(L^*A_p)^{(s)}$, $p=1,\ldots,m$, $\forall\, s \in Z_+^n$, $|s| \leq 2$. Moreover, we have

$$(23)\quad (L^*A)^{(\mathcal{J})} = \left(\mathcal{E}_j\,\partial/\partial u_j + i\sum\nolimits_\ell \beta_{j\ell}\,\partial/\partial u_\ell\right),$$

where $\mathcal{J} = (0,\ldots,0,1,0,\ldots,0)$ with 1 only on the j-th place, and $(L^*A)^{(\mathcal{J})}$ is the corresponding operator derivative of the operator L^*A. Let us remind that the operator A belongs to the linear hull of $A_1,\ldots,A_m$. That is why there exist and are continuous all strong generalized derivatives $(\mathcal{E}_j(\partial/\partial u_j)f)$, $\forall\, j$, and for $\forall f \in \mathcal{W}_\infty^{L^*A}$. Since $\varphi^* \in \mathcal{W}_\infty^{L^*A}$ and since φ^* does not have the continuous $\partial/\partial u_n$ derivative, hence $\mathcal{E}_n = 0$.

Let r be the number of $\mathcal{E}_j \neq 0$. We proved that $r < n$. Furthermore, since φ^* is real, then (22) and (23) imply the existence and the continuity of all strong generalized derivatives

$$\left\{\sum\nolimits_\ell \beta_{j\ell}(\partial/\partial u_\ell)\right\}\varphi^*$$

in the supremum norm, as well as the existence and the continuity of the strong generalized derivatives

$$\left\{\sum_{\ell,(\mathcal{E}_\ell=0)} \beta_{j\ell}(\partial/\partial u_\ell)\right\}\varphi^*.$$

But the function φ^* have not a continuous $\partial/\partial u_n$ derivative on $\mathring{K}^*$, after its choice. Therefore

$$\operatorname{rang}\left(\beta_{j\ell}\right)_{(\mathcal{E}_\ell=0)} < n-r.$$

That is why a part of the derivatives $\partial/\partial u_l$ with $\mathcal{E}_l = 0$, including $\partial/\partial u_n$, are linear real combination of the other derivatives $\partial/\partial u_l$ with $\mathcal{E}_\ell = 0$. Therefore we have

$$L^*A^{[2]} = (L^*A)^{[2]} = \sum_{j=1}^{n-1} \mathcal{E}_j\,\partial^2/\partial u_j^2 + i\,Q(\partial/\partial u_\ell)$$

where Q is a quadratic form in less than n variables and Q does not depend on $\partial/\partial u_n$. After Definition 2 we must find a linear real nondegenerated transformation, which transforms the operator A(in the studied case N=2) in the kind either

$$\sum_{|s|=2} a_s D^s + b D_t + P(D)$$ if k=1 or in the kind

$$\sum_{|s|=2} a_s D^s + \sum_{|s|\leq 1} b_s D^s + P(D)$$ if k=2 ,

where P is a polynomial of D of degree not larger than 1 .But such is the transformation L^* ($D_t = \partial/\partial u_n$ here), i.e. the operator A is a generalized parabolic operator. ∎

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A NOTE ON THE AUTOMORPHISMS OF THE TORI

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This note transforms the necessary and sufficient conditions for automorphisms among algebraic elliptic curves and gives to these conditions a more constructive equivalent form, althought not a concise one. The comparison of the two forms of these conditions gives some results on integers.

Let the Riemann surface of genius I be with periods ω_1 and ω_2, $\tau = \omega_2/\omega_1$, $\mathrm{Im}\,\tau > 0$. Then this torus will be denoted by $T = T(\tau)$.

Theorem 1. The torus $T(\tau)$ eventually has nontrivial automorphisms when $\tau = \omega_2/\omega_1 = a + ib$, $a, b \in \mathbb{R}$, is with a and b^2 rationals and iff either 1). $\tau = (s + i)/r$, where $s, r > 0$, $(s^2 + 1)/r$ are integers, or

2). $\tau = (2s + \gamma\varepsilon + i\sqrt{3})/2r$ with s, $r > 0$, $(s^2 + \gamma\varepsilon s + 1)/r$ are integers, $\gamma = \pm 1$, $\varepsilon = \pm 1$. The automorphisms are realized in the case 1) by the transformations

$$P = \varepsilon s x - \varepsilon y(s^2+1)/r, \quad Q = \varepsilon r x - \varepsilon s y.$$

The nontrivial automorphisms are realized in the case 2) by the transformations $P = \varepsilon s x - y(s^2 + \gamma\varepsilon s + 1)/\varepsilon r$, $Q = \varepsilon r x - y(\varepsilon s + \gamma)$.

Corollary ([1, 2]). The tori $T(i)$ and $T((\varepsilon + i\sqrt{3})/2)$, $\varepsilon = \pm 1$, have nontrivial automorphisms (which evidently follows from Theorem 1 if $s = 0$, $r = 1$ (for the both cases 1) and 2))).

Theorem 2. 1. The number $(s^2 + 1)/r$ is integer (where r, s are integers, $r \neq 0$) if and only if $r = m^2 + n^2$, $s = (kr - n)/m$ for some integers k, $m \neq 0$, and $l = (kn - 1)/m$. Moreover, then $(s^2 + 1)/r = k^2 + l^2$.

2. The number $(s^2 + \varepsilon s + 1)/r$ is integer (where s, r are integers, $r > 0$), if and only if $r = m^2 + mn + n^2$, $s = [kr - n - m(\varepsilon + 1)/2]/m$, $\varepsilon = \pm 1$, for some integers $m \neq 0$, k, n and $l = (kn - 1)/m$. Moreover, then we have $(s^2 + \varepsilon s + 1)/r = k^2 + kl + l^2$.

Proof of Theorem 1. It is well known that the conditions

(1) $(\tau k + l)/(\tau m + n) = \delta$, $kn - lm = 1$, $k, l, m, n \in \mathbb{Z}$, $\mathrm{Im}\,\tau > 0$, $\mathrm{Im}\,\delta > 0$

are necessary and sufficient conditions for an automorphism between two algebraic elliptic curves $T(\tau)$ and $T(\delta)$ correspondingly with $\mathrm{Im}\,\tau > 0$, $\mathrm{Im}\,\delta > 0$ (see [1,2]). Here Z is the set of all integers. We receive from (1) that the necessary and sufficient conditions for an automorphism of the torus $T(\tau)$ are (see [1]) the following

(2)
$$(\tau k + l)/(\tau m + n) = \tau,\ kn - lm = 1,\ k, l, m, n \in \mathbb{Z},\ \mathrm{Im}\,\tau > 0$$
$$(\text{I.e. } m\tau^2 + (n-k)\tau - l = 0).$$

Let $\tau = a + ib$, $a, b \in \mathbb{R}$. Then $b > 0$, since $\mathrm{Im}\,\tau > 0$. I.

I. If we assume $m = 0$ in the system (2), then we receive $\tau(n - \overline{k}) = l$. Since $\mathrm{Im}\,\tau > 0$, hence $l = 0$ and $n = k$. This is, there only exist the trivial automorphisms $P = x$, $Q = y$ and $P^o = -x$, $Q^o = -y$.

II. Now let $m \neq 0$. I.e.

(3)
$$\tau_{1,2} = \left(k - n \pm \sqrt{(k-n)^2 + 4lm}\right)/2m = \left[k - n \pm \sqrt{(k+n)^2 - 4}\right]/2m.$$

The conditions (2) are also equivalent to the conditions

(4)
$$\left|\begin{array}{l} m(a^2 - b^2) + a(n-k) - l = 0, \\ 2mab + (n-k)b = 0 \\ kn - lm = 1, \end{array}\right. \qquad k, l, m, n \in \mathbb{Z},\ b > 0,\ m \neq 0.$$

Therefore we obtain the equivalent system

(5)
$$\begin{array}{l} a = (k-n)/2m \\ b^2 = \left[4 - (n+k)^2\right]/4m^2 \\ k, m, n \in \mathbb{Z},\ m \neq 0,\ (kn-1)/m \in \mathbb{Z},\ b > 0. \end{array}$$

Remark. The system (5) yields $a^2 + b^2 = -l/m$.

It follows from (5) (as well as from (3)) that a and b^2 are rationals.

The system (5) implies $|k + n| < 2$, since $b \in \mathbb{R}$. This is, $k + n$ can be equal only to -1 , 0 , 1 . 1. If $k + n = 0$ then $a = k/m$, $b^2 = 1/m^2$, $\tau = k/m \pm i/m$, $l = -(k^2 + 1)/m$. The nontrivial automorphisms are realized by the transformations:

$$P = kx - y(k^2+1)/m,\quad Q = mx - ky .$$

Thus we get the assertion of Theorem 1.1). ~~[illegible]~~ , when $n = \varepsilon k$, $r = \varepsilon m$, $\varepsilon = \mathrm{sign}\ m = \pm 1$.

2. Let $k+n = -1$. Then $l = -(k^2 + k + 1)/m \in Z$, $a = (2k + 1)/2m$, $b = \sqrt{3}/2|m|$, $\tau = (2k + 1 \pm i\sqrt{3})/2m$. The transformations

$$P = kx - (k^2 + k + 1)y/m, \quad Q = mx - (k+1)y$$

realize nontrivial automorphisms. When $s = \varepsilon k$, $r = \varepsilon m$, ε = sign m, we get the assertion of Theorem 1.2 with $\gamma = 1$.

3. If $k+n = 1$, then we have $l = -(k^2 - k + 1)/m \in Z$, $a = (2k - 1)/2m$, $b = \sqrt{3}/2|m|$, $\tau = (2k - 1 \pm i\sqrt{3})/2m$. The nontrivial automorphisms are realized by the transformations:

$$P = kx - y(k^2 - k + 1)/m, \quad Q = mx - (k-1)y.$$

Thus we get the assertion of Theorem 1.2 with $\gamma = -1$, when $s = \varepsilon k$, $r = \varepsilon m$, $\varepsilon = \text{sign } m = \pm 1$.

Proof of Theorem 2.1. I. At first let $\tau = s/r + i/r$, where $r > 0$, s, $(s^2 + 1)/r$ are integers. In the case $r = 1$, we receive the assertion of the theorem with $n = 0$, $m = 1$, $s = k$, $l = -1$. Further on, let $r \neq 1$. Then there exist nontrivial automorphisms of the Riemann surface $T(s/r + i/r)$ of genius I after Theorem 1. Therefore the torus $T(s/r + i/r)$ must be isomorphous either to the torus $T(i)$ or to the torus $T(1/2 + i\sqrt{3}/2)$ after [1,2]. Let us assume that the torus $T(s/r + i/r)$ is isomorphous with the torus $T(1/2 + i\sqrt{3}/2)$. Then we have

$$(6) \quad \left| \begin{array}{l} [k(1/2 + i\sqrt{3}/2) + l] / [m(1/2 + i\sqrt{3}/2) + n] = s/r + i/r \\ kn - lm = 1 \end{array} \right.$$

for some integers k, l, m and n, in accordance with (1). The system (6) implies

$$[2km + 2ln + kn + ml + i\sqrt{3}]/2(m^2 + mn + n^2) = s/r + i/r$$

Thus an isomorphism between $T(s/r + i/r)$ and $T(1/2 + i\sqrt{3}/2)$ involves that $\sqrt{3} = 2(m^2 + mn + n^2)/r$ with m, n, r integers, which is impossible. Therefore the torus $T(s/r + i/r)$ is isomorphous to the torus $T(i)$. This is, we have

$$(7) \quad \left| \begin{array}{l} (ki + l)/(mi + n) = s/r + i/r \\ kn - lm = 1 \end{array} \right.$$

for some integers k, l, m, n (eventually depending on s). The system (7) gives $(km + ln + i)/(m^2 + n^2) = s/r + i/r$

Comparing the real and imaginar parts of the latter equation and again using (7), we receive an equivalent system -

(8) $$\left| \begin{array}{l} r = m^2 + n^2, \ s = km + ln, \ k, l, m, n \in \mathbb{Z}, \\ kn - lm = 1 \end{array} \right.$$

The system (8) implies $(s^2 + 1)/r = k^2 + l^2$. Let us assume $m=0$. Then the system (8) can be transformed in the system

$$\left| \begin{array}{l} r = n^2, \ s = ln, \ kn = 1 \\ k, l, n \in \mathbb{Z}, \end{array} \right.$$

in the case $m=0$. Therefore $k = n = 1$ and $r = 1$. But we have $r \neq 1$. That is why (8) involves $m \neq 0$ and $s = (km + ln) = (kr - n)/m$, $k , l , m, n \in Z$, $m \neq 0$.

Inversely, let the integers $r = m^2 + n^2$, $r \neq 0,1$, $m \neq 0$, $s = (kr - n)/m$, $k, n \quad Z$, $l = (kn - 1)/m \quad Z$. Then we have $kn - lm = 1$ and $s = (kr - n)/m = km + n(kn - 1)/m = km + ln$ .Therefore we obtain that the number $(s^2 + 1)/r = \left[(km + ln)^2 + 1\right]/(m^2 + n^2) = k^2 + l^2$ is integer . The case $r=1$ is evident with $m = 1$, $n = 0$, $l = -1$, $s = k$. This complete the proof of Theorem 2,Point 1 .

Proof of Point 2. The case $r = 1$ is evident with $m = \pm 1$, $n = 0$, $l = -1$, $s = k - (\varepsilon + 1)/2$, $\varepsilon = \operatorname{sign} m$. Further on let $r \neq 1$.

I. At first let the number $(s^2 + \varepsilon s + 1)/r$ be integer,where $r > 0$, $r \neq 1$, r and s are integers, $\varepsilon = \pm 1$. Let us choose $\tau = (2s + \varepsilon + i\sqrt{3}).(1/2r)$. Then the torus $T(\tau)$ has nontrivial automorphisms after Theorem 1 . Therefore the torus $T(\tau)$ must be isomorphous either to the torus $T(i)$ or to the torus $T(1/2 + i\sqrt{3}/2)$, according to [1,2]. Let us assume that $T(\tau)$ is isomorphous with the torus $T(i)$. Then there exist integers k , l , m , n with

(9) $$\left| \begin{array}{l} (ki + l)/(mi + n) = \tau = (2s + \varepsilon + i\sqrt{3})/2r, \\ kn - lm = 1 \\ k, l, m, n \in \mathbb{Z} \end{array} \right.$$

The system (9) involves

$$(km + ln + i)/(m^2 + n^2) = (2s + \varepsilon + i\sqrt{3})/2r.$$

Comparing the imaginar parts of the latter equation, we receive $2r/(m^2 + n^2) = \sqrt{3}$ with the integers r , m , n, which is impossible. That is why the torus $T(\tau)$ is isomorphous with the torus $T(1/2 + i\sqrt{3}/2)$

Such an isomorphism implies the existence of the system

(10) $$\left| \begin{array}{l} \left[k(1/2+i\sqrt{3}/2)+l\right] / \left[m(1+i\sqrt{3})/2+n\right] = \tau = (2s+\varepsilon+i\sqrt{3})/2\tau, \\ kn-lm=1,\ k,l,m,n \in \mathbb{Z}. \end{array} \right.$$

for some integers k, l , m, n . The system (10) is equivalent with the system

(10') $$\left| \begin{array}{l} (2ln+kn+lm+2km+i\sqrt{3})/(m^2+mn+n^2) = (2s+\varepsilon+i\sqrt{3})/\tau \\ kn-lm=1,\ k,l,m,n \in \mathbb{Z}. \end{array} \right.$$

Comparing the real and imaginar parts of the first equation of (10'),we receive the following equivalent system

(11) $$\left| \begin{array}{l} (2ln+kn+lm+2km)/(m^2+mn+n^2) = (2s+\varepsilon)/\tau, \\ \tau = m^2+mn+n^2,\ kn-lm=1,\ k,l,m,n \in \mathbb{Z} \end{array} \right.$$

This is, we obtain the equivalent system

(11') $$\left| \begin{array}{l} \tau = m^2+mn+n^2,\ 2s+\varepsilon = 2ln+kn+lm+2km \\ kn-lm=1,\ k,l,m,n \in \mathbb{Z}. \end{array} \right.$$

Thus we have only the following two cases ,depending on ε :

a). $\varepsilon=1$, $2s'+1 = 2ln+2km+kn+lm$.

b). $\varepsilon=-1$, $2s''-1 = 2ln+2km+kn+lm$

a). We receive in the case ε = 1 that s' = ln + km + lm and

$$(s'^2+s'+1)/\tau = \left[(ln+km+lm)^2+(ln+km+lm)+1\right]/(m^2+mn+n^2) = k^2+kl+l^2.$$

b). We obtain in the case ε = -1 that s" = ln + km+kn and

$$(s''^2-s''+1)/\tau = \left[(ln+km+kn)^2-(ln+km+kn)+1\right]/(m^2+mn+n^2) = k^2+kl+l^2 .$$

Let us assume m = 0 . Then we receive from (11') that

$$\left| \begin{array}{l} \tau = n^2 \\ 2s+\varepsilon = 2ln+kn,\ kn=1,\ k,l,n \in \mathbb{Z}. \end{array} \right.$$

The latter system involves k = n = 1 ; r = 1 ; 2s + ε = 2ln + 1 ; l$\in$ Z. But since r $\neq$ 1, hence we have m $\neq$ 0 . Therefore l = (kn - 1)/m. This involves in the case a) that s' = ln + km +lm = (kr - m - n)/m and in the case b) that s" = ln + km + kn = (kr - n)/m . Thus we get

II. Inversely, let us have the integers $r = m^2 + mn + n^2$, $r \neq 1$, $r>0$, $s = \left[kr - n - (\varepsilon + 1)m/2\right]/m$, where $m \neq 0$, k, n, $l = (kn - 1)/m$ also are integers. Then we shall prove that the number $(s^2 + \varepsilon s + 1)/r$ is an integer too, $\varepsilon = \pm 1$.

Let at first $\varepsilon = 1$. Then $s = (kr - m - n)/m$ and $(s^2+s+1)/r =$

$$\left[(kr-m-n)^2 + m(kn-m-n) + m^2\right]/m^2 r = (k^2r^2+r-krm-$$

$$-2krn)/rm^2 = k^2 + \left[(kn-1)^2 + km(kn-1)\right] = k^2+kl+l^2.$$

Let now $\varepsilon = -1$. Then we have $s = (kr - n)/m$ and then $(s^2-s+1)/r =$

$$\left[(kr-n)^2-(kr-n)+1\right]/m^2r = \left[k^2r-2kn-km+1\right]/m^2 =$$

$$k^2 + \left[(kn-1)^2+(kn-1)km\right]/m^2 = k^2+kl+l^2. \quad \blacksquare$$

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ON A MODEL OF THE REAL NUMBERS

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This article exposes a new model of the real numbers, constructed by the rational numbers analogously of the well known models of Cantor-Méray, Dedekind, Bachmann and others [1-3]. The model, proposed here, is more natural from gnosiological (i.e. epistemological) and ontological points of view [4-7]. The problem to construct the real numbers by the intervals of the real line in the sense of the method, built here, has been proposed by Whitehead and Russell in the beginning of this century. The idea of Whitehead for such constructions by psychologically more primary objects (for instance by the events for the instants, or by the intervals for the real numbers) does not concern the instants only. This idea also includes the real numbers. This idea has found a general recognition since the events, or the intervals are psychologically more primary conceptions, while the instants or the real numbers are intuitive-mental constructions. That is why it is worth-while the conceptions of the instants or of the real numbers to be built by logico-mathematical ways from the more primary psychologically objects, as the events or the intervals correspondingly.

The exposed here model of the real numbers has resemblances with the other models of the real numbers, as well as the other models of the real numbers have resemblances, although each of these models has its importance. The model of Cantor-Méray and the Dedekind's model of the real numbers are considered as the most different [1-3]. But each of these constructions can evidently be reduced to the other. Although each of them is independently and thoroughly exposed in the literature. Here we shall show how easy and clearly the construction of Cantor-Méray of the real numbers by fundamental sequences can be reduced to the Dedekind's construction of the real numbers by sections.

Let us remind the basic definitions.

Definition. A sequence $\{r_n\}$ is called fundamental if for every natural number m there exists a natural number n_m, such that

$|r_n - r_{n'}| < \frac{1}{m}$ for $\forall$ n and $\forall$ n' with $n \geqslant n_m$, $n' \geqslant n_m$.

Definition. The nonempty disjoint subsets A and B of the set of the rational numbers Q form a section A I B of Q if they have the following properties:

I. If $p \in A$ and $p' < p$, then $p' \in A$, p , $p' \in Q$;

II. If $q \in B$ and $q' > q$, then $q' \in B$, q , $q' \in Q$;

III. The set $Q - (A \cup B)$ does not contain more than one rational number.

Now we shall easily reduce the construction of Cantor-Méray of the real numbers by fundamental sequences to the construction of the real numbers by sections. I.e. we must show how does any fixed fundamental sequence $\{r_n\}$ of rational numbers determine a corresponding Dedekind's section. $A \mid B$ of Q.

Let $A = \left\{ p : p \in Q , p \leq r_n - 1/m \text{ for } \forall m \text{ with } \forall n \geq n_m , m, n, n_m \in Z_+ \right\}$

and $B = \left\{ q : q \in Q , q \geq r_n + 1/m \text{ for } \forall m , \forall n \text{ with } n \geq n_m, m,n,n_m \in Z_+ \right\}$.

Evidently, $A \mid B$ is a Dedekind's section by the construction: Clearly $A \mid B$ has the properties I and II of a Dedekind's section. III. If $a \overline{\in} A \cup B$, $a \in Q$, then we have $r_n - 1/m < a < r_n + 1/m$ for $\forall m$, $\forall n$, $n \geq n_m$. But this cannot be satisfied from more than one rational number, since the sequence $\{r_n\}$ is fundamental. ∎

Although each of these two constructions of the real numbers is independently exposed and has its advantages. Therefore the proposed here model of the real numbers has its rights to be exposed also.

Let Q be the set of the rational numbers with their natural order "$\leq$", the addition "+" and the multiplication "." .Let p and q be arbitrarily fixed rational numbers $p \leq q$ .We shall denote the set of all rational numbers r with $p \leq r \leq q$ by $\langle p,q \rangle$ and we shall call it a rational segment.We shall say that two rational segments $\langle p,q \rangle$ and $\langle r,s \rangle$ are intersecting if they have at least one common element,i.e. $\langle p,q \rangle \cap \langle r,s \rangle \neq \emptyset$. Let $\mathbb{R}^*$ denote the set of all classes of mutually intersecting rational segments, i.e., $\{\langle p_i,q_i \rangle\}_{i \in I} \in \mathbb{R}^*$ if and only if $\langle p_{i'},q_{i'} \rangle \cap \langle p_{i''},q_{i''} \rangle \neq \emptyset$ for any two elements i' , $i'' \in I$.Here p_i ,$q_i \in Q$, $p_i \leq q_i$; I is a complex of indexes. We introduce a partial order in $\mathbb{R}^*$ by inclusions (in the sense of the Set Theory).of the classes of $\mathbb{R}^*$: Let Γ_1 and Γ_2 belong to $\mathbb{R}^*$ .We shall deem that Γ_2 follows Γ_1 , $\Gamma_1 \prec \Gamma_2$ iff $\Gamma_1 \subseteq \Gamma_2$. The complex $\mathbb{R}^*$ is partially ordered by this relation.We shall call the maximal elements of $\mathbb{R}^*$ real classes or real numbers.Coarsely,these are the classes of all mutually intersecting rational segments.We shall prove the existence of such maximal elements in $\mathbb{R}^*$. Further we shall denote these maximal classes of $\mathbb{R}^*$ of mutually intersecting rational segments (i.e. the real classes) by small Greek letters.The complex of all real classes,i.e. of all real numbers, we shall denote by $\mathfrak{R}$.

I.Existence. It is evident by Proposition 1 that $\mathfrak{R}$ is not empty,i.e. has at least one element.This article proves the following results on the existence of real classes.

Proposition 1. Let p be an arbitrarily fixed rational number.The complex of all rational segments containing the rational segment $\langle p,p\rangle$ is a maximal element of $\mathfrak{R}$,which is denoted by $\pi = \{\langle p,p\rangle\}$,i.e. $\pi \in \mathfrak{R}$. π will be called generated by $p \in Q$.

Let us scrutinize the subcomplex $\mathfrak{Q}$ of $\mathfrak{R}$,consisting of all maximal elements of $\mathfrak{R}^*$ any of which is generated by some rational segment $\langle p,p\rangle$, $p \in Q$. Evidently, the set $\mathfrak{Q}$ is isomorphous to the set Q of all rational numbers.We shall call the classes of $\mathfrak{Q}$ rational classes or rational numbers.

II. Order. The natural order "$<$" in Q induces an order in the complex $\mathfrak{R}$: Let the real classes (numbers) μ and $\nu \in \mathfrak{R}$.We shall say that μ is less than the real number ν (or that ν is larger than μ), $\mu < \nu$, if there exist rational segments $\langle p,q\rangle \in \mu$,$\langle r,s\rangle \in \nu$,such that $q < r$.We shall use the notation $\mu = \nu$ if the maximal class of μ coincides with the maximal class ν of rational segments.

Proposition 2. If the real class (i.e. real number) α is less than the real class β , $\alpha < \beta$,then it is not true neither $\beta < \alpha$, nor $\alpha = \beta$.

Proposition 3. The order of $\mathfrak{R}$ is transitive.

Theorem 4. The order of $\mathfrak{R}$ is a linear order.

Theorem 5. For any arbitrarily fixed different real classes π and τ of $\mathfrak{R}$ with $\pi < \tau$, there exists a rational class $\varkappa \in \mathfrak{Q}$ such that $\pi < \varkappa < \tau$.

Theorem 6. The complex $\mathfrak{R}$ of the real classes is a continuum,i.e., satisfies Dedekind's Postulate.This is: let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two nonempty disjoint parts of $\mathfrak{R}$ ($\mathfrak{R}_1 \subset \mathfrak{R}$, $\mathfrak{R}_2 \subset \mathfrak{R}$, $\mathfrak{R}_1 \cap \mathfrak{R}_2 = \emptyset$) such that each real class of $\mathfrak{R}$ belongs either to $\mathfrak{R}_1$ or to $\mathfrak{R}_2$ ($\mathfrak{R}_1 \cup \mathfrak{R}_2 = \mathfrak{R}$); each element of $\mathfrak{R}_1$ is less than any real class of $\mathfrak{R}_2$.Then there exists at least one real class such that every real class, less than τ, belongs to $\mathfrak{R}_1$ and any real class, larger than τ, belongs to $\mathfrak{R}_2$.

III. The addition and multiplication in the complex.

Addition. Let α, β be arbitrarily fixed real numbers of $\mathfrak{R}$. Let the rational segments of α and β be denoted by $\langle p,q \rangle$ and $\langle r,s \rangle$ correspondingly. The real class, containing all rational segments of the kind $\langle p+r, q+s \rangle$ for some $\langle p,q \rangle \in \alpha$, $\langle r,s \rangle \in \beta$, will be denoted by $\alpha + \beta$ and will be called a sum of α and β. The real class $\Diamond = \{\langle 0,0 \rangle\}$, $0 \in Q$, will be called the zero of $\mathfrak{R}$. The real class $\mathbf{1} = \{\langle 1,1 \rangle\}$, $1 \in Q$, will be called the unity of $\mathfrak{R}$.

Theorem 7. The addition in $\mathfrak{R}$ is uniquely defined. Moreover, it is true that (i). $\alpha + \beta = \beta + \alpha$; (iii). $\alpha + \Diamond = \alpha$; (ii). $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$; (iv). The equation $\alpha + \xi = \Diamond$ has a solution. The solution of $\alpha + \xi = \Diamond$ (which is uniquely determined) will be denoted by $-\alpha$. Here $\alpha, \beta, \gamma \in \mathfrak{R}$.

Multiplication. Let $\alpha, \beta \in \mathfrak{R}$ be arbitrarily fixed. The product $\alpha . \beta$ is the unique real class, containing all rational segments $\langle u,v \rangle$, $u \leq v$, such that for a fixed u there are rational segments $\langle p,q \rangle \in \alpha$, $\langle r,s \rangle \in \beta$ with u less or equal of the product of any rationals $k \in \langle p,q \rangle$, $l \in \langle r,s \rangle$, $u \leq kl$. Let v, $\langle p,q \rangle \in \alpha$, $\langle r,s \rangle \in \beta$ be arbitrarily fixed. Then there are rationals $k^\circ \in \langle p,q \rangle$, $l^\circ \in \langle r,s \rangle$ with $k^\circ l^\circ \leq v$.

Theorem 8. The multiplication in $\mathfrak{R}$ is uniquely defined and has the following properties: (i). $\alpha . \beta = \beta . \alpha$; (ii). $(\alpha . \beta) . \gamma = \alpha . (\beta . \gamma)$ (iii). $\alpha . (\beta + \gamma) = \alpha . \beta + \alpha . \gamma$; (iv). $\alpha . \mathbf{1} = \alpha$; (v). $\alpha . \Diamond = \Diamond$; (vi). The equation $\alpha . \xi = \mathbf{1}$ with $\alpha \neq \Diamond$ has at least one solution. The solution (which is unique) of the equation $\alpha . \xi = \mathbf{1}$ will be denoted by $\mathbf{1}/\alpha$.

IV. Positive real numbers. A real class (number) α is positive iff $\alpha > \Diamond$. It is almost evident that we have:

Proposition 9. (i). The zero $\Diamond$ is not positive;

(ii). If $\alpha \neq 0$ one and only one of the real classes α and $-\alpha$ is positive.

(iii). If α and β are positive then $\alpha+\beta$ and $\alpha.\beta$ are also positive.

Theorems 4-9 imply that $\mathcal{R}$ is a continuously ordered body. It is well known that the only continuously ordered body up to isomorphism is the body of the real numbers.

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Proofs. The proof of Proposition 1 is almost evident.

