SOME CLASSES OF NONCOMMUTATIVE RINGS AND ABELIAN GROUPS

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ABSTRACT and MAIN PURPOSES

The dissertation deals with some aspects of ring theory and commutative group theory as, specifically, it will be focussed on some structural and characterization results concerning the specific ring and group structures. In the machinery that we will use and develop in order to prove the established results, the following directions in modern algebra will be implemented in some of the proofs:

(i) *matrix theory and computations*

(ii) *homological algebra*

(iii) *set theory and formal logic*

(iv) *graph theory*

(v) *number theory*

A brief outline of some of the main results includes the following:

Result 1. *A ring R is invo-clean $\iff$ R decomposes as $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is a nil-clean ring of characteristic ≤ 8, and either $R_2 = \{0\}$ or R_2 is embedding in a direct product (i.e., it is a subdirect product) of a family of copies of the field $\mathbb{F}_3$. In particular, if R is strongly invo-clean, then $R_1/J(R_1)$ is Boolean with nil $J(R_1)$ whenever R_1 is non-zero.*

Result 2. *A ring R is uniquely weakly nil-clean $\iff$ R is decomposable as $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or $R_1/J(R_1)$ is Boolean with nil $J(R_1)$, and either $R_2 = \{0\}$ or $R_2/J(R_2) \cong \mathbb{Z}_3$ with nil $J(R_2)$.*

Result 3. *The weakly nil-clean index of $\mathbb{T}_2(\mathbb{Z}_p)$ is equal to p, while for $\mathbb{T}_3(\mathbb{Z}_p)$ it is p^2, whenever p is a prime number; the weakly nil-clean index for $\mathbb{M}_2(\mathbb{Z}_3)$ equals to* 5.

Result 4. *A ring R is strongly n-torsion clean for some $n \in \mathbb{N}$ $\iff$ R is strongly clean and $U(R)$ is of finite exponent. In particular, if n is odd, then R is a clean ring in which orders of all units are odd, bounded by n and there exists a unit of order n $\iff$ R is a subdirect product of copies of the fields $\mathbb{F}_{2^{k_i}}$, where $i \in [1,t]$ for some integer $t \geq 1$ such that there exist integers $k_1, \cdots, k_t \geq 1$ with $n = LCM(2^{k_1} - 1, \ldots, 2^{k_t} - 1)$.*

Result 5. *If G is a locally finite group and R is an arbitrary ring, then the group ring $R[G]$ is UU $\iff$ R is UU and G is* 2*-torsion.*

Result 6. *Let $G = A \oplus B$ be a group. Then*

(1) G is socle-regular $\iff$ A is socle-regular, provided B is separable.

(2) A is socle-regular, provided G is socle-regular, that is, a direct summand of a socle-regular group is again a socle-regular group.

(3) Krylov transitive groups are themselves socle-regular with irreversible implication.

(4) There is a weakly transitive group which is not socle-regular.

(5) Any totally projective group of length $\leq \omega^2$ is strongly projectively fully transitive.

(6) If G is a group such that the first Ulm subgroup $p^\omega G$ is elementary, then G is fully transitive $\iff$ the square $G \oplus G$ is strongly projectively fully transitive $\iff$ the square $G \oplus G$ is strongly commutator fully transitive.

(7) Any totally projective group of length $< \omega^2$ is commutator socle-regular.

(8) A direct summand of a commutator socle-regular group is not necessarily commutator socle-regular; a direct summand of a commutator fully transitive group need not be commutator fully transitive too.

(9) Both projective socle-regularity and commutator socle-regularity notions are independent to transitivity and full transitivity.

(10) Commutator fully transitive groups are always commutator socle-regular.

(11) A direct summand of a fully transitive torsion-free IFI-group is again a fully transitive IFI-group.

(12) If G is an IFI-group, then $G \oplus G$ is also an IFI-group.

(13) Any strongly irreducible group G such that $|G/pG| \leq p$ for each prime p is an IFI-group.

Result 7. *Suppose that G is a group such that the factor-group $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective. If $p^{\omega+1}G$ is countable, then G is the direct sum of a $p^{\omega+1}$-projective group and a countable group. Moreover, there is a group G for which $G/p^{\omega+2}G$ is $p^{\omega+2}$-projective and $p^{\omega+2}G$ is countable, but G is not a direct sum of a $p^{\omega+2}$-projective group and a countable group. In particular, if $0 < n < \omega$, then the class of $\omega + n$-totally $p^{\omega+n}$-projective groups is not closed under (finite) direct sums.*

Result 8. *Suppose G is a group, n is an arbitrary natural and λ is an arbitrary ordinal. Then G is n-simply presented $\iff$ both $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented.*

Result 9. *For every $n \in \mathbb{N}$ a direct summand of an n-simply presented group is again an n-simply presented group, provided that the complement is a countable group.*

Result 10. *Let $n \in \mathbb{N}$. Then the following two points hold:*

(a) Nicely ω_1-n-simply presented groups of length $< \omega^2$ are n-simply presented.

(b) Suppose G is a group whose quotient $G/p^\lambda G$ is n-simply presented for some ordinal λ. Then G is nicely ω_1-n-simply presented $\iff$ $p^\lambda G$ is nicely ω_1-n-simply presented.

So, the main purpose of this dissertation is to promote some new ideas in certain contemporary subjects of algebra as well as to demonstrate a new insight of ideas and methods in some branches which could be of further interest for future developments. This will be subsequently achieved in the next sections and their subsections. Our strategically point of view is in developing of a modern technology which will be approachable in many cases in both ring theory and commutative group theory.

CONTENTS

Chapter I. Introduction and Fundamentals

Everywhere in the text of this dissertation, although it is concretely specified in each section, all rings into consideration will be associative unital (sometimes called *unitary*) and all groups unless it is explicitly stated something else (e.g., the unit groups of rings and the groups which form the group rings) will be assumed additive Abelian.

The motivation in writing up this dissertation is to illustrate the study of two different at first glance topics in the modern algebra, which topics actually possess a few close relationships each to other. In fact, one evidence for the existence of such a transversal is the endomorphism ring of Abelian groups. The key approach is that the ring structure unambiguously helps us to decide how the investigated groups are situated into some well-behaved classes of groups.

Specifically, these two subjects are relevant to the following two omnibuses:

(1) *Weakly Exchange Rings with Applications to Group Rings*

We here demonstrate the role of weakly exchange rings and weakly clean as being a common expansion of the classical exchange rings and clean rings to the general theory of rings and modules having numerous applications in the area of (not necessarily commutative) group rings. The new moments are in proving up that some complete descriptions of these ring classes do exist, including also some new dealings with the long-known classes of nil-clean rings, weakly nil-clean rings, invo-clean rings and some their modifications.

(2) *Generalizations of (Fully) Transitive and Simply Presented Abelian Groups*

We here show the majority of some new classes of commutative groups (e.g., the classes of (strongly) socle-regular p-groups and (strongly) n-simply presented p-groups) to the general point of view in the theory of Abelian groups. The new moments are in showing up that some complete descriptions of these group classes do exist, including also some new treatments of the well-known classes of Krylov transitive groups, weakly transitive groups, IFI-groups, n-balanced projective groups, n-simply presented groups, ω_1-n-simply presented groups and some their variations.

To be more concrete, the principally known more important results pertaining to the comments alluded to above, on which results we will somewhat do improvements below, are these:

On **(1)** we have that:

• There exist various characterization results on clean, exchange, nil-clean, weakly nil-clean and some other closely related sorts of rings (see, for more account, [1], [2], [11], [12], [39], [40], [41], [64], [68], [77], [80], [93], [94], [95], [96], [110], etc. some other sources listed below in the literature).

Indeed, invoking the classical source [93], where the pivotal concepts of *clean* and *exchange* rings were defined, what can be more importantly mentioned is that *a ring R is clean (resp., exchange) iff the quotient $R/J(R)$ is clean (resp., exchange) and all idempotents lift modulo $J(R)$ (that is, given $r \in R$ with $r^2 - r \in J(R)$, there is $e \in Id(R)$ having the property that $e - r \in J(R)$).* This was somewhat strengthened in [12] (see also [40] for the commutative case) for the class of weakly nil-clean rings (in fact, it was proven there that any weakly nil-clean ring is necessarily clean, and clean rings are exchange). Same type of results appeared for nil-clean rings in [41] (let us remember that nil-clean rings are always weakly nil-clean). Likewise, further refinements were definitely obtained in [94], [64], [80] and [110], respectively, where under new points of view a new insight in the global structure of clean and exchange rings, arising from some new conditions, was established.

• There exist certain matrix computations regarding to what extent the structure of matrix ring will heavily depend on the structure of the former ring (see, for more information, [4], [11], [81]).

Indeed, inspired by the definition of the notion *clean index* of a ring, which somewhat reaches one of the best knowledge for the class of clean rings, in [4] was defined the concept of *nil-clean index* of a ring. Some interesting results in that matter were proved. Furthermore, in regard to [4], we define in the corresponding subsection below the more general setting of *weakly nil-clean index* of a ring which is of merit investigation being a successful instrument for the full characterization of *uniquely weakly nil-clean* rings (compare also with [D5]) – in that way, some concrete computations were done for certain special full and triangular matrix rings. Also, being involved with certain extremely difficult matrix questions in ring theory, some recent progress was made in [32].