Proof of Proposition 2. Since $\alpha \prec \beta$ hence there exist segments $\langle p, q\rangle \in \alpha$, $\langle r,s\rangle \in \beta$ with $q<r$. Then we have $\alpha \neq \beta$. Let us assume that it holds $\beta \prec \alpha$ also, and let $\langle p^\circ,q^\circ\rangle \in \alpha$, $\langle r^\circ,s^\circ\rangle \in \beta$ be with $s^\circ < p^\circ$. After the definition of the real classes, there exist rationals a, $b \in Q$, $a \in (\langle p,q\rangle \cap \langle p^\circ,q^\circ\rangle)$, $b \in (\langle r,s\rangle \cap \langle r^\circ,s^\circ\rangle)$. Since $q<r$ and $a \leq q$, $r \leq b$, then we receive $a<b$. Since $s^\circ < p^\circ$ and $p^\circ \leq a$, $b \leq s^\circ$, then we obtain $b<a$. The obtained contradiction proves that the assumption $\beta \prec \alpha$ is not true. ■

Proof of Proposition 3. Let $\alpha \prec \beta$ and $\beta \prec \gamma$, where $\alpha, \beta, \gamma \in \mathcal{R}$. Therefore there exist rational segments $\langle p,q\rangle \in \alpha$, $\langle r,s\rangle \in \beta$, $\langle r^\circ,s^\circ\rangle \in \beta$, $\langle t^\circ,u^\circ\rangle \in \gamma$ such that $q<r$ and $s^\circ < t^\circ$. We have $\langle r,s\rangle \cap \langle r^\circ,s^\circ\rangle \neq \emptyset$ after the definition of the real classes. Thus $r \leq s^\circ$. We receive $q<r$, $r \leq s^\circ$, $s^\circ < t^\circ$. Hence $q<t^\circ$, which signifies that $\alpha \prec \gamma$. ■

Proof of Theorem 4. Now it is sufficient to show that if α and β are real classes with $\alpha \neq \beta$ then we have either $\alpha \prec \beta$ or $\beta \prec \alpha$. Let us fix arbitrarily two different real classes α and β. Then there exists at least one rational segment $\langle r,s\rangle \in \beta$, $\langle r,s\rangle \notin \alpha$. $\langle r,s\rangle \notin \alpha$ involves the existence of a rational segment $\langle p,q\rangle \in \alpha$ with $\langle p,q\rangle \cap \langle r,s\rangle = \emptyset$. (Since if any rational segment of α has nonempty intersection with $\langle r, s\rangle$, hence $\langle r,s\rangle$ must belong to α after the maximality of α). The following two relations between the rationals q and r are possible- either $q<r$ or $r<q$. We have $\alpha \prec \beta$ in the case $q<r$. The case $r<q$ implies $s<p$, since $\langle p,q\rangle \cap \langle r,s\rangle = \emptyset$. Then $\beta \prec \alpha$. ■

Proof of Theorem 5. There exist rational segments $\langle p,q\rangle \in \pi$, $\langle r,s\rangle \in \tau$ with $q<r$ after the relation $\pi \prec \tau$. Let u be a rational number

with $q < u < r$. Let $\varkappa \in \mathbb{Q}$ be the rational class, generated by $\langle u,u\rangle$. It is evident that we have $\pi < \varkappa < \sigma$, $\pi \neq \varkappa$, $\sigma \neq \varkappa$. ■

Proof of Theorem 6. $\mathfrak{R}$ is a continuum: Let $\pi \in \mathfrak{R}_1$, $\varrho \in \mathfrak{R}_2$. As $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are nonempty we can choose and fix such real classes. Let $\langle p,q\rangle \in \pi$, $\langle r,s\rangle \in \varrho$ be arbitrary. Let us study the class $\mathfrak{t}$ of all rational segments $\langle p,s\rangle$ when π and ϱ range $\mathfrak{R}_1$ and $\mathfrak{R}_2$ correspondingly, i.e.

(1) $\mathfrak{t} = \Big\{\langle p,s\rangle$: we have either $\Big(\{\langle p,p\rangle\} \in \mathfrak{R}_1, \{\langle s,s\rangle\} \in \mathfrak{R}_2\Big)$ or $\Big[\{\langle s,s\rangle\} \in \mathfrak{R}_1$ and $\{\langle u,u\rangle\} \in \mathfrak{R}_2$ for any rational $u > s\Big]$ or $\Big[\{\langle p,p\rangle\} \in \mathfrak{R}_2$ and $\{\langle v,v\rangle\} \in \mathfrak{R}_1$ for any rational $v < p\Big]\Big\}$.

We shall prove that $\mathfrak{t} \in \mathbb{R}^*$. Let $\langle p_1,s_1\rangle, \langle p_2,s_2\rangle \in \mathfrak{t}$ be arbitrarily fixed. Then $p_1 \leq s_2$ and $p_2 \leq s_1$ after all possible cases of (1). Therefore we have $\langle p_1,s_1\rangle \cap \langle p_2,s_2\rangle \neq \emptyset$ This is, $\mathfrak{t} \in \mathbb{R}^*$. Moreover, $\mathfrak{t}$ is a maximal element of $\mathbb{R}^*$ since: Let us assume that the class $\mathfrak{t}^\circ \in \mathbb{R}^*$, $\mathfrak{t}^\circ \supset \mathfrak{t}$ and let the rational segment $\langle p^\circ,s^\circ\rangle \in \mathfrak{t}^\circ$. There are only the following possibilities (i)-(iii) for $\langle p^\circ,s^\circ\rangle$: (i). $\{\langle p^\circ,p^\circ\rangle\} \in \mathfrak{R}_1$, $\{\langle s^\circ,s^\circ\rangle\} \in \mathfrak{R}_2$. That is why $\langle p^\circ,s^\circ\rangle \in \mathfrak{t}$ in this case. (ii). $\{\langle s^\circ,s^\circ\rangle\} \in \mathfrak{R}_1$. Since $\mathfrak{t}^\circ \supset \mathfrak{t}$, hence $\langle p^\circ,s^\circ\rangle$ has a nonempty intersection with any segment $\langle p,s\rangle \in \mathfrak{t}$. Then $p \leq s^\circ$ for any rational p with $\langle p,s\rangle \in \mathfrak{t}$. Therefore $\{\langle p,p\rangle\} \in \mathfrak{R}_1$ for any rational p with $\langle p,s\rangle \in \mathfrak{t}$ in this case. Let u be a rational, $u > s^\circ$. We have either $\{\langle u,u\rangle\} \in \mathfrak{R}_1$ or $\{\langle u,u\rangle\} \in \mathfrak{R}_2$ since $\mathfrak{R}_1 \cup \mathfrak{R}_2 = \mathfrak{R}$. But the case $\{\langle u,u\rangle\} \in \mathfrak{R}_1$ is impossible since then $\langle u,t\rangle \in \mathfrak{t}$ for any rational t with $\{\langle t,t\rangle\} \in \mathfrak{R}_2$. (Such t exists after $\mathfrak{R}_2 \neq \emptyset$). But we have $\langle p^\circ,s^\circ\rangle \cap \langle u,t\rangle = \emptyset$. We have proved that $p \leq s^\circ$ for any rational p with $\langle p,s\rangle \in \mathfrak{t}$ for some s. Thus $u \leq s^\circ$ whereas we have chosen $u > s^\circ$. Therefore it remains only $\{\langle u,u\rangle\} \in \mathfrak{R}_2$ for any $u > s^\circ$. Then $\langle p^\circ,s^\circ\rangle \in \mathfrak{t}$ also in the case (ii) after (1). (iii). $\{\langle p^\circ,p^\circ\rangle\} \in \mathfrak{R}_2$. Since $\mathfrak{t}^\circ \supset \mathfrak{t}$, hence $\langle p^\circ,s^\circ\rangle$ has a nonempty intersection with any segment $\langle p,s\rangle \in \mathfrak{t}$. Then $p^\circ \leq s$ for any rational s with $\langle p,s\rangle \in \mathfrak{t}$ for some p . Now let the rational $v < p^\circ$. We have either $\{\langle v,v\rangle\} \in \mathfrak{R}_1$ or $\{\langle v,v\rangle\} \in \mathfrak{R}_2$. The case $\{\langle v,v\rangle\} \in \mathfrak{R}_2$ is impossible since then $\langle a,v\rangle \in \mathfrak{t}$ for any rational a with $\{\langle a,a\rangle\} \in \mathfrak{R}_1$ after (1). (Such rationals a exist after $\mathfrak{R}_1 \neq \emptyset$).

We have proved that $\langle a,v\rangle \in \mathfrak{t}$ implies $p^\circ \leqq v$. But we chose $v < p^\circ$. Thus it remains only $\langle v,v\rangle \in \mathfrak{K}_1$ for any rational $v < p^\circ$. Therefore $\langle p^\circ,s^\circ\rangle \in \mathfrak{t}$ also, after (1). That is why $\mathfrak{t}^\circ = \mathfrak{t}$ for any $\mathfrak{t}^\circ \supset \mathfrak{t}$, $\mathfrak{t}^\circ \in \mathbb{R}^*$. Thus $\mathfrak{t}$ is a maximal element of $\mathbb{R}^*$, i.e. $\mathfrak{t}$ is a real class of $\mathfrak{K}$. Let us denote this real class by τ. We shall prove that the real number τ separates the complexes $\mathfrak{K}_1$ and $\mathfrak{K}_2$. This is, if $\alpha \angle \tau$, then $\alpha \in \mathfrak{K}_1$ and if $\tau < \beta$ then $\beta \in \mathfrak{K}_2$: Let $\alpha \angle \tau$. Then there exist rational segments $\langle a,c\rangle \in \alpha$, $\langle p,s\rangle \in \tau = \mathfrak{t}$, such that $c \angle p$. Therefore $\{\langle c,c\rangle\} \in \mathfrak{K}_1$ after (1). That is why $\alpha \in \mathfrak{K}_1$ too. Let now $\tau \angle \beta$. Then there exist rational segments $\langle b,d\rangle \in \beta$, $\langle p,s\rangle \in \tau = \mathfrak{t}$ with $s < b$. We have $\{\langle b,b\rangle\} \in \mathfrak{K}_2$ after (1). That is why we receive $\beta \in \mathfrak{K}_2$ also. ∎

Lemma 10. The segments $\langle a,b\rangle$ and $\langle c,d\rangle$ have nonempty intersection iff it simultaneously holds $a \leq d$ and $c \leq b$.

This Lemma is almost evident.

Lemma 11. I. If α is a real class, then for any fixed integer $n > 0$, there exists a rational segment $\langle p,q\rangle \in \alpha$ with $q-p \angle 1/n$.

II. If α and β are real classes, such that for any fixed integer $n > 0$ there exist rational segments (eventually depending on the choice of n) $\langle p_\alpha,q_\alpha\rangle \in \alpha$, $\langle p_\beta,q_\beta\rangle \in \beta$ with $|q_\beta - p_\alpha| < 1/n$, $|q_\alpha - p_\beta| < 1/n$, then we have $\alpha = \beta$.

Proof. I. Let us assume the contrary. Let $n^\circ > 0$ be the least positive integer with $q-p \geqq 1/n^\circ$ for any $\langle p,q\rangle \in \alpha$. Then either $n^\circ = 1$ or $n^\circ > 1$ and $q^\circ - p^\circ < 1/(n^\circ-1)$ for some $\langle p^\circ,q^\circ\rangle \in \alpha$. a). Let $n^\circ > 1$ and $r^\circ = (p^\circ + q^\circ)/2$. The assumption $\langle r^\circ,r^\circ\rangle \cap \langle p',q'\rangle = \emptyset$ for some rational segment $\langle p',q'\rangle \in \alpha$ is impossible since then either $r^\circ < p'$ or $q' \angle r^\circ$. But $r^\circ \angle p'$ involves $\langle p',q^\circ\rangle \in \alpha$ (after the maximality of α in $\mathbb{R}^*$) whereas we have $q^\circ - p' < (q^\circ-p^\circ)/2 < 1/(n^\circ-1) < 1/n^\circ$. The case $q' < r^\circ$ involves $\langle p^\circ,q'\rangle \in \alpha$ whereas we have $q' - p^\circ \angle 1/n^\circ$. Therefore the obtained contradiction proves $\langle r^\circ,r^\circ\rangle \cap \langle p,q\rangle \neq \emptyset$ for any segment $\langle p,q\rangle \in \alpha$. This implies $\langle r^\circ,r^\circ\rangle \in \alpha$ after the maximality of α in $\mathbb{R}^*$. But $0 = r^\circ - r^\circ$ and thus cannot be equal or larger than $1/n^\circ$. b). Let $n^\circ = 1$ and $n^\wedge$ be the larger integer with $q-p \geqslant n^\wedge$ for any $\langle p,q\rangle \in \alpha$. Then we have $q^\wedge - p^\wedge < (n^\wedge$

$+ 1)$ for some $\langle p^\wedge,q^\wedge\rangle \in \alpha$. Let $r^\wedge=(p^\wedge+q^\wedge)/2$. The assumption $\langle r^\wedge,r^\wedge\rangle \cap \langle p'', q''\rangle = \emptyset$ for some $\langle p'',q''\rangle \in \alpha$ is impossible since then either $r^\wedge < p''$ or $q'' < r^\wedge$. The case $r^\wedge < p''$ involves $\langle p'',q^\wedge\rangle \in \alpha$ (after the maximality of α in $\mathbb{R}^*$ and $\langle p'',q''\rangle \in \alpha$, $\langle p^\wedge,q^\wedge\rangle \in \alpha$). But we have $q^\wedge-p'' < (q^\wedge-p^\wedge)/2 < (n^\wedge+1)/2 < n^\wedge$. The case $q'' < r^\wedge$ involves $\langle p^\wedge,q''\rangle \in \alpha$ (after the maximality of α in $\mathbb{R}^*$). But we have $q''-p^\wedge < (q^\wedge-p^\wedge)/2 < (n^\wedge+1)/2 < n^\wedge$. The obtained contradiction proves that $\langle r^\wedge,r^\wedge\rangle \cap \langle p,q\rangle \neq \emptyset$ for any $\langle p,q\rangle \in \alpha$. Thus we receive $\langle r^\wedge,r^\wedge\rangle \in \alpha$. But $0 = r^\wedge - r^\wedge$ is not larger or equal of $n^\wedge \geq 1$. The obtained contradictions in the cases a) and b) prove Lemma 11.I.

II. Let us assume the contrary, i.e. that $\alpha \neq \beta$. Then either $\alpha < \beta$ or $\beta < \alpha$ after Theorem 4. In the case $\alpha < \beta$ there exist segments $\langle p^\circ_\alpha, q^\circ_\alpha\rangle \in \alpha$, $\langle p^\circ_\beta, q^\circ_\beta\rangle \in \beta$ with $q^\circ_\alpha < p^\circ_\beta$. Then we have $p_\alpha \leq q^\circ_\alpha < p^\circ_\beta \leq q_\beta$ for each segments $\langle p_\alpha, q_\alpha\rangle \in \alpha$, $\langle p_\beta, q_\beta\rangle \in \beta$. Let $a = p^\circ_\beta - q^\circ_\alpha$. Thus $|q_\beta - p_\alpha| \geq a > 0$, which contradicts the condition of Lemma 11.II. In the case $\beta < \alpha$ we have $q^{\cdot\cdot}_\beta < p^{\cdot\cdot}_\alpha$ for some segments $\langle p^{\cdot\cdot}_\alpha, q^{\cdot\cdot}_\alpha\rangle \in \alpha$, $\langle p^{\cdot\cdot}_\beta, q^{\cdot\cdot}_\beta\rangle \in \beta$. Then $p_\beta \leq q^{\cdot\cdot}_\beta < p^{\cdot\cdot}_\alpha \leq q_\alpha$ for each segments $\langle p_\alpha, q_\alpha\rangle \in \alpha$, $\langle p_\beta, q_\beta\rangle \in \beta$. This contradicts again the condition of Lemma 11.II. Thus it remains only $\alpha = \beta$.

Proof of Theorem 7. We shall prove that the addition in $\mathcal{R}$ is uniquely defined. I. Let us denote by $[\alpha+\beta]$ the set of all rational segments of the kind $\langle p+r, q+s\rangle$ for some $\langle p,q\rangle \in \alpha$, $\langle r,s\rangle \in \beta$. We shall show that $[\alpha+\beta] \in \mathbb{R}^*$. Let $\langle p_1,q_1\rangle$ and $\langle p_2, q_2\rangle$ belong to α; $\langle r_1,s_1\rangle$, $\langle r_2,s_2\rangle$ belong to β. Then we have $p_1 \leq q_2$, $p_2 \leq q_1$; $r_1 \leq s_2$, $r_2 \leq s_1$ after Lemma 10 and $\alpha, \beta \in \mathbb{R}^*$. Therefore we receive $p_1 + r_1 \leq q_2 + s_2$, $p_2+r_2 \leq q_1+s_1$. Then the intersection of the segments $\langle p_1+r_1, q_1+s_1\rangle$ and $\langle p_2+r_2, q_2+s_2\rangle$ is nonempty after Lemma 10. Thus we obtain $[\alpha+\beta] \in \mathbb{R}^*$. II. We shall now scrutinize whether $[\alpha+\beta] \in \mathcal{R}_s$. Let $\langle e,f\rangle$ be a rational segment with nonempty intersection with any segment of $[\alpha+\beta]$. There are only the following cases: a). For any fixed such segment $\langle e,f\rangle$ we can choose a segment $\langle p+r, q+s\rangle$, $\langle p,q\rangle \in \alpha$, $\langle r,s\rangle \in \beta$ with $e \leq p+r$, $q+s \leq f$. Let $g = p+r-e \geq 0$, $h = f - (q+s) \geq 0$. Let $p'=p-g$, $q'=q+h$. Then $\langle p',q'\rangle \in \alpha$ since $\langle p',q'\rangle \supset \langle p,q\rangle$ and α is a maximal element in $\mathbb{R}^*$. We have $p'+r=e$, $q'+s = f$. Thus $\langle e,f\rangle = \langle p'+r, q'+s\rangle \in [\alpha+\beta]$. Therefore $[\alpha+\beta]$ is a maximal element of $\mathbb{R}^*$ in this case and $[\alpha+\beta] = \alpha+\beta$ also. b).There exists a rational segment $\langle e^o,f^o\rangle$ with a nonempty intersection with any segment of $[\alpha+\beta]$ but $p+r < e^o$ for any $\langle p,q\rangle \in \alpha$, $\langle r,s\rangle \in \beta$. Moreover, we have $e^o \leq q+s$ after Lemma 10.Therefore the segment $\langle e^o,e^o\rangle$ has a nonempty intersection with any segment of $[\alpha+\beta]$ .Then $\alpha+\beta = \{\langle e^o,e^o\rangle\}$. Since we can choose $\langle p,q\rangle$,$\langle r,s\rangle$ such that $0 < q+s-(p+r) = (q-p)+(s-r) < 1/n$ for any natural n after Lemma 11.I., hence $\alpha+\beta = \{\langle e^o,e^o\rangle\}$ is uniqely defined. c).There exists a rational segment $\langle e^o,f^o\rangle$ with a nonempty intersection with any segment of $[\alpha+\beta]$ but $f^o < q+s$ for any $\langle p,q\rangle \in \alpha$, $\langle r,s\rangle \in \beta$ .Moreover we have $p+r \leq f^o$ after Lemma 10.Therefore the segment $\langle f^o,f^o\rangle$ has a nonempty intersection with any segment of $[\alpha+\beta]$. Then $\alpha+\beta = \{\langle f^o,f^o\rangle\}$. Since we can choose $\langle p,q\rangle$, $\langle r,s\rangle$ such that $0 < q+s-(p+r) = (q-p)+(s-r) < 1/n$ for any natural n after Lemma 11.I., hence $\alpha+\beta = \{\langle f^o,f^o\rangle\}$ is uniquely defined in this case.Thus the addition $\alpha+\beta$ in $\mathcal{R}$ is uniquely defined in all possible cases.

Further on, (i). $\alpha+\beta = \beta+\alpha$ as the addition of the rational numbers of Q is commutative. (ii). $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ as the addition in Q is associative. (iii).The zero $\Diamond$ is generated by $\langle 0,0\rangle$, i.e. $\Diamond = \{\langle 0,0\rangle\}$.That is why if $\langle p,q\rangle \in \alpha$,then $\langle p,q\rangle = \langle p+0,q+0\rangle \in (\alpha + \Diamond)$. This is, the class of the rational segments $\alpha \subset \alpha+\Diamond$ .Then it follows by the maximality of α in $\mathbb{R}^*$ that $\alpha = \alpha+\Diamond$. (iv). Let us choose $\xi^o = \{\langle -q,-p\rangle, \forall \langle p,q\rangle \in \alpha\}$ .We have $\xi^o \in \mathbb{R}^*$ since $\alpha \in \mathbb{R}^*$.Moreover, $\xi^o \in \mathcal{R}$ since $\alpha \in \mathcal{R}$ .Further on, $\alpha+\xi^o = \Diamond$ after Lemma 11.I. and II. . ∎

Proof of Theorem 8. Let us denote all described segments $\langle u,v\rangle$ in the definition of $\alpha.\beta$ by $[\alpha\beta]$ .We shall show that $[\alpha\beta] \in \mathbb{R}^*$. Let the rational segments $\langle u_1,v_1\rangle,\langle u_2,v_2\rangle \in [\alpha\beta]$ with their corresponding segments $\langle p_1,q_1\rangle,\langle p_2,q_2\rangle \in \alpha$ and $\langle r_1,s_1\rangle,\langle r_2,s_2\rangle \in \beta$ such that $u_1 \leq k_1l_1$, $u_2 \leq k_2l_2$ for each rationals $k_1 \in \langle p_1,q_1\rangle$, $l_1 \in \langle r_1,s_1\rangle$, $k_2 \in \langle p_2,q_2\rangle$, $l_2 \in \langle r_2,s_2\rangle$. There exist rationals $k_1^o \in \langle p_1,q_1\rangle$, $k_2^o \in \langle p_2,$

$q_2>$, $l_1^\circ \in <r_1,s_1>$, $l_2^\circ \in <r_2,s_2>$ with $k_1^\circ l_1^\circ \leqslant v_2$, $k_2^\circ\, l_2^\circ \leqslant v_1$ after the construction of the multiplication in $\mathcal{R}_s$. Thus we receive $u_1 \leqslant k_1^\circ\, l_1^\circ \leqslant v_2$ and $u_2 \leqslant k_2^\circ\, l_2^\circ \leqslant v_1$. The Lemma 10 involves that $<u_1,v_1> \bigcap <u_2, v_2> \neq \emptyset$ .Therefore $[\alpha\beta] \in \mathbb{R}^*$ .Now we want to show that the real class,containing $[\alpha\,\beta]$ is uniquely defined.Let $<e,f>$ be a rational segment with nonempty intersection with any of the segments of $[\alpha\,\beta]$. There are the following cases only: a).For any fixed such segment $<e,f>$ there exists a segment $<u,v> \in [\alpha\beta]$ with $e \leq u$, $v \leq f$ .Let the segments $<p',q'> \in \alpha$, $<r',s'> \in \beta$ correspond to u .This is, $u \leqslant kl$ for any rationals $k \in <p',q'>$, $l \in <r',s'>$.Since $e \leqslant u$,then $e \leq kl$ for any rationals $k \in <p',q'>$, $l \in <r',s'>$.We have also the existence of rationals $k^\circ \in <p'',q''>$, $l^\circ \in <r'',s''>$ with $k^\circ l^\circ \leqslant v$ for any fixed $<p'',q''> \in \alpha$,$<r'',s''> \in \beta$.Thus we receive $k^\circ l^\circ \leqslant v \leq f$. Therefore $<e,f> \in [\alpha\,\beta]$ according to the construction of $[\alpha\,\beta]$ .Therefore $[\alpha\beta]$ is the unique maximal element $\alpha\,.\,\beta$ of $\mathbb{R}^*$ in this case. b).There exists a rational segment $<e^\circ,f^\circ>$ with a nonempty intersection with any segment of $[\alpha\,\beta]$ but we have $u < e^\circ$ for any $<u,v> \in [\alpha\,\beta]$ .Moreover,it holds $e^\circ \leq v$ after Lemma 10. Then the segment $<e^\circ,e^\circ>$ has a nonempty intersection with any segment of $[\alpha\,\beta]$ .That is why $\alpha\,.\,\beta = \{<e^\circ,e^\circ>\}$.Since we can choose $<p,q> \in \alpha$ and $<r,s> \in \beta$ with $0 \leq q-p < 1/n$, $0 \leqslant s-r < 1/n$ for any fixed natural n, hence $\alpha\,.\,\beta = \{<e^\circ,e^\circ>\}$ is uniquely defined in this case c). There exists a rational segment $<e^\circ,f^\circ>$ with a nonempty intersection with any segment of $[\alpha\,\beta]$ but $f^\circ < v$ for each $<u, v> \in [\alpha\,\beta]$ .Moreover, we have $u \leqslant f^\circ$ after Lemma 10. Then the segment $<f^\circ,f^\circ>$ has a nonempty intersection with any segment of $[\alpha\,\beta]$. That is why $\alpha\,.\,\beta = \{<f^\circ,f^\circ>\}$ .Since we can choose $<p,q> \in \alpha$, $<r,s> \in \beta$ with $0 \leq q-p < 1/n$, $0 \leq s-r < 1/n$ for any fixed natural integer n (after Lemma 11.I.), hence $\alpha\,.\,\beta = \{<f^\circ,f^\circ>\}$ is uniquely defined. Therefore the product $\alpha\,.\,\beta$ is well and uniquely defined in all possible cases.

Further on, (i).The product $\alpha\,.\,\beta$ is uniquely determined by the class $[\alpha\,\beta]$ .But we have $[\alpha\,\beta] = [\beta\,\alpha]$ by the commutativity of the multiplication of the rational numbers of Q.Thus $\alpha . \beta = \beta . \gamma$ also. (ii).We receive $(\alpha . \beta).\gamma = \alpha.(\beta.\gamma)$ since $[\alpha(\beta.\gamma)]$ and $[(\alpha.\beta).\gamma]$ define the same real class. (iii).We have $\alpha.(\beta+\gamma) = \alpha.\beta + \beta.\gamma$ since $[\alpha(\beta+\gamma)]$ and $[\alpha.\beta + \alpha.\gamma]$ define the same real class. (iv). $\alpha.\mathbb{1}$ is a real class,defined by $[\alpha.\mathbb{1}] \supset \alpha$ .But α is a maximal element of $\mathbb{R}^*$ and $\alpha.\mathbb{1} \supset [\alpha.\mathbb{1}]$.That is why $\alpha.\mathbb{1} = \alpha$. (v).We have $[\alpha.\Diamond] \supset \langle 0,0\rangle$. Therefore $\alpha.\Diamond = \{\langle 0,0\rangle\} = \Diamond$. (vi).We have $\alpha \neq \Diamond$ .Then it holds either $\alpha > \Diamond$ or $\alpha < \Diamond$ after Theorem 4. In the case $\alpha > \Diamond$ there is a segment $\langle p,q\rangle \in \alpha$ with $p > 0$. In the case $\alpha < \Diamond$ there is a segment $\langle p,q\rangle \in \alpha$ with $q < 0$ .Let us denote by $\widehat{[\alpha]}$ all such segments $\langle p,q\rangle \in \alpha$ with $pq > 0$ in the both case. Let $\widehat{[\xi^*]}$ be the class of rational segments $\widehat{[\xi^*]} = \{\langle 1/q , 1/p\rangle, \langle p,q\rangle \in \alpha , pq > 0\}$.Evidently $\widehat{[\xi^*]} \in \mathbb{R}^*$ .Let ξ^* be the real class,containing $\widehat{[\xi^*]}$ .Then $\alpha . \xi^* = \mathbb{1}$ after Lemma 11.I. and II. ■

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TWO MODELS OF TIME WITH WALKER'S DEFINITION OF INSTANTS BY EVENTS

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This article constructs two models of Time, using Walker's definition of instants by events. It follows from either of the proposed systems of axioms on the events, that the instants, constructed by events after Walker's definition, compose an open-ended linear continuum with a "dense" sequence of instants. I.e. Time continuum has the properties, characterizing the real line. Here the exposition is based only on Walker's definition of instants without mix up Russell's definition of instants. The used here systems of axioms are simplier than previous in the literature and treat only events.

The attempts of mathematical constructions of the instants of Time by events derived fromm Russell and Whitehead [1,2]. Such constructions of Time are elaborated also by Robbs [3], N.Wiener [4], Walker [5], Whitrow [6], Thomason [7]. The articles [8-10] constructed two models of Time, based on Russell's definition of the instants by events. The physiologists, psichologists and philosophers are in accord that the conception of the events is more primary and fundamental whereas the instants are intuitive-mental constructions. Russell and Whitehead have posed the problem to obtain the construction of the instants from the events by a logico-mathematical way [1,2,6]). The proposed here two different models, based on Walker's definition, have more simple requirements about the events (cf. [6,7]). (For instance here only the relations $\prec$ and $\odot$ are required among the events, whereas the article [7] needs the relations $\prec_1$, $\prec_0$, $\prec$, $\odot$ among the events). Here the constructions and proofs use only the Walker's definition of the instants (without a mixing of Russell's definition of the instants (cf. [7]).

The constructed two models of Time with the Walker's definition of instants (see [5-7]) are based on two different systems of axioms on the events. It follows from either of these systems, that the instants, constructed by events after Walker's definition [5-7], have the discussed in the literature [6,7] properties of the continuum of Time of Mathematical Physics. This is the instants compose an open-ended linear continuum with a "dense" sequence of instants, which are characterizing properties of the real line. The second model here is introduced not only to show a new possibility of the construction of Time. The second model of Time avoids the conceptual imperfection of the first simpler model of Time here. All events are finite in the first model, whereas the second model admits unbounded events also. The first system of axioms on the events is satisfied for instance by all nonempty compact segments of the real line. The second system of axioms on the events is satisfied for instance by all nonempty open intervals of the real line. The axiom $\mathfrak{A}$ is from [2,4,5].

Let us denote by $\mathcal{S}$ the whole complex of all events.

The first model of Time with walker's definition of instants by events. The first model of Time here consists of Walker's definition [5-7] of instants by events and by the following axioms on the events.

Axiom $\mathfrak{A}$ (B.Russell [2]). 1. $\mathcal{S} \neq \emptyset$. For any two events either one of them is "before" ("earlier than"), ($\prec$), the other or in the oposite case they are "simultaneous" (at least partially) (i.e. they "overlap", i.e. are contemporary"), ($\odot$). This is, for any two events $a, b \in \mathcal{S}$ one and only one of the following statements is true: either $a \prec b$ or $b \prec a$, or $a \odot b$; We have $a \odot a$ for $\forall a \in \mathcal{S}$.

2. If $a \prec b$, $b \odot c$, $c \prec d$, then $a \prec d$ for any events a, b , c, $d \in \mathcal{S}$.

It follows from Axiom $\mathfrak{A}$ that the relation $\prec$ is transitive, i.e. if $a \prec b$ and $b \prec d$, then $a \prec d$,where $a, b, d \in \mathcal{S}$. It also follows that if $a \odot b$, then $b \odot a$. Thus the set $\mathcal{S}$ of all events is partially ordered by the relation $\prec$.

Axiom $\mathfrak{B}$. There exists a sequence K of events from $\mathcal{S}$,such that for any arbitrarily fixed events a, $b \in \mathcal{S}$ with $a \prec b$ there is an event $k \in K$ with $a \prec k \prec b$.