Being closely familiar with the general *theory of matrices* and its computational aspects, we shall try to give a comprehensive presentation of its use in the contemporary directions of the associative rings, especially in their structural characterizations – see, e.g., [99] and [107].

• There exist results focussed on the isomorphic structure of group rings which entirely relies on the group structure of the basis and on the ring structure of the initial ring (see, for more concrete news, [40], [73], [84], [85], [92], [103]).

Indeed, May gave in [84] a complete description of the nil radical of an arbitrary group ring in terms of special elements, whereas Karpilovsky somewhat enlarged that to the Jacobson radical of such a ring. On the other vein, Nicholson explored in [92] local group rings, while in [85] McGovern et al. found a necessary and sufficient condition for a commutative group ring to be nil-clean (for a general necessary and sufficient condition in that way, we refer to [103]). This was substantially strengthened in [40] by the present author of the dissertation along with McGovern to the larger class of weakly nil-clean rings (see [77] too).

On **(2)** we have that:

• There are a series of results dealing with the characterization of both classical classes of transitive and fully transitive groups and their non-trivial extensions.

In fact, the classical properties of transitivity and full transitivity for Abelian groups were firstly defined by Kaplansky in [71] as a common extension of some well-studied classes of primary Abelian groups. Both the definitions entirely rely on the manner how two arbitrary elements of the group are situated, by mapping one to other via an existing group endomorphism, depending on their Ulm sequences in the full group. Likewise, the independence of these two notions was firstly showed by Corner in [29]. Namely, he exhibited an Abelian p-group which is fully transitive but not transitive as well as an Abelian 2-group which is transitive but not fully transitive – note the remarkable fact that every transitive group which is not fully transitive is necessarily a 2-group, a fact first shown by Kaplansky in [71, Theorem 26]. Despite this Corner's result, there is a connection between the two concepts: in fact, Files and Goldsmith showed in [43] that an Abelian p-group G is fully transitive if, and only if, the square's Abelian p-group $G \oplus G$ is transitive. This critical fact will somewhat be refined in one of our subsections. Furthermore, major works on transitive and fully transitive groups were produced in [52] and [61], respectively.

Further very general notions of transitivity were introduced by Goldsmith and Strüngmann in their seminal papers [50] and [51], namely they defined the so-called *Krylov transitive* and *weakly transitive* Abelian p-groups. They proved there that these two concepts are independent each to other as well.

Some recent advantage in the topic was done in [10, Theorem 2.5] by showing that there is a Krylov transitive 2-group that is neither transitive nor fully transitive nor weakly transitive, thus answering a question posed by Danchev-Goldsmith in [D9]. In proving that, they establish the surprising fact stated in [10, Proposition 2.4] that if the Abelian p-group $G \oplus G$ is Krylov transitive, then the Abelian p-group G is fully transitive. Some other effective results could be found in [87, 88] as well.

Nevertheless, among the existing unsettled things of the problematic in the corresponding literature, stated in the reference list of the bibliography, left-open were the questions of what can be said for the structure of the former group, provided its endomorphism ring is (additively) generated by commutators. In other words, all endomorphism are representable as a finite sum of products of commutators. We will be trying to give in the current study some satisfactory affirmative answer in this subject. Our solution will depend heavily on the structure of the first Ulm subgroup of the whole group, determined by the action of the full endomorphism ring on this subgroup (see [D12], [D13] and [D14]).

Our major goals here are to promote a new insight in the structure of the afore-defined (projective, commutator) transitive-like groups and to demonstrate their capability for the classical concepts of transitivity and full transitivity due to Kaplansky in his famous red-book leading to the publication of the monograph [71].

The methods we have developed in order to establish these results are certain innovations in the representation of projective and commutator endomorphisms in terms of matrices, by strengthening the methodology utilized in [D14]. They are concerned with ingenious computations involving number theory and some other not too classical instruments and tricks.

• There are a series of results which deal with the relationships between characteristic, fully invariant and projection-invariant subgroups of Abelian groups (see, e.g., [54], [55, 56], [57], [88]).

In fact, Grinshpon in [54] and Grinshpon et al. in [55],[56] consider those groups (namely, torsion, torsion-free and mixed groups) whose fully invariant subgroups have finite Ulm-Kaplansky invariants and are also endowed with some additional properties. We shall extend this by examining the groups for which all fully invariant subgroups are isomorphic (see [D15], too) as well as we shall consider some other relations and combinations between appropriate group classes.

• There are a series of results pertaining to the generalization of totally projective and simply presented Abelian p-groups in various aspects by considering their purely algebraic structure as well as their homological behavior (see, cf. [44, 47], [62, 63], [66], [74, 75], [76], [97, 98]).

In fact, giving a brief outline of the most important of them, it is a Nunke's achievement in [98] proving the reduction criterion that a group G is totally projective (resp., simply presented) iff so are both the groups $p^\alpha G$ and $G/p^\alpha G$. We considerably supersede that in this dissertation to the class of n-simply presented groups as our proof is rather difficult and long equipping more than ten pages (compare with [D11] as well). On the other hand, concerning their homological shape, totally projective groups are known to be balanced projective with respect to all short-exact sequences (cf. [44]). This will also be improved here for the class of n-balanced projectives, whenever $n \geq 1$ (see [D11] and [76]). Further generalizations are given in [D16]. Some numerical invariants involving set theory machinery were given in [3].

Being closely familiar with the theories of *homological algebra* and *set theory*, we shall try to give a detailed presentation of their usage in the contemporary directions of the commutative groups – see, for instance, [97, 98] and [100]. The theory of valuated groups also plays a crucial role in the structural aspects of Abelian groups – see, for example, [101].

Further details in both points (1) and (2) are stated in each of the subsequent sections and their subsections separately.

As for the fundamental notions, notations and terminology, we will follow mainly those from the classical monograph series of [79], [102] as well as of [44, 47], [71] and [78]. Nevertheless, for readers' convenience and for the sake of completeness, they will be stated in details in the duration.

Chapter II. Background and Conventions

For the present dissertation, a ring R will be assumed to be an associative ring with identity 1 which differs from the zero element 0. We shall use in the sequel the notation $Id(R)$ to denote the set of all idempotents of R, $Nil(R)$ to denote the set of all nilpotents of R, and $U(R)$ to denote the set of all units of R. We also shall use $\mathbb{M}_n(R)$ to denote the ring of all $n \times n$ matrices with entries in R (also called the full matrix ring) and $\mathbb{T}_n(R)$ to denote the ring of all $n \times n$ upper triangular matrices with elements from R (also called the upper triangular matrix ring), whenever $n \in \mathbb{N}$, the set of all positive integers (also termed naturals). Almost all other ring-theoretical notions and terminologies with which we have played will be in agreement with those from [79] as the more profit ones will be explicitly stated and formulated in each separate section and subsection from the corresponding chapters. About the conventions in writing up the text, we shall use "**wnc**" to denote the "**weakly nil-clean**" index of a ring as well as "**a JU-ring**" will mean "**a ring with Jacobson units**".

Likewise, all our groups in Section 2 titled "Applications to Group Rings" of Chapter III "Noncommutative Rings", where group rings are considered, will be written multiplicatively – surely, same appears and for the unit group of an arbitrary ring.

Concerning Abelian group theory, all our groups with which we will play are assumed to be additively written. The notion and notation will follow in general those established in [44, 47] with some little exceptions which will be specified and clarified when needed in the text. About the conventions in writing, used throughout the dissertation, we shall abbreviate "**a dsc-group**" for "**a direct sum of countable groups**" as well as "**projectives**" for "**projective groups**". Besides, abbreviating "**a cft-group**" means "**a commutator fully transitive group**" as well as "**a scft-group**" means "**a strongly commutator fully transitive group**".

Also, we will henceforth use somewhere in the text, where it is possible and better for usage, the widely accepted shorthand abbreviation "**iff**" for the standard phrase "**if and only if**". As for the latter, we shall somewhere write "**if, and only if,**" whenever the text is more specific in the sense that it needs more specifications in the meaning.

Chapter III. Noncommutative Rings

Our main results of this branch are distributed into two sections as follows:

1. Weakly Exchange Rings

Here, for the sake of completeness and for the convenience of the readers, we shall consider below a few more subsections like these:

1.1. **On weakly exchange rings.** The following fundamental notion was defined in [93].

Definition 1.1. A ring R is called *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$.

Likewise, in [93] it was pointed out the fundamental fact that R is clean iff $R/J(R)$ is clean and all idempotents lift modulo $J(R)$.

The "clean" concept was generalized there to the following one:

Definition 1.2. A ring R is said to be *exchange* if, for every $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

It was obtained in [93] that R is an exchange ring iff $R/J(R)$ is an exchange ring and all idempotents lift modulo $J(R)$. Also, it was established there that Definitions 1.1 and 1.2 are equivalent for abelian rings (that are rings for which each idempotent lies in the center of the former ring). However, there is an exchange ring that is not clean.

On the other hand, it was introduced in [2] the notion of *weakly clean* rings but only in a commutative version. We shall do that in the general way as follows:

Definition 1.3. A ring R is called *weakly clean* if each $r \in R$ can be expressed as either $r = u + e$ or $r = u - e$, where $u \in U(R)$ and $e \in Id(R)$.

Evidently, all clean rings are weakly clean, whereas the converse does not hold even in the commutative aspect (see, e.g., [2]). However, every weakly clean ring of characteristic 2 is clean, and vice versa. One of our goals here is to improve this observation by requiring that 2 lies in $J(R)$, which supersedes the condition $2 = 0$.

Definition 1.4. A ring R is called *weakly exchange* if, for any $x \in R$, there exists $e \in Id(R)$ such that $e \in xR$ and either $1-e \in (1-x)R$ or $1-e \in (1+x)R$.

It was established in [24] that the notions of being weakly exchange and weakly clean do coincide for abelian rings, thus extending the aforementioned facts from [93] (see also [109]).

Apparently, all exchange rings are weakly exchange, while the converse does not hold even in the commutative variant as some simple examples demonstrably show. However, every weakly exchange ring of characteristic 2 is exchange, and visa versa. One of our aims here is to enlarge this observation by requiring that 2 lies in $J(R)$, which is weaker than the condition $2 = 0$.

So, we come to our first basic result in the current subsection.

Theorem 1.5. *A ring R is weakly exchange if $R/J(R)$ is weakly exchange and all idempotents in R lift modulo $J(R)$. In addition, if $2 \in J(R)$, then the converse is true.*

We are now arriving at the following:

Theorem 1.6. *A ring R is weakly clean if $R/J(R)$ is weakly clean and all idempotents in R lift modulo $J(R)$. In addition, if $2 \in J(R)$, then the converse is true.*

2. Rings with Jacobson units

It is well known that the inclusion $1 + J(R) \subseteq U(R)$ or, equivalently, $J(R) \subseteq 1 + U(R)$ holds. However, these containments could be strict, so that it is rather natural to state the following:

Definition 2.1. A ring R is called a *JU ring* or a ring with *Jacobson units* if the equality $U(R) = 1 + J(R)$ holds.

Obviously, this is tantamount to the equality $J(R) = 1+U(R)$. In an equivalent form, since one can show

$$U(R)/(1 + J(R)) \cong U(R/J(R)),$$

we observe in the presence of this isomorphism that all JU rings are just those rings R for which $U(R/J(R)) = \{1\}$.

The leitmotif of the next chief result listed below is to describe explicitly exchange JU rings. The intersection between these two classes, however, gives nothing new. Specifically, the following is valid:

Theorem 2.2. *A ring R is an exchange JU ring if, and only if, it is J-clean.*

2.1. **On exchange π-UU unital rings.** We begin here with recalling some useful concepts as follows:

Definition 2.3. A ring R is said to be *UU* if $U(R) = 1 + Nil(R)$.

Definition 2.4. A ring R is said to be *exchange* if, for each $r \in R$, there is an idempotent $e \in rR$ such that $1 - e \in (1 - r)R$.

It was proved in [39] that a ring R is an exchange UU ring iff $J(R)$ is nil and $R/J(R)$ is Boolean.

Before proceed by proving our chief result, we need a few more technicalities. Generalizing Definition 2.3, one can state the following.

Definition 2.5. Let $n \in \mathbb{N}$. A ring R is called *n-UU* if the inclusion $U^n(R) \subseteq 1 + Nil(R)$ holds.

Clearly, UU rings just coincide with 1-UU rings.
This can be substantially expanded to the following:

Definition 2.6. A ring R is called *π-UU* if, for any $u \in U(R)$, there exists $i \in \mathbb{N}$ such that $u^i \subseteq 1 + Nil(R)$.

These rings play a key, if not (at least) facilitate, role in developing a new modern theory of *periodic rings* (see, e.g., [32]).

We shall now restate and reproof the main result from [1] by giving a more convenient form and more transparent proof arising from well-known recent results in [39] and [D4] (compare with the subsequent subsection, too), respectively. Actually, a new substantial achievement, including new points with more strategic estimations, arises as follows:

Theorem 2.7. *Suppose that R is a ring. Then the following five items are equivalent:*

(a) R is exchange 2-UU.

(b) $J(R)$ is nil and $R/J(R)$ is commutative invo-clean.

(c) $J(R)$ is nil and $R/J(R) \cong B \times C$, where $B \subseteq \prod_\lambda \mathbb{Z}_2$ and $C \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

(d) $J(R)$ is nil and $R/J(R)$ is tripotent.

(e) $J(R)$ is nil and $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

The next construction manifestly demonstrates that the theorem is *no* longer true for n-UU rings when $n > 2$.

Example 2.8. Consider the full matrix 2×2 ring $R = \mathbb{M}_2(\mathbb{Z}_2)$. It was proved in [11] that R is nil-clean and hence exchange. Moreover, R is a 3-UU ring. However, it is easily checked that $J(R) = \{0\}$ and that R is even not tripotent (whence not Boolean). In fact, $U(R)$ has 6 elements satisfying the following identities:

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

2.2. **Weakly nil-clean index and uniquely weakly nil-clean rings.** In [41] a ring R is said to be *nil-clean* if each element $a \in R$ can be represented as $a = b + e$, where $b \in Nil(R)$ and $e \in Id(R)$; note that this is equivalent to the representation that, for every $a \in R$, we have $a = b - e$. If this presentation is unique, the ring R is called *uniquely nil-clean*. It is not too hard to check that this is tantamount to the requirement that the existing idempotent e is unique (see, e.g., [23] and [41]).

On the other vein, in [40] and [12] was stated the definition of a *weakly nil-clean* ring as such a ring R for which any element $a \in R$ is of the form $a = b + e$ or $a = b - e$, where $b \in Nil(R)$ and $e \in Id(R)$. Moreover, a ring R is said to be *uniquely weakly nil-clean* if the existing idempotent e is unique.

Our further work is motivated by the notions of *unique nil-cleanness* and *weak nil-cleanness* as we will combine them into a new concept. So, the aim here is to explore some variations of unique weak nil-cleanness in order to enlarge the principal known results on unique nil-cleanness from [41] and [23]. In doing that, we set and explore in details the weakly nil-clean index of rings and discuss the original notion of uniquely weakly nil-clean rings stated in Problem 3 of [40]. We shall also investigate here some other aspects of unique weak nil-cleanness which arise from its specific definition.

For any $a \in R$, let $\mathcal{E}(a) = \{e \in R \mid e^2 = e,\ a - e \in U(R)\}$ and then the *clean index* of R, denoted as $c(R)$, is defined in [81] by $c(R) = sup\{|\mathcal{E}(a)| : a \in R\}$. For any $a \in R$, set $\eta(a) = \{e \in R \mid e^2 = e$ and $a - e \in Nil(R)\}$ and then the *nil-clean index* of R, denoted as $Nin(R)$, is defined in [4] by $sup\{|\eta(a)| : a \in R\}$. In this way, for a more comprehensive investigation of these two notions and, especially, as a natural generalization of the nil-clean index, we also define the concept of *weakly nil-clean index* of a ring. Thereby, as it will be showed below, a ring is uniquely weakly nil-clean if and only if it is weakly nil-clean of weakly nil-clean index 1.

In [81] the clean index $c(R)$ of a ring R was defined and studied. Imitating this, in [4] was introduced the *nil-clean index* $Nin(R)$ of R and some detailed study was given.

In parallel to these two notions, we proceed by stating the following concepts.

Definition 2.9. Let R be a ring and $a \in R$. We define the set

$$\alpha(a) = \{e \in R : e^2 = e \text{ and } a - e \text{ or } a + e \text{ is a nilpotent}\}.$$

Definition 2.10. For an element $a \in R$ the *weakly nil-clean index* of a, abbreviated as $wnc(a)$, is defined to be the cardinality of the set $\alpha(a)$.

Definition 2.11. We define the *weakly nil-clean index* of a ring R as follows:

$$wnc(R) = \sup\{|\alpha(a)| : a \in R\}.$$

Our basic theorem here is the following one:

Theorem 2.12. *The following are equivalent for a ring R:*

(1) *R is uniquely weakly nil-clean;*
(2) *R is abelian weakly nil-clean;*
(3) *$R \cong R_1 \times R_2$, where R_1 is either $\{0\}$ or an abelian nil-clean ring and R_2 is either $\{0\}$ or a local weakly nil-clean ring such that $J(R_2)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$.*

2.3. **n-Torsion clean rings.** Our notations and notions here are in agreement with those from [79]. For instance, for such a ring R, the symbol $U(R)$ denotes the group of units, $\mathrm{Id}(R)$ the set of idempotents and $J(R)$ the Jacobson radical of R, respectively. Besides, the finite field with m elements will be denoted by $\mathbb{F}_m$, and $\mathbb{M}_k(R)$ will stand for the $k \times k$ matrix ring over R; $k \in \mathbb{N}$. For an element u of a group G, the letter $o(u)$ will denote the order of u. And finally, the symbol $LCM(n_1, \dots, n_k)$ will be reserved for the least common multiple of $n_1, \dots, n_k \in \mathbb{N}$.