Axiom $\mathfrak{C}$. For any arbitrarily fixed event $a \in \mathcal{S}$ there are events b , $c \in \mathcal{S}$ such that $b \prec a \prec c$.

Axiom $\mathfrak{D}$. Whenever $c \prec a \prec b \prec d$ (a, b, c, $d \in \mathcal{S}$), then there is an event s , simultaneous with a and b, $a \odot s$, $b \odot s$,for which $c \prec s \prec d$.

Remark. There exist complexes $\mathcal{S}$,satisfying Axioms $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ .Such is the set of all compacts (i.e. closed and finite) nonempty segments of the real line.

We shall formalize the Walker's construction of the instants by events :

Definition of the instants (after Walker [5]). Let (P,Q,R) be a triple of subsets P, Q, R of $\mathcal{S}$, such that (i). P,Q,R are nonempty, $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$. (ii). Each event of P is before any event of R.

(iii). Any event of Q is simultaneous with an event of P and with an event of R.

There exist such triples after Axioms $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

Let $\mathcal{W}$ be the complex of all such triples (P,Q,R). We introduce a partial order in $\mathcal{W}$ by inclusions (in the sense of the Set Theory) of the triples of $\mathcal{W}$: Let $w_1, w_2 \in \mathcal{W}$, $w_1 = (P_1, Q_1, R_1)$, $w_2 = (P_2, Q_2, R_2)$. We shall deem that w_2 follows ($\prec'$) w_1, $w_1 \prec' w_2$, iff $P_1 \subset P_2$, $Q_1 \subset Q_2$, $R_1 \subset R_2$.

The maximal elements of $\mathcal{W}$ will be called instants (moments) (after Walker) and will be denoted by small Greek letters. The class of all instants will be denoted by $\mathcal{SW}$.

Theorem 1. $\mathcal{SW}$ is not empty, i.e. $\mathcal{SW}$ has at least one element.

Remark. If $\alpha = (P,Q,R) \in \mathcal{SW}$ and the event a is simultaneous with any event $q \in Q$, $q \odot a$, then we shall say that the instant α belongs to the event a, $\alpha \in a$.

Theorem 2. For any fixed event $a \in \mathcal{S}$ there exists an instant α, belonging to a, $\alpha \in a$.

Theorem 3. Let a and b be arbitrarily fixed simultaneous events, $a, b \in \mathcal{S}$. Then there exists at least one instant γ with $\gamma \in a$, $\gamma \in b$.

Theorems 1-3 are formulated and proved separately because these results have been widely discussed in the literature (see [2, 3, 7]). In some articles these results are axioms (cf. [23, 6]).

The order in $\mathcal{SW}$. We shall say that the instant α is before (earlier than), ($\prec$), the instant β, if there are events $q_\alpha \in Q_\alpha$, $q_\beta \in Q_\beta$, with $q_\alpha \prec q_\beta$, where $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$. If any two events $q_\alpha \in Q_\alpha$ and $q_\beta \in Q_\beta$ are simultaneous, $q_\alpha \odot q_\beta$, then we shall say that $\alpha = \beta$. (It is not necessary to have $P_\alpha = P_\beta$, $Q_\alpha = Q_\beta$, $R_\alpha = R_\beta$).

Proposition 4. The relation "=" among instants is transitive, i.e. if $\alpha = \beta$, $\beta = \gamma$ with $\alpha, \beta, \gamma \in \mathcal{SW}$, then $\alpha = \gamma$. Moreover, $\alpha = \alpha$ for any instant $\alpha \in \mathcal{SW}$, this is any two events $q'_\alpha, q''_\alpha \in Q_\alpha$ are simultaneous, $q'_\alpha \odot q''_\alpha$, where $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$ is an arbitrarily fixed instant of $\mathcal{SW}$.

Proposition 5. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$ be an arbitrary fixed instant of $\mathcal{SW}$.

Then we have a). $P_\alpha \cup Q_\alpha \cup R_\alpha = \mathcal{S}$;

b). $P_\alpha \cap Q_\alpha = \emptyset$; $Q_\alpha \cap R_\alpha = \emptyset$; $P_\alpha \cap R_\alpha = \emptyset$;

c). Any two events q'_α and q''_α of Q_α are simultaneous , $q'_\alpha \odot q''_\alpha$.

Proposition 6. If the instant α is before the instant β, $\alpha \prec \beta$,then it is not true that $\beta \prec \alpha$.

Proposition 7. Time order of the instants of $\mathcal{W}$ is a linear order.

Moreover, the axioms $\mathcal{A}-\mathcal{D}$ ensure all desired [2,6,7] properties of Time continuum for the class $\mathcal{W}$ of all instants. This is, $\mathcal{W}$ is an open-ended linear continuum with a "dense" sequence of instants, which are characterizing properties of the real line. Thus, the properties of Time continuum T, used in Mathematical Physics, are the following after [6,7]:

1. T is linearly ordered ;

2. T is a "dense" set, i.e. if the instant π is earlier than the instant $\varkappa$, then there exists at least one instant ϱ between π and $\varkappa$, $\pi \neq \varrho$, $\varkappa \neq \varrho$.

3. T satisfies Dedekind's postulate, this is: If T_1 and T_2 are two non-empty disjoint parts of T, such that each instant of T belongs either to T_1 or to T_2 and each instant of T_1 is before any instant of T_2 , then there exists at least one instant $\tau \in T$, such that every instant before τ belongs to T_1 and every instant after τ belongs to T_2 .

4. T contains a countable subset G, such that for any two different instants π and $\varkappa$ of T there exists at least one instant ϱ of G , which is between π and $\varkappa$, $\pi \neq \varrho$, $\varkappa \neq \varrho$.

The property 4 immediately implies the property 2 of T. These four properties of T are satisfied also by a model of Time, which has an earliest and a last moments, i.e. by a model of Time with a begining and an end Therefore **one** more property must be added [7,8-10]:

5. For any arbitrarily fixed moment τ of T there exist instants τ_1 and τ_2 , such that τ is between τ_1 and τ_2 , $\tau \neq \tau_1$, $\tau \neq \tau_2$.

Axioms $\mathcal{A}-\mathcal{D}$ ensure the following theorem.

Theorem 9. The complex $\mathcal{W}$ of all instants is an open-ended linear continuum with a dense sequence of instants, i.e. $\mathcal{W}$ has the properties 1-5 of Time continuum T of Mathematical Physics, which properties are characterizing for the real line.

The second model of Time with Walker's definition of instants by events.

The second model of Time, based on Walker's definition of the instants, proposed here, permits a more free interpretation of the events, which more completely corresponds to our conception of the events. It permits non-bounded events, while in the first, more simple model, all events are bounded. This second model has the following axioms $\mathcal{A}^*$, $\mathcal{B}^*$ and $\mathcal{D}^*$ on the complex of all events $\mathcal{S}$.

Axiom $\mathcal{A}^*$. Point 1 and Poind 2 of the Axiom $\mathcal{A}$ of the first model (i.e. Axiom $\mathcal{A}^*$ consists of Points 1 and 2 of the Axiom $\mathcal{A}$ and the following requirements 3 and 4).

Point 3. If a is an arbitrarily fixed event of $\mathcal{S}$,then there exist events b , c , d , simultaneous with a, for which $c \prec b \prec d$. (We shall use the notation $b \sqsubset a$ in the case of Point 3 of Axiom $\mathcal{A}^*$.)

Point 4. If m and n are arbitrarily fixed simultaneous events of $\mathcal{S}$, $m \odot n$, then there exists an event $p \in \mathcal{S}$, such that $p \sqsubset m$, $p \sqsubset n$.

Proposition 1'. If the events a , b are in the relation $b \sqsubset a$ and if the event m is simultaneous with b , then we have $a \odot m$ also.

Proposition 2'. If the events a and b are in the relation $b \sqsubset a$, then it is not true $a \sqsubset b$.

Proposition 3'. If we have $a' \prec b'$, $a \sqsubset a'$, $b \sqsubset b'$, then $a \prec b$ also , where $a , a' , b , b' \in \mathcal{S}$.

Axiom $\mathcal{B}^*$. There exists a sequence K of events from $\mathcal{S}$, such that if the events $a \sqsubset a'$, $b \sqsubset b'$ are arbitrarily fixed with $a' \prec b'$, then there exists an event $k \in K$, for which $a \prec k \prec b$.

Axiom $\mathcal{D}^*$. Whenever we have $c \prec a \prec b \prec d$ (a , b , c , d are events of $\mathcal{S}$), then there exists an event s , simultaneous with a and b , $a \odot s$, $b \odot s$, and with $c \prec s \prec d$.

Remark. Axioms $\mathcal{A}^*$, $\mathcal{B}^*$, $\mathcal{D}^*$ are satisfied for instance by the complex of all nonempty open intervals of the real line.

CONSTRUCTION OF THE INSTANTS (received, formalizing appropriately Walker's definition). Let $\mathcal{W}'$ be the complex of all triples (P,Q,R) of sets of events with the properties: (i'). $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$;

(ii'). Each event of P is before any event of R ;

(iii'). Each event of Q is simultaneous with an event of P and with an event of R ;

(iv'). For any event q of Q there exists an event $q^\circ \in Q$ with $q^\circ \sqsubset q$.

There exist such triples (P,Q,R) after Axioms $\mathcal{A}^*, \mathcal{B}^*, \mathcal{D}^*$.

We introduce a partial order in $\mathcal{W}'$ by inclusions of its triples. This is, let $W_1 \in \mathcal{W}'$, $W_2 \in \mathcal{W}'$, $W_1 = (P_1, Q_1, R_1)$, $W_2 = (P_2, Q_2, R_2)$. We shall say that W_2 follows W_1 , $W_1 \prec W_2$, if $P_1 \subset P_2$, $Q_1 \subset Q_2$, $R_1 \subset R_2$,and we shall denote it by $W_1 \prec W_2$ .The relation $\prec$ introduces a partial order in $\mathcal{W}'$.

The maximal elements of $\mathcal{W}'$ will be called instants (moments), and will be denoted by small Greek letters. The class of all instants will be denoted by $\mathfrak{W}'$.

Theorem 4'.The class of all instants $\mathfrak{W}'$ is nonempty, i.e. there exists at least one instant of $\mathfrak{W}'$.

Proposition 5'.Let $\alpha = (P_\alpha , Q_\alpha , R_\alpha)$ be an arbitrarily fixed instant of $\mathfrak{W}'$. Then we have a). $P_\alpha \cup Q_\alpha \cup R_\alpha = \mathcal{S}$;

b). $P_\alpha \cap Q_\alpha = \emptyset$, $Q_\alpha \cap R_\alpha = \emptyset$; $P_\alpha \cap R_\alpha = \emptyset$;

c). Any two events q'_α , q''_α of Q_α are simultaneous, $q'_\alpha \odot q''_\alpha$.

Definition. Let the instant $\alpha \in \mathfrak{W}'$ be $\alpha = (P,Q,R)$. If a is an event for which exists an event $a^\circ \sqsubset a$, such that a° is simultaneous with any event $q \in Q$, then we shall say that the instant α belongs to a , $\alpha \in a$.

If any instant of the event a is before (resp.after) an instant β , then we shall say that the event a is before (resp.after) the instant β , $a \prec \beta$ (resp. $\beta \prec a$).

Theorem 6'. If a is an arbitrarily fixed event, then there is at least one instant α , belonging to a .

Theorem 7'. If m and n are simultaneous events, $m \odot n$, then there is at least one instant γ with $\gamma \in m$, $\gamma \in n$.

TIME ORDER IN $\mathfrak{W}'$. Let the instants $\mu = (P_1 , Q_1 , R_1) \in \mathfrak{W}'$, $\nu =$

$(P_2, Q_2, R_2) \in {}^{S}W^{R}$. We shall say that μ is before (earlier than) ν , $\mu \prec \nu$, if there exist events $q_1 \in Q_1$, $q_2 \in Q_2$ with $q_1 \prec q_2$. We shall deem that $\mu = \nu$, if each event of Q_1 is simultaneous with any event of Q_2, i.e. $q_1 \odot q_2$ for any two events $q_1 \in Q_1$, $q_2 \in Q_2$. Thus it is not necessary for $\mu = \nu$ to have $P_1 = P_2$, $Q_1 = Q_2$, $R_1 = R_2$!

Proposition 8'. The relation "=" is transitive and reflexive, i.e. if $\mu = \nu$ and $\nu = \tau$, then $\mu = \tau$; $\mu = \mu$, where $\mu, \nu, \tau \in {}^{S}W^{R}$.

Theorem 9'. Time order in ${}^{S}W^{R'}$ is a linear order.

Proposition 10'. Let α be an instant. Then there exist instants β and γ with $\beta \prec \alpha \prec \gamma$.

Theorem 11'. ${}^{S}W^{R}$ is an open-ended linear continuum with a dense sequence of instants. This is, ${}^{S}W^{R}$ has the properties 1-5 of Time continuum T of Mathematical Physics, which properties are characterizing for the real line.

PROOFS FOR THE FIRST MODEL OF TIME (WITH WALKER'S DEFINITION OF INSTANTS)

We shall use Zorn's Lemma in the proofs of Theorems 1-3 . Let us remind it:

Lemma of Zorn (Zorn [9]). Let X be a partially ordered nonempty set. If any linearly ordered subset A of X is upper bounded in X, then X contains at least one maximal element.

Proof of Theorem 1. Evidently, $\mathbb{W}$ is a partially ordered complex. We shall prove that $\mathbb{W}$ is not empty. There exists at least one event a after Axiom $\mathcal{A}$. There is an event b with $a \prec b$ after Axiom $\mathcal{C}$. Applying Axiom $\mathcal{D}$, we get an event s with $a \odot s$, $b \odot s$. We have that the triple $(\{a\}, \{s\}, \{b\}) \in \mathbb{W}$.

Now, let $\mathbb{A}$ be a linearly ordered subset of $\mathbb{W}$, $\mathbb{A} = \{(P_i, Q_i, R_i), i \in I\}$, where I is a complex of indexes. Let us scrutinize the triple

$$w^{*} = (P, Q, R), \quad P = \bigcup_{i \in I} P_i, \quad Q = \bigcup_{i \in I} Q_i, \quad R = \bigcup_{i \in I} R_i .$$

We shall prove that $w^{*} \in \mathbb{W}$,i.e. are satisfied the requirements (i)-(ii) (iii). (i). If $\mathbb{A} \neq \emptyset$, then $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$;

(ii). Let $p \in P$, $r \in R$. We shall prove that $p \prec r$ .Evidently, $p = p_i$, and $r = r_{i''}$ for some $i', i'' \in I$. Since $\mathbb{A}$ is linearly ordered, then one of the triple $\mathcal{W}_{i'} = (P_{i'}, Q_{i'}, R_{i'})$, $\mathcal{W}_{i''} = (P_{i''}, Q_{i''}, R_{i''})$ follows the other. If $\mathcal{W}_{i''} \preceq \mathcal{W}_{i'}$, then we have $p_{i'} \in P_{i'}$, $r_{i''} \in R_{i'}$, which implies $p \prec r$. If $\mathcal{W}_{i'} \preceq \mathcal{W}_{i''}$, then we have $p_{i'} \in P_{i''}$, $r_{i''} \in R_{i''}$ which implies $p \prec r$ again.

(iii). Now, let $q \in Q$. Thus $q = q_{i^\circ}$ for some $i^\circ \in I$. Since $\mathcal{W}^\circ = (P_{i^\circ}, Q_{i^\circ}, R_{i^\circ}) \in \mathbb{W}$, then there exist events $p_{i^\circ} \in P_{i^\circ}$, $r_{i^\circ} \in R_{i^\circ}$ with $p_{i^\circ} \odot q$, $r_{i^\circ} \odot q$. But we have also $p_{i^\circ} \in P$, $r_{i^\circ} \in R$, by the construction of $\mathcal{W}^*$. Therefore the triple $\mathcal{W}^* \in \mathbb{W}$.

Evidently, this triple $\mathcal{W}^*$ upper bounds $\mathbb{A}$. Therefore there exists at least one maximal element α of $\mathbb{W}$ after Zorn's Lemma. We have $\alpha \in \mathbb{W}$ by the definition of $\mathbb{W}$. ∎

Proof of Theorem 2. Let us fix an event $b \prec a$. Such an event b exists after Axiom $\mathcal{A}$. Let s be an event, simultaneous with a and b. Such an event exists after Axioms $\mathcal{G}$ and $\mathcal{D}$. Then the triple $\mathcal{W}^\circ = (a, s, b) \in \mathbb{W}$. Let $\mathbb{V}$ be the subset of $\mathbb{W}$ which contains all triples $\mathcal{W} \in \mathbb{W}$ with $\mathcal{W} = (P, Q, R) \preceq \mathcal{W}^\circ = (a, s, b)$ and such that any event of P is not after the event a, i.e. either $p \odot a$ or $p \prec a$ for any event $p \in P$.

Evidently, $\mathbb{V}$ is not empty partially ordered subset of $\mathbb{W}$ with the same relation of order $\preceq$. We shall see that $\mathbb{V}$ satisfies the requirement of Zorn's Lemma. Let $\mathbb{A} = \{(P_i, Q_i, R_i), i \in I\}$ be nonempty linearly ordered subset of $\mathbb{V}$, where I is a complex of indexes. Let us scrutinize the triple $\mathcal{W}^* = (P, Q, R)$ with

$$P = \bigcup_{i \in I} P_i, \quad Q = \bigcup_{i \in I} Q_i, \quad R = \bigcup_{i \in I} R_i .$$

We have $\mathcal{W}^* \in \mathbb{W}$ since the requirements (i)-(iii) are satisfied: We have

(i). $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$ as $\mathbb{A}$ is not empty;

(ii). If $p \in P$, $r \in R$, then $p \in P_{i'}$, $r \in R_{i''}$ for some $i', i'' \in I$. $\mathbb{A}$ is linearly ordered. This is, we have either $\mathcal{W}_{i'} \preceq \mathcal{W}_{i''}$ and $p \in P_{i''}$, $r \in R_{i''}$, so $p \prec r$, or $\mathcal{W}_{i''} \preceq \mathcal{W}_{i'}$ and $p \in P_{i'}$, $r \in R_{i'}$, thus $p \prec r$ also.

(iii). Let $q \in Q$ be arbitrarily fixed. Then q belongs to some Q_{i°, $i^\circ \in I$, $\mathcal{W}_{i^\circ} = (P_{i^\circ}, Q_{i^\circ}, R_{i^\circ}) \in \mathbb{W}$.

That is why there exist events $p \in P_{i^\circ}$, $r \in R_{i^\circ}$, simultaneous with q , $p \odot q$, $r \odot q$. This complete the proof of $w^* \in \mathcal{W}$.

Moreover, $w^* \in \mathbb{V}$ since we have: 1).It is true that $w^* \succeq w^\circ$, as $w_i \succeq w^\circ$ for $\forall\, i \in I$; 2). Let the event $p \in P$ be arbitrarily fixed. Then $p \in P_{i^\circ}$ for some $i^\circ \in I$. Since $(P_{i^\circ} , Q_{i^\circ} , R_{i^\circ}) \in \mathbb{V}$, hence the event p cannot be after the event a , i.e. either $p \odot a$ or $p \prec a$.

Evidently, the triple w^* upper bounds $\mathbb{A}$. Then $\mathbb{V}$ contains at least one maximal element α after Zorn's Lemma. Obviously, α is a maximal element of $\mathcal{W}$ also. This is, α is an instant of $\mathcal{W}$. Moreover, we have $\alpha \in a$ after the construction of $\mathbb{V}$. ∎

Proof of Theorem 3. Let us denote by D_a (resp. by D_b) all events d_a (resp. d_b) which are after the event a (resp. b). There exist the following three cases only: I. We have $a \prec d_b$, $b \prec d_a$ for any $d_a \in D_a$, $\forall\, d_b \in D_b$. Let us denote $a = c$, $d_a = d$ for some arbitrarily fixed event $d_a \in D_a$, and let s be an event simultaneous with a and d . We have $c \prec d$, $c \odot s$, $d \odot s$.

II. There is an event $d'_a \in D_a$ with $b \odot d'_a$. then we have $a \prec d'_a$, $d'_a \odot b$, $b \prec d_b$. It follows applying Axiom $\mathcal{A}$ .Point 2, that $a \prec d_b$ for $\forall\, d_b \in D_b$. Let us denote a by c , b by s and $d'_a = d$. We have $c \prec d$, $c \odot s$, $d \odot s$.

III. There is an event $d'_b \in D_b$ with $a \odot d'_b$. Then we have $b \prec d'_b$, $d'_b \odot a$, $a \prec d_a$. Thus we obtain $b \prec d_a$ for $\forall\, d_a \in D_a$, applying Axiom $\mathcal{A}$. Point 2. Let us denote b by c , a by s and d'_b by d . We have $c \prec d$, $c \odot s$, $d \odot s$. Thus we receive in each of the cases I-III that the triple $w^0 = (P^\circ , Q^\circ , R^\circ)$ with $P^\circ = \{c\}$, $Q^\circ = \{s\}$, $R^\circ = \{d\}$ belongs to $\mathcal{W}$.

Let us scrutinize the partially ordered complex $\mathbb{V} = \{ w = (P,Q,R) \in \mathcal{W} , w \succeq w^\circ$ and any event of P is not after the event c $\}$ with the order, induced by the order of $\mathcal{W}$.

$\mathbb{V}$ satisfies the requirement of Zorn's Lemma since the following holds Let $\mathbb{A} = \{ w_i \}_{i \in I}$ be a nonempty linearly ordered subset of $\mathbb{V}$, where $w_i = (P_i , Q_i , R_i)$ and I is a complex of indexes.

Let us denote by w^* the triple $w^* = (P,Q,R)$ with

$P = \bigcup_{i\in I} P_i$; $Q = \bigcup_{i\in I} Q_i$, $R = \bigcup_{i\in I} R_i$.

We shall prove that the triple $\mathcal{W}^* \in \mathcal{W}$,since the requirements (i)-(iii) hold for $\mathcal{W}^*$: (i). Since $\mathbb{A}$ is not empty we have $P \neq \emptyset$,$Q \neq \emptyset$, $R \neq \emptyset$.

(ii). If $p \in P$, $r \in R$, then $p \in P_{i'}$, $r \in R_{i''}$ for some i' , $i'' \in I$. In the case $\mathcal{W}_{i'} \preceq \mathcal{W}_{i''}$, we receive $p \in P_{i''}$, $r \in R_{i''}$,and therefore $p \prec r$. In the case $\mathcal{W}_{i''} \preceq \mathcal{W}_{i'}$, we obtain $p \in P_{i'}$, $r \in R_{i'}$ and thus $p \prec r$.

(iii). Let $q \in Q$, then $q \in Q_{i^\circ}$ for some $i^\circ \in I$. That is why there exist events $p \in P_{i^\circ}$, $r \in R_{i^\circ}$, such that $q \odot p$, $q \odot r$. It follows by the construction of $\mathcal{W}^*$ that $p \in P$, $r \in R$. Thus we obtain $\mathcal{W}^* \in \mathcal{W}$.

Moreover, $\mathcal{W}^* \in \mathcal{V}$ since the following holds: 1. We have $c \in P$, $s \in Q$, $d \in R$ as $c \in P_i$, $s \in Q_i$, $d \in R_i$ for $\forall i \in I$. Thus we receive $\mathcal{W}^* \succeq \mathcal{W}^\circ$.

2. Any event $p \in P$ cannot be after the event c since $p \in P_{i^\circ}$ for some $i^\circ \in I$ and $\mathcal{W}_{i^\circ} \in \mathcal{V}$. Thus we obtain $\mathcal{W}^* \in \mathcal{V}$.

Applying Zorn's Lemma to $\mathcal{V}$, we receive that there exists a maximal element γ of $\mathcal{V}$. Evidently γ is a maximal element of $\mathcal{W}$ also. Thus γ is an instant, for which $\gamma \in a$, $\gamma \in b$, after the construction of $\mathcal{V}$. ∎

Proof of Proposition 5. Point a).Let a be an arbitrarily fixed event of $\mathcal{S}$. At first we shall scrutinize the following cases I-III : There exists an event $p_0 \in P_\alpha$, such that $a \prec p_0$. Then $a \in P_\alpha$ in this case, after the maximality of the instant α in $\mathcal{W}$.

II. There exists an event $r_0 \in R_\alpha$ with $r_0 \prec a$. Then $a \in R_\alpha$,after the maximality of the instant α of $\mathcal{W}$.

III. There exist events $p_1 \in P_\alpha$, $r_1 \in R_\alpha$ with $p_1 \odot a$, $r_1 \odot a$ .In this case $a \in Q_\alpha$,again after the maximality of the instant α in $\mathcal{W}$.

Let a be an event, for which are not satisfied the requirement of any of the cases I-III. Then we have either $p^\circ \odot a$ or $p^\circ \prec a$ for each event $p^\circ \in P_\alpha$. If $a \odot p^\circ$ for some $p^\circ \in P_\alpha$,then $a \prec r$ for $\forall r \in R_\alpha$,since the event a does not satisfy the requirements of the cases II and III. Therefore $a \in P_\alpha$, after the construction of α .

Now let us eliminate the previous case also. This is, now the event a is not simultaneous with any of the events of P_α . Since the case I also is eliminated for the event a, then $a \succ p$ for $\forall p \in P_\alpha$. Therefore $a \in R_\alpha$ in this case, after the construction of α and after the maximality of α in $\mathcal{W}$. Thus we receive $P_\alpha \sqcup Q_\alpha \sqcup R_\alpha = \mathcal{S}$.

The assertion of Point b) is evident after the construction of $\mathcal{W}$.

Point c). Let the events $q'_\alpha, q''_\alpha \in Q_\alpha$. Then there exist events $p', p'' \in P_\alpha$, $r', r'' \in R_\alpha$ with $p' \odot q'_\alpha$, $r' \odot q'_\alpha$, $p'' \odot q''_\alpha$, $r'' \odot q''_\alpha$, after the construction of $\mathcal{W}$.

Let us assume that $q'_\alpha \prec q''_\alpha$. Then we have $q'_\alpha \prec q''_\alpha$, $q''_\alpha \odot p''$, $p'' \prec r$, $\forall r \in R_\alpha$. Applying Axiom $\mathcal{A}$. Point 2, we obtain that $q'_\alpha \prec r$ for $\forall r \in R_\alpha$, which contradicts $\alpha \in \mathcal{W}$. Thus the assumption $q'_\alpha \prec q''_\alpha$ is not true.

Now, let us assume $q''_\alpha \prec q'_\alpha$. Since we have $q''_\alpha \prec q'_\alpha$, $q'_\alpha \odot p'$, $p' \prec r$, $\forall r \in R_\alpha$, hence $q''_\alpha \prec r$ for $\forall r \in R_\alpha$. This contradicts $\alpha \in \mathcal{W}$ again.

Thus it remains possible only the relation $q'_\alpha \odot q''_\alpha$. ∎

Proof of Proposition 4. We have $\alpha = \alpha$ after the definition of the relation "=" and after the already proved Point c) of Proposition 5.

Also, if $\alpha = \beta$, then $\beta = \alpha$, after the definition.

Now let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$, $\alpha = \beta$, $\beta = \gamma$. Let us assume $\alpha \neq \gamma$. Then there exist events $q^*_\alpha \in Q_\alpha$, $q^*_\gamma \in Q_\gamma$ which are not simultaneous. Then we have either $q^*_\alpha \prec q^*_\gamma$ or $q^*_\gamma \prec q^*_\alpha$. Since if $\mu = \nu$ for some instants μ, ν, then also $\nu = \mu$, hence it is sufficient to reject the possibility $q^*_\alpha \prec q^*_\gamma$. If $q^*_\alpha \prec q^*_\gamma$, then there exists an event d with $q^*_\alpha \prec d \prec q^*_\gamma$, after Axiom $\mathcal{B}$.

We shall prove that $d \in Q_\beta$. Let us assume $p_\beta \prec d$ for $\forall p_\beta \in P_\beta$. Then we have $d \notin P_\beta$, $d \notin Q_\beta$ and $d \in R_\beta$, after the maximality of the instant β in $\mathcal{W}$. Therefore there exists an event s with $s \odot d$, $s \odot p^*_\beta$ for some fixed $p^*_\beta \in P_\beta$ and $s \prec q^*_\gamma$, after Axiom $\mathcal{D}$. The maximality of β in $\mathcal{W}$ implies $s \in Q_\beta$. But $\beta = \gamma$, which yields $s \odot q^*_\gamma$. The obtained contradiction proves that $d \notin R_\beta$ and $d \odot p^*_\beta$ for some event $p^*_\beta \in P_\beta$.

Now, let us assume that $d \prec r_\beta$ for $\forall r_\beta \in R_\beta$. Then $d \in P_\beta$, after the maximality of β in $\mathcal{W}$, and there exists an event v with $q^*_\alpha \prec v$, $v \odot d$, $v \odot r^*_\beta$ for some arbitrarily fixed event r^*_β of R_β. The last two relation involve $v \in Q_\beta$, after the maximality of β in $\mathcal{W}$. Therefore we must have

$v \odot q^*_\alpha$, since $\alpha = \beta$, whereas it holds $q^*_\alpha \prec v$ .The obtained contradiction proves that $d \notin P_\beta$ and $d \odot r^*_\beta$ for some $r^*_\beta \in R_\beta$.

Thus we have $d \odot p^*_\beta$, $d \odot r^*_\beta$, $p^*_\beta \in P_\beta$, $r^*_\beta \in R_\beta$. Therefore $d \in Q_\beta$, after the maximality of β in $\mathcal{W}$.

Then the relations $d \in Q_\beta$ and $\alpha = \beta$ prove that $d \odot q^*_\alpha$, while we have $q^*_\alpha \prec d$. The obtained contradiction implies the impossibility of $q^*_\alpha \prec q^*_\gamma$. After the symmetricity of "=", this is sufficient to assert $\alpha = \gamma$. ∎

Proof of Proposition 6. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$ be instants, $\alpha \prec \beta$, with $q'_\alpha \prec q'_\beta$, $q'_\alpha \in Q_\alpha$, $q'_\beta \in Q_\beta$. Let us assume that simultaneously we have $\beta \prec \alpha$ with $q''_\beta \prec q''_\alpha$, $q''_\alpha \in Q_\alpha$, $q''_\beta \in Q_\beta$. Therefore we have $q'_\alpha \prec q'_\beta$, $q'_\beta \odot q''_\beta$, $q''_\beta \prec q''_\alpha$ .Axiom $\mathcal{A}$.Point 2 involves $q'_\alpha \prec q''_\alpha$, whereas we have $q'_\alpha \odot q''_\alpha$, after Proposition 5.Point c). The obtained contradiction proves that the assumption $\beta \prec \alpha$ is not true. ∎

Proof of Proposition 7. Let us have $\alpha \prec \beta$, $\beta \prec \gamma$ for some instants $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$,belonging to $\mathcal{W}$, with $q_\alpha \prec q'_\beta$, $q''_\beta \prec q_\gamma$ for some events $q_\alpha \in Q_\alpha$, q'_β ,$q''_\beta \in Q_\beta$, $q_\gamma \in Q_\gamma$ .We have $q'_\beta \odot q''_\beta$ after Proposition 5. Point c).Therefore Axiom $\mathcal{A}$. Point 2 implies $q_\alpha \prec q_\gamma$. This signifies that $\alpha \prec \gamma$. ∎

Proof of Theorem 8. It is sufficient to show after the proofs of Propositions 6 and 7, that if μ and ν are arbitrarily fixed instants with $\mu \neq \nu$ then we have either $\mu \prec \nu$ or $\nu \prec \mu$. Let $\mu = (P_\mu, Q_\mu, R_\mu)$, $\nu = (P_\nu, Q_\nu, R_\nu)$. Since $\mu \neq \nu$, then there exists at least one pair of events $q^*_\mu \in Q_\mu$, $q^*_\nu \in Q^*_\nu$, which are not simultaneous. Then we have only two possibilities: 1. Either $q^*_\mu \prec q^*_\nu$, or 2. $q^*_\nu \prec q^*_\mu$. In the first case we have $\mu \prec \nu$ after the definition of the order of $\mathcal{W}$. In the second case we have $\nu \prec \mu$. ∎

Lemma 10. Let the instant $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$ belongs to the event q . If the event b is before q , $b \prec q$, then $b \in P_\alpha$. If c is an event after q , $q \prec c$, then $c \in R_\alpha$.

Proof. Since $\alpha \in q$, hence $b \notin Q_\alpha$, $c \notin Q_\alpha$, as q must be simultaneous with any event of Q_α . We shall prove that $b \notin R_\alpha$. Let us assume the contrary. Then $p \prec b$ for $\forall\, p \in P_\alpha$. Let us fix one event $p' \in P_\alpha$. Since we have $p' \prec b \prec q$, hence there is an event s with $s \odot p'$, $s \odot b$, $s \prec q$. But α is an instant, i.e. α is a maximal element of $\mathcal{W}$. Hence $s \in Q_\alpha$, as $s \odot p'$, $s \odot b$, $p' \in P_\alpha$, $b \in R_\alpha$. Moreover, $\alpha \in q$, thus q must be simultaneous with any element of Q_α. So $q \odot s$ and $s \prec q$. The obtained contradiction proves that $b \notin R_\alpha$. Since we have also $b \notin Q_\alpha$, then $b \in P_\alpha$, according to the maximality of α in $\mathcal{W}$.

We shall prove now, that $c \notin P_\alpha$. Let us assume the contrary. Thus $c \prec r$, $\forall\, r \in R_\alpha$. Let us fix one event $r' \in R_\alpha$. There exists an event s' with $s' \odot r'$, $s' \odot c$, $s' \succ q$, since $q \prec c \prec r'$ and Axiom $\mathcal{D}$ holds. We get $s' \in Q_\alpha$ according to the maximality of α in $\mathcal{W}$ and $s' \odot c$, $c \in P_\alpha$, $s' \odot r'$, $r' \in R_\alpha$. Therefore we receive $q \odot s'$ and $q \prec s'$. The obtained contradiction proves that $c \notin P_\alpha$. But $c \notin Q_\alpha$. Then $c \in R_\alpha$ after the maximality of α in $\mathcal{W}$. ■

Lemma 11. Let the instant α belongs to the event q. If β is an instant belonging to the event b before q, $b \prec q$, then $\beta \prec \alpha$.

If γ is an instant, belonging to the event c after q, $q \prec c$, then $\alpha \prec \gamma$.

Proof. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. Applying the axiom $\mathcal{B}$, we get the existence of events a', a", b', b", c', c", k, l with $b' \prec b \prec b'' \prec k \prec a' \prec q \prec a'' \prec l \prec c' \prec c \prec c''$. We have $b' \in P_\beta$, $b'' \in R_\beta$, $a' \in P_\alpha$, $a'' \in R_\alpha$, $c' \in P_\gamma$, $c'' \in R_\gamma$ after Lemma 10. Therefore there exist events s', s", s"' with

$s' \odot b'$	$s'' \odot a'$	$s''' \odot c'$
$s' \odot b''$	$s'' \odot a''$	$s''' \odot c''$
$s' \prec k$	$k \prec s'' \prec l$	$l \prec s'''$.

Thus we receive $s' \in Q_\beta$, $s'' \in Q_\alpha$, $s''' \in Q_\gamma$, $s' \prec s'' \prec s'''$. Hence $\beta \prec \alpha \prec \gamma$. ■

Lemma 12. Let α, β, γ be instants with $\alpha \prec \beta \prec \gamma$. Then there exist events a, b', b", c with $\alpha \in a$, $\beta \in b'$, $\beta \in b''$, $\gamma \in c$ and $a \prec b'$, $b'' \prec c$. We have for any such events a and c that $a \prec c$.

Proof. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. It follows by the definition of the relation "$\prec$", the existence of events $a \in Q_\alpha$, b', $b'' \in Q_\beta$, $c \in Q_\gamma$ with $a \prec b'$, $b'' \prec c$. Now, let a, b', b'', c be arbitrarily fixed events, satisfying the requirements of Lemma 12. We must prove $a \prec c$. It is sufficient to prove the impossibility of the cases $c \prec a$ and $a \odot c$. Let us assume $c \prec a$. Then we get $\gamma \prec \alpha$, according Lemma 11. But $\gamma \prec \alpha$ contradicts the conditions of Lemma 12. Thus the assumption $c \prec a$ is not true. Now, let us assume $a \odot c$. Then Theorem 3 involves the existence of an instant $\xi \in a$ and $\xi \in c$. Since $a \prec b'$, $\xi \in a$, $\beta \in b'$, hence $\xi \prec \beta$ after Lemma 11. As $b'' \prec c$, $\beta \in b''$, $\xi \in c$, then $\beta \prec \xi$. Therefore $\xi \prec \beta \prec \xi$. The obtained contradiction proves that the case $a \odot c$ is not possible. Thus it remains only $a \prec c$.

Proof of Theorem 9. I. $\mathfrak{W}^2$ is linearly ordered after Theorem 8.

II. $\mathfrak{W}^2$ has the property 4 of T: Let K be a fixed sequence of events from Axiom $\mathfrak{B}$. Each arbitrarily fixed event $k \in K$ defines at least one instant $\varkappa \in k$, according to Theorem 2. Let us fix arbitrarily such an instant $\varkappa \in k$. Let $\mathfrak{K}$ be the sequence of these instants $\varkappa$, ($k \longmapsto \varkappa$), when k ranges K.

Let α and β be instants of $\mathfrak{W}^2$ with $\alpha \prec \beta$, $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$ and let $q_\alpha \prec q_\beta$ for some events $q_\alpha \in Q_\alpha$, $q_\beta \in Q_\beta$. Axiom $\mathfrak{B}$ implies the existence of an event $k^\circ \in K$ with $q_\alpha \prec k^\circ \prec q_\beta$. Let $\varkappa^\circ \in k^\circ$ be the chosen instant of $\mathfrak{K}$, corresponding to k°. Further on, we have $\alpha \in q_\alpha$, $\beta \in q_\beta$, $\varkappa^\circ \in k^\circ$. Lemma 11 implies $\alpha \prec \varkappa^\circ \prec \beta$. Thus the sequence $\mathfrak{K}$ is a dense sequence of instants.

III. $\mathfrak{W}^2$ has the property 5 of T, i.e. $\mathfrak{W}^2$ is open-ended: Let α be an arbitrarily fixed instant of $\mathfrak{W}^2$, $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$. Let q be an arbitrarily fixed event of Q_α. There exist events m and n with $m \prec q \prec n$, after Axiom $\mathfrak{C}$. Theorem 2 implies the existence of instants $\mu \in m$, $\nu \in n$. Then we have $\mu \prec \alpha \prec \nu$ after Lemma 11 as $\alpha \in q$. Therefore $\mathfrak{W}^2$ has the property 5 of T.

IV. We shall prove that $\mathfrak{W}^2$ is a continuum: Let $\mathfrak{W}^2_1$ and $\mathfrak{W}^2_2$ be two disjoint ($\mathfrak{W}^2_1 \cap \mathfrak{W}^2_2 = \emptyset$) nonempty parts of $\mathfrak{W}^2$, whose union is $\mathfrak{W}^2$, ($\mathfrak{W}^2_1 \cup \mathfrak{W}^2_2 = \mathfrak{W}^2$), and each instant of $\mathfrak{W}^2_1$ is before any instant of $\mathfrak{W}^2_2$. We must prove the existence of an instant γ, such that each instant before γ to belong to $\mathfrak{W}^2_1$ and each instant after γ to belong to $\mathfrak{W}^2_2$.

Let $\mathcal{E} \in \mathcal{W}_1$, $\varkappa \in \mathcal{W}_2$. Since $\mathcal{W}_1$ and $\mathcal{W}_2$ are nonempty, we can choo- and fix such instants. Let $\varrho \prec \mathcal{E}$ and $\varkappa \prec \delta$. Such instants ϱ , δ exist after the proved property 5 of $\mathcal{W}$. Let r , e , m , k , n , d be events with $\varrho \in$ r , $\mathcal{E} \in$ e , m$\in$K , n$\in$K , $\varkappa \in$ k , $\delta \in$ d and r $\prec$ m $\prec$ e , k $\prec$ n $\prec$ d . Here K is the sequence from Axiom $\mathfrak{B}$. Let μ , ν be arbitrarily fixed instants of m and n correspondingly, $\mu \in$ m , $\nu \in$ n .

Further on, the events m and n have the following properties:

1°. m $\prec$ n (after Lemma 12).

2°. Each instant of m belongs to $\mathcal{W}_1$ (since we have m $\prec$ e , $\mathcal{E} \in$ e , $\mathcal{E} \in \mathcal{W}_1$).

3°. Each instant of n belongs to $\mathcal{W}_2$ (since we have k $\prec$ n, $\varkappa \in$ k , $\varkappa \in \mathcal{W}_2$).

Any such a pair of events a and b , having the properties 1°-3° will be denoted by a & b . Let us fix such a pair a&b . Then there exists at least one event s , simultaneous with a and b , a$\odot$s , b$\odot$s , according to Axiom $\mathfrak{D}$.

Let us construct the following class of events

$$\mathbb{Q} = \{ s : \exists a , b \in \mathcal{S} , a \& b , s \odot a , s \odot b \} .$$

Let $\mathbb{P}$ be the set of all events a , corresponding to events s of $\mathbb{Q}$; let $\mathbb{R}$ be the set of all events b , corresponding to events s of $\mathbb{Q}$. We want to prove that the triple $\Gamma = (\mathbb{P}, \mathbb{Q}, \mathbb{R}) \in \mathcal{W}$,i.e. has the properties (i)-(iii): (i).we have shown the existence of events m , n,s with m & n , s$\odot$m , s$\odot$n . Therefore $\mathbb{R} \neq \emptyset$, $\mathbb{Q} \neq \emptyset$, $\mathbb{P} \neq \emptyset$;

(ii). let the events p $\in \mathbb{P}$, r $\in \mathbb{R}$ be arbitrarily fixed. We must prove that p $\prec$ r . There exist events m° , **n** with p & **n** , m° & r , by the construction of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$ .Now we shall prove that the cases r $\prec$ p and p$\odot$r are impossible. Let us assume r $\prec$ p . Let ξ , η be instants with $\xi \in$ p , $\eta \in$ r (cf. Theorem 2). Since we have assumed r $\prec$ p , then $\eta \prec \xi$ after Lemma 11 . But p & **n** implies $\xi \in \mathcal{W}_1$ after the property 2° of p & **n** . The relation m° & r involves $\eta \in \mathcal{W}_2$,according to the property 3° of m° & r . Therefore $\xi \prec \eta$ after the choice of $\mathcal{W}_1$ and $\mathcal{W}_2$.

The obtained contradiction proves that the relation $r \prec p$ is not possible. Now, let us assume $p \odot r$. Let the instant $\xi \in p$, $\xi \in r$, according to Theorem 3. As we have p & n, then $\xi \in \mathcal{W}_1^2$. But since $m°$ & r and $\xi \in r$, then $\xi \in \mathcal{W}_2^2$ by the property 3° of " & ". This contradicts $\mathcal{W}_1^2 \cap \mathcal{W}_2^2 = \emptyset$. Thus the assumption $p \odot r$ is not true. Then it remains only $p \prec r$.

(iii). This requirement is satisfied evidently by the construction of $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{R}$. Thus we obtain $\Gamma \in \mathbb{W}$.

Let $\mathbb{V}$ be the subset of $\mathbb{W}$ of all triples $w \in \mathbb{W}$ with $w \succeq \Gamma$.

$\mathbb{V}$ satisfies the requirement of Zorn's Lemma: Let $\mathbb{A} = \{ w_i = (P_i, Q_i, R_i), i \in I \}$ be a linearly ordered nonempty subset of $\mathbb{V}$. Then the triple $w^* = (P, Q, R)$ with

$$P = \bigcup_{i \in I} P_i, \quad Q = \bigcup_{i \in I} Q_i, \quad R = \bigcup_{i \in I} R_i \text{ belongs to } \mathbb{V}.$$

Moreover, w^* upper bounds $\mathbb{A}$. After Zoren's Lemma $\mathbb{V}$ has at least one maximal element γ. It is evident that γ is a maximal element of $\mathbb{W}$ also. Thus γ is an instant, $\gamma \in \mathcal{W}$.

We shall prove that any instant ξ with $\xi \prec \gamma$ belongs to $\mathcal{W}_1^2$ and that any instant η with $\gamma \prec \eta$ belongs to $\mathcal{W}_2^2$. Now, let $\xi \prec \gamma$. Let us assume the contrary, i.e., that $\xi \in \mathcal{W}_2^2$. Then we must have $\gamma \in \mathcal{W}_2^2$ by the choice of $\mathcal{W}_1^2$ and $\mathcal{W}_2^2$. Since $\xi \prec \gamma$, hence there are events r, $\xi \in r$, $n \in K$, g, $\gamma \in g$, for which $r \prec n \prec g$. Let ν be some fixed instant of n. Let m be an event of $\mathcal{W}_1^2$, this is, if $\mu \in m$, then $\mu \in \mathcal{W}_1^2$. We have shown that such events exist. We shall prove that the pair of events m and n has the properties 1°- 3°, i.e., m & n:

Let us scrutinize the order between m and n. If we assume $n \prec m$, since $\mu \in m$, $\nu \in n$, then $\nu \prec \mu$. But this is impossible because $\mu \in \mathcal{W}_1^2$, $\nu \in \mathcal{W}_2^2$. Thus the assumption $n \prec m$ is not true. Now, let us assume $m \odot n$. Then there exists an instant $\mu^* \in m$, $\mu^* \in n$, after Theorem 3. $\mu^* \in m$ involves $\mu^* \in \mathcal{W}_1^2$; $\mu^* \in n$ implies $\mathcal{W}_2^2 \ni \mu^*$. But we have $\mathcal{W}_1^2 \cap \mathcal{W}_2^2 = \emptyset$. The contradiction proves that the assumption $m \odot n$ is not true. Thus, it remains $m \prec n$.

Moreover, since any instant of m belongs to $\mathcal{W}_1^2$ and any instant of n belongs to $\mathcal{W}_2^2$, hence we have m & n.

Let s be an event, simultaneous with m and n , $s \odot m$, $s \odot n$, and let $s \prec g$. Such an event s exists after Axiom $\mathfrak{D}$. Since $s \in \mathbb{Q}$, then $\gamma \in s$. By the other hand we have $s \prec g$ and $\gamma \in g$. This is, we must have $\gamma \prec \gamma$ after Lemma 11 , since $\gamma \in s$ and $\gamma \in g$. This contradiction proves that the assumption $\xi \in \mathfrak{W}_2$ is not true. Therefore $\xi \in \mathfrak{W}_1$ for any instant ξ with $\xi \prec \gamma$.

Now, let η be an instant after γ . We must prove that $\eta \in \mathfrak{W}_2$. Let us assume the contrary, i.e., that $\eta \in \mathfrak{W}_1$. Then $\gamma \in \mathfrak{W}_1$ as $\gamma \prec \eta$ and after the choice of $\mathfrak{W}_1$ and $\mathfrak{W}_2$. Therefore there exist events u , v with $\gamma \in u$, $l \in K$, $\eta \in v$ and $u \prec l \prec v$ .Let w be an event of $\mathfrak{W}_2$, i.e.any instant $\sigma \in w$ belongs to $\mathfrak{W}_2$. We have shown that such events w exist.

Let us scrutinize the pair l , w of events. We shall show that l & w :

Let us investigate the order between the events l and w . We shall prove that $l \prec w$ .Let us assume that $w \prec l$, and let ϑ , σ be instants with $\vartheta \in l$, $\sigma \in w$. The assumption $w \prec l$ involves $\sigma \prec \vartheta$. But we have $l \prec v$, $\vartheta \in l$, $\eta \in v$, $\vartheta \in \mathfrak{W}_1$, whereas $w \subset \mathfrak{W}_2$, and therefore $\vartheta \prec \sigma$ .The obtained contradiction proves the impossibility of the assumption $w \prec l$. Now, let us assume $l \odot w$ .Then there exists an instant $\omega \in l$, $\omega \in w$,after Theorem 3 .Since $\omega \in l$ and $l \prec v$, hence $\omega \in \mathfrak{W}_1$ .But as $\omega \in w$ and $w \subset \mathfrak{W}_2$, then $\omega \in \mathfrak{W}_2$. This contradicts $\mathfrak{W}_1 \cap \mathfrak{W}_2 = \emptyset$. Thus the assumption $l \odot w$ is not true. Therefore, it remains $l \prec w$ .Moreover, we have $l \subset \mathfrak{W}_1$, $w \subset \mathfrak{W}_2$. Thus l & w .

Let t be an event,simultaneous with l and w , $t \odot l$, $t \odot w$, and $u \prec t$ Such an event t exists after Axiom $\mathfrak{D}$. Since $t \in \mathbb{Q}$, hence $\gamma \in t$. Therefore $\gamma \prec \gamma$,as $u \prec t$, $\gamma \in u$ .The obtained contradiction proves that the assumption $\eta \in \mathfrak{W}_1$ is not true. Therefore $\eta \in \mathfrak{W}_2$.

Thus Theorem 9 is true. ∎

Proposition 13. The instant γ ,separating $\mathfrak{W}_1$ and $\mathfrak{W}_2$, determined by $\pi = (\mathbb{P}, \mathbb{Q}, \mathbb{R})$, is unique.

Proof. Let us assume that there are at least two such different instants. We shall denote the earlier of them by γ_1 and the other by γ_2, i.e. $\gamma_1 \prec \gamma_2$. Then there exists an instant τ with $\gamma_1 \prec \tau \prec \gamma_2$, after the proved already properties of $\mathcal{W}'$. It follows from $\gamma_1 \prec \tau$ that $\tau \in \mathcal{W}'_2$. But since $\tau \prec \gamma_2$, hence we have $\tau \in \mathcal{W}'_1$. This contradicts $\mathcal{W}'_1 \cap \mathcal{W}'_2 = \emptyset$. The obtained contradiction proves the uniqueness of γ. ■

PROOFS FOR THE SECOND MODEL OF TIME (WITH WALKER'S DEFINITION OF INSTANTS)

Proof of Proposition 1'. The relation $b \sqsubset a$ implies the existence of events c and d with $c \odot a$, $d \odot a$, $c \prec b \prec d$. We want to prove that $a \odot m$. We shall eliminate the possibilities $m \prec a$ and $a \prec m$. Let us assume at first that $m \prec a$. Since we have $m \prec a$, $a \odot c$, $c \prec b$, hence we get $m \prec b$ after Axiom $\mathcal{A}^*$. Point 2. This contradicts $b \odot m$. Thus we proved that the assumption $m \prec a$ is not true. Let us assume now that $a \prec m$. Since we have $a \prec m$, $m \odot b$, $b \prec d$, we receive $a \prec d$, which contradicts $a \odot d$. Thus, the assumption $a \prec m$ is not true also. Therefore it holds $a \odot m$. ■

Proof of Proposition 2'. Let us assume the contrary, i.e. that we have at the same time $b \sqsubset a$ and $a \sqsubset b$ for some events a and b. Axiom $\mathcal{A}^*$. Point 3 implies the existence of events c, c' with $a \odot c$, $c \prec b$, $b \odot c'$, $c' \prec a$. Thus we receive $c' \prec a$, $a \odot c$, $c \prec b$. Therefore we have $c' \prec b$ which contradicts the condition $c' \odot b$. The obtained contradiction proves Proposition 2'. ■

Proof of Proposition 3'. We shall eliminate the other possibilities - $a \odot b$ or $b \prec a$. Let us assume $a \odot b$. Since $a \sqsubset a'$, $b \sqsubset b'$, hence we have $b \odot a'$, $a \odot b'$, after Proposition 1'. As $b \sqsubset b'$ and $b \odot a'$, then Proposition 1' yields $b' \odot a'$, which contradicts the condition of Proposition 3'. Thus the assumption $a \odot b$ is not true.

Now, let us assume $b \prec a$. Since we have also $a \odot a'$, $a' \prec b'$, hence Axiom $\mathcal{A}^*$. Point 2 implies $b \prec b'$, whereas $b \sqsubset b'$ gives $b \odot b'$. Thus the assumption $b \prec a$ is not true. Therefore it only remains $a \prec b$. ■

Proof of Theorem 4'. Evidently, the relation "follow", $\prec$, partially orders the class $\mathcal{W}'$ of the triples. Now, we shall prove that $\mathcal{W}'$ is not empty. There exists at least one event a_0 after Axiom $\mathcal{A}^*$. Point 1. There are events a_1, c_0, d_0 with $a_1 \sqsubset a_0$, $c_0 \prec a_1 \prec d_0$, $c_0 \odot a_0$, $d_0 \odot a_0$, after Axiom $\mathcal{A}^*$. Point 3. Applying again Axiom $\mathcal{A}^*$. Point 3, we we get events a_2 , c_1 , d_1 with $a_2 \sqsubset a_1$, $c_1 \prec a_2 \prec d_1$, $c_1 \odot a_1$, $d_1 \odot a_1$

Inductively, if we have chosen the events a_n , c_{n-1} , d_{n-1} , then there exist events a_{n+1} , c_n , d_n with $a_{n+1} \sqsubset a_n$, $c_n \prec a_{n+1} \prec d_n$, $a_n \odot c_n$, $a_n \odot d_n$. Let us scrutinize the triple $\mathcal{W}' = (P', Q', R')$ with

$$P' = \bigcup_{i=0}^{\infty} \{c_i\} \ , \ Q' = \bigcup_{i=0}^{\infty} \{a_i\} \ , \ R' = \bigcup_{i=0}^{\infty} \{d_i\} .$$

The triple $\mathcal{W}'$ satisfies the requirements (i') - (iv') :

(i'). $P' \neq \emptyset$, $Q' \neq \emptyset$, $R' \neq \emptyset$ after Axiom $\mathcal{A}^*$. Points 1 and 3.

(ii'). Let $p \in P'$, $r \in R'$. Then $p = c_{n'}$, $r = d_{n''}$ for some integers n' and n'' . That is why we have

$$c_{n'} \prec a_{n'+1} , \quad a_{n'+1} \odot a_{n''+1} , \, a_{n''+1} \prec d_{n''} .$$

This is $p = c_{n'} \prec d_{n''} = r$ after Axiom $\mathcal{A}^*$. Point 2 .

(iii'). Let $q \in Q'$. Then $q = a_{n^\circ}$ for some integer n° . We have

Since $c_{n^\circ} \in P'$, $d_{n^\circ} \in R'$, then the requirement (iii') is satisfied.

(iv'). Let $q \in Q'$. Then $q = a_{n^\circ}$ for some integer n° . We have

$$a_{n^\circ+1} \sqsubset a_{n^\circ} \text{ by construction and } a_{n^\circ+1} \in Q' \text{ also.}$$

Thus $\mathcal{W}' \in \mathbb{W}'$.

In the proof of the property (ii') of $\mathcal{W}'$, we used $a_{n'} \odot a_{n''}$. Moreover, all events of Q' are simultaneous. This may be proved by induction: We have $a_0 \odot a_1$ and $a_n \odot a_{n+1}$ as well, for any nonnegative integer n . Let us fix arbitrary such an integer n . We shall prove at first that $a_n \odot a_{n+2}$. Let us assume that $a_n \prec a_{n+2}$. Then we have

$$a_n \prec a_{n+2} , \quad a_{n+2} \odot a_{n+1} , \, a_{n+1} \prec d_n .$$

We receive $a_n \prec d_n$ after Axiom $\mathcal{A}^*$. Point 2 , whereas we have $a_n \odot d_n$ by construction. Therefore the assumption $a_n \prec a_{n+2}$ is not true.

Now, let us assume $a_{n+2} \prec a_n$. This implies

$$c_n \prec a_{n+1} , \quad a_{n+1} \odot a_{n+2} , \, a_{n+2} \prec a_n .$$

Thus we get $c_n \prec a_n$, contradicting $a_n \odot c_n$. This is why the assumption $a_{n+2} \prec a_n$ is not true also. Therefore $a_n \odot a_{n+2}$ for any integer $n \geq 0$.

Let us suppose that we have already proved

$$a_n \odot a_{n+1}, a_n \odot a_{n+2}, \ldots, a_n \odot a_{n+k^\circ}$$

for $\forall n$, $k^\circ \in Z_+$ be fixed .We shall prove that $a_n \odot a_{n+k^\circ+1}$ also. Let us assume $a_n \prec a_{n+k^\circ+1}$. Then we have

$$a_n \prec a_{n+k^\circ+1} \, , \, a_{n+k^\circ+1} \odot a_{n+1}, \, a_{n+1} \prec d_n .$$

This is $a_n \prec d_n$, which contradicts the construction of $\mathcal{W}'$. Thus the assumption $a_n \prec a_{n+k^\circ+1}$ is not true.Now, let us assume $a_{n+k^\circ+1} \prec a_n$. This implies

$$c_n \prec a_{n+1} \, , \, a_{n+1} \odot a_{n+k^\circ+1} \, , a_{n+k^\circ+1} \prec a_n .$$

Therefore we get $c_n \prec a_n$, contradicting the construction of $\mathcal{W}'$. Thus the assumption $a_{n+k^\circ+1} \prec a_n$ is not true.This is why $a_n \odot a_{n+k^\circ+1}$ for $\forall n$, $\forall k^\circ$. Hence we receive by induction $a_{n'} \odot a_{n''}$ for $\forall n'$, $\forall n'' \in Z_+$.

Now, we shall verify that satisfies the requirement of Zorn's Lemma. Let $\mathbb{A} = \{ w_i \}_{i \in I}$ be an arbitrarily fixed nonempty linearly ordered subset of $\mathcal{W}'$, where $w_i = (P_i, Q_i, R_i)$, I is a complex of indexes. Let us scrutinize the triple

$$w^\circ = (P, Q, R) \text{ with } P = \bigcup_{i \in I} P_i \, , \, Q = \bigcup_{i \in I} Q_i \, , \, R = \bigcup_{i \in I} R_i .$$

The triple $w^\circ \in \mathcal{W}'$ since w° has the properties (i')-(iv'):

(i') $P \neq \emptyset$, $Q \neq \emptyset$, $R \neq \emptyset$ as $\mathbb{A}$ is nonempty .

(ii'). Let $p \in P$, $r \in R$ be arbitrarily fixed events . Then $p \in P_{i'}$, $r \in R_{i''}$ for some i' , $i'' \in I$. If $w_{i'} \preceq w_{i''}$, hence $p \in P_{i''} \supset P_{i'}$ and it holds $p \prec r$. If $w_{i''} \preceq w_{i'}$, then $r \in R_{i'} \supset R_{i''}$ and then $p \prec r$ as well.

(iii'). Let $q \in Q$. Then $q \in Q_{i^\circ}$ for some $i^\circ \in I$. That is why there are events $p \in P_{i^\circ} \subset P$, $r \in R_{i^\circ} \subset R$, such that $q \odot p$, $q \odot r$.

(iv'). Let $q \in Q$. Then $q \in Q_{i^\circ}$ for some $i^\circ \in I$. Hence there is an event $q^* \in Q_{i^\circ} \subset Q$ with $q^* \sqsubset q$.

Evidently the triple W^o upper bounds the set $\mathbb{A}$ .Then Zorn's Lemma implies the existence of a maximal element $\alpha \in \mathcal{W}'$, i.e. of an instant $\alpha \in \mathcal{W}^{\iota\,\prime}$. ∎

Lemma 12'. Let us have $p^* \prec r^*$, $s^* \odot p^*$, $s^* \odot r^*$ for some events p^*, r^*, $s^* \in \mathcal{S}$. Then there exist the following sequences of events

(1)

$$p^* \sqsupset a = a_1 \sqsupset a_2 \sqsupset a_3 \sqsupset \dots \sqsupset a_n \sqsupset a_{n+1} \sqsupset \dots$$

$$r^* \sqsupset b = b_1 \sqsupset b_2 \sqsupset b_3 \sqsupset \dots \sqsupset b_n \sqsupset b_{n+1} \sqsupset \dots$$

$$s^* = s_1 \sqsupset s_2 \sqsupset s_3 \sqsupset \dots \sqsupset s_n \sqsupset s_{n+1} \sqsupset \dots$$

with $s_n \odot a_n$, $s_n \odot b_n$, $n = 1,2,\dots$. Moreover, if $p^* \in P_\alpha$, $r^* \in R_\alpha$ for an instant $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, then always when are satisfied the relations (1) for events a_n , b_n , s_n it follows that $a_n \in P_\alpha$, $b_n \in R_\alpha$, $s_n \in Q_\alpha$, $n = 1,2,\dots$.

Lemma.13'. If we have $t \sqsubset u$ and $u \prec v$, then $t \prec v$ also. If it holds $t \sqsubset u$ and $w \prec u$, then $w \prec t$ as well . Here t, u , v and w are events .

Proof. Since $t \sqsubset u$, hence there are events k , l with $k \prec t \prec l$, $k \odot u$, $l \odot u$.

Let us establish the order between t and v , elliminating the possibilities $t \odot v$, or $v \prec t$. Let us assume $t \odot v$. Therefore we have $u \prec v$, $v \odot t$, $t \prec l$. Applying Axiom $\mathcal{A}^*$ .Point 2 we get $u \prec l$, whereas it is true $u \odot l$. Thus the assumption $t \odot v$ is not true. Now , let us assume $v \prec t$. Then $v \prec t$, $t \odot u$, $u \prec v$ and Axiom $\mathcal{A}^*$. Point 2 involve $v \prec v$. That is why the assumption $v \prec t$ is not true also . Therefore it remains $t \prec v$ only .