We will say a nil ideal I of R is nil of index k if, for any $r \in I$, we have $r^k = 0$ and k is the minimal natural number with this property. Likewise, we will say that I is nil of bounded index if it is nil of index k, for some fixed k.

Let us recall that a ring R is said to be *clean* if, for every $r \in R$, there are $u \in U(R)$ and $e \in \mathrm{Id}(R)$ with $r = e + u$. If, in addition, the commutativity condition $ue = eu$ is satisfied, the clean ring R is called *strongly clean*. These rings were introduced by Nicholson in [93] and [94]. Both clean rings and their various specializations or generalizations are intensively studied since then (see, for example, [12], [23], [D4], [40], [41] and references within).

A decomposition $r = e + u$ of an element r in a ring R will be called *n-torsion clean decomposition* of r if $e \in \mathrm{Id}(R)$ and $u \in U(R)$ is n-torsion, i.e. $u^n = 1$. We will say that such a decomposition of r is *strongly n-torsion clean*, if additionally e and u commute.

The aim of this article is to investigate in detail the following proper subclasses of (strongly) clean rings:

Definition 2.13. A ring R is said to be (*strongly*) *n-torsion clean* if there is $n \in \mathbb{N}$ such that every element of R has a (strongly) n-torsion clean decomposition and n is the smallest possible natural number with the above property.

It is easy to see that boolean rings are precisely the rings which are (strongly) 1-torsion clean. Thus the classes introduced above can be treated as natural generalizations of boolean rings.

Let us notice that in [D4] the class of (*strongly) invo-clean* rings was investigated. In our terminology, (strongly) invo-clean rings are precisely rings which are either (strongly) 1-torsion clean or (strongly) 2-torsion clean.

It is clear that every clean ring having the unit group of bounded exponent s is n-torsion clean for some n with $1 \leq n \leq s$. We will see below that n has to divide s, but does not have to be equal to s. Let us also observe that a homomorphic image of an n-torsion clean ring is always m-torsion clean, for some $m \leq n$. Hoverer, it is not clear whether n is a multiple of m. Notice that finite rings are always clean, so they are n-torsion clean for suitable n and it would be of interest to compute n for some classes of finite rings; for instance, for matrix rings over finite fields.

Now we are ready to state the following result.

Theorem 2.14. *Let $n \in \mathbb{N}$. Suppose R is a strongly n-torsion clean ring. Then:*

(1) *R is a PI-ring satisfying the polynomial identity $(x^n-1)((x-1)^n-1)=0$;*
(2) *R has finite characteristic* $\mathrm{char}(R)=|1\cdot\mathbb{Z}|$*;*
(3) *$J(R)$ is a nil ideal of index smaller than* $(\mathrm{char}(R))^n$*;*
(4) *When n is odd, then R is a reduced ring of characteristic 2 and $J(R)=0$;*
(5) *If R is an algebra over a field F, then:*
(i) $J(R)$ is a nil ideal of index bounded by n;
(ii) either R is abelian (i.e., all idempotents of R are central) or $\mathrm{char}(F)$ *divides n.*

our next major result is the following one:

Theorem 2.15. *For a ring R, the following conditions are equivalent:*

(1) *There exists $n \in \mathbb{N}$ such that R is an n-torsion clean abelian ring.*
(2) (a) $\mathrm{char}(R)$ *is finite;*
(b) *The Jacobson radical $J(R)$ is nil of bounded index;*
(c) *Idempotents lift uniquely modulo $J(R)$;*
(d) *$R/J(R)$ is a subdirect product of finite fields F_i, where i ranges over some index set I, such that $LCM(|F_i|-1 \mid i \in I)$ exists.*
(3) *R is an abelian clean ring such that the unit group $U(R)$ is of finite exponent.*

In parallel to Theorem 2.15, one can state the following:

Theorem 2.16. *For a ring R, the following conditions are equivalent:*

(1) *R is strongly n-torsion clean, for some $n \in \mathbb{N}$.*
(2) *R is strongly clean and $U(R)$ is of finite exponent.*

We now have at our disposal all the necessary information to present a satisfactory structural characterization of strongly n-torsion clean rings for all odd naturals n.

Theorem 2.17. *Suppose $n \in \mathbb{N}$ is odd. For a ring R, the following conditions are equivalent:*

(1) *R is a strongly n-torsion clean ring;*

(2) *There exist integers* $k_1, \dots, k_t \geq 1$ *such that* $n = LCM(2^{k_1}-1, \dots, 2^{k_t}-1)$ *and* R *is a subdirect product of copies of fields* $\mathbb{F}_{2^{k_i}}$, $1 \leq i \leq t$*;*

(3) R *is a clean ring in which orders of all units are odd, bounded by* n *and there exists a unit of order* n.

3. Applications to Group Rings

Here, as usual, the symbol $R[G]$ stands for the group ring of an arbitrary multiplicative group G over an arbitrary unital ring R, and $\omega(R[G])$ is its standard augmentation ideal, generated by the elements $1 - g$, where g runs over G.

Imitating [39], we state the following:

Definition 3.1. A ring R is said to be UU if its unit group $U(R)$ satisfies the equality $U(R) = 1 + Nil(R)$, where $Nil(R)$ is the set of all nilpotent elements of R.

Our basic statement here is the following one:

Theorem 3.2. *Let* G *be a group and* R *a ring.*

(i) If $R[G]$ *is UU, then* R *is* UU *and* G *is a* 2*-group.*

(ii) If G *is locally finite, then* $R[G]$ *is UU if, and only if,* R *is UU and* G *is a* 2*-group.*

(iii) If H *is a normal subgroup of* G *such that* H *is locally normal and if* $R[G]$ *is UU, then* $R[G/H]$ *is UU.*

Chapter IV. Abelian Groups

Our chief results of this branch are distributed into two sections as follows:

4. Generalizations of transitive and fully transitive Abelian groups

We shall distinguish here six subsections as follows:

4.1. **On the socles of fully invariant subgroups of Abelian p-groups.** The classification of all the fully invariant subgroups of a reduced Abelian p-group is a difficult and long-standing problem, not withstanding the progress made by Kaplansky in the 1950s utilizing the notion of a fully transitive group - see Σ18 in [71]. Further progress was made for the special class of so-called *large* subgroups by Pierce in [100, Theorem 2.7]. A somewhat less ambitious programme is to try to characterize the socles of fully invariant subgroups and this is the subject of our discussions here. Despite the seeming simplification engendered by restricting attention to socles, the situation is still complicated once one moves away from fully transitive groups. We will show by means of examples that full transitivity is not the real core of the problem. We remark at the outset that the consideration of reduced groups only, is not a serious restriction; see the Note after Lemma **??** below. *Hence, in the sequel, we shall assume that our groups are always reduced p-groups for some arbitrary but a fixed prime p.*

Our notation is standard and follows [44, 71], an exception being that maps are written on the right. Finally we recall the notion of a U-sequence from [71]: a U-sequence relative to a p-group G is a monotone increasing sequence of ordinals $\{\alpha_i\}(i \geq 0)$ (each less than the length of the group G) except that it is permitted that the sequence be ∞ from some point on but that if a gap occurs between α_n and α_{n+1}, the α_n^{th} Ulm invariant of G is non-zero.

We introduce two additional concepts, the first of which shall be the primary focus our interest:

(i) A group G is said to be *socle-regular* if for all fully invariant subgroups F of G, there exists an ordinal α (depending on F) such that $F[p] = (p^\alpha G)[p]$.

(ii) Suppose that H is an arbitrary subgroup of the group G. Set $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min(H[p])$; clearly $H[p] \leq (p^\alpha G)[p]$.

Our first result states thus:

Theorem 4.1. *If G is a fully transitive group, then G is socle-regular.*

The following is also of some interest and importance.

Theorem 4.2. *Suppose that $A = G \oplus H$ where H is separable, then A is socle-regular if, and only if, G is socle-regular.*

We can also show that direct powers of a single socle-regular group are again socle-regular. In fact we have the stronger:

Theorem 4.3. *The group G is socle-regular if, and only if, the direct sum $G^{(\kappa)}$ is socle-regular for any cardinal κ.*

Once we drop the hypothesis of full transitivity, it is possible to exhibit groups of varying levels of complexity which are not socle-regular. Our first result shows that this failure can happen at the next stage beyond separability. We give two examples, the first based on the well-known realization theorem of Corner in [27], while the second is essentially due to Megibben in [87] – compare also with Chapter III above for some similar results on noncommutative rings pertaining to the endomorphism ring of groups of the present type.

Theorem 4.4. *There exist groups of length $\omega + 1$ which are not socle-regular.*

Note that the elongations of socle-regular groups by socle-regular groups need not be socle-regular. We can however obtain some additional information in the special situation where the quotient $G/p^{\omega}G$ is a direct sum of cyclic groups.