Now, we shall prove $w \prec t$, elliminating the cases $t \odot w$ and $t \prec w$ .If we assume $w \odot t$, then $k \prec t$, $t \odot w$, $w \prec u$ and Axiom $\mathcal{A}^*$ .Point 2 yield $k \prec u$, while we have $k \odot u$. Thus the assumption $t \odot w$ is not true. Now , let us assume $t \prec w$. Then $w \prec u$, $u \odot t$, $t \prec w$ involve $w \prec w$ .The obtained contradiction proves $w \prec t$. ∎

Proof of Lemma 12'.Since $p^* \odot s^*$, then there exists an event a_1, f_1 with $a_1 \sqsubset s_1$, $a_1 \sqsubset p^*$, $f_1 \prec a_1$, $f_1 \odot s_1$. The relation $s^* \odot r^*$ implies the existence of events b_1 , g_1 with $b_1 \sqsubset r^*$, $b_1 \sqsubset s_1$, $b_1 \prec g_1$, $g_1 \odot s^*$.

Moreover, we have $a_1 \prec b_1$ after Proposition 3' and $a_1 \sqsubset p^*$, $b_1 \sqsubset r^*$, $p^* \prec r^*$. Thus we get $f_1 \prec a_1 \prec b_1 \prec g_1$. Then there exists an event s_2 with $s_2 \odot a_1$, $s_2 \odot b_1$, $f_1 \prec s_2 \prec g_1$, after Axiom $\mathcal{D}^*$. We shall prove that $s_2 \sqsubset s_1$. Since we have $f_1 \odot s_1$, $g_1 \odot s_1$, $f_1 \prec s_2 \prec g_1$, hence it suffices to show $s_1 \odot s_2$. If we assume $s_2 \prec s_1$, then $f_1 \prec a_1$, $a_1 \odot s_2$, $s_2 \prec s_1$ give $f_1 \prec s_1$, whereas we have $f_1 \odot s_1$. Thus the assumption $s_1 \succ s_2$ is not true. Now, let us assume $s_1 \prec s_2$, then $s_1 \prec s_2$, $s_2 \odot b_1$, $b_1 \prec g_1$ imply $s_1 \prec g_1$, while we have $s_1 \odot g_1$. Thus it remains $s_1 \odot s_2$. Moreover, then we get $s_2 \sqsubset s_1$.

Let us suppose that we have already construced the sequences of events

$$p^* \sqsupset a_1 \sqsupset a_2 \sqsupset \ldots \sqsupset a_{n-1}, \quad r^* \sqsupset b_1 \sqsupset b_2 \sqsupset \ldots \sqsupset b_{n-1},$$

$$s_1 \sqsupset s_2 \sqsupset \ldots \sqsupset s_{n-1} \text{ with } a_{n-1} \sqsubset s_{n-1}, \quad b_{n-1} \sqsubset s_{n-1},$$

$$f_{n-1} \prec a_{n-1} \prec b_{n-1} \prec g_{n-1}, \quad s_{n-1} \odot f_{n-1}, \quad s_{n-1} \odot g_{n-1}$$

for some events f_{n-1}, g_{n-1}. Then there exists an event s_n with

$$f_{n-1} \prec s_n \prec g_{n-1}, \quad s_n \odot a_{n-1}, \quad s_n \odot b_{n-1}.$$

Therefore we have $s_n \sqsubset s_{n-1}$ since

$$s_{n-1} \odot f_{n-1}, \quad s_{n-1} \odot g_{n-1}, \quad f_{n-1} \prec s_n \prec g_{n-1}, \quad s_n \odot s_{n-1}.$$

Now, there exist events a_n, b_n, f_n, g_n with

$$a_n \sqsubset a_{n-1}, \; a_n \sqsubset s_n, \; b_n \sqsubset b_{n-1}, \; b_n \sqsubset s_n, \; f_n \prec a_n \prec b_n \prec g_n, \; f_n \odot s_n, \; g_n \odot s_n$$

Thus we inductively get the sequences (1).

Further on, when $p^* \in P_\alpha$, then $p^* \prec r$ for $\forall\, r \in R_\alpha$. Lemma 13' implies $a_n \prec r$ for $\forall\, r \in R_\alpha$ also. Thus $a_n \in P_\alpha$ according to the maximality of α in $\mathbb{W}'$. Since $r^* \in R_\alpha$ and $b_n \sqsubset r^*$, hence $b_n \in R_\alpha$ also by the maximality of α in $\mathbb{W}'$. Then the conditions (1) involves $s_n \in Q_\alpha$ after the maximality of α in $\mathbb{W}'$, $n = 1, 2, \ldots$. ■

Proof of Proposition 5'. Point a). At first we shall investigate the following cases I - III for an event a. I. There exists an event $p_0 \in P_\alpha$ with $a \prec p_0$. Then we have $a \in P_\alpha$ after the maximality of α in $\mathbb{W}'$.

II. There exists an event $r_o \in R_\alpha$ with $r_o \prec a$. Then we have $a \in R_\alpha$, after the maximality of α in $\mathcal{W}'$.

III. There exist events $p^* \in P_\alpha$, $r^* \in R_\alpha$ with $a \odot p^*$, $a \odot r^*$. Then we have $a \in Q_\alpha$ after Lemma 12' and the construction of α.

Now let the event a do not satisfy any of the cases I-III.Then we have $p \odot a$ or $p \prec a$ for any fixed event $p \in P_\alpha$. If we have $a \odot p^*$ for some $p^* \in P_\alpha$, since ~~of~~ the conditions of the cases I-III are not satisfied for the event a, hence $a \prec r$ for $\forall\, r \in R_\alpha$. Therefore $a \in P_\alpha$ after the construction of α .

Now let us exclude the precedent case together the cases I-III for the event a . This is, the event is not simultaneous with any event of P_α. Since the case I is already eliminated for the event a , hence $a \succ p$ for $\forall\, p \in P_\alpha$. Therefore $a \in R_\alpha$ in this case after the construction of and after the maximality of α in $\mathcal{W}'$.That is why we get $P_\alpha \cup Q_\alpha \cup R_\alpha = \mathcal{S}$.

The assertion of Point b) is evident by the construction of $\mathcal{W}'$.

Point c). We shall show that it is not possible one event of Q_α to be before some other event of Q_α. Let us assume the contrary. Let us denote the earlier event of two arbitrarily fixed different events of Q_α by q' and the other - by q'' . I.e. $q' \prec q''$. Then there are events $p'' \in P_\alpha$, $r' \in R_\alpha$ with $q' \odot r'$, $q'' \odot p''$. Thus we receive $q' \prec q''$, $q'' \odot p''$, $p'' \prec r'$ Axiom $\mathcal{A}^*$.Point 2 yields $q' \prec r'$. The obtained contradiction proves Point c). ∎

Proof of Theorem 6'. There exist events a_1 , c_o , d_o with

$$a_1 \subset a_0 = a\ ,\ c_0 \prec a_1 \prec d_0\ ,\ a_0 \odot c_0\ ,\ a_0 \odot d_0$$

after Axiom $\mathcal{A}^*$. Point 3 . Let us have chosen events a_{n+1}, c_n , d_n with the corresponding properties. Then we can choose events a_{n+2}, c_{n+1}, d_{n+1} with

$$a_{n+2} \subset a_{n+1}\ ,\ c_{n+1} \prec a_{n+2} \prec d_{n+1}\ ,\ c_{n+1} \odot a_{n+1}\ ,\ d_{n+1} \odot a_{n+1},$$

applying Axiom $\mathcal{A}^*$ .Point 3. Let the triple $w' = (P', Q', R')$ be with

$$P' = \bigcup_{j=0}^{\infty} \{c_j\}\ ,\ Q' = \bigcup_{j=0}^{\infty} \{a_j\}\ ,\ R' = \bigcup_{j=0}^{\infty} \{d_j\}\ .$$

We have proved in Theorem 4' that such a triple $w' \in \mathcal{W}'$. Let $\mathcal{V}$ be the subset of $\mathcal{W}'$ of all triples w of $\mathcal{W}'$ with $w' \lesssim w$. The relation $\prec$

of $\mathbb{W}'$ induces a semiorder "$\prec$" in $\mathbb{V}$. Let $\mathbb{A} = \{w_i\}_{i\in I}$, $w_i = (P_i, Q_i, R_i)$, be a linearly ordered nonempty subset of $\mathbb{V}$. Let $w^o = (P, Q, R)$ be with

$$P = \bigcup_{i\in I} P_i, \quad Q = \bigcup_{i\in I} Q_i, \quad R = \bigcup_{i\in I} R_i .$$

We have shown in the proof of Theorem 4' that such a triple $w^o \in \mathbb{W}'$. It remains to prove that $w^o \in \mathbb{V}$. This is we must show that $w^o \succeq w'$. We have $w^o \succeq w_i$ and $w_i \succeq w'$, $\forall i \in I$, thus $w^o \succeq w'$ also.

Evidently, w^o upper bounds $\mathbb{A}$. Therefore Zorn's Lemma is applyable to $\mathbb{V}$. It yields the existence of a maximal element α of $\mathbb{V}$. Obviously α is a maximal element of $\mathbb{W}'$ also. Thus α is an instant.

Moreover, since $\alpha \succeq w'$, hence $\alpha \in$ a after Proposition 5'. ∎

Proof of Theorem 7'. We have $m \odot n$. Axiom $\mathcal{A}^*$.Point 4 yields the existence of an event q with $q \subset m$, $q \subset n$. Let w' be the triple from the proof of Theorem 4' for which $a_o = q$. The set $\mathbb{V}$ of all triples $w \in \mathbb{W}'$ such that $w \succeq w'$ is nonempty and satisfies the requirement of Zorn's Lemma. Then we receive the existence of an instant $\gamma \succeq w' = (P', Q', R')$ with $q = a_o \in Q'$. Let $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. Therefore each event $q_\gamma \in Q_\gamma \supset Q'$ is simultaneous with q after Proposition 5'.Point c), $q_\gamma \odot q$. Since $q \subset m$, $q \subset n$, hence we receive, applying Proposition 1', that $q_\gamma \odot m$ and $q_\gamma \odot n$ for $\forall q_\gamma \in Q_\gamma$. Therefore we have $\gamma \in m$ and $\gamma \in n$. ∎

Proof of Proposition 8'. We have $\alpha = \alpha$ after Proposition 5'.Point c).

Let us have $\alpha = \beta$ and $\beta = \gamma$ with $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. Let us assume $\alpha \neq \gamma$, this is, that there exist events $q^*_\alpha \in Q_\alpha$ and $q^*_\gamma \in Q_\gamma$ which are not simultaneous. Then we have either $q^*_\alpha \prec q^*_\gamma$ or $q^*_\gamma \prec q^*_\alpha$. Since if $\sigma = \omega$ for some instants σ, ω, then also $\omega = \sigma$ hence it is sufficient to reject the possibility $q^*_\alpha \prec q^*_\gamma$. If $q^*_\alpha \prec q^*_\gamma$, then there exist events q^o_α, q^o_γ, d with $q^o_\alpha \subset q^*_\alpha$, $q^o_\alpha \in Q_\alpha$, $q^o_\gamma \subset q^*_\gamma$, $q^o_\gamma \in Q_\gamma$, $q^o_\alpha \prec d \prec q^o_\gamma$.

We shall prove that $d \in Q_\beta$. Let us assume $p_\beta \prec d$ for $\forall p_\beta \in P_\beta$. Then we have $d \notin P_\beta$, $d \notin Q_\beta$ and therefore $d \in R_\beta$, after the maximality of the instant β in $\mathbb{W}'$. Therefore there exists an event s with $s \odot d$,

$s \odot p^*_\beta$ for some fixed event $p^*_\beta \in P_\beta$ and $s \prec q^\circ_\gamma$, after Axiom $\mathcal{D}^*$. The maximality of β and Lemma 12' imply $s \in Q_\beta$. But $\beta = \gamma$ which yields $s \odot q^\circ_\gamma$. The obtained contradiction proves that $d \notin R_\beta$ and $d \odot p^*_\beta$ for some event $p^*_\beta \in P_\beta$.

Now, let us assume that $d \prec r_\beta$ for $\forall r_\beta \in R_\beta$. Then $d \in P_\beta$,after the maximality of β in $\mathcal{W}'$, and there exists an event v with $q^\circ_\alpha \prec v$, $v \odot d$, $v \odot r^*_\beta$ for some arbitrarily fixed event r^*_β of R_β. The last two relations and Lemma 12' involve $v \in Q_\beta$, after the maximality of β in $\mathcal{W}'$. Therefore we must have $v \odot q^\circ_\alpha$, since $\alpha = \beta$, whereas it holds $q^\circ_\alpha \prec v$. The obtained contradiction proves that $d \notin P_\beta$ and $d \odot r^*_\beta$ for some $r^*_\beta \in R_\beta$.

Thus we have $d \odot p^*_\beta$, $d \odot r^*_\beta$, $p^*_\beta \in P_\beta$, $r^*_\beta \in R_\beta$. Therefore $d \in Q_\beta$ after Lemma 12' and the maximality of β in $\mathcal{W}'$.

Then the relations $d \in Q_\beta$ and $\alpha = \beta$ prove that $d \odot q^\circ_\alpha$, while we have $q^\circ_\alpha \prec d$. The obtained contradiction implies the impossibility of $q^*_\alpha \prec q^*_\gamma$. After the symmetricity of "=", this is sufficient to assert $\alpha = \gamma$. ∎

Proof of Theorem 9'. I. We shall prove that if we have for some instants $\alpha \prec \beta$, then it cannot be true $\beta \prec \alpha$. Let us assume the contrary, i.e. that we have at the same time $\alpha \prec \beta$ and $\beta \prec \alpha$ for some instants α and β . Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$ with $q'_\alpha \prec q'_\beta$, $q''_\beta \prec q''_\alpha$, for some events q'_α , $q''_\alpha \in Q_\alpha$, q'_β , $q''_\beta \in Q_\beta$. Thus we obtain $q'_\alpha \prec q'_\beta$, $q'_\beta \odot q''_\beta$, $q''_\beta \prec q''_\alpha$, using Proposition 5'.Point c). Therefore $q'_\alpha \prec q''_\alpha$, according Axiom $\mathcal{A}^*$.Point 2 . But Proposition 5'.Point c) implies $q'_\alpha \odot q''_\alpha$. The obtained contradiction proves that we cannot have at the same time $\alpha \prec \beta$ and $\beta \prec \alpha$ for any two instants .

II. We shall show that the order in $\mathcal{W}'$ is transitive. Let us have $\alpha \prec \beta$ and $\beta \prec \gamma$ for some instants α , β , γ . We want to prove that $\alpha \prec \gamma$ Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$ .Then there exist events $q_\alpha \in Q_\alpha$, q'_β , $q''_\beta \in Q_\beta$, $q_\gamma \in Q_\gamma$ with $q_\alpha \prec q'_\beta$, $q''_\beta \prec q_\gamma$. Therefore $q_\alpha \prec q'_\beta$, $q'_\beta \odot q''_\beta$, $q''_\beta \prec q_\gamma$, using Proposition 5'.Point c). That is why we obtain $q_\alpha \prec q_\gamma$, which signifies that $\alpha \prec \gamma$.

III. Let μ , $\nu \in \mathcal{W}'$ be arbitrarily fixed instants. We must prove that it holds one and only one of the relations - either $\mu \prec \nu$ or $\nu \prec \mu$ or $\mu = \nu$. Let $\mu = (P_\mu, Q_\mu, R_\mu)$, $\nu = (P_\nu, Q_\nu, R_\nu)$.

1. If any two events of Q_μ and Q_ν are simultaneous, then $\mu = \nu$.

2. Let now there exist events $q_\mu \in Q_\mu$, $q_\nu \in Q_\nu$, which are not simultaneous. Then $\mu \neq \nu$. Moreover, Axiom $\mathcal{A}^*$ .Point 1 involves that it is possible one and only one of the relations - either $q_\mu \prec q_\nu$ or $q_\nu \prec q_\mu$. In the case $q_\mu \prec q_\nu$, we have $\mu \prec \nu$ by the definition . We have $\nu \prec \mu$ after the definition in the case $q_\nu \prec q_\mu$. ■

Lemma 14'. Let the instant α belong to the event q . If b is ~~an~~ event before q , $b \prec q$, then $b \in P_\alpha$, where $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$. If c is an event after q , $q \prec c$, then $c \in R_\alpha$.

Proof. Since $\alpha \in q$, $b \prec q$, then $b \notin Q_\alpha$. We shall prove that $b \notin R_\alpha$. Let us assume the contrary. Then $p \prec b$ for $\forall p \in P_\alpha$. Let us fix an event $p^* \in P_\alpha$. Let a , m be events with $a \sqsubset p^*$, $m \prec a$, $m \odot p^*$, $a \odot p^*$(cf. ~~Axiom~~ Axiom $\mathcal{A}^*$ .Point 3). We have $a \prec b$ after Lemma 13'. Therefore $m \prec a \prec b \prec q$ and there is an event s with $s \odot a$, $s \odot b$, $m \prec s \prec q$ according to Axiom $\mathcal{D}^*$. Proposition 1' implies $s \odot p^*$. α is an instant , i.e. α is a maximal element of $\mathcal{W}'$. Hence $s \in Q_\alpha$, since $s \odot p^*$, $s \odot b$, $b \in R_\alpha$ and after Lemma 12'.

Moreover, we have $s \prec q$. But $\alpha \in q$. Thus the event q must be simultaneous with any element of Q_α . So $s \odot q$ and $s \prec q$. The obtained contradiction proves that $b \notin R_\alpha$. Since we have also $b \notin Q_\alpha$, then $b \in P_\alpha$, according to the maximality of α in $\mathcal{W}'$.

Now, we shall prove that $c \notin P_\alpha$. Let us assume the contrary, i.e. $c \in P_\alpha$. Thus $c \prec r$ for $\forall r \in R_\alpha$. Let us fix one event $r^* \in R_\alpha$. There are events d , n with $d \sqsubset r^*$, $d \odot r^*$, $n \odot r^*$, $d \prec n$, after Axiom $\mathcal{A}^*$. Point 3 . We ~~shall~~ show that $c \prec d$ in accord with $c \prec r^*$, $d \sqsubset r^*$, and Lemma 13'.The relations $q \prec c \prec d \prec n$ and Axiom $\mathcal{D}^*$ imply the existence of an event s' with $q \prec s' \prec n$, $s' \odot c$, $s' \odot d$. Moreover, we have $s' \odot r^*$, according Proposition 1' . Then Lemma 12' and the maximality of α in $\mathcal{W}'$ involve $s' \in Q_\alpha$, as $s' \odot c$, $c \in P_\alpha$, $s' \odot r^*$, $q \prec s' \prec n$. But $\alpha \in q$, then $q \odot s'$, since $s' \in Q_\alpha$. The contradiction with $q \prec s'$ proves that the assumption $c \in P_\alpha$ is not true. Thus $c \notin P_\alpha$. We also have $c \notin Q_\alpha$, since $q \prec c$ and $\alpha \in q$. Then $c \in R_\alpha$ after the maximality of α in $\mathfrak{W}'$. ■

Lemma 15'. Let the instant α belong to the event a. If β is an instant, belonging to the event b before a, $b \prec a$, then $\beta \prec \alpha$.

If γ is an instant, belonging to the event c after a, $a \prec c$, then $\alpha \prec \gamma$.

Proof. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. Let a', b', c', m, n, b°, c° be events with $a' \sqsubset a$, a' be simultaneous with any event of Q_α, $m \prec a' \prec n$, $a \odot m$, $a \odot n$, $b' \sqsubset b$, b' be simultaneous with any event of Q_β, $b^\circ \prec b'$, $b^\circ \odot b$, $c' \sqsubset c$, c' be simultaneous with any event of Q_γ, $c' \prec c^\circ$, $c^\circ \odot c$. Such events exist after the definition of "an instant belongs to an event".

I. Let at first $b \prec a$. Axiom $\mathfrak{B}^*$ implies the existence of events b_1, b_2, b_3 with $b' \prec b_1 \prec b_2 \prec b_3 \prec a'$. Then we have $b^\circ \in P_\beta$, $b_1 \in R_\beta$, $b_3 \in P_\alpha$, $n \in R_\alpha$, after analogous proofs as in Lemma 14'. Axiom $\mathfrak{D}^*$ involves the existence of events s_1 and s_2 with $s_1 \odot b^\circ$, $s_1 \odot b_1$, $s_1 \prec b_2$, $s_2 \odot b_3$, $s_2 \odot n$, $b_2 \prec s_2$. Therefore Lemma 12' and the maximality of the instants. give $s_1 \in Q_\beta$, $s_2 \in Q_\alpha$, $s_1 \prec s_2$. Thus we receive $\beta \prec \alpha$.

II. Let now $a \prec c$. Axiom $\mathfrak{B}^*$ implies the existence of events c_1, c_2, c_3 with $a' \prec c_1 \prec c_2 \prec c_3 \prec c'$. Then we have $m \in P_\alpha$, $c_1 \in R_\alpha$, $c_3 \in P_\gamma$, $c^\circ \in R_\gamma$, according to analogous proofs as in Lemma 14'. Axiom $\mathfrak{D}^*$ involves the existence of events s' and s'' with $s' \odot m$, $s' \odot c_1$, $s' \prec c_2$, $s'' \odot c_3$, $s'' \odot c^\circ$, $c_2 \prec s''$. Therefore $s' \in Q_\alpha$ and $s'' \in Q_\gamma$ after Lemma 12' and $s' \prec s''$ after the indicated relations. Thus we obtain $\alpha \prec \gamma$. ■

Proof of Proposition 10'. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$ and the event $q \in Q_\alpha$. There exists an event $q' \sqsubset q$, $q' \in Q_\alpha$, after the construction of the triples of $\mathfrak{W}'$. The definition of the relation $\sqsubset$ for events implies the existence of events b and c with $b \prec q' \prec c$. Theorem 6' involves the existence of instants β, γ, $\beta \in b$, $\gamma \in c$. Since $b \prec q' \prec c$, hence $\beta \prec \alpha \prec \gamma$ after Lemma 15'. ■

Lemma 16'. Let α, β, γ be instants with $\alpha \prec \beta \prec \gamma$. Then there exist events a, b', b", c, for which $\alpha \in a$, $\beta \in b'$, $\beta \in b''$, $\gamma \in c$ and $a \prec c$. We have for any such events that $a \prec c$.

Proof. Let $\alpha = (P_\alpha, Q_\alpha, R_\alpha)$, $\beta = (P_\beta, Q_\beta, R_\beta)$, $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$. It follows by the definition of the order "$\prec$" that there exist events $a \in Q_\alpha$, $b' \in Q_\beta$, $b'' \in Q_\beta$, $c \in Q_\gamma$ with $a \prec b'$, $b'' \prec c$, $b' \odot b''$, i.e. $a \prec c$.

Now, let a, b', b", c be arbitrarily fixed events, satisfying the requirements of Lemma 16'. We must prove $a \prec c$. It is sufficient to show that it is not possible to have either $c \prec a$ or $a \odot c$. The case $c \prec a$ is impossible, since then $\gamma \prec \alpha$, according Lemma 15'. But $\gamma \prec \alpha$ contradicts the conditions of Lemma 16'. The case $a \odot c$ is impossible, since then there is an instant $\xi \in a$ and $\xi \in c$, after Theorem 7'. As $a \prec b'$, $\xi \in a$, $\beta \in b'$, hence $\xi \prec \beta$. Since $b'' \prec c$, $\beta \in b''$, $\xi \in c$, then $\beta \prec \xi$. Therefore $\xi \prec \beta \prec \xi$. The obtained contradiction rejects the case $a \odot c$. Thus it only remains $a \prec c$. ■

Proof of Theorem 11'. I. $\mathfrak{W}'$ is linearly ordered (Property 1 of T) after Theorem 9';

$\mathfrak{W}'$ is open-ended (Property 5 of T) according to Proposition 10'.

II. Let us prove that $\mathfrak{W}'$ has a dense sequence of instants (Property 4 of Time continuum T): Let K be the sequence of events from Axiom $\mathfrak{B}^*$. Any arbitrarily fixed event $k \in K$ defines at least one instant $\varkappa \in k$, after Theorem 6'. We chose and fix such an instant $\varkappa \in k$. Let $\mathfrak{K}$ be the sequence of the instants $\varkappa$, when the event k ranges the sequence K.

Now, let $\mu \prec \nu$ be two arbitrarily fixed instants, $\mu = (P_\mu, Q_\mu, R_\mu)$, $\nu = (P_\nu, Q_\nu, R_\nu)$ with $q_\mu \prec q_\nu$ for some events $q_\mu \in Q_\mu$, $q_\nu \in Q_\nu$. There exist events $q^\circ_\mu \sqsubset q_\mu$, $q^\circ_\mu \in Q_\mu$ and $q^\circ_\nu \sqsubset q_\nu$, $q^\circ_\nu \in Q_\nu$ after the construction of $\mathfrak{W}'$ and $\mathfrak{W}'$. Then we get an event $k^\circ \in K$ with $q^\circ_\mu \prec k^\circ \prec q^\circ_\nu$, according to Axiom $\mathfrak{B}^*$. Let $\varkappa^\circ \in \mathfrak{K}$ be the instant, corresponding to the event k°, $\varkappa^\circ \in k^\circ$. Thus we have $\mu \prec \varkappa^\circ \prec \nu$ after Lemma 15'. This proves Property 4 of $\mathfrak{W}'$.

III. It remains to prove that the complex of all instants $\mathfrak{W}'$ is continuous, i.e., satisfies the Dedekind's postulate. Let $\mathfrak{W}^l_1$ and $\mathfrak{W}^l_2$ be two nonempty disjoint parts of $\mathfrak{W}'$, which sum is $\mathfrak{W}'$, and let each instant of $\mathfrak{W}_1$ be before any instant of $\mathfrak{W}^l_2$. We must prove the existence of an instant γ, such that any instant before γ belongs to $\mathfrak{W}^l_1$ and any instant after γ belongs to $\mathfrak{W}^l_2$. We shall use the following Lemma.

Lemma 17'. There exist events a and b, such that any instant of a belongs to $\mathfrak{W}^l_1$ (we shall denote this by $a \subset \mathfrak{W}^l_1$) and each instant of b belongs to $\mathfrak{W}^l_2$ ($b \subset \mathfrak{W}^l_2$). Moreover, there are instants μ, ξ, ϱ, ν, such that $a \prec \mu \prec \xi$, $\xi \in \mathfrak{W}^l_1$ and $\varrho \prec \nu \prec b$, $\varrho \in \mathfrak{W}^l_2$. (We shall use the notation a £ b if the requirements of Lemma 17' are satisfied for a pair of events a and b).

Proof. $\mathfrak{W}^l_1$ and $\mathfrak{W}^l_2$ are nonempty. Let $\xi \in \mathfrak{W}^l_1$ and $\varrho \in \mathfrak{W}^l_2$. There exist instants μ, μ', ν, ν' with $\mu' \prec \mu \prec \xi$, $\varrho \prec \nu \prec \nu'$ after Proposition 10'. Since $\mu' \prec \mu$, $\nu \prec \nu'$, hence there are events $q_{\mu'} \in Q_{\mu'}$, $q_\mu \in Q_\mu$, $q_\nu \in Q_\nu$, $q_{\nu'} \in Q_\nu$ with $q_{\mu'} \prec q_\mu$, $q_\nu \prec q_{\nu'}$ where $\mu' = (P_{\mu'}, Q_{\mu'}, R_{\mu'})$, $\nu' = (P_{\nu'}, Q_{\nu'}, R_{\nu'})$, $\mu = (P_\mu, Q_\mu, R_\mu)$, $\nu = (P_\nu, Q_\nu, R_\nu)$. Let us denote $q_{\mu'} = a$ and $q_{\nu'} = b$. Then we have $a \subset \mathfrak{W}^l_1$, $b \subset \mathfrak{W}^l_2$, $a \prec \mu \prec \xi$, $\xi \in \mathfrak{W}^l_1$, $\varrho \prec \nu \prec b$, $\varrho \in \mathfrak{W}^l_2$. ∎

Now, let us construct the classes of events $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ as it follows:

$$\mathbb{Q} = \{ s : s \in \mathcal{S}, \exists\, a \pounds b, s \odot a, s \odot b, a, b \in \mathcal{S} \},$$

$\mathbb{P}$ consists of all events $a \subset \mathfrak{W}^l_1$, corresponding to events s of $\mathbb{Q}$, $\mathbb{R}$ consists of all events $b \subset \mathfrak{W}^l_2$, corresponding to events s of $\mathbb{Q}$. The triple $\Gamma = (\mathbb{P}, \mathbb{Q}, \mathbb{R})$ belongs to $\mathfrak{W}'$ since the requirements (i')-(iv') are satisfied: (i'). $\mathbb{P} \neq \emptyset$, $\mathbb{Q} \neq \emptyset$, $\mathbb{R} \neq \emptyset$ after Lemma 17'.

(ii'). Each event of $\mathbb{P}$ is before any event of $\mathbb{R}$ as a £ b for any events $a \in \mathbb{P}$, $b \in \mathbb{R}$. This is, let $p \in \mathbb{P}$, $r \in \mathbb{R}$. The case $p \odot r$ is impossible, since then there exists an instant $\zeta \in p$, $\zeta \in r$, after The Theorem 7'. Then $\zeta \in \mathfrak{W}^l_1$ as $p \subset \mathfrak{W}^l_1$ and simultaneously $\zeta \in \mathfrak{W}^l_2$ as $r \subset \mathfrak{W}^l_2$. But we have $\mathfrak{W}^l_1 \cap \mathfrak{W}^l_2 = \emptyset$.

The case $r \prec p$ is impossible since then there are instants $\theta \in p$, $\sigma \in r$, after Theorem 6'. We have $\theta \in \mathfrak{W}^l_1$ as $p \subset \mathfrak{W}^l_1$ and we have also

$\sigma \in \mathfrak{W}_2^2$ as $r \subset \mathfrak{W}_2$. We get $\sigma \prec \theta$ after the assumption $r \prec p$ and Lemma 15'. But this contradicts the choice of $\mathfrak{W}_1^2$ and $\mathfrak{W}_2^2$. Thus it only remains $p \prec r$.

(iii'). This requirement is satisfied by the construction of Γ.