Theorem 4.5. *Let G be a group such that $G/p^{\omega}G$ is a direct sum of cyclic groups. Then G is socle-regular if, and only if, $p^{\omega}G$ is socle-regular.*

4.2. **On socle-regularity and some notions of transitivity for Abelian p-groups.** Early work in the theory of infinite Abelian p-groups focused on issues such as classification by cardinal invariants. This led initially to the rich theory known now as Ulm's theorem and, in some sense, culminated in deep classification of the class of groups known variously as *simply presented, totally projective* or *Axiom 3* groups. Such groups are, of necessity, somewhat special. On the other hand, there was also interest in properties of groups that were held by "the majority" of Abelian p-groups. Within this latter category, the extensive classes of transitive and fully transitive groups were prominent. Recently, the present authors introduced two new classes of p-groups which, respectively, properly contained the corresponding classes of transitive and fully transitive groups: these are the socle-regular and strongly socle-regular groups developed in [D8] and [35] – see also Chapter III for some related results on ring theory which could be interpreted on the endomorphism ring of abelian groups of these special kinds. The present subsection looks further at the interconnections between these classes and some other recent notions of transitivity.

Throughout, all groups will be additively written, reduced Abelian p-groups; standard concepts relating to such groups may be found in [44, 47] or [71]. We follow the notation of these texts but write mappings on the right. To avoid subsequent need for definitions of fundamental ideas, we mention that the *height* of an element x in the group G (written like $h_G(x)$) is the ordinal α if $x \in p^\alpha G \setminus p^{\alpha+1}G$ with the usual convention that $h(0) = \infty$. The *Ulm sequence* of x with respect to G is the sequence of ordinals or symbols ∞ given by $U_G(x) = (h_G(x), h_G(px), h_G(p^2x), \dots)$; the collection of such sequences may be partially ordered pointwise. Finally we recall an *ad hoc* notion introduced in [D8] which continues to be useful here: suppose that H is an arbitrary subgroup of the group G. Set $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min(H[p])$; clearly $H[p] \leq (p^\alpha G)[p]$.

We will now explore some various notions of transitivity. The notions of transitivity and full transitivity for Abelian p-groups were introduced by Kaplansky in [70] and became a topic of ongoing interest in Abelian group theory with the publication of Kaplansky's famous "little red book" [71]. Recall that a p-group G is said to be *transitive* (resp., *fully transitive*) if for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$ (resp., $U_G(x) \leq U_G(y)$) there is an automorphism (endomorphism) ϕ of G with $x\phi = y$. In recent times two addition notions of transitivity have been introduced: in [51] a group G is said to be *Krylov transitive* if, for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$, there is an endomorphism ϕ of G with $x\phi = y$. Finally, a group G was said in [51] to be *weakly transitive* if, given $x, y \in G$ and endomorphisms ϕ, ψ of G with $x\phi = y,\ y\psi = x$, there is an automorphism θ of G with $x\theta = y$. Notice in this last concept that although there is no explicit reference to Ulm sequences, the existence of the endomorphisms ϕ, ψ ensures that $U_G(x) = U_G(y)$.

To avoid a great deal of repetition, we find it convenient to use the expression *G is *-transitive* to mean that G has a fixed one of the the four transitivity properties discussed above.

In [27], Antony Corner showed that transitivity and full transitivity of a group G are determined by the action of the endomorphism ring on the first Ulm subgroup $p^\omega G$. Following his example, we say that if Φ is a unital subring of the endomorphism ring $\mathrm{End}(G)$ of G and if H is a Φ-invariant subgroup of G, then

(i) Φ is transitive on H if, for any x, y in H with $U_G(x) = U_G(y)$, there is a unit $\phi \in \Phi$ with $x\phi = y$;

(ii) Φ is Krylov transitive on H if, for any x, y in H with $U_G(x) = U_G(y)$, there is an element $\phi \in \Phi$ with $x\phi = y$;

(iii) Φ is fully transitive on H if, for any x, y in H with $U_G(x) \leq U_G(y)$, there is an element $\phi \in \Phi$ with $x\phi = y$;

(iv) Φ is weakly transitive on H if, for any x, y in H and elements $\phi, \psi \in \Phi$ with $x\phi = y$ and $y\psi = x$, there is a unit $\theta \in \Phi$ with $x\theta = y$.

Our first result is an analogue for Krylov transitivity of part of a well-known result of Kaplansky [71, Theorem 26], the other part being contained in (ii) and (iii) above.

Theorem 4.6. *Suppose G is a Krylov transitive reduced p-group and that G has at most two Ulm invariants equal to* 1*, and if it has exactly two, they correspond to successive ordinals, then G is fully transitive.*

Our next assertion shows that Krylov transitive groups behave nicely when "squared", provided that the lattice of Ulm sequences of the first Ulm subgroup is a chain.

Theorem 4.7. *Suppose G is a group such that all elements of $p^\omega G$ have comparable Ulm sequences. Then $G \oplus G$ is Krylov transitive if, and only if, G is Krylov transitive. This property may fail if there are elements of $p^\omega G$ with incomparable Ulm sequences.*

4.3. **On projectively fully transitive Abelian p-groups.** In 1952 Kaplansky, [70], began his investigations into the fully invariant and characteristic subgroups of an Abelian p-group. He followed this up in his now famous "*little red book*", Infinite Abelian Groups, [71], and introduced the notions of transitive and fully transitive p-groups in a natural way arising from his investigations in [70]; these notions have been of interest in Abelian group theory ever since. There is another notion, closely related to full invariance, which has also been studied: projection invariance. Recall that a subgroup H of the group G is said to be *projection-invariant in G* if $\pi(H) \leq H$ for all idempotent endomorphisms π of G. Significant work on this topic was produced by Hausen [57] and Megibben [88], concentrating in the main in establishing when projection-invariant subgroups are actually fully invariant; the socles of such subgroups have been investigated by the present authors in [36]. In this work we follow a somewhat different path and explore a new notion of transitivity which we shall call projective full transitivity. Recall that a group G is said to be fully transitive if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there is an endomorphism ϕ of G with $\phi(x) = y$. Our modification is to say that G is *projectively fully transitive* if the endomorphism ϕ can be chosen to be in the subring of the full endomorphism ring generated by the idempotent endomorphisms; clearly a projectively fully transitive group is always fully transitive.

We shall establish a number of basic properties of projectively fully transitive groups; in particular we shall show that this class of groups is properly contained in the class of fully transitive groups. Moreover, the class is large but is not closed under the taking of direct summands, unlike the situation which pertains for fully transitive groups. Recent work on various types of transitivity - see, for example, [D9] - has revealed the role played by 'squares' of a group in this connection and similar properties re-appear here (compare also with Chapter III concerning some ring-theoretic results which might be translated for the endomorphism ring of abelian groups of the mentioned above sorts).

To simplify the notation and to avoid risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism **ring** of G and $\mathrm{End}(G)$ for the endomorphism **group** of G. We shall denote by $\mathrm{Proj}(G)$ the **subring** of $\mathrm{E}(G)$ generated by the idempotents of $\mathrm{E}(G)$; thus an element $\phi \in \mathrm{Proj}(G)$ will have the form $\phi = \sum_{\text{finite}} \pm\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$, where each π_{i_j} is an idempotent in $\mathrm{E}(G)$.

In the final part of the subsection, we shall examine briefly an apparently stronger notion. Following Hausen, [57], we let $\Pi(G)$ denote the **subgroup** of the endomorphism group $\mathrm{End}(G)$ generated by the idempotent endomorphisms; so $\phi \in \Pi(G)$ has the form $\phi = \sum_{i=1}^{n} \pm\pi_i$ for some finite n, where each π_i is an idempotent endomorphism. Then a group G is said to be *strongly projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \le U_G(y)$, there exists $\phi \in \Pi(G)$ with $\phi(x) = y$; clearly a strongly projectively fully transitive group is projectively fully transitive. Our results here are somewhat sketchier.

Throughout, the word group will denote an additively written Abelian p-group. In this context our notation is standard and follows Fuchs [44, 47] and Kaplansky [71, 72]; mappings are written on the left.

Since it is clear that a fully transitive group G is projectively fully transitive if $\mathrm{E}(G) = \mathrm{Proj}(G)$ (and similarly it is strongly projectively fully transitive if $\mathrm{End}(G) = \Pi(G)$), we consider firstly this situation. To simplify our terminology we shall say that a group G is an *idempotent-generated* group (or IG-group) if $\mathrm{E}(G) = \mathrm{Proj}(G)$; we say that G is an *idempotent-sum* group (or IS-group) if $\mathrm{End}(G) = \Pi(G)$. If $\mathrm{E}(G)$ is commutative, then it is obvious that $\mathrm{Proj}(G) = \Pi(G)$ so that the IG-groups are then precisely the IS-groups; in general an IS-group is always an IG-group. However, this situation is rather rare for a primary group: it follows from results of Szele and Szendrei - see Exercise 6, p. 227 in [44] - that groups with commutative endomorphism ring are precisely subgroups of $\mathbb{Z}(p^\infty)$ and it is easy to see that any cyclic group is an IS-group, while the quasi-cyclic group $\mathbb{Z}(p^\infty)$ is not even an IG-group.