(iv'). If $s \in \mathbb{Q}$, then there is a pair of events $a^* \pounds b^*$, $a^* \odot s$, $b^* \odot s$. Moreover ~~we have $a^* \prec b^*$, after the choice~~ of $\mathfrak{W}_1^2, \mathfrak{W}_2^2$ and $a^* \subset \mathfrak{W}_1^2$, $b^* \subset \mathfrak{W}_2^2$ as $a^* \pounds b^*$. Thus Lemma 12' is applyable with $a^* = p^*$, $b^* = r^*$, $s = s_1$. Therefore we get the sequences of events

(2)

$$a^* = p^* \supset a_1 \supset a_2 \supset \dots \supset a_n \supset \dots$$

$$b^* = r^* \supset b_1 \supset b_2 \supset \dots \supset b_n \supset \dots$$

$$s = s_1 \supset s_2 \supset s_3 \supset \dots \supset s_n \supset \dots \quad \text{with } a_n \odot s_n \text{ , } b_n \odot s_n \text{ ,} n=1,2,\dots$$

Since we have $a^\circ \pounds b^\circ$ then the sequences (2) and Lemma 13' involve

$$a_n \pounds b_n \text{ , } n = 1,2,\dots .$$

Therefore the requirement (iv') also holds for the triple Γ. That is why $\Gamma \in \mathfrak{W}'$.

Let $\mathbb{V}$ be the subset of $\mathfrak{W}'$ of all triples $\mathfrak{w} \in \mathfrak{W}'$, with $\mathfrak{w} \succeq \Gamma$. $\mathbb{V}$ is nonempty and satisfies the requirement of Zorn's Lemma, after a proof, analogous to the proof of Theorem 6'. Zorn's Lemma implies that $\mathbb{V}$ has at least one maximal element γ. It is evident that γ is a maximal element of $\mathfrak{W}'$ also. Thus γ is an instant with $Q_\gamma \supseteq \mathbb{Q}$, where $\gamma = (P_\gamma, Q_\gamma, R_\gamma)$.

We shall prove that γ separates $\mathfrak{W}_1^2$ and $\mathfrak{W}_2^2$ in the sense of Dedekind's postulate. Let α be an instant before γ. We must prove that $\alpha \in \mathfrak{W}_1^2$. Let us assume the contrary, i.e. that $\alpha \in \mathfrak{W}_2^2$. Then we have also $\gamma \in \mathfrak{W}_2^2$. Let δ, ε be instants with $\alpha \prec \delta \prec \varepsilon \prec \gamma$, according to the Property 4 of $\mathfrak{W}^{2'}$. Let e and g be events with $\varepsilon \in e$, $\gamma \in g$, $e \prec g$. Since $\delta \prec \varepsilon$, hence there exist events q_δ and q_ε with $q_\delta \prec q_\varepsilon$, $q_\delta \in Q_\delta$, $q_\varepsilon \in Q_\varepsilon$, where $\delta = (P_\delta, Q_\delta, R_\delta)$, $\varepsilon = (P_\varepsilon, Q_\varepsilon, R_\varepsilon)$. Then there are events q_δ°, q_ε°, k with $q_\delta^\circ \in Q_\delta$, $q_\delta^\circ \sqsubset q_\delta$, $q_\varepsilon^\circ \in Q_\varepsilon$, $q_\varepsilon^\circ \sqsubset q_\varepsilon$, $k \in K$ and $q_\delta^\circ \prec k \prec q_\varepsilon^\circ$ after the construction of $\mathfrak{W}'$ and after Axiom $\mathfrak{B}^*$. Since

$\delta \in q^{\circ}_{\delta}$, $\varepsilon \in q^{\circ}_{\varepsilon}$, hence $\delta \prec k \prec \varepsilon$. Thus $k \subset \mathcal{W}^{\ell}_{2}$. Let $a \in \mathbb{P}$. Then we have a £ k . Let s be an event, simultaneous with a and k, $s \prec g$. Such an event s exists after Axiom $\mathcal{D}^{*}$, Proposition 1' and Lemma 16'. Evidently, $s \in \mathbb{Q}$. Further on, since $\gamma \in g$ and $s \in \mathbb{Q}$ with $\gamma \succeq \Gamma$, hence $g \odot s$, whereas we have $s \prec g$. The obtained contradiction proves that $\alpha \in \mathcal{W}^{\ell}_{1}$.

Now , let β be an instant after γ , $\gamma \prec \beta$. We must prove $\beta \in \mathcal{W}^{\ell}_{2}$ Let us assume the contrary, i.e. that $\beta \in \mathcal{W}^{\ell}_{1}$. Then also we have $\gamma \in \mathcal{W}^{\ell}_{1}$. Let δ , τ be instants with $\gamma \prec \delta \prec \tau \prec \beta$ after Property 4 of $\mathcal{W}^{\ell\prime}$. Let l , h be events with $\gamma \in h$, $\delta \in l$, $h \prec l$. Since $\delta \prec \tau$, hence there are events $q_{\delta} \in Q_{\delta}$, $q_{\tau} \in Q_{\tau}$ with $q_{\delta} \prec q_{\tau}$, where $\delta = (P_{\delta}, Q_{\delta}, R_{\delta})$, $\tau = (P_{\tau}, Q_{\tau}, R_{\tau})$. Therefore there also exist events q°_{δ} , q°_{τ} , m with $q^{\circ}_{\delta} \in Q_{\delta}$, $q^{\circ}_{\delta} \sqsubset q_{\delta}$, $q^{\circ}_{\tau} \in Q_{\tau}$, $q^{\circ}_{\tau} \sqsubset q_{\tau}$, $q^{\circ}_{\delta} \prec m \prec q^{\circ}_{\tau}$, $m \in K$, according to the construction of $\mathcal{W}'$ and Axiom $\mathcal{B}^{*}$. As $\delta \in q^{\circ}_{\delta}$, $\tau \in q^{\circ}_{\tau}$, then $\delta \prec m \prec \tau$. Thus we get $m \subset \mathcal{W}^{\ell}_{1}$ since $\tau \prec \beta$, $\beta \in \mathcal{W}^{\ell}_{1}$. Let n be an event of $\mathbb{R}$, $n \in \mathbb{R}$. Then we have m £ n . We also have $h \prec l$, $\delta \in l$, $\delta \prec m$, i.e. $h \prec m$. Let s^ be an event, simultaneous with m and n and $h \prec$ s^. Such an event s^ exists after Axiom $\mathcal{D}^{*}$. Therefore s^ $\in \mathbb{Q}$, in accord with the construction of $\mathbb{Q}$. Since $\gamma \succeq \Gamma$, then $Q_{\gamma} \supseteq \mathbb{Q}$. Moreover we have $\gamma \in h$ and s^ $\in \mathbb{Q} \subseteq Q_{\gamma}$. Hence $h \odot$ s^. But this contradicts the choice of s^ with $h \prec$ s^. Thus the assumption $\beta \in \mathcal{W}^{\ell}_{1}$ is not true. That is why $\beta \in \mathcal{W}^{\ell}_{2}$. ■

Remark. The instant γ , separating $\mathcal{W}^{\ell}_{1}$ and $\mathcal{W}^{\ell}_{2}$ is unique. The proof coincides with the proof of Proposition 13.

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ON THE MEASUREMENT OF TIME IN MATHEMATICAL TIME'S MODELS

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This article proposes a measurement of Time in the mathematical models of Time. Any measurement consists in an establishment of a correspondence between the measured object and a number, or a vector, or some other mathematical quantity. Here we construct an isomorphism between the moments in any fixed mathematical model of Time and the real numbers,which isomorphism preserves the order. This construction includes a one-to-one correspondence between all moments of Time and the real numbers.for each fixed mathematical model of Time. Moreover this correspondence preserves the order,i.e. it maps larger real numbers to the later moments.

The developpement of the axiomatic theories of Time has been begun by Bertrand Russell, Whitehead, Norbert Wiener, Gerald J.Whitrow. This article on the measurement of Time in the axiomatic theories of Time formulates only exact mathematical results, almost without commentaries, following Newton's motto "hypotheses non fingo". The present here result is very small in comparison with the immensity and grandiosity of the problem of Time.

There exist mathematical models of Time (see Russell [1], Whitrow [2], Walker [3], Thomason [4], Madguerova [5]) in which theories of Time are constructed by axioms over the events and by the definitions of Time's moments. These definitions are based on the events and are due of Russell [1] and Walker [3](see also Whitrow [2]).

The indicated models realized the idea of Whitehead [6] and Russell [1] to receive the fundamental properties of Time by axioms on the events. Really, these models prove [4,5] the basic properties of Time continuum, used in Mathematical Physics. This is, these models prove that all instants (i.e. moments) of Time form a linearly ordered open-ended continuum with everywhere dense sequence of instants. Here we further develop these models(see [5]) of Time, proposing a measurement of Time in all of them together. The necessity of a precision of the conception of the measurement of Time for mathematical models of Time is noticed by Whitrow [2], N.Wiener [7], A.Wintner [7], V.A.Uspensky (on Conference of Logic,Varna, 1986). Any measurement consists in an establishment of a correspondence between the measured object and a number, or a vector,or some other mathematical quantity. Here we establish a one-to-one correspondence between all moments of Time and the real numbers for each of these Time models. Moreover, this correspondence preserves the order, i.e. it maps larger real numbers to the later moments. (This correspondence even is an isomorphism.)

The constructed correspondence can evidently be changed in many aspects. The possibility of many kind of measurements of Time reflects the real relativity of the measurement of Time, depending on the choice of the "clocks", i.e.depending on the choice of the "periodical processes" and their comparisons and confrontations. The choice of the clock would reflect on the choice of the basic dense sequence $\mathcal{K}$ of instants of Time.

The sequence $\mathcal{K}$ is constructed in any model of Time by a given (from the axioms) sequence of events K. This is why it is not difficult and is almost evident to substitute the proposed here construction for a measurement of Time by such one, based on the sequence K of events, avoiding the aidding sequence $\mathcal{K}$ of instants.

Since we shall mainly use the everywhere dense sequence $\mathcal{K}$ of instants, constructed in all mathematical models of Time (see [5,4,2]), for the exposition of the measurement here, that is why we shall not in details remind the voluminous mathematical models of Time.

The proposed measurement of Time here is for any arbitrarily chosen coordinate system of account and is based on the events(of K) in this system. This assures the compatibility of the measurement of Time with Theory of Relativity, as the existence of the events and their order do not depend on the choice of the co-ordinate system of account, although their perception can. depend.

Coarsely, we can choose a suitable sequence of "periodical"events for K. As an example, K can consist of the motions of an eternal clock pendulum, whose motions are reduced to fragments. We can choose for $\mathcal{K}$ the instants of the fixed positions of the pendulum. Then the proposed here construction of a measurement will coincide with the usual measurement of Time.

The construction of a measurement in any co-ordinate system of account is necessary for the comparison of different co-ordinate systems of account. The different measurement of Time in different co-ordinate systems of account can as usual be assured and obtained, postulating the Lorentz's formulas (or Newton's formulas). Thus we have :

Theorem. We shall construct an one-to-one correspondence between all instants of Time in any fixed mathematical model of Time and the real numbers, which maps larger numbers to the later instants. Moreover, this correspondence is an isomorphism between the instants and the real numbers, preserving the order.

Proof. There exists a dense sequence $\mathcal{K}$ of instants of Time after the axioms on the events in any of the mathematical models of Time [2,4,5]. At first we shall map an instant γ of Time from the sequence $\mathcal{K}$ to each rational number r ($\gamma \longmapsto r$), (which will be denoted by $M(\gamma) = r$) in such a way that a later moment of Time will correspond to a larger number.

We shall divide the proof of Theorem for the sake of the clearness of the exposition, formulating and proving its parts as Propositions 1 and 2:

Proposition 1. We can construct subsequences of instants of $\mathcal{K}$

$$\rho_0 \prec \rho_1 \prec \rho_2 \prec \ldots \prec \rho_m \prec \ldots \text{ and } \rho_0 \succ \rho_{-1} \succ \rho_{-2} \succ \ldots \succ \rho_{-m} \succ \ldots$$

such that if γ is an arbitrarily fixed instant, then there are instants ρ_k, ρ_l with $\rho_k \prec \gamma \prec \rho_l$. Moreover we can choose two successive instants of the indicated subsequences ρ_k and ρ_{k+1} with $\rho_k \preceq \gamma \preceq \rho_{k+1}$. We have the map $M(\rho_k) = k$.

Proof of Proposition 1(which is a part of the proof of Theorem). Let us arbitrarily choose and fix a "null" ("zero") moment of Time among the members of the sequence $\mathcal{K}$ (i.e. a moment α of $\mathcal{K}$ to which we shall map the zero number, $M(\alpha) = 0$). For instance, let us choose the first member $\varkappa_0$ of $\mathcal{K} = \{\varkappa_0, \varkappa_1, \ldots, \varkappa_p, \ldots\}$ to correspond to real zero. This correspondence will be denoted by $M(\varkappa_0) = 0$.

There exist moments of $\mathcal{K}$ after the moment $\varkappa_0 = \rho_0$ and other moments of $\mathcal{K}$ before $\varkappa_0$, according to the density of the sequence $\mathcal{K}$. For instance, let us have $\varkappa_{n_0^{**}} \prec \varkappa_0 \prec \varkappa_{n_0^*}$, $\varkappa_{n_0^{**}} \in \mathcal{K}$, $\varkappa_{n_0^*} \in \mathcal{K}$. The relation "the moment μ is before the moment ν, or ν is after μ" will be as usual denoted by $\mu \prec \nu$. Among the instants of $\mathcal{K}$, which are after $\varkappa_0$, there exists an instant $\varkappa_{n'_1} \in \mathcal{K}$, which is with the least number as a member of $\mathcal{K}$, i.e. $\varkappa_{n'_1} \succ \varkappa_0$ and if we have $\varkappa_0 \prec \varkappa_{n^*}, \varkappa_{n^*} \in \mathcal{K}$ then $n^* \geq n'_1$. This follows from the properties of the integers, since the instants $\varkappa_0, \varkappa_1, \dots, \varkappa_{n_0^*}$ are a finite number and among them there exist $\varkappa_{n'_1}$.

Let 10^{u_1} be with the least integer u_1 for which $n'_1 \leq 10^{u_1}$. The moments $\varkappa_0, \varkappa_1, \dots, \varkappa_{10^{u_1}}$ of the sequence $\mathcal{K}$ again are a finite number. Let $\varkappa_{n_1^*} = \varrho_1$ be the last instant among the instants $\varkappa_0, \varkappa_1, \dots, \varkappa_{10^{u_1}}$. Then we have

$$\varkappa_0 = \varrho_0 \prec \varkappa_{n'_1} \preceq \varkappa_{n_1^*} = \varrho_1 .$$

We shall map the integer 1 to ϱ_1, $M(\varrho_1) = 1$.

Respectively, there exists an instant $\varkappa_{n''_1}$ of $\mathcal{K}$ with the least number among the instants of $\mathcal{K}$ which are before ϱ_0. Let 10^{v_1} be with v_1 a positive integer for which $n''_1 \leq 10^{v_1}$ and moreover, let v_1 be the least integer of this kind. Let

$$\varkappa_{n_1^{**}} = \varrho_{-1}$$

be the earliest moment of the instants $\varkappa_0, \varkappa_1, \dots, \varkappa_{10^{v_1}}$. Then we have

$$\varrho_{-1} \preceq \varkappa_{n''_1} \prec \varkappa_0 = \varrho_0 .$$

We shall map the number -1 to ϱ_{-1}, $M(\varrho_{-1}) = -1$.

Recursively, let us have chosen analogously the moments ϱ_m and ϱ_{-m} of $\mathcal{K}$ and have constructed the map $M(\varrho_m) = m$, $M(\varrho_{-m}) = -m$ together with the sequences

$$10^{u_1}, 10^{u_2}, \dots, 10^{u_m}, 1 \leq u_1 < u_2 < \dots < u_m; 10^{v_1}, \dots, 10^{v_m}, 1 \leq v_1 < \dots < v_m,$$

where $u_1, \dots, u_m$ and $v_1, \dots, v_m$ are integers. Then there exist moments of Time $\varkappa_{n'_{m+1}}$ and $\varkappa_{n''_{m+1}}$ of $\mathcal{K}$ which are with

$$\varkappa_{n''_{m+1}} \prec \varrho_{-m} \prec \varrho_m \prec \varkappa_{n'_{m+1}}$$

in accord with the properties of $\mathcal{K}$. Moreover, let $\varkappa_{n''_{m+1}}$ and $\varkappa_{n'_{m+1}}$ be such moments with the least possible numbers as members of $\mathcal{K}$. Then we also have

$$n'_{m+1} > 10^{u_m}, \quad n''_{m+1} > 10^{v_m}.$$

Now let u_{m+1} and v_{m+1} be the least integers with

$$n'_{m+1} \leq 10^{u_{m+1}}, \quad n''_{m+1} \leq 10^{v_{m+1}}.$$

Then we have $u_m < u_{m+1}$, $v_m < v_{m+1}$. Let $\varkappa_{n^*_{m+1}} = \rho_{m+1}$; $\varkappa_{n^{**}_{m+1}} = \rho_{-m-1}$ be the members of $\mathcal{K}$ which correspondingly are the last moment among the finite number of moments of $\mathcal{K}$: $\varkappa_0, \varkappa_1, \dots, \varkappa_{10^{u_{m+1}}}$ (i.e. with numbers in $\mathcal{K}$ not larger than $10^{u_{m+1}}$) for ρ_{m+1} and the earliest moment among the finite number of moments of $\mathcal{K}$:

$\varkappa_0, \varkappa_1, \dots, \varkappa_{10^{v_{m+1}}}$ (i.e. with numbers in $\mathcal{K}$ not larger than $10^{v_{m+1}}$) for ρ_{-m-1}. Thus we have

$$10^{u_m} < n^*_{m+1} \leq 10^{u_{m+1}}, \quad 10^{v_m} < n^{**}_{m+1} \leq 10^{v_{m+1}}.$$

Let us pose $M(\rho_{m+1}) = m+1$, $M(\rho_{-m-1}) = -m-1$. We also have

$$\rho_{-m-1} \prec \rho_{-m} \prec \dots \prec \rho_{-1} \prec \rho_0 \prec \rho_1 \prec \dots \prec \rho_m \prec \rho_{m+1}.$$

Further on, let γ be an arbitrarily fixed instant. There exist instant $\varkappa_p$ and $\varkappa_\tau$ of $\mathcal{K}$ with $\varkappa_p \prec \gamma \prec \varkappa_\tau$, according to the properties of $\mathcal{K}$. We can always choose $u_{m'}$ and $v_{m''}$ with

$$p \leq 10^{v_{m''}}, \quad \tau \leq 10^{u_{m'}}$$

Then we receive $\rho_{-m''} \prec \gamma \prec \rho_{m'}$ in accord with the choice of $\rho_{-m''}$ and $\rho_{m'}$. Moreover γ can be bounded among two successive members of the constructed sequences,

$$\rho_r \preceq \gamma \preceq \rho_{r+1}.$$

This can be shown in the following way: Let l_o and s_o be the least non negative integers with $\varrho_{-s_o} \preceq \gamma \preceq \varrho_{l_o}$. If $l_o \neq 0$, we have

$$\varrho_{l_o-1} \preceq \gamma \preceq \varrho_{l_o}.$$

If $s_o \neq 0$, we have

$$\varrho_{-s_o} \preceq \gamma \preceq \varrho_{-s_o+1}.$$

If $l_o = s_o = 0$, then $\gamma = \varrho_o$. That is why any fixed moment γ is between two successive mumbers of the indicated subsequences of $\mathcal{K}$:

$$\varrho_0 = \varkappa_0, \; \varrho_1 = \varkappa_{n_1^*}, \; \ldots, \; \varrho_m, \; \ldots \quad \text{and}$$

$$\varrho_0 = \varkappa_0, \; \varrho_{-1} = \varkappa_{n_1^{**}}, \; \ldots, \; \varrho_{-m}, \; \ldots .$$

This finishes the proof of Proposition 1, which is a part of the proof of Theorem and will also be used further.We shall use and the following :

Proposition 2. We can map an instant $\varrho_{m/k}$ $(= \varkappa_{n'})$ from the sequence $\mathcal{K}$ to each rational number m/k, which map is denoted by $M(\varrho_{m/k}) = m/k$ We have $\varrho_{m_1/k_1} \prec \varrho_{m_2/k_2}$ if and only if $m_1/k_1 < m_2/k_2$. Here m, k, m_1, m_2, k_1, k_2 are integers, $k, k_1, k_2 \neq 0$.

Proof of Proposition 2. Let us fix an integer k. We have the existence of a moment $\varkappa$ of $\mathcal{K}$. Then there exists a moment $\varkappa_{n^*}$ of $\mathcal{K}$ with the least number n^* and with $\varrho_k \prec \varkappa_{n^*} \prec \varrho_{k+1}$. Let us pose $M(\varkappa_{n^*}) = k + 1/2 = k\frac{1}{2}$, and let us denote $\varkappa_{n^*} = \varrho_{k+1/2}$.

Further on, there exist members of $\mathcal{K}$, $\varkappa_{n_1}$ and $\varkappa_{n_2}$ with the least numbers n_1 and n_2 for which

$$\varrho_k \prec \varkappa_{n_1} \prec \varrho_{k+1/2}, \quad \varrho_{k+1/2} \prec \varkappa_{n_2} \prec \varrho_{k+1}.$$

We shall pose $M(\varkappa_{n_1}) = k + 1/3$, $M(\varkappa_{n_2}) = k + 2/3$, $\varkappa_{n_1} = \varrho_{k+1/3}$, $\varkappa_{n_2} = \varrho_{k+2/3}$.

Let us suppose that we have determined $\varrho_{k+l/q}$ for $l < q$, where l, q were positive integers and l/q was nonreducible, and have posed

$$M(\varrho_{k+l/q}) = k + l/q.$$

Let us suppose that $\rho_{k+p/r} \prec \rho_{k+s/t}$ by construction if and only if p/r < s/t , where p, r , s , t are positive integers, $r \leq q$, $t \leq q$.

Now, let 1/(q+1) ,..., q/(q+1) be nonreducible fractions. Any rational a = k + l/(q+1) , 0 < l < (q+1) , is between two successive rationals of the kind k + p/r < a < k + s/t with $0 \leq p \leq r$, $0 < s \leq t$; p, r , s , t - integers, $0 < r \leq q$, $0 < t \leq q$, such that there is not any rational of that kind between k+p/r and k+ s/t . We have supposed that we had already determined $\rho_{k+p/r}$ and $\rho_{k+s/t}$, $\rho_{k+p/r} \prec \rho_{k+s/t}$. There exist moments $\varkappa$, members of the sequence $\mathcal{K}_0$, which are between them:

$\rho_{k+p/r} \prec \varkappa \prec \rho_{k+s/t}$. Let $\varkappa_{n_0}$ be the member of $\mathcal{K}$ wi with the least number n_0 for which $\rho_{k+p/r} \prec \varkappa_{n_0} \prec \rho_{k+s/t}$. Let us determine $M(\varkappa_{n_0})$ = k + l/(q+1) and let us denote $\varkappa_{n_0} = \rho_{k+l/(q+1)}$

Moreover, then we have $\rho_{k+u/v} \prec \rho_{k+w/z}$ if and only if k+u/v < k+w/z , where u,v, w , z are nonnegative integers, $0 < v \leq q+1$, $0 < z \leq q+1$, $u \leq v$, $w \leq z$. By induction this finishes the proof of Proposition 2, which is a part of the proof of Theorem and will be used further.

Let now γ be an arbitrary chosen moment, which does not coincide with any moment of the kind $\rho_{m/k}$ for any rational m/k , $k \neq 0$. What is the convenient value of $M(\gamma)$?

We have $\rho_m \prec \gamma \prec \rho_{m+1}$ for some fixed integer m in accord with Proposition 1 . Let us denote $\rho_m = \varphi_1$, $\rho_{m+1} = \psi_1$. We compare γ with $\rho_{m+1/2}$. We have either

$$\gamma \prec \rho_{m+1/2} \quad \text{or} \quad \rho_{m+1/2} \prec \gamma .$$

In the first case we shall denote $\rho_m = \varphi_2$, $\rho_{m+1/2} = \psi_2$. In the secon case we shall denote $\rho_{m+1/2} = \varphi_2$, $\rho_{m+1} = \psi_2$.

Moreover, all rational numbers m+l/q , $l \leq q$, l , q -positive integer: between the integers m and m+1 can be ordered in a sequence as usual b: the increasing of the dominator q , where l and q are nonreducible. I.e. we have the sequence m , m+1 , m+1/2 , m+1/3 , m+2/3 , m+1/4 , m+3/4 , m+1/5 , m+2/5 , m+3/5 , m+4/5 , m+1/6 ,...

This sequence determine the corresponding sequence of instants

$$\varrho_m, \varrho_{m+1}, \varrho_{m+1/2}, \varrho_{m+1/3}, \varrho_{m+2/3}, \varrho_{m+1/4}, \dots .$$

Further on, let us suppose that we have determined the instants φ_{k-1} and ψ_{k-1} with $\varphi_{k-1} \prec \gamma \prec \psi_{k-1}$.

Then we establish the order among the instants

$$\gamma, \varrho_m, \varrho_{m+1}, \varrho_{m+1/2}, \dots, \varrho_{m+1/k}, \dots, \varrho_{m+(k-1)/k}.$$

This is, we exactly determine between which two successive instants of the sequence

$$\varrho_m, \varrho_{m+1}, \varrho_{m+1/2}, \dots, \varrho_{m+1/k}, \dots, \varrho_{m+(k-1)/k}$$

is the instant γ . The earlier of these two instants we denote by φ_k and the later instant by ψ_k . Thus we have $\varphi_k \prec \gamma \prec \psi_k$.

In such a way we receive two sequences of instants

(1)

$$\varphi_1 \preceq \varphi_2 \preceq \dots \preceq \varphi_k \preceq \dots \quad \text{and} \quad \psi_1 \succeq \psi_2 \succeq \dots \succeq \psi_k \succeq \dots$$

with the properties $\varphi_k \prec \gamma \prec \psi_k$, $\varphi_k = \varrho_{r_k}$, $\psi_k = \varrho_{p_k}$,

where p_k , r_k are rational numbers, $m \leq r_k < m+1$, $m < p_k \leq m+1$. Moreover, the sequences of rational numbers $\{p_k\}$, $\{r_k\}$ are monotonous.The sequence $\{r_k\}$ is increasing, the sequence $\{p_k\}$ is decreasing. These sequences are also bounded by their construction: $m \leq r_k < p_k \leq m+1$. It follows that these sequences are convergent. Since $p_k - r_k \leq 1/k$ for $\forall k$ hence the both sequences have a same limit. I.e. there exists

$$\lim_{k\to\infty} r_k = g = \lim_{k\to\infty} p_k .$$

We shall map the real number g to the moment of Time γ , $M(\gamma) = g$.

Further on we shall deem that $\varphi_k = \psi_k = \gamma$ for $\forall k$ in the case $\gamma =$

$\rho_{r/s}$, where r/s is a rational number.

Thus we have maped a real number g to any instant γ , $M(\gamma) = g$.

Moreover, let γ and δ be two different instants with $\gamma \prec \delta$. We want to show that $M(\gamma) < M(\delta)$. Let the corresponding monotonous sequences $\{\varphi_r\}$, $\{\psi_r\}$ according to (1) are $\{\varphi_r^*\}$, $\{\psi_r^*\}$ for γ and $\{\varphi_r^{**}\}$, $\{\psi_r^{**}\}$ for δ , with

$$\varphi_r^* \preceq \gamma \preceq \psi_r^* , \quad \varphi_r^{**} \preceq \delta \preceq \psi_r^{**} .$$

Then we have $\varphi_r^* \preceq \gamma \prec \delta \preceq \psi_r^{**}$. I.e. we receive $M(\gamma) \leq M(\delta)$ since

$$\lim_r M(\varphi_r^*) = M(\gamma) , \quad \lim_r M(\psi_r^{**}) = M(\delta).$$

Let us assume that $M(\gamma) = M(\delta)$ for some different instants γ and δ, $\gamma \prec \delta$. This signifies that there does not exist any instant of the kind $\rho_{n+k/l}$ (where k , l , n are integers, $l \neq 0$) between the instants γ and δ . It follows that the instants γ and δ both are between instants of the kind ρ_m and ρ_{m+1} for some fixed integer m . Moreover, the both γ and δ are together not of the kind ρ_r for some rational number r with $M(\rho_r) = r$, according to the construction of the map M . Since $\gamma \prec \delta$,then there exist instants $\varkappa$ of the sequence $\mathcal{K}$ with $\gamma \prec \varkappa \prec \delta$. Moreover, there there is an instant $\varkappa_{p_0}$ of $\mathcal{K}$ with $\gamma \prec \varkappa_{p_0} \prec \delta$, which number p_0 is the least. Therefore we have

$$\varphi_r^* \prec \varkappa_{p_0} \prec \psi_r^{**} \quad \text{for } \forall k .$$

Since the both instants γ and δ do not simultaneously correspond to rational numbers, hence either the sequence $\{\varphi_r^*\}$ or the sequence $\{\psi_r^{**}\}$ has infinitely many different members. These members are also members $\varkappa_{w_j}$ of the sequence $\mathcal{K}$ (according to the construction of the sequences $\{\varphi_r^*\}$ and $\{\psi_r^{**}\}$), chosen to correspond to different rational numbers. Moreover each such instant $\varkappa_{w_j}$ has been chosen with the least number w_j in the sequence $\mathcal{K}$ and also satisfying the corresponding determinative inequalities. Therefore we receive that $\varkappa_{p_0}$ must be equal to some of the members either of the sequence $\{\varphi_r^*\}$ or of the sequence $\{\psi_r^{**}\}$, whereas we have $\varphi^{*}_{k} \prec \mathcal{X}_{p^0} \prec \psi^{**}_{k}$ for $\forall k$. That is why the obtained contradiction proves the impossibility of $M(\gamma) = M(\delta)$. It remains $M(\gamma) \prec M(\delta)$. This finishes the construction of the map M , comparing a real number to each instant and preserving the order.

Now we want to show that there exists an instant γ for any real number a , such that $M(\gamma) = a$. If a is a rational number, we have $\gamma = \rho_a$, in accord with the construction of the map M .

Further, let the number a do not be a rational. The real number a realizes a Dedekind's section $Q_1 \mid Q_2$ of the rational numbers Q , where

$Q_1 = \left\{ q \in Q \text{ , } q \leq a \right\}$ and $Q_2 = \left\{ q \in Q \text{ , } q > a \right\}$.

Let us scrutinize the corresponding sets of instants

$$\mathfrak{M}_1 = \{\alpha : \alpha \in \mathfrak{M},\ M(\alpha) \leq a\} \text{ and } \mathfrak{M}_2 = \{\alpha : \alpha \in \mathfrak{M},\ M(\alpha) > a\}$$

where $\mathfrak{M}$ is the linearly ordered continuum of all instants (see [5]). We shall show that the sets $\mathfrak{M}_1$ and $\mathfrak{M}_2$ realize a Dedekind's section $\mathfrak{M}_1 \mid \mathfrak{M}_2$ of $\mathfrak{M}$. This is true since we evidently have: 1. $\mathfrak{M}_1 \neq \emptyset$, $\mathfrak{M}_2 \neq \emptyset$, because if the rational number r is with $r \leq a$, then $\rho_r \in \mathfrak{M}_1$ if the rational number r is with $r > a$, then $\rho_r \in \mathfrak{M}_2$.

2. If the instant τ is before an instant α of $\mathfrak{M}_1$, $\tau \prec \alpha$, $\alpha \in \mathfrak{M}_1$, then we have $M(\tau) < M(\alpha) \leq a$, i.e. we get $\tau \in \mathfrak{M}_1$; If the instant τ^* is after an instant α^* of $\mathfrak{M}_2$, $\alpha^* \prec \tau^*$, $\alpha^* \in \mathfrak{M}_2$, then $M(\tau^*) > M(\alpha^*) > a$. Therefore $\tau^* \in \mathfrak{M}_2$ in accord with the construction of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, and because the map M preserves the order.

3. We have $\mathfrak{M}_1 \cup \mathfrak{M}_2 = \mathfrak{M}$ since the map M is defined on $\mathfrak{M}$ and after the construction of $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Thus we have for any $\alpha' \in \mathfrak{M}$ that either $M(\alpha') \leq a$ and then $\alpha' \in \mathfrak{M}_1$, or $M(\alpha') > a$ and then $\alpha' \in \mathfrak{M}_2$.

4. We have $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \emptyset$ by the construction and by the properties of the map M .

Therefore we receive that $\mathfrak{M}_1 \mid \mathfrak{M}_2$ is a Dedekind's section of $\mathfrak{M}$.

But $\mathfrak{M}$ is a continuum after [5]. Then there exists an instant $\alpha^0 \in \mathfrak{M}$ dividing $\mathfrak{M}_1$ and $\mathfrak{M}_2$, i.e. if $\mu \prec \alpha^0$, then $\mu \in \mathfrak{M}_1$, if $\nu \succ \alpha^0$, then $\nu \in \mathfrak{M}_2$, where μ and ν are instants.

Moreover, we have $M(\alpha^0) = a$ after (1) and after the construction of

$\mathfrak{M}_1$, $\mathfrak{M}_2$ and the map M . One of the proof is the following: If we assume that $M(\alpha^0) = b < a$, then there exists a rational number r with $b < r < a$. Then the instant $\varrho_r \in \mathfrak{M}_1$,since $M(\varrho_r) = r < a$. Simultaneously we obtain $\varrho_r \in \mathfrak{M}_2$, since $\varrho_r > \alpha^0$ because the map M preserves the order and $M(\alpha^0) = b < r = M(\varrho_r)$. But we have $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \emptyset$. The received contradiction proves the impossibility of $M(\alpha^0) = b < a$.

Let us assume now that $M(\alpha^0) = b > a$. Then there exists a rational number p with $a < p < b$. Therefore the instant ϱ_p must belong to $\mathfrak{M}_1$, since $M(\varrho_p) = p < b = M(\alpha^0)$. Simultaneously we have $\varrho_p \in \mathfrak{M}_2$, since $a < M(\varrho_p) = p$. But we have $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \emptyset$. The obtained contradiction proves the impossibility of $a < b = M(\alpha^0)$. Thus it remains $a = b = M(\alpha^0)$.

This finishes the proof of Theorem on the measurement of Time.

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ON THE DERIVATIVES OF A COMPOSITE FUNCTION

A. S. Madguerova

This article gives the formula of the n-th derivative of composite functions. The previous results are well known to be remind. Professor V.N. Vragov denoted on the Conference of Mathematics and its Applications, Varna, 1989, that the coefficients in this formula had not been determined yet.

Theorem. Let f and φ be defined and infinitely differentiable functions on the intervals Δ and Δ correspondingly, $\Delta \subset R^1$, $\Delta \subset R^1$. Let the values of the function φ on Δ belong to Δ, $\varphi(\Delta) \subset \Delta$. Then the n-th derivative $g^{(n)}$ of the function $g = f\circ\varphi = f(\varphi)$ on Δ has the kind

(1)

$$g^{(n)} = f^{(n)}(\varphi')^n + f^{(n-1)}P_{n-1,n}(\varphi',\varphi'') + f^{(n-2)}P_{n-2,n}(\varphi',\varphi'',\varphi''') + f^{(n-3)}P_{n-3,n}(\varphi',\varphi'',\varphi''',\varphi^{(4)}) + f^{(n-4)}P_{n-4,n}(\varphi',\dots,\varphi^{(5)}) + \dots + f^{(n-k)}P_{n-k,n}(\varphi',\dots,\varphi^{(k)},\varphi^{(k+1)}) + \dots + f''P_{2,n}(\varphi',\dots,\varphi^{(n-1)}) + f'P_{1,n} = \sum_{j=1}^{n} f^{(j)}P_{j,n},$$

where n is a positive integer; $P_{j,n}$ are homogeneous polynomials of degree j. Moreover, we have (i). $P_{1,n} = \varphi^{(n)}$ and $P_{n,n} = (\varphi')^n$; (ii). $P_{n-1,n} = n(n-1)(\varphi')^{n-2}\varphi''/2$ if $n > 1$; (iii). If $(\varphi^{(k_1)})^{l_1}(\varphi^{(k_2)})^{l_2}\dots(\varphi^{(k_p)})^{l_p}$ is an item in some $P_{s,n}$, then we hav $k_1 l_1 + k_2 l_2 + \dots + k_p l_p = n$ and $l_1 + l_2 + \dots + l_p = s$;

(iv). $P_{n-k,n+1} = (P_{n-k,n})' + \varphi' P_{n-k-1,n}$, where $n \geq 3$, $n \geq k+1$;

(v). The numerical coefficient befor the item $f^{(t)}(\varphi')^{n-t}(\varphi'')^{\alpha_2}(\varphi''')^{\alpha_3}\dots(\varphi^{(p)})^{\alpha_p}$ in $g^{(n)}$ is $\dfrac{n(n-1)\dots(n-t+1)}{(2!)^{\alpha_2}\alpha_2!(3!)^{\alpha_3}\alpha_3!\dots(p!)^{\alpha_p}\alpha_p!}$, if $\alpha_2 > 0, \alpha_3 > 0, \dots, \alpha_p > 0$, $n-t \geq 0$;

(vi). We have

$$P_{n-k,n} = \frac{n(n-1)\dots(n-2k+1)}{(2!)^k k!}(\varphi')^{n-2k}(\varphi'')^k + \frac{n(n-1)\dots(n-2k+2)}{(2!)^{k-2}(k-2)!\,3!}\cdot(\varphi')^{n-2k+1}(\varphi'')^{k-2}\varphi''' + \frac{n(n-1)\dots(n-2k+3)}{(2!)^{k-4}(k-4)!(3!)^2 2!}(\varphi')^{n-2k+2}(\varphi'')^{k-4}(\varphi''')^2 +$$

$$+ \frac{n(n-1)\dots(n-2k+3)}{(2!)^{k-3}(k-3)!\,4!}(\varphi')^{n-2k+2}(\varphi'')^{k-3}\varphi^{(4)} + \frac{n(n-1)\dots(n-2k+4)}{(2!)^{k-6}(k-6)!(3!)^3 3!}(\varphi')^{n-2k+3}(\varphi'')^{k-6}(\varphi''')^3 +$$

$$+ \frac{n(n-1)\dots(n-2k+4)}{(2!)^{k-5}(k-5)!\,3!\,4!}(\varphi')^{n-2k+3}(\varphi'')^{k-5}(\varphi''')\varphi^{(4)} + \frac{n(n-1)\dots(n-2k+4)}{(2!)^{k-4}(k-4)!\,5!}(\varphi')^{n-2k+3}(\varphi'')^{k-4}\varphi^{(5)}$$

$$+ \frac{n(n-1)\dots(n-2k+5)}{(2!)^{k-8}(k-8)!(3!)^4 4!}(\varphi')^{n-2k+4}(\varphi'')^{k-8}(\varphi''')^4 +$$

$$\frac{(n-1)\dots(n-2r+5)}{2^{r-7}(r-7)!(3!)^2 4!}(\varphi')^{n-2r+4}(\varphi'')^{r-7}(\varphi''')^2\varphi^{(4)} + \frac{n(n-1)\dots(n-2r+5)}{2^{r-6}(r-6)!(4!)^2 2!}(\varphi')^{n-2r+4}(\varphi'')^{r-6}(\varphi^{(4)})^2 +$$

$$\frac{(n-1)\dots(n-2r+5)}{2^{r-6}(r-6)!3!5!}(\varphi')^{n-2r+4}(\varphi'')^{r-6}\varphi'''\varphi^{(5)} + \frac{n(n-1)\dots(n-2r+5)}{2^{r-5}(r-5)!6!}(\varphi')^{n-2r+4}(\varphi'')^{r-5}\varphi^{(6)}$$

$$\frac{(n-1)\dots(n-2r+6)}{2^{r-10}(r-10)!(3!)^5 5!}(\varphi')^{n-2r+5}(\varphi'')^{r-10}(\varphi''')^5 + \frac{n(n-1)\dots(n-2r+6)}{2^{r-9}(r-9)!(3!)^3 3!4!}(\varphi')^{n-2r+5}(\varphi'')^{r-9}(\varphi''')^3\varphi^{(4)} +$$

$$\frac{(n-1)\dots(n-2r+6)}{2^{r-8}(r-8)!3!(4!)^2 2!}(\varphi')^{n-2r+5}(\varphi'')^{r-8}\varphi'''(\varphi^{(4)})^2 + \frac{n(n-1)\dots(n-2r+6)}{2^{r-8}(r-8)!(3!)^2 5!2!}(\varphi')^{n-2r+5}(\varphi'')^{r-8}(\varphi''')^2\varphi^{(5)}$$

$$\frac{(n-1)\dots(n-2r+6)}{2^{r-7}(r-7)!4!5!}(\varphi')^{n-2r+5}(\varphi'')^{r-7}\varphi^{(4)}\varphi^{(5)} + \frac{n(n-1)\dots(n-2r+6)}{2^{r-7}(r-7)!3!6!}(\varphi')^{n-2r+5}(\varphi'')^{r-7}\varphi'''\varphi^{(6)} +$$

$$\frac{(n-1)\dots(n-2r+6)}{2^{r-6}(r-6)!7!}(\varphi')^{n-2r+5}(\varphi'')^{r-6}\varphi^{(7)} + \frac{n(n-1)\dots(n-2r+7)}{2^{r-12}(r-12)!(3!)^6 6!}(\varphi')^{n-2r+6}(\varphi'')^{r-12}(\varphi''')^6 +$$

$$\frac{(n-1)\dots(n-2r+7)}{(r-11)!(3!)^4 4!4!}(\varphi')^{n-2r+6}(\varphi'')^{r-11}(\varphi''')^4\varphi^{(4)} + \frac{n(n-1)\dots(n-2r+7)}{2^{r-10}(r-10)!(3!)^2 2!(4!)^2 2!}(\varphi')^{n-2r+6}(\varphi'')^{r-10}(\varphi''')^2(\varphi^{(4)})^2 +$$

$$\frac{(n-1)\dots(n-2r+7)(\varphi')^{n-2r+6}}{2^{r-10}(r-10)!(3!)^3 3!5!}(\varphi'')^{r-10}(\varphi''')^3\varphi^{(5)} + \frac{n(n-1)\dots(n-2r+7)}{2^{r-9}(r-9)!(4!)^3 3!}(\varphi')^{n-2r+6}(\varphi'')^{r-9}(\varphi^{(4)})^3 +$$

$$\frac{(n-1)\dots(n-2r+7)}{2^{r-9}(r-9)!3!4!5!}(\varphi')^{n-2r+6}(\varphi'')^{r-9}\varphi'''\varphi^{(4)}\varphi^{(5)} + \frac{n(n-1)\dots(n-2r+7)}{2^{r-9}(r-9)!(3!)^2 2!6!}(\varphi')^{n-2r+6}(\varphi'')^{r-9}(\varphi''')^2\varphi^{(6)} +$$

$$\frac{(n-1)\dots(n-2r+7)}{(r-8)!(5!)^2 2!}(\varphi')^{n-2r+6}(\varphi'')^{r-8}(\varphi^{(5)})^2 + \frac{n(n-1)\dots(n-2r+7)}{2^{r-8}(r-8)!4!6!}(\varphi')^{n-2r+6}(\varphi'')^{r-8}\varphi^{(4)}\varphi^{(6)} +$$

$$\frac{(n-1)\dots n-2r+7}{2^{r-8}(r-8)!3!7!}(\varphi')^{n-2r+6}(\varphi'')^{r-8}\varphi'''\varphi^{(7)} + \frac{n(n-1)\dots(n-2r+7)}{2^{r-7}(r-7)!8!}(\varphi')^{n-2r+6}(\varphi'')^{r-7}\varphi^{(8)} + \dots +$$

$$\frac{(n-1)\dots(n-2r+\tau+1)}{2^{\tau}(r-2\tau)!(3!)^{\tau}\tau!}(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau}(\varphi''')^{\tau} + \frac{n(n-1)\dots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}}{2^{r-2\tau+1}(r-2\tau+1)!(3!)^{\tau-2}(\tau-2)!4!}(\varphi'')^{r-2\tau+1}(\varphi''')^{\tau-2}\varphi^{(4)}$$

$$\frac{(n-1)\dots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-3}\varphi^{(5)}}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-3}(\tau-3)!5!} +$$

$$\frac{(n-1)\dots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-4}(\tau-4)!(4!)^2 2}(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-4}(\varphi^{(4)})^2 + \dots +$$

$$\frac{(n-1)\dots(n-2r+\tau+1)}{2^{r-2\tau+q}(r-2\tau+q)!(3!)^{l_3}(4!)^{l_4}\dots((q+3)!)^{l_{q+3}}\,l_3!\dots}(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+q}(\varphi''')^{l_3}(\varphi^{(4)})^{l_4}\dots(\varphi^{(q+3)})^{l_{q+3}} + \dots +$$

$\frac{n(n-1)\dots(n-2k+r+1)(\varphi')^{n-2k+r}}{2^{k-r-2}(k-r-2)!\,(p+4)!\,(r-p)!}(\varphi'')^{k-r-2}\varphi^{(p+4)}\varphi^{(r-p)}$ (with $-2 \le p \le \frac{r-4}{2}$) $+ \dots +$

$\frac{n(n-1)\dots(n-2k+r+1)(\varphi')^{n-2k+r}}{2^{k-r-2}(k-r-2)!\,7!\,(r-3)!}(\varphi'')^{k-r-2}\varphi^{(7)}\varphi^{(r-3)} + \frac{n(n-1)\dots(n-2k+r+1)(\varphi')^{n-2k+r}}{2^{k-r-2}(k-r-2)!\,6!\,(r-2)!}(\varphi'')^{k-r-2}\varphi^{(6)}\varphi^{(r-2)}$

$\frac{n(n-1)\dots(n-2k+r+1)(\varphi')^{n-2k+r}}{2^{k-r-2}(k-r-2)!\,5!\,(r-1)!}(\varphi'')^{k-r-2}\varphi^{(5)}\varphi^{(r-1)} + \frac{n(n-1)\dots(n-2k+r+1)}{2^{k-r-2}(k-r-2)!\,4!\,r!}(\varphi')^{n-2k+r}(\varphi'')^{k-r-2}\varphi^{(4)}\varphi^{(r)}$

$\frac{n(n-1)\dots(n-2k+r+1)(\varphi')^{n-2k+r}}{2^{k-r-2}(k-r-2)!\,3!\,(r+1)!}(\varphi'')^{k-r-2}\varphi^{(3)}\varphi^{(r+1)} + \frac{n(n-1)\dots(n-2k+r+1)}{2^{k-r-1}(k-r-1)!\,(r+2)!}(\varphi')^{n-2k+r}(\varphi'')^{k-r-1}\varphi^{(r+2)} +$

$+\frac{n(n-1)\dots(n-k-2)}{a!\,(p^*+3-a)!\,(k-p^*)!}(\varphi')^{n-k-3}\varphi^{(a)}\varphi^{(p^*+3-a)}\varphi^{(k-p^*)} + \dots + \frac{n(n-1)\dots(n-k-2)}{2!\,(p^{**}+1)!\,(k-p^{**})!}(\varphi')^{n-k-3}\varphi''\varphi^{(p^{**}+1)}\varphi^{(k-p^{**})} + \dots +$

$+\frac{n(n-1)\dots(n-k-2)}{3!\,4!\,(k-4)!}(\varphi')^{n-k-3}\varphi'''\varphi^{(4)}\varphi^{(k-4)} + \frac{n(n-1)\dots(n-k-2)}{2!\,5!\,(k-4)!}(\varphi')^{n-k-3}\varphi''\varphi^{(5)}\varphi^{(k-4)} +$

$\frac{n(n-1)\dots(n-k-2)}{(3!)^2\,2!\,(k-3)!}(\varphi')^{n-k-3}(\varphi''')^2\varphi^{(k-3)} + \frac{n(n-1)\dots(n-k-2)}{2!\,4!\,(k-3)!}(\varphi')^{n-k-3}\varphi''\varphi^{(4)}\varphi^{(k-3)} +$

$\frac{n(n-1)\dots(n-k-2)}{2!\,3!\,(k-2)!}(\varphi')^{n-k-3}\varphi''.\varphi'''.\varphi^{(k-2)} + \frac{n(n-1)\dots(n-k-2)}{2^2 2\,(k-1)!}(\varphi')^{n-k-3}(\varphi'')^2\varphi^{(k-1)} +$

$n(n-1)\dots(n-k-1)(\varphi')^{n-k-2}\Big[\frac{1}{(\hat p+2)!\,(k-\hat p)!}\varphi^{(\hat p+2)}\varphi^{(k-\hat p)} + \dots + \frac{1}{5!\,(k-3)!}\varphi^{(5)}\varphi^{(k-3)} + \frac{\varphi^{(4)}\varphi^{(k-2)}}{4!\,(k-2)!} +$

$\frac{\varphi'''\varphi^{(k-1)}}{3!\,(k-1)!} + \frac{\varphi''\varphi^{(k)}}{2!\,k!}\Big] + \frac{n(n-1)\dots(n-k)}{(k+1)!}(\varphi')^{n-k-1}\varphi^{(k+1)}$. Here $n-2k+r \geqslant 0$; $k-2r+q \geqslant 0$; $0 \le q \le r-1$. If either $n-2k+r < 0$ or $k-2r+q < 0$, then the corresponding item vanishes! As usual we deem $0! = 1$. The integers k, r, q are nonnegative. Here l_3, l_4, l_{q+3}, a, p, p^*, p^{**}, $\hat p$ are integers satisfying $l_3 \geqslant 0, \dots, l_{q+3} \geqslant 0$, $l_3 \leqslant r-q-1$; if $s \geqslant 4$, then $l_s \leqslant q-s+4$; $l_3 + \dots + l_{q+3} = r-q$; $3l_3 + \dots + (q+3)l_{q+3} = 3r-2q$; $2 \leqslant p+4 \leqslant r-p$; $2 \le a \le p^* + 3 - a \leqslant k-p^*$; $2 \leqslant p^{**} + 1 \le k - p^{**}$; $2 \le \hat p + 2 \le k - \hat p$.

For instance we have if $n=2w$ after this formula that

$P_{2,n} = P_{2,2w} = \frac{n!}{2!(n-2)!}\varphi''\varphi^{(n-2)} + \frac{n!}{3!(n-3)!}\varphi'''\varphi^{(n-3)} + \frac{n!}{4!(n-4)!}\varphi^{(4)}\varphi^{(n-4)} + \dots + \frac{n!}{(w!)^2 2}(\varphi^{(w)})^2 +$ $+ n\varphi'\varphi^{(n-1)}$. If $n = 2w+1$, then $P_{2,n} = \frac{n!}{2!(n-2)!}\varphi''\varphi^{(n-2)} + \frac{n!}{3!(n-3)!}\varphi'''\varphi^{(n-3)} + \dots +$ $\frac{n!}{w!(w+1)!}\varphi^{(w)}\varphi^{(w+1)} + n\varphi'\varphi^{(n-1)}$. Also we have after (vi) that

$$P_{3,n}=\frac{n!\,(\varphi'')^2}{2^2 2\,(n-4)!}\varphi^{(n-4)}+\frac{n!\,\varphi''\varphi'''}{2!\,3!\,(n-5)!}\varphi^{(n-5)}+\frac{n!\,\varphi''\varphi^{(4)}}{2!\,4!\,(n-6)!}\varphi^{(n-6)}+\frac{n!\,(\varphi''')^2\varphi^{(n-6)}}{(3!)^2\,2!\,(n-6)!}+$$
$$\frac{n!\,\varphi''\varphi^{(5)}}{2!\,5!\,(n-7)!}\varphi^{(n-7)}+\frac{n!\,\varphi'''\varphi^{(4)}}{3!\,4!\,(n-7)!}\varphi^{(n-7)}+\dots+\frac{n!\,\varphi''\varphi^{(u+1)}}{2!\,(u+1)!\,(n-3-u)!}\varphi^{(n-3-u)}+\dots+\frac{n!\,\varphi^{(a)}\varphi^{(p^++3-a)}}{a!\,(p^++3-a)!\,(n-3-p^+)!}\cdot$$
$$\cdot\varphi^{(n-3-p)}+n!\,\varphi'\left[\frac{\varphi''\varphi^{(n-3)}}{2!\,(n-3)!}+\frac{\varphi'''\varphi^{(n-4)}}{3!\,(n-4)!}+\frac{\varphi^{(4)}\varphi^{(n-5)}}{4!\,(n-5)!}+\dots+\right.$$
$$\left.+\frac{\varphi^{(z+2)}\varphi^{(n-3-z)}}{(z+2)!\,(n-3-z)!}\right]+\frac{n(n-1)}{2}(\varphi')^2\varphi^{(n-2)},$$

the integers u,a, p^+ ,z satisfy the inequalities:
$2\le u+1\le n-3-u$; $2\le a\le p^++3-a\le n-3-p^+$; $2\le z+2\le n-3-z$. If the written of some derivative is negative, then the corresponding item vanishes.

Proof.of Points (i)-(iv).and the formula (1). If n=1, $g'=f'.\varphi'$;if n=2, $g''=f''.(\varphi')^2+f'.\varphi''$ acording to the formula (1) and the points (i)-(iii If n=3, then $g'''=f'''.(\varphi')^3+f''.\varphi'.\varphi''+f'.\varphi'''$, according to the formula (1) and the points (i)-(iv). Now let us suppose that the formula (1) together with the points (i)-(iv) are true for an arbitrarily fixed positive integer n . Then it follows from (1) and the points (i)-(iv), that

$$g^{(n+1)}=f^{(n+1)}(\varphi')^{n+1}+f^{(n)}\left\{\left[(\varphi')^n\right]'+\varphi'\left[\frac{n(n-1)}{2}(\varphi')^{n-2}\varphi''\right]\right\}+$$
$$+f^{(n-1)}\left[\frac{n(n-1)(n-2)}{2}(\varphi')^{n-3}(\varphi'')^2+\frac{n(n-1)}{2}(\varphi')^{n-2}\varphi'''+\varphi' P_{n-2,n}(\varphi',\varphi'',\varphi''')\right]+$$
$$f^{(n-2)}\left[P'_{n-2,n}(\varphi',\varphi'',\varphi''')+\varphi' P_{n-3,n}(\varphi',\varphi'',\varphi''',\varphi^{(4)})\right]+f^{(n-3)}\left[P'_{n-3,n}(\varphi',\dots,\varphi^{(4)})+\varphi' P_{n-4,n}(\varphi',\dots,\varphi^{(5)})\right]$$
$$+f^{(n-r+2)}\left[P'_{n-r+2,n}(\varphi',\dots,\varphi^{(r-1)})+\varphi' P_{n-r+1,n}(\varphi',\dots,\varphi^{(r)})\right]+$$
$$+f^{(n-r+1)}\left[P'_{n-r+1,n}(\varphi',\dots,\varphi^{(r)})+\varphi' P_{n-r,n}(\varphi',\dots,\varphi^{(r+1)})\right]+$$
$$+f^{(n-r)}\left[P'_{n-r,n}(\varphi',\dots,\varphi^{(r+1)})+\varphi' P_{n-r-1,n}(\varphi',\dots,\varphi^{(r+2)})\right]+$$
$$+f^{(n-r-1)}\left[P'_{n-r-1,n}(\varphi',\dots,\varphi^{(r+2)})+\varphi' P_{n-r-2,n}(\varphi',\dots,\varphi^{(r+3)})\right]+\dots+$$
$$+f''\left[P'_{2,n}(\varphi',\dots,\varphi^{(n-1)})+\varphi'\varphi^{(n)}\right]+f'\varphi^{(n+1)}=f^{(n+1)}(\varphi')^{n+1}+\frac{n(n+1)}{2}f^{(n)}(\varphi')^{n-1}\varphi''+$$
$$f^{(n-1)}P_{n-1,n+1}(\varphi',\varphi'',\varphi''')+\dots+f^{(n-r+1)}P_{n-r+1,n+1}(\varphi',\dots,\varphi^{(r+1)})+$$
$$f^{(n-r)}P_{n-r,n+1}(\varphi',\dots,\varphi^{(r+2)})+\dots+f''P_{2,n+1}(\varphi',\varphi'',\dots,\varphi^{(n)})+f'\varphi^{(n+1)}.$$

Thus we receive that the derivative $g^{(n+1)}$ has the kind (1) and the properties (i)-(iv). Therefore we inductively the proof of formula (1), including the points (i)-(iv). Further we shall use the following Proposition, proving the points (v) and (vi) for k=2 and $n \geq 3$:

Proposition. We have $P_{n-2,n} = \frac{n(n-1)(n-2)(n-3)}{2^2.2}(\varphi')^{n-4}(\varphi'')^2 + \frac{n(n-1)(n-2)}{3!}(\varphi')^{n-3}\varphi'''$ $n \geq 3$.

Proof of Proposition. We shall proceed inductively. The assertion is true for n=4 since we have $g^{(4)} = f^{(4)}(\varphi')^4 + 6f'''(\varphi')^2\varphi'' + f''[3(\varphi'')^2 + 4\varphi'\varphi'''] + f'\varphi^{(4)}$ (as well for $n=3$)

Let us suppose that the assertion od Proposition is tue for an arbitrarily fixed integer $n \geq 4$. Let us try to involve the corresponding formula for n+1. We receive that the coefficients before the members with the derivative $f^{(n-1)}$ in the expression of $g^{(n+1)}$ have the following kind after the proved point (iv) of the Theorem: $P_{n-1,n+1}(\varphi',\varphi'',\varphi''') = \left[\frac{n(n-1)}{2}(\varphi')^{n-2}\varphi''\right]' + \varphi' P_{n-2,n} =$

$$= (\varphi')^{n-3}(\varphi'')^2\left[\frac{n(n-1)(n-2)}{2} + \frac{n(n-1)(n-2)(n-3)}{8}\right] + (\varphi')^{n-2}\varphi'''\left[\frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6}\right] =$$

$$= (\varphi')^{n-3}(\varphi'')^2\frac{(n+1)n(n-1)(n-2)}{2^2.2} + (\varphi')^{n-2}\varphi'''\frac{(n+1)n(n-1)}{3!}.$$ This is, we get the assertion of the Proposition.for $g^{(n+1)}$. Therefore the Proposition is true by Induction.