We will now carefully consider the class of projectively fully transitive groups as follows:

In the classical theory of transitive and fully transitive groups, it is usual to restrict consideration to reduced groups. However, it is not difficult to extend the theory to non-reduced groups. This is normally achieved by modifying the definition of an Ulm sequence for an element of a divisible group - see [71, p.57] - so that if D is divisible and $x \in D$, then $U_D(x) = (0, \dots 0, \infty, \dots)$ where the symbol ∞ occurs at precisely the $(n+1)^{st}$ place if x has order p^n; with this understanding it is easy to show that divisible groups are fully transitive – see, for example, [71, Exercise 71] or [15, Proposition 2.1]. In fact, we can show even that any divisible group is necessarily a projectively fully transitive group. Recall once again from the introduction that a group G is said to be *projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there exists $\phi \in \mathrm{Proj}(G)$ with $\phi(x) = y$; clearly a projectively fully transitive group is fully transitive.

Theorem 4.8. *If D is a divisible group, then D is a projectively fully transitive group.*

Recall [43, Definition 1] that the groups G_1, G_2 form a *fully transitive pair* if, for every $x \in G_i, y \in G_j (i, j \in \{1, 2\})$ with $U_G(x) \leq U_G(y)$, there exists $\alpha \in \mathrm{Hom}(G_i, G_j)$ with $\alpha(x) = y$. Note that $\{G_1, G_2\}$ is a fully transitive pair if, and only if, if $G_1 \oplus G_2$ is fully transitive – see [43, Proposition 1].

Summarizing, we have:

Theorem 4.9. *If $G = D \oplus R$, where D is divisible and R is reduced, then G is projectively fully transitive if, and only if, R is projectively fully transitive.*

We thus come to the following.

Theorem 4.10. *Suppose $\kappa > 1$. Then the following are equivalent:*
(i) G is fully transitive;
(ii) $G^{(\kappa)}$ is fully transitive;
(iii) $G^{(\kappa)}$ is transitive;
(iv) $G^{(\kappa)}$ is projectively fully transitive.

Three other results of interest are these:

Theorem 4.11. *Suppose that α is an ordinal strictly less than ω^2 and $G/p^\alpha G$ is totally projective. If $p^\alpha G$ is projectively fully transitive, then so also is G.*

Theorem 4.12. *A group $G = D \oplus R$, where D is divisible and R is reduced, is strongly projectively fully transitive if, and only if, R is strongly projectively fully transitive.*

Theorem 4.13. *(i) If G is strongly projectively fully transitive, then $p^\beta G$ is strongly projectively fully transitive for all ordinals β;*

(ii) if $p^n G$ is strongly projectively fully transitive for some finite n, then G is strongly projectively fully transitive;

(iii) if α is an ordinal strictly less than ω^2 and $G/p^\alpha G$ is totally projective, then if $p^\alpha G$ is strongly projectively fully transitive, so also is G;

(iv) if A, B are strongly projectively fully transitive and $\{A, B\}$ is a fully transitive pair, then $A \oplus B$ is strongly projectively fully transitive;

(v) if G is strongly projectively fully transitive, then $G^{(\kappa)}$ is strongly projectively fully transitive for any cardinal κ;

(vi) if G is totally projective of length $\leq \omega^2$, then G is strongly projectively fully transitive;

(vii) if λ is cofinal with ω and G is a C_λ-group of length $\lambda \leq \omega^2$, then G is strongly projectively fully transitive.

4.4. **On commutator socle-regular Abelian p-groups.** Throughout our discussion, we shall focus on additively written Abelian p-groups, where p is a prime fixed for the rest of the present work, although many of the topics we investigate can be considered in a much wider context. The notion of a fully invariant subgroup of a group is, of course, a classical notion in algebra, as is the weaker notion of a characteristic subgroup. Kaplansky devoted a section of his famous "Little Red Book"[71] (see also [70]) to the study of such subgroups and, arising from this, he introduced the much-studied classes of transitive and fully transitive groups – see, for example, [27, 28, 29, 43]. Recall that a group G is said to be *transitive* (respectively, *fully transitive*) if given $x, y \in G$ with Ulm sequences $\mathrm{U}_G(x) = \mathrm{U}_G(y)$ (respectively, $\mathrm{U}_G(x) \leq \mathrm{U}_G(y)$), there exists an automorphism (respectively, an endomorphism) ϕ such that $\phi(x) = y$. But there are several other weaker notions which have been of interest: recall that a subgroup H of a group G is said to be *projection invariant* in G if $\pi(H) \leq H$ for all idempotent endomorphisms π of G – see, for instance, [57, 88, 36] as well as [D9] – while a subgroup H of G is said to be *commutator invariant* in G if $[\phi, \psi](H) \leq H$ for all $\phi, \psi \in \mathrm{E}(G)$, where, as usual, $[\phi, \psi]$ denotes the additive commutator $\phi\psi - \psi\phi$. These two notions are independent of each other; in fact, there is a commutator invariant subgroup that is not projection invariant, and a projection invariant subgroup which is not commutator invariant. For the first case, consider the group $A = \langle a \rangle \oplus \langle b \rangle$ such that $\mathrm{o}(a) = p$ and $\mathrm{o}(b) = p^3$ with a proper subgroup $H = \langle a + pb \rangle$. It was established in [19] that H is commutator invariant in A but not a fully invariant subgroup. With the aid of [88] we also deduce that H is not projection invariant in A because in finite groups full invariance and projection invariance coincide. For the second case, the group G of Example **??** below will suffice; see the note immediately following the proof of Example **??** as well.

In [D8] and [35] the authors generalized the classes of transitive and fully transitive groups by focusing on the possible socles of characteristic and fully invariant subgroups (see [D9] too). In [36] full invariance was replaced by projection invariance and the current work continues this theme by replacing full invariance with commutator invariance. Our interest in this was sparked by the timely appearance of Chekhlov's interesting paper [19].

We show that in relation to commutator socle-regularity, one can restrict attention to reduced groups: if $A = D \oplus R$, where D is divisible and R is reduced, then A is commutator socle-regular if, and only if, R is commutator socle-regular- Theorem **??**. Using realization results of Corner, we establish a useful method of constructing groups whose commutator socle-regularity is precisely determined by that of its first Ulm subgroup. We then exploit this result to show, *inter alia*, that for groups G with $G/p^\alpha G$ totally projective and $\alpha < \omega^2$, commutator socle-regularity of G is determined by that of $p^\alpha G$ - Theorem 4.14; on the other hand we construct groups G, K with $p^\omega G = p^\omega K$ but K is commutator socle-regular while G is not - Example **??**.

Next, we relate the various notions of socle-regularity that have previously been investigated in [D8], [D9] and [35, 36] with commutator socle-regularity. Our principal results show that the notions are equivalent when the group involved is the direct sum of at least two copies of a fixed group - Theorem **??** - but we provide examples showing that the notions are, in fact, different in general. It follows easily from this that summands of commutator socle-regular groups need not be commutator socle-regular- Corollary **??**. However, we also show that the addition of a separable summand to a group does not influence commutator socle-regularity - Theorem 4.15.

Our interest here will focus on the Abelian p-groups involved but we should point out that a ring-theoretic perspective is also possible: Kaplansky in [72] raised the notion of rings in which every element is a sum of additive commutators - the so-called *commutator rings*. These too have been the subject of a great deal of interest; see, e.g., the recent significant work of Mesyan in [90].

We re-iterate that all groups throughout the current work are additively written Abelian p-groups, where p is an arbitrary but fixed prime. Our notation and terminology not explicitly stated herein are standard and follow mainly those in [44, 47]. As usual, $\mathrm{E}(G)$ denotes the *endomorphism ring* of a group G. We close this introduction by recalling an important result of A.L.S. Corner from [28, Theorem 6.1] which we shall use repeatedly in the sequel: *If H is a countable bounded p-group and Φ is a countable subring of $E(H)$, then H may be imbedded as the subgroup $p^\omega G$ of a p-group G such that $E(G)$ acts on H as Φ and with the property that each $\phi \in \Phi$ extends to an endomorphism ϕ^* of G.* The mapping $\phi \mapsto \phi^*$ may even be taken as a semigroup homomorphism between the respective multiplicative semigroups of the rings; we shall need this semigroup property only in Example **??**. We shall also exploit the groups constructed by Corner using this imbedding result: there is a fully transitive non-transitive p-group with first Ulm subgroup elementary of countably infinite rank and a transitive 2-group which is not fully transitive having a finite first Ulm subgroup which is the direct sum of cycles of order 2 and 8 - see Sections 3 & 4 in [29] and [51] for further details as well as Chapter III for some related results in ring theory relevant to the endomorphism ring of such abelian groups.

The construction of examples in this area invariably leads one to considerable amounts of reasonably straightforward but somewhat laborious calculations. These calculations have been recorded separately in an Appendix in order not to interfere with the presentation of results.

In the upcoming lenes we investigate some of the fundamental properties of the class of commutator socle-regular groups; we begin with the appropriate definitions.

Definition 1. A subgroup C of a group G is said to be *commutator invariant* if $f(C) \leq C$ for every $f \in \mathrm{E}(G)$ which is of the form $f = [\phi, \psi] = \phi\psi - \psi\phi$, where $\phi, \psi \in \mathrm{E}(G)$.

Clearly each fully invariant subgroup is commutator invariant, whereas the converse fails (see, e.g., [19]). Nevertheless, in some concrete situations, commutator invariant subgroups are fully invariant. Specifically, the following result from [19] holds:

Definition 2. A group G is said to be *commutator socle-regular* if, for each commutator invariant subgroup C of G, there exists an ordinal α (depending on C) such that $C[p] = (p^\alpha G)[p]$.

Our first result here asserts as follows:

Theorem 4.14. *(i) If G is a group such that either $p^\omega G = \{0\}$ or $p^\omega G \cong \mathbb{Z}(p^n)$ for some finite n, then G is commutator socle-regular;*

(ii) A group G is commutator socle-regular if, and only if, p^nG is commutator socle-regular for some $n \in \mathbb{N}$;

(iii) If G is a group such that $G/p^\alpha G$ is totally projective for some ordinal $\alpha < \omega^2$, then G is commutator socle-regular if, and only if, $p^\alpha G$ is commutator socle-regular;

(iv) Totally projective groups of length $< \omega^2$ are commutator socle-regular.

In a certain specific case the following direct summand property holds:

Theorem 4.15. *Suppose that $A = G \oplus H$ and H is separable. Then A is commutator socle-regular if, and only if, G is commutator socle-regular.*

4.5. **On commutator fully transitive Abelian groups.** Throughout the present subsection, let all groups be additive Abelian groups and let all unexplained notions and notations follow those from [44, 47] and [71].

To simplify the notation, and to avoid any risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism **ring** of a group G, and $\mathrm{End}(G) = \mathrm{E}(G)^+$ for the endomorphism **group** of a group G. Likewise, the endomorphism ψ is called *commutator* if it can be represented as $\psi = [\alpha, \beta] = \alpha\beta - \beta\alpha$ for some endomorphisms α, β of G. Commutators of endomorphisms rings of groups and certain other questions connected with them were studied in the papers from [16] to [22].

Moreover, we shall denote by $\mathrm{Comm}(G)$ the **subring** of $\mathrm{E}(G)$ containing the same identity and generated by the commutator endomorphisms. In view of the equality $[\alpha, \beta] = -[\beta, \alpha]$, an element $\phi \in \mathrm{Comm}(G)$ will have the form $\phi = \sum_{\text{finite}} c_{i_1} c_{i_2} \dots c_{i_k}$, where every c_{i_j} is a commutator in $\mathrm{E}(G)$ for $i_j \in \mathbb{N}$ and $1 \leq j \leq k \in \mathbb{N}$.

Analogically, we let $\mathrm{comm}(G)$ denote the **subgroup** of $\mathrm{End}(G)$ generated by the commutator endomorphisms; so $\varphi \in \mathrm{comm}(G)$ has the form $\varphi = \sum_{i=1}^{n} c_i$ for some finite n, where each c_i is a commutator in $\mathrm{End}(G)$. Since 1 can be represented as a finite sum of finite products of commutators, it is immediately seen that the same holds for $c_i = 1 \cdot c_i = c_i \cdot 1$ and thus $\mathrm{comm}(G) \subseteq \mathrm{Comm}(G)$.

As usual, mimicking [78, Section 27], $H_G(g)$ denotes the *height matrix* of the element g of a group G. In case that the group G is a p-group, instead of $H_G(g)$, it can be considered the *Ulm indicator* $U_G(g)$ of the element g, while if the group G is torsion-free it can be considered the *characteristic* $\chi_G(g)$. Also, $o(g)$ will denote the *order* of the element g, i.e., the least $n \in \mathbb{N}$ with $ng = 0$ or ∞ if such an n does not exist. We also define the relation $\preceq$ as follows: for $m, n \in \mathbb{N} \cup \{\infty\}$ we suppose that $m \preceq n \Leftrightarrow$ either $n \mid m$ or $m = \infty$.

Let R be an associative unital ring, let G be a group, and let $\phi\colon R \to \mathrm{E}(G)$ be a ring homomorphism. We shall define the action of R on G by the equality $r(g) = \phi(r)(g)$. Analogously as above, we denote by $\mathrm{Comm}(R)$ and $\mathrm{comm}(R)$ the subring of R and the subgroup of R^+, respectively, generated by all commutators of R. So, we come to the following notion:

Main Definition. A group G is said to be *R-commutator fully transitive* if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists $\varphi \in \mathrm{Comm}(R)$ with $\varphi(x) = y$. If φ is chosen from $\mathrm{comm}(R)$, then the group is called *R-strongly commutator fully transitive*.

In what follows we will consider several times the examined group as a module on its endomorphism ring. In particular, when $R = \mathrm{E}(G)$ and $R^+ = \mathrm{End}(G)$, one can obtain the following two concepts:

Definition 1. A group G is said to be *commutator fully transitive* (briefly written as a *cft-group*) if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists $\phi \in \mathrm{Comm}(G)$ with $\phi(x) = y$.

Definition 2. A group G is said to be *strongly commutator fully transitive* (briefly written as a *scft-group*) if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists $\varphi \in \mathrm{comm}(G)$ with $\varphi(x) = y$.

Note that if the group is reduced, then the condition $o(x) \preceq o(y)$ in both Definitions 1 and 2 can be eliminated in conjunction with [54, Proposition 2.23]. However, the later usage of that condition is basically motivated by the existence of divisible direct factors. It is also clear that any scft-group is a cft-group.

Notice that in [D12] were studied the so-termed *projectively fully transitive* p-groups,i.e., the p-groups G having the property that, for any $x, y \in G$ with $U_G(x) \leqslant U_G(y)$, there exists $\varphi \in Proj(G)$ such that $\varphi(x) = y$, where $Proj(G)$ is the subring of $\mathrm{E}(G)$ generated by the idempotents of $\mathrm{E}(G)$. There were also explored *strongly projectively fully transitive p-groups* defined in a similar way replacing $Proj(G)$ by $\Pi(G)$, which is the subgroup of $\mathrm{End}(G)$ generated by all the idempotents additively. We shall often cite and use in what follows some results of [D12].

Once again, throughout the text, the word group will denote an *additively* written *Abelian* group. In this context, our terminology not explicitly explained herein is standard and follows the excellent monographs of Fuchs [44, 47] and the book of Kaplansky [71], where all mappings are written on the left. A good source in this subject is [15] too. Likewise, if A, B are groups and $H \subseteq A$, then let $\mathrm{Hom}(A,B)H = \sum_{f\in \mathrm{Hom}(A,B)} f(H)$. Standardly, for this subsection $\mathbb{Z}_n$ denotes the cyclic group of order n, whereas the ring of integers modulo n is denoted by $\mathbb{Z}_{(n)}$.

Our work is motivated mainly by [D12] and [D13]. Here we wish to consider the situation when the projection endomorphisms are replaced by commutator endomorphisms and thus to find the similarity and the discrepancy in both of them. We just emphasize that there is no absolute analogy in both cases.

It is clear that if $\mathrm{Comm}(G) = \mathrm{E}(G)$ (resp., $\mathrm{comm}(G) = \mathrm{End}(G)$), then the fully transitive group G is a cft-group (resp., a scft-group), so we consider firstly this situation. We shall say that a group G is a *commutator-generated* group (or a CG-group for short) if $\mathrm{Comm}(G) = \mathrm{E}(G)$; reciprocally, we say that G is a *commutator-sum* group (or a CS-group for short) if $\mathrm{comm}(G) = \mathrm{End}(G)$. It is self-evident that a CS-group is a CG-group because $\mathrm{End}(G) \subseteq \mathrm{E}(G)$. Likewise, it is apparent that a group with commutative endomorphism ring is neither a CG-group nor a CS-group; for a more concrete information concerning groups with commutative endomorphism ring, we refer the interested reader to both [108] and [106] – compare also with results from Chapter III.

We can now state the following:

Theorem 4.16. *Let $\kappa > 1$ and let G be either a p-group or a torsion-free homogeneous group. Then the following condition are equivalent:*

(a) G is fully transitive;

(b) $G^{(\kappa)}$ is fully transitive;

(c) $G^{(\kappa)}$ is cft.

4.6. **On abelian groups having all proper fully invariant subgroups isomorphic.** Throughout the present subsection, let all groups into consideration be *additively* written and *abelian*. Our notations and terminology from group theory are mainly standard and follow those from [44, 47] and [71]. For instance, if p is a prime integer and G is an arbitrary group, $p^nG = \{p^ng \mid g \in G\}$ denotes the *p^n-th power subgroup* of G consisting of all elements of p-height greater than or equal to $n \in \mathbb{N}$, $G[p^n] = \{g \in G \mid p^ng = 0, n \in \mathbb{N}\}$ denotes the *p^n-socle* of G, and $G_p = \cup_{n<\omega} G[p^n]$ denotes the *p-component* of the *torsion part* $tG = \oplus_p G_p$ of G.

On the other hand, if G is a torsion-free group and $a \in G$, then let $\chi_G(a)$ denote the *characteristic* and let $\tau_G(a)$ denote the *type* of a, respectively. Specifically, the class of equivalence in the set of all characteristics is just called *type* and we write τ. If $\chi_G(a) \in \tau$, then we write $\tau_G(a) = \tau$, and so $\tau(G) = \{\tau_G(a) \mid 0 \neq a \in G\}$ is the set of types of all non-zero elements of G. The set $G(\tau) = \{g \in G \mid \tau(g) \geqslant \tau\}$

forms a pure fully invariant subgroup of the torsion-free group G. Recall that a torsion-free group G is called *homogeneous* if all its non-zero elements have the same type.