Proof of Points (v) and (vi) of Theorem. These Points are true for k=2,1 $\forall$ $n \geq 3$, after the Proposition and the point (ii). Let us suppose that the pointspoints (v) and (vi) are true for some integer k for any $n > k$. We want to prove (v) and (vi) for k+1 and $\forall$ $n > (k+1)$. Moreover let us suppose that the points (v),(vi) are true for k+1 for some fixed integer $n > k+1$. Then we have $P_{n-r,n+1} = P'_{n-r,n} + \varphi' P_{n-r-1,n} =$

$$(\varphi')^{n-2r-1}(\varphi'')^{r+1}\left[(n-2r)\frac{n(n-1)\ldots(n-2r+1)}{2^r\, r!} + \frac{n(n-1)\ldots(n-2r-1)}{2^{r+1}(r+1)!}\right] +$$

$$(\varphi')^{n-2r}(\varphi'')^{r-1}\varphi'''\left[r\frac{n(n-1)\ldots(n-2r+1)}{2^r\, r!} + (n-2r+1)\frac{n(n-1)\ldots(n-2r+2)}{2^{r-2}(r-2)!\,3!} + \frac{n(n-1)\ldots(n-2r)}{2^{r-1}(r-1)!\,3!}\right] +$$

$$\varphi')^{n-2r+1}(\varphi'')^{r-3}(\varphi''')^2\left[(r-2)\frac{n(n-1)\ldots(n-2r+2)}{2^{r-2}(r-2)!\,3!} + \frac{n(n-1)\ldots(n-2r+1)}{2^{r-3}(r-3)!\,(3!)^2 2} + (n-2r+2)\frac{n(n-1)\ldots(n-2r+3)}{2^{r-4}(r-4)!\,(3!)^2 2}\right] +$$

$$\varphi')^{n-2r+1}(\varphi'')^{r-2}\varphi^{(4)}\left[\frac{n(n-1)\ldots(n-2r+2)}{2^{r-2}(r-2)!\,3!} + (n-2r+2)\frac{n(n-1)\ldots(n-2r+3)}{2^{r-3}(r-3)!\,4!} + \frac{n(n-1)\ldots(n-2r+1)}{2^{r-2}(r-2)!\,4!}\right] +$$

$$+(\varphi')^{n-2k+2}(\varphi'')^{k-5}(\varphi''')^{3}\left[(k-4)\frac{n(n-1)\dots(n-2k+3)}{2^{k-4}(k-4)!(3!)^{2}2}+\frac{n(n-1)\dots(n-2k+2)}{2^{k-5}(k-5)!(3!)^{3}3!}+(n-2k+3)\frac{n(n-1)\dots(n-2k+4)}{2^{k-6}(k-6)!(3!)^{4}}\right]+$$

$$+(\varphi')^{n-2k+2}(\varphi'')^{k-4}\varphi'''\varphi^{(4)}\left[2\frac{n(n-1)\dots(n-2k+3)}{2^{k-4}(k-4)!(3!)^{2}2}+(n-2k+3)\frac{n(n-1)\dots(n-2k+4)}{2^{k-5}(k-5)!3!4!}+(k-3)\frac{n(n-1)\dots(n-2k+3)}{2^{k-3}(k-3)!4!}+\right.$$

$$\left.+\frac{n(n-1)\dots(n-2k+2)}{2^{k-4}(k-4)!3!4!}\right]+$$

$$+(\varphi')^{n-2k+2}(\varphi'')^{k-3}\varphi^{(5)}\left[\frac{n(n-1)\dots(n-2k+3)}{2^{k-3}(k-3)!4!}+(n-2k+3)\frac{n(n-1)\dots(n-2k+4)}{2^{k-4}(k-4)!5!}+\frac{n(n-1)\dots(n-2k+2)}{2^{k-3}(k-3)!5!}\right]+$$

$$+(\varphi')^{n-2k+3}(\varphi'')^{k-7}(\varphi''')^{4}\left[(k-6)\frac{n(n-1)\dots(n-2k+4)}{2^{k-6}(k-6)!(3!)^{3}3!}+(n-2k+4)\frac{n(n-1)\dots(n-2k+5)}{2^{k-8}(k-8)!(3!)^{4}4!}+\frac{n(n-1)\dots(n-2k+3)}{2^{k-7}(k-7)!(3!)^{4}4!}\right]+$$

$$+(\varphi')^{n-2k+3}(\varphi'')^{k-6}(\varphi''')^{2}\varphi^{(4)}\left[3\cdot\frac{n(n-1)\dots(n-2k+4)}{2^{k-6}(k-6)!(3!)^{3}3!}+(k-5)\frac{n(n-1)\dots(n-2k+4)}{2^{k-5}(k-5)!3!4!}+\right.$$

$$\left.(n-2k+4)\cdot\frac{n(n-1)\dots(n-2k+5)}{2^{k-7}(k-7)!(3!)^{2}2.4!}+\frac{n(n-1)\dots(n-2k+3)}{2^{k-6}(k-6)!(3!)^{2}2.4!}\right]$$

$$+(\varphi')^{n-2k+3}(\varphi'')^{k-5}(\varphi^{(4)})^{2}\left[\frac{n(n-1)\dots(n-2k+4)}{2^{k-5}(k-5)!3!4!}+(n-2k+4)\frac{n(n-1)\dots(n-2k+5)}{2^{k-6}(k-6)!(4!)^{2}2}+\frac{n(n-1)\dots(n-2k+3)}{2^{k-5}(k-5)!(4!)^{2}2}\right]$$

$$+(\varphi')^{n-2k+3}(\varphi'')^{k-5}\varphi'''\varphi^{(5)}\left[\frac{n(n-1)\dots(n-2k+4)}{2^{k-5}(k-5)!3!4!}+(k-4)\frac{n(n-1)\dots(n-2k+4)}{2^{k-4}(k-4)!5!}+\right.$$

$$\left.(n-2k+4)\frac{n(n-1)\dots(n-2k+5)}{2^{k-6}(k-6)!3!5!}+\frac{n(n-1)\dots(n-2k+3)}{2^{k-5}(k-5)!3!5!}\right]+$$

$$+(\varphi')^{n-2k+3}(\varphi'')^{k-4}\varphi^{(6)}\left[\frac{n(n-1)\dots(n-2k+4)}{2^{k-4}(k-4)!5!}+(n-2k+4)\frac{n(n-1)\dots(n-2k+5)}{2^{k-5}(k-5)!6!}+\frac{n(n-1)\dots(n-2k+3)}{2^{k-4}(k-4)!6!}\right]+$$

$$+(\varphi')^{n-2k+4}(\varphi'')^{k-9}(\varphi''')^{5}\left[(k-8)\frac{n(n-1)\dots(n-2k+5)}{2^{k-8}(k-8)!(3!)^{4}4!}+(n-2k+5)\frac{n(n-1)\dots(n-2k+6)}{2^{k-10}(k-10)!(3!)^{5}5!}+\frac{n(n-1)\dots(n-2k+4)}{2^{k-9}(k-9)!(3!)^{5}5!}\right]+$$

$$+(\varphi')^{n-2k+4}(\varphi'')^{k-8}(\varphi''')^{3}\varphi^{(4)}\left[4\frac{n(n-1)\dots(n-2k+5)}{2^{k-8}(k-8)!(3!)^{4}4!}+(k-7)\frac{n(n-1)\dots(n-2k+5)}{2^{k-7}(k-7)!(3!)^{2}2.4!}+\right.$$

$$\left.(n-2k+5)\frac{n(n-1)\dots(n-2k+6)}{2^{k-9}(k-9)!(3!)^{3}3!4!}+\frac{n(n-1)\dots(n-2k+4)}{2^{k-8}(k-8)!(3!)^{3}3!4!}\right]$$

$$+(\varphi')^{n-2k+4}(\varphi'')^{k-7}\varphi'''(\varphi^{(4)})^{2}\left[2\frac{n(n-1)\dots(n-2k+5)}{2^{k-7}(k-7)!(3!)^{2}2.4!}+(k-6)\frac{n(n-1)\dots(n-2k+5)}{2^{k-6}(k-6)!(4!)^{2}2}+\right.$$

$$\left.(n-2k+5)\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!3!(4!)^{2}2}+\frac{n(n-1)\dots(n-2k+4)}{2^{k-7}(k-7)!3!(4!)^{2}2}\right]+$$

$$+(\varphi')^{n-2k+4}(\varphi'')^{k-7}(\varphi''')^{2}\varphi^{(5)}\left[\frac{n(n-1)\dots(n-2k+5)}{2^{k-7}(k-7)!(3!)^{2}2.4!}+(k-6)\frac{n(n-1)\dots(n-2k+5)}{2^{k-6}(k-6)!3!5!}+\right.$$

$$\left.(n-2k+5)\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!(3!)^{2}2.5!}+\frac{n(n-1)\dots(n-2k+4)}{2^{k-7}(k-7)!(3!)^{2}2.5!}\right]+$$

$$+(\varphi')^{n-2k+4}(\varphi'')^{k-6}\varphi^{(4)}\varphi^{(5)}\Big[2\frac{n(n-1)\dots(n-2k+5)}{2^{k-6}(k-6)!(4!)^2 2}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-6}(k-6)!\,3!\,5!}+$$

$$+(n-2k+5)\frac{n(n-1)\dots(n-2k+6)}{2^{k-7}(k-7)!\,4!\,5!}+\frac{n(n-1)\dots(n-2k+4)}{2^{k-6}(k-6)!\,4!\,5!}\Big]+$$

$$+(\varphi')^{n-2k+4}(\varphi'')^{k-6}\varphi'''\varphi^{(6)}\Big[\frac{n(n-1)\dots(n-2k+5)}{2^{k-6}(k-6)!\,3!\,5!}+(k-5)\frac{n(n-1)\dots(n-2k+5)}{2^{k-5}(k-5)!\,6!}+$$

$$+(n-2k+5)\frac{n(n-1)\dots(n-2k+6)}{2^{k-7}(k-7)!\,3!\,6!}+\frac{n(n-1)\dots(n-2k+4)}{2^{k-6}(k-6)!\,3!\,6!}\Big]+$$

$$+(\varphi')^{n-2k+4}(\varphi'')^{k-5}\varphi^{(7)}\Big[\frac{n(n-1)\dots(n-2k+5)}{2^{k-5}(k-5)!\,6!}+(n-2k+5)\frac{n(n-1)\dots(n-2k+6)}{2^{k-6}(k-6)!\,7!}+\frac{n(n-1)\dots(n-2k+4)}{2^{k-5}(k-5)!\,7!}\Big]$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{k-11}(\varphi''')^{6}\Big[(k-10)\frac{n(n-1)\dots(n-2k+6)}{2^{k-10}(k-10)!(3!)^5 5!}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-11}(k-11)!(3!)^6 6!}+$$

$$+(n-2k+6)\frac{n(n-1)\dots(n-2k+7)}{2^{k-12}(k-12)!(3!)^6 6!}\Big]+$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{k-10}(\varphi''')^{4}\varphi^{(4)}\Big[5\frac{n(n-1)\dots(n-2k+6)}{2^{k-10}(k-10)!(3!)^5 5!}+(k-9)\frac{n(n-1)\dots(n-2k+6)}{2^{k-9}(k-9)!(3!)^3 3!\,4!}+$$

$$+(n-2k+6)\frac{n(n-1)\dots(n-2k+7)}{2^{k-11}(k-11)!(3!)^4 4!\,4!}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-10}(k-10)!(3!)^4 4!\,4!}\Big]+$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{k-9}(\varphi''')^{2}(\varphi^{(4)})^2\Big[3\frac{n(n-1)\dots(n-2k+6)}{2^{k-9}(k-9)!(3!)^3 3!\,4!}+(k-8)\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!\,3!(4!)^2 2}+$$

$$+(n-2k+6)\frac{n(n-1)\dots(n-2k+7)}{2^{k-10}(k-10)!(3!)^2 2(4!)^2 2}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-9}(k-9)!(3!)^2 2(4!)^2 2}\Big]+$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{k-9}(\varphi''')^{3}\varphi^{(5)}\Big[\frac{n(n-1)\dots(n-2k+6)}{2^{k-9}(k-9)!(3!)^3 3!\,4!}+(k-8)\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!(3!)^2 2.5!}+$$

$$+(n-2k+6)\frac{n(n-1)\dots(n-2k+7)}{2^{k-10}(k-10)!(3!)^3 3!\,5!}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-9}(k-9)!(3!)^3 3!\,5!}\Big]+$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{k-8}(\varphi^{(4)})^{3}\Big[\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!\,3!(4!)^2 2}+(n-2k+6)\frac{n(n-1)\dots(n-2k+7)}{2^{k-9}(k-9)!(4!)^3 3!}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-8}(k-8)!(4!)^3 3!}\Big]+$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{k-8}\varphi'''\varphi^{(4)}\varphi^{(5)}\Big[2\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!\,3!(4!)^2 2}+2\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!(3!)^2 2.5!}+(k-7)\frac{n(n-1)\dots(n-2k+6)}{2^{k-7}(k-7)!\,4!\,5!}+$$

$$+(n-2k+6)\frac{n(n-1)\dots(n-2k+7)}{2^{k-9}(k-9)!\,3!\,4!\,5!}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-8}(k-8)!\,3!\,4!\,5!}\Big]+$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{k-8}(\varphi''')^{2}\varphi^{(6)}\Big[\frac{n(n-1)\dots(n-2k+6)}{2^{k-8}(k-8)!(3!)^2 2.5!}+(k-7)\frac{n(n-1)\dots(n-2k+6)}{2^{k-7}(k-7)!\,3!\,6!}+$$

$$+(n-2k+6)\frac{n(n-1)\dots(n-2k+7)}{2^{k-9}(k-9)!(3!)^2 2.6!}+\frac{n(n-1)\dots(n-2k+5)}{2^{k-8}(k-8)!(3!)^2 2\,6!}\Big]+$$

$$+(\varphi')^{n-2r+5}(\varphi'')^{r-7}(\varphi^{(5)})^2\left[\frac{n(n-1)\ldots(n-2k+6)}{2^{r-7}(r-7)!\,4!\,5!}+(n-2r+6)\frac{n(n-1)\ldots(n-2k+7)}{2^{r-8}(r-8)!\,(5!)^2 2}+\frac{n(n-1)\ldots(n-2k+5)}{2^{r-7}(r-7)!\,(5!)^2 2}\right]+$$

$$+(\varphi')^{n-2r+5}(\varphi'')^{r-7}\varphi^{(4)}\varphi^{(6)}\left[\frac{n(n-1)\ldots(n-2k+6)}{2^{r-7}(r-7)!\,4!\,5!}+\frac{n(n-1)\ldots(n-2k+6)}{2^{r-7}(r-7)!\,3!\,6!}+(n-2r+6)\frac{n(n-1)\ldots(n-2k+7)}{2^{r-8}(r-8)!\,4!\,6!}+\right.$$

$$\left.+\frac{n(n-1)\ldots(n-2k+5)}{2^{r-7}(r-7)!\,4!\,6!}\right]+$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{r-7}\varphi'''\varphi^{(7)}\left[\frac{n(n-1)\ldots(n-2k+6)}{2^{r-7}(r-7)!\,3!\,6!}+(r-6)\frac{n(n-1)\ldots(n-2r+6)}{2^{r-6}(r-6)!\,7!}+\right.$$

$$\left.(n-2r+6)\frac{n(n-1)\ldots(n-2k+7)}{2^{r-8}(r-8)!\,3!\,7!}+\frac{n(n-1)\ldots(n-2r+5)}{2^{r-7}(r-7)!\,3!\,7!}\right]-$$

$$+(\varphi')^{n-2k+5}(\varphi'')^{r-6}\varphi^{(8)}\left[\frac{n(n-1)\ldots(n-2k+6)}{2^{r-6}(r-6)!\,7!}+\frac{n(n-1)\ldots(n-2k+7)}{2^{r-7}(r-7)!\,8!}(n-2k+6)+\frac{n(n-1)\ldots(n-2k+5)}{2^{r-6}(r-6)!\,8!}\right]+$$

$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$+(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau-1}(\varphi''')^{\tau+1}\left[(r-2\tau)\frac{n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau}(r-2\tau)!\,(3!)^{\tau}\tau!}+(n-2r+\tau+1)\frac{n(n-1)\ldots(n-2r+\tau+2)}{2^{r-2\tau-2}(r-2\tau-2)!\,(3!)^{\tau+1}(\tau+1)!}+\right.$$

$$\left.+\frac{n(n-1)\ldots(n-2r+\tau)}{2^{r-2\tau-1}(r-2\tau-1)!\,(3!)^{\tau+1}(\tau+1)!}\right]$$

$$+(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau}(\varphi''')^{\tau-1}(\varphi^{4})\left[\tau\frac{n(n-1)\ldots(n-2k+\tau+1)}{2^{r-2\tau}(r-2\tau)!\,(3!)^{\tau}\tau!}+(r-2\tau+1)\frac{n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau+1}(r-2\tau+1)!\,(3!)^{\tau-2}(\tau-2)!\,4!}+\right.$$

$$\left.+(n-2r+\tau+1)\frac{n(n-1)\ldots(n-2r+\tau+2)}{2^{r-2\tau-1}(r-2\tau-1)!\,(3!)^{\tau-1}(\tau-1)!\,4!}+\frac{n(n-1)\ldots(n-2r+\tau)}{2^{r-2\tau}(r-2\tau)!\,(3!)^{\tau-1}(\tau-1)!\,4!}\right]+$$

$$+(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+1}(\varphi''')^{\tau-3}(\varphi^{(4)})^2\left[(\tau-2)\frac{n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau+1}(r-2\tau+1)!\,(3!)^{\tau-2}(\tau-2)!\,4!}+(r-2\tau+2)\cdot\right.$$

$$\left.\cdot\frac{n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!\,(3!)^{\tau-4}(\tau-4)!\,(4!)^2 2}+\frac{(n-2r+\tau+1)\cdot n(n-1)\ldots(n-2r+\tau+2)}{2^{r-2\tau}(r-2\tau)!\,(3!)^{\tau-3}(\tau-3)!\,(4!)^2 2}+\frac{n(n-1)\ldots(n-2r+\tau)}{2^{r-2\tau+1}(r-2\tau+1)!\,(3!)^{\tau-3}(\tau-3)!\,(4!)^2}\right]$$

$$+(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+1}(\varphi''')^{\tau-2}\varphi^{(5)}\left[\frac{n(n-1)\ldots(n-2k+\tau+1)}{2^{r-2\tau+1}(r-2\tau+1)!\,(3!)^{\tau-2}(\tau-2)!\,4!}+\frac{(r-2\tau+2)\cdot n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!\,(3!)^{\tau-3}(\tau-3)!\,5!}+\right.$$

$$\left.\frac{(n-2r+\tau+1)\cdot n(n-1)\ldots(n-2r+\tau+2)}{2^{r-2\tau}(r-2\tau)!\,(3!)^{\tau-2}(\tau-2)!\,5!}+\frac{n(n-1)\ldots(n-2r+\tau)}{2^{r-2\tau+1}(r-2\tau+1)!\,(3!)^{\tau-2}(\tau-2)!\,5!}\right]+$$

$$+(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-4}\varphi^{(4)}\varphi^{(5)}\left[\frac{(\tau-3)\cdot n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!\,(3!)^{\tau-3}(\tau-3)!\,5!}+\right.$$

$$2\frac{n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!\,(\tau-4)!\,(3!)^{\tau-4}(4!)^2 2}+\frac{(n-2r+\tau+1)\cdot n(n-1)\ldots(n-2r+\tau+2)}{2^{r-2\tau+1}(r-2\tau+1)!\,(3!)^{\tau-4}(\tau-4)!\,4!\,5!}+\frac{n(n-1)\ldots(n-2r+\tau)}{2^{r-2\tau+2}(r-2\tau+2)!\,(3!)^{\tau-4}}\cdot\frac{1}{(\tau-4)!\,4!\,5!}$$

$$\left.+(r-2\tau+3)\cdot\frac{n(n-1)\ldots(n-2r+\tau+1)}{2^{r-2\tau+3}(r-2\tau+3)!\,(3!)^{\tau-5}(\tau-5)!\,4!\,5!}\right]$$

$$(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-3}\varphi^{(6)}\Big[\frac{n(n-1)\dots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-3}(\tau-3)!5!}+\frac{(r-2\tau+3).n(n-1)\dots}{2^{r-2\tau+3}(r-2\tau+3)!}\cdot$$

$$\cdot\frac{(n-2r+\tau+1)}{(3!)^{\tau-4}(\tau-4)!6!}+\frac{(n-2r+\tau+1).n(n-1)\dots(n-2r+\tau+2)}{2^{r-2\tau+1}(r-2\tau+1)!(3!)^{\tau-3}(\tau-3)!6!}+\frac{n(n-1)\dots(n-2r+\tau)}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-3}(\tau-3)!6!}\Big]+$$

$$(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-5}(\varphi^{(4)})^3\Big[\frac{(\tau-4).n(n-1)\dots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-4}(\tau-4)!(4!)^2 2}+\frac{(r-2\tau+3).n(n-1)\dots}{2^{r-2\tau+3}(r-2\tau+3)!}\cdot$$

$$\cdot\frac{(n-2r+\tau+1)}{(3!)^{\tau-6}(\tau-6)!(4!)^3 3!}+\frac{n(n-1)\dots(n-2r+\tau+2)(n-2r+\tau+1)}{2^{r-2\tau+1}(r-2\tau+1)!(3!)^{\tau-5}(\tau-5)!(4!)^3 3!}+\frac{n(n-1)\dots(n-2r+\tau)}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-5}(\tau-5)!(4!)^3 3!}\Big]$$

$$- - - - - - - - - - - - - - - - - - - +$$

$$+(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+q}(\varphi''')^{l_3}(\varphi^{(4)})^{l_4}\dots(\varphi^{(q+3)})^{l_{q+3}}\Big[\frac{n(n-1)\dots(n-2r+\tau)}{2^{r-2\tau+q}(r-2\tau+q)!(3!)^{l_3}l_3!\dots(q+3)!^{l_{q+3}}l_{q+3}!}+$$

$$+\frac{(n-2r+\tau+1).n(n-1)\dots(n-2r+\tau+2)}{2^{r-2\tau+q-1}(r-2\tau+q-1)!(3!)^{l_3}l_3!\dots((q+3)!)^{l_{q+3}}l_{q+3}!}+\frac{n(n-1)\dots(n-2r+\tau+1)}{2^{r-2\tau+q}(r-2\tau+q)!(3!)^{l_3+1}(l_3+1)!}\cdot$$

$$\cdot\frac{(l_3+1)}{(4!)^{l_4-1}(l_4-1)!(5!)^{l_5}l_5!\dots((q+3)!)^{l_{q+3}}l_{q+3}}+\dots+\frac{n(n-1)\dots(n-2r+\tau+1)\cdot(l_{q+2}+1)}{2^{r-2\tau+q}(r-2\tau+q)!(3!)^{l_3}l_3!\dots l_{q+1}!(q+1)!^{l_{q+1}}(q+2)!^{l_{q+2}+1}}\cdot$$

$$\cdot\frac{1}{(l_{q+2}+1)!(q+3)!^{l_{q+3}-1}(l_{q+3}-1)!}+\frac{(r-2\tau+q+1).n(n-1)\dots(n-2r+\tau+1)}{2^{r-2\tau+q+1}(r+2\tau+q+1)!(3!)^{l_3-1}(l_3-1)!4!^{l_4}l_4!\dots(q+3)^{l_{q+3}}l_{q+3}!}\Big]$$

$$+ - - - - - - - - - - - - - - - - - - +$$

$$+(\varphi')^{n-2r+\tau}(\varphi'')^{r-\tau-1}(\varphi''')^{\tau+3}\Big[\frac{n(n-1)\dots(n-2r+\tau)}{2^{r-\tau-1}(r-\tau-1)!(\tau+3)!}+\frac{(n-2r+\tau+1).n(n-1)\dots(n-2r+\tau+2)}{2^{r-\tau-2}(r-\tau-2)!(\tau+3)!}+$$

$$+(\tau+2)\frac{n(n-1)\dots(n-2r+\tau+1)}{2^{r-\tau-1}(r-\tau-1)!(\tau+2)!}\Big]+\dots\dots\dots+$$

$$+(\varphi')^{n-r-1}\varphi^{(r+2)}\Big[\frac{n(n-1)\dots(n-r)}{(r+1)!}+\frac{n(n-1)\dots(n-r-1)}{(r+2)!}\Big]=$$

$$(\varphi')^{n-2r-1}(\varphi'')^{r+1}\cdot\frac{(n+1)n(n-1)\dots(n-2r)}{2^{r+1}(r+1)!}+\frac{(n+1)n(n-1)\dots(n-2r+1)}{2^{r-1}(r-1)!\,3!}(\varphi')^{n-2r}(\varphi'')^{r-1}\varphi'''+$$

$$+\frac{(n+1)n(n-1)\dots(n-2r+2)}{2^{r-3}(r-3)!(3!)^2 2}(\varphi')^{n-2r+1}(\varphi'')^{r-3}(\varphi''')^2+\frac{(n+1)n\dots(n-2r+2)}{2^{r-2}(r-2)!4!}(\varphi')^{n-2r+1}(\varphi'')^{r-2}\varphi^{(4)}+$$

$$+\frac{(n+1)n\dots(n-2r+3)}{2^{r-5}(r-5)!(3!)^3 3!}(\varphi')^{n-2r+2}(\varphi'')^{r-5}(\varphi''')^3+\frac{(n+1)n\dots(n-2r+3)}{2^{r-4}(r-4)!3!4!}(\varphi')^{n-2r+2}(\varphi'')^{r-4}\varphi'''\varphi^{(4)}+$$

$$+\frac{(n+1)n\dots(n-2r+3)}{2^{r-3}(r-3)!5!}(\varphi')^{n-2r+2}(\varphi'')^{r-3}\varphi^{(5)}+\frac{(n+1)n\dots(n-2r+4)}{2^{r-7}(r-7)!(3!)^4 4!}(\varphi')^{n-2r+3}(\varphi'')^{r-7}(\varphi''')^4+$$

$$+\frac{(n+1)n\dots(n-2r+4)}{2^{r-6}(r-6)!(3!)^2 2.4!}(\varphi')^{n-2r+3}(\varphi'')^{r-6}(\varphi''')^2\varphi^{(4)}+\frac{(n+1)n\dots(n-2r+4)}{2^{r-5}(r-5)!4!^2 2}(\varphi')^{n-2r+3}(\varphi'')^{r-5}(\varphi^{(4)})^2+$$

$$+\frac{(n+1)n\dots(n-2r+4)}{2^{r-5}(r-5)!3!5!}(\varphi')^{n-2r+3}(\varphi'')^{r-5}\varphi'''\varphi^{(5)}+\frac{(n+1)n\dots(n-2r+4)}{2^{r-4}(r-4)!6!}(\varphi')^{n-2r+3}(\varphi'')^{r-4}\varphi^{(6)}+$$

$$+\frac{(n+1)n\ldots(n-2r+5)}{2^{r-9}(r-9)!(3!)^5 5!}(\varphi')^{n-2r+4}(\varphi'')^{r-9}(\varphi''')^5+\frac{(n+1)n\ldots(n-2r+5)}{2^{r-8}(r-8)!(3!)^3 3!4!}(\varphi')^{n-2r+4}(\varphi'')^{r-8}(\varphi''')^3\varphi^{(4)}+$$

$$+\frac{(n+1)n\ldots(n-2r+5)}{2^{r-7}(r-7)!3!(4!)^2 2}(\varphi')^{n-2r+4}(\varphi'')^{r-7}\varphi'''(\varphi^{(4)})^2+\frac{(n+1)n\ldots(n-2r+5)}{2^{r-7}(r-7)!(3!)^2 2.5!}(\varphi')^{n-2r+4}(\varphi'')^{r-7}(\varphi''')^2\varphi^{(5)}+$$

$$+\frac{(n+1)n\ldots(n-2r+5)}{2^{r-6}(r-6)!4!5!}(\varphi')^{n-2r+4}(\varphi'')^{r-6}\varphi^{(4)}\varphi^{(5)}+\frac{(n+1)n\ldots(n-2r+5)}{2^{r-6}(r-6)!3!6!}(\varphi')^{n-2r+4}(\varphi'')^{r-6}\varphi'''\varphi^{(6)}+$$

$$+\frac{(n+1)n\ldots(n-2r+5)}{2^{r-5}(r-5)!7!}(\varphi')^{n-2r+4}(\varphi'')^{r-5}\varphi^{(7)}+\frac{(n+1)n\ldots(n-2r+6)}{2^{r-11}(r-11)!(3!)^6 6!}(\varphi')^{n-2r+5}(\varphi'')^{r-11}(\varphi''')^6$$

$$\frac{(n+1)n\ldots(n-2r+6)}{2^{r-10}(r-10)!(3!)^4 4!4!}(\varphi')^{n-2r+5}(\varphi'')^{r-10}(\varphi''')^4\varphi^{(4)}+\frac{(n+1)n\ldots(n-2r+6)(\varphi')^{n-2r+5}}{2^{r-9}(r-9)!(3!)^2 2(4!)^2 2}(\varphi'')^{r-9}(\varphi''')^2(\varphi^{(4)})^2$$

$$\frac{(n+1)n\ldots(n-2r+6)}{2^{r-9}(r-9)!(3!)^3 3!5!}(\varphi')^{n-2r+5}(\varphi'')^{r-9}(\varphi''')^3\varphi^{(5)}+\frac{(n+1)n\ldots(n-2r+6)}{2^{r-8}(r-8)!(4!)^3 3!}(\varphi')^{n-2r+5}(\varphi'')^{r-8}(\varphi^{(4)})^3+$$

$$\frac{(n+1)n\ldots(n-2r+6)}{2^{r-8}(r-8)!3!4!5!}(\varphi')^{n-2r+5}(\varphi'')^{r-8}\varphi'''\varphi^{(4)}\varphi^{(5)}+\frac{(n+1)n\ldots(n-2r+6)(\varphi')^{n-2r+5}}{2^{r-8}(r-8)!(3!)^2 2.6!}(\varphi'')^{r-8}(\varphi''')^2\varphi^{(6)}$$

$$\frac{(n+1)n\ldots(n-2r+6)}{2^{r-7}(r-7)!(5!)^2 2}(\varphi')^{n-2r+5}(\varphi'')^{r-7}(\varphi^{(5)})^2+\frac{(n+1)n\ldots(n-2r+6)}{2^{r-7}(r-7)!4!6!}(\varphi')^{n-2r+5}(\varphi'')^{r-7}\varphi^{(4)}\varphi^{(6)}+$$

$$\frac{(n+1)n\ldots(n-2r+6)}{2^{r-7}(r-7)!3!7!}(\varphi')^{n-2r+5}(\varphi'')^{r-7}\varphi'''\varphi^{(7)}+\frac{(n+1)n\ldots(n-2r+6)}{2^{r-6}(r-6)!8!}(\varphi')^{n-2r+5}(\varphi'')^{r-6}\varphi^{(8)}+\ldots$$

$$\frac{(n+1)n\ldots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}}{2^{r-2\tau-1}(r-2\tau-1)!(3!)^{\tau+1}(\tau+1)!}(\varphi'')^{r-2\tau-1}(\varphi''')^{\tau+1}+\frac{(n+1)n\ldots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}}{2^{r-2\tau}(r-2\tau)!(3!)^{\tau-1}(\tau-1)!4!}(\varphi'')^{r-2\tau}(\varphi''')^{\tau-1}\varphi^{(4)}$$

$$\frac{(n+1)n\ldots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}}{2^{r-2\tau+1}(r-2\tau+1)!(3!)^{\tau-3}(\tau-3)!(4!)^2 2}(\varphi'')^{r-2\tau+1}(\varphi''')^{\tau-3}(\varphi^{(4)})^2+\frac{(n+1)n\ldots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-4}4!5!}\cdot$$

$$\cdot\frac{(\varphi')^{n-2r+\tau}}{(\tau-4)!}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-4}\varphi^{(4)}\varphi^{(5)}+\frac{(n+1)n\ldots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}}{2^{r-2\tau+1}(r-2\tau+1)!(3!)^{\tau-2}(\tau-2)!5!}(\varphi'')^{r-2\tau+1}\cdot$$

$$\cdot(\varphi''')^{\tau-2}\varphi^{(5)}+\frac{(n+1)n\ldots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-3}(\tau-3)!6!}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-3}(\varphi)^{(6)}+$$

$$+\frac{(n+1)n\ldots(n-2r+\tau+1)}{2^{r-2\tau+2}(r-2\tau+2)!(3!)^{\tau-5}(\tau-5)!(4!)^3 3!}(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+2}(\varphi''')^{\tau-5}(\varphi^{(4)})^3+\ldots$$

$$+\frac{(n+1)n\ldots(n-2r+\tau+1)(\varphi')^{n-2r+\tau}(\varphi'')^{r-2\tau+q}(\varphi''')^{l_3}(\varphi^{(4)})^{l_4}\ldots(\varphi^{(q+3)})^{l_{q+3}}}{2^{r-2\tau+q}(r-2\tau+q)!(3!)^{l_3}(4!)^{l_4}\ldots(q+3)!^{l_{q+3}}l_3!l_4!\ldots l_{q+3}!}+\ldots+$$

$$+\frac{(n+1)n\ldots(n-2r+\tau+1)}{2^{r-\tau-1}(r-\tau-1)!(3!)^{\tau+3}(\tau+3)!}(\varphi')^{n-2r+\tau}(\varphi'')^{r-\tau-1}(\varphi''')^{\tau+3}+\ldots+$$

$$+\frac{(n+1)n\ldots(n-r)}{(r+2)!}(\varphi')^{n-r-1}\varphi^{(r+2)}$$

. We have used here that $l_1+2l_2+3l_3+\ldots+(q+3)l_{q+3}=n+1$ in this case. Thus, we received the formulae of the points

(v) and (vi) for n+1 . Since we have already proved the point (iv), hence is sufficient to assert by induction the formulae (v) and (vi). ∎