Concerning ring theory, suppose that all rings which we consider are *associative* with *identity* element. For any ring R, the letter R^+ will denote its *additive group.* To simplify the notation and to avoid a risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism ring of G and $\mathrm{End}(G) = \mathrm{E}(G)^+$ for the endomorphism group of G.

As usual, a subgroup F of a group G is called *fully invariant* if $\phi(F) \subseteq F$ for any $\phi \in \mathrm{E}(G)$. In addition, if ϕ is an invertible endomorphism (= an automorphism), then F is called *a characteristic* subgroup, while if ϕ is an idempotent endomorphism (= a projection), then F is called *a projection invariant* subgroup.

Classical examples of important fully invariant subgroups of an arbitrary group G are the defined above subgroups p^nG and $G[p^n]$ for any natural n as well as tG and the maximal divisible subgroup dG of G; actually dG is a fully invariant direct summand of G (see, for instance, [44]).

We shall say that a group G has only *trivial fully invariant subgroups* if $\{0\}$ and G are the only ones. Same appears for characteristic and projection invariant subgroups, respectively.

The following notions are our major tools.

Definition 1. A non-zero group G is said to be an *IFI-group* if either it has only trivial fully invariant subgroups, or all its non-trivial fully invariant subgroups are isomorphic otherwise.

Definition 2. A non-zero group G is said to be an *IC-group* if either it has only trivial characteristic subgroups, or all its non-trivial characteristic subgroups are isomorphic otherwise.

Definition 3. A non-zero group G is said to be an *IPI-group* if either it has only trivial projection invariant subgroups, or all its non-trivial projection invariant subgroups are isomorphic otherwise.

Note that Definition 3 implies Definition 1 and Definition 2 implies Definition 1. In other words, any IPI-group is an IFI-group and any IC-group is an IFI-group; in fact every fully invariant subgroup is both characteristic and projection invariant.

Definition 4. A non-zero group G is called a *strongly IFI-group* if either it has only trivial fully invariant subgroups, or all its non-zero fully invariant subgroups are isomorphic otherwise.

Definition 5. A non-zero group G is called a *strongly IC-group* if either it has only trivial characteristic subgroups, or all its non-zero characteristic subgroups are isomorphic otherwise.

Definition 6. A non-zero group G is called a *strongly IPI-group* if either it has only trivial projection invariant subgroups, or all its non-zero projection invariant subgroups are isomorphic otherwise.

Notice that Definition 6 implies Definition 4 and Definition 5 implies Definition 4.

On the other hand, it is obvious that Definition 4 implies Definition 1, whereas the converse fails as the next example shows: In fact, construct the group $G \cong \mathbb{Z}(p) \oplus \oplus_{\aleph_o}\mathbb{Z}(p^2)$. Since it is fairly clear that $G \neq pG$, $G \neq G[p]$ and $G = G[p^2]$, we deduce that $pG \cong \oplus_{\aleph_0}\mathbb{Z}(p) \cong G[p]$ that are the only proper fully invariant subgroups of G. However, $G \ncong G[p]$, as required. Thus there exists a p-primary IFI-group which is not a strongly IFI-group, as asserted.

However, in the torsion-free case, Definitions 1 and 4 are tantamount (see Proposition **??** below).

Moreover, each subgroup of an indecomposable group is projection invariant, so that an indecomposable group is an IPI-group if and only if it is either a cyclic group of order p for some prime p, or is isomorphic to the additive group of integers $\mathbb{Z}$.

It is worthwhile noticing in the current context that in [55] and [56] were studied p-groups which are isomorphic to their fixed proper fully invariant subgroup (see also cf. [54]) as well as in [5] were examined the so-called *IP-groups* that are isomorphic to their fixed pure subgroup.

Our purpose here is to explore some crucial properties of the defined above new classes of groups. The chief results are stated and proved in the next section.

As usual, $\oplus_m G = G^{(m)}$ will denote the *external* direct sum of m copies of the group G, where m is some ordinal (finite or infinite). The following statement asserts that in a special case the three classes from Definitions 1, 2 and 3 do coincide.

Theorem 4.17. *Let G be a p-group and let $m \geqslant 2$ be an ordinal. Then $G^{(m)}$ is an IFI-group if, and only if, G is an IC-group if and only if G is an IPI-group.*

In accordance to the last statement, since divisible groups are well-classified (cf. [44]), we will henceforth consider only *reduced* groups.

Theorem 4.18. *The following two points hold:*

(i) A non-zero group G is an IFI-group if, and only if, one of the following holds:

- *For some prime p either $pG = \{0\}$, or $p^2G = \{0\}$ with $r(G) = r(pG)$.*
- *G is a homogeneous torsion-free IFI-group of an idempotent type.*

(ii) A non-zero torsion group G is a strongly IFI-group if, and only if, it is an elementary p-group for some prime p.

So, we proceed with the following statement.

Theorem 4.19. *Suppose A is an irreducible endofinite torsion-free group, the center C of $E(A)$ is a principal ideal domain and the module ${}_CA$ has rank $\leqslant \aleph_0$. Then A is an IFI-group. Besides, if the group A is decomposable, then it is both an IC-group and an IP-group.*

The next statement also describes certain cases of IFI-groups.

Theorem 4.20. *For a torsion-free group G of finite rank, for which the center C of $E(G)$ is a ring satisfying property $(*)$, the following four conditions are equivalent:*

(1) G is an IFI-group;

(2) G is an irreducible endofinite group and C is a principal ideal E-ring;

(3) $G \cong (C^+)^{(n)}$, where n is some natural number and C^+ is a strongly indecomposable E-group of finite rank;

(4) G is a homogeneous fully transitive group of idempotent type.

5. Generalizations of simply presented Abelian p-groups

We shall distinguish here three subsections as follows:

5.1. **An application of set theory to $(\omega{+}n)$-totally $p^{\omega+n}$-projective abelian p-groups.** By the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration. Our group theoretic terminology and notation will generally follow that found in [44, 47]. In particular, $p^\omega G$ denotes the first Ulm subgroup of a group G consisting of all elements of infinite height, and $p^{\omega+n}G = p^n(p^\omega G)$. The cyclic group of order p^k will be denoted by $\mathbb{Z}_{p^k}$ and the infinite cocyclic group will be denoted by $\mathbb{Z}_{p^\infty}$. We will say a group G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. A group G is a *dsc-group* if it is isomorphic to a direct sum of countable groups. In particular, we are not assuming that our dsc-groups are necessarily reduced; in fact, they are a direct sum of a divisible group and a reduced group where the second summand is a dsc-group in the sense of [44]. Following [65] and [67], a group G is said to be a Σ-*group* if one (and hence every) *high* subgroup of G is Σ-cyclic (where a subgroup X of G is high if it is maximal with respect to the property $X \cap p^\omega G = \{0\}$).

It was asked in [65] and [67] whether or not subgroups of Σ-groups are again Σ-groups. In general, a subgroup of a Σ-group is not necessarily a Σ-group (see Example 2 of [86]). We will say G is a *totally* Σ-*group* if every subgroup of G is also a Σ-group. Our first objective is to give several different characterizations of this class (Theorem 5.1). For example, G is a totally Σ-group iff it is the direct sum of a countable group and a Σ-cyclic group. Alternatively, we will say that G is ω-*totally* Σ-*cyclic* if every separable subgroup S of G is Σ-cyclic. It is elementary that G is a totally Σ-group iff it is ω-totally Σ-cyclic.

The class of ω-totally Σ-cyclic groups can be described in other ways. For example, it coincides with the class of ω-*totally pure-complete groups*, i.e., those groups all of whose separable subgroups are pure-complete (where a group X is pure-complete if for every subgroup $S \subseteq X[p]$ there is a pure subgroup $P \subseteq X$ such that $P[p] = S$). It also coincides with the class of $\omega + n$-*totally dsc-groups*, i.e., those groups all of whose $p^{\omega+n}$-bounded subgroups are dsc-groups.

Expanding slightly on the example of Megibben in [86], if H is any group (e.g., a torsion-complete group), then there is a group G such that $p^\omega G = H$ and $G/p^\omega G$ is Σ-cyclic. Since for any high subgroup Z of G there is an embedding $Z \to G/p^\omega G$, Z must be Σ-cyclic, so that G will be a Σ-group containing H. On the other hand, if H is not countable, then G will not be a totally Σ-group. We sharpen this observation by showing that any separable group S can be embedded as a subgroup in a group G of length $\omega + 1$ which is a Σ-group but not a totally Σ-group.

More generally, if $\mathbf{C}$ is a class of groups and α is an ordinal, we will say that G is α-*totally* $\mathbf{C}$ if every p^α-bounded subgroup of G is a member of $\mathbf{C}$. Again, it is elementary that G is α-totally $\mathbf{C}$ iff every subgroup of G has the property that all of its p^α-*high* subgroups are in $\mathbf{C}$ (where a subgroup X of a group Y is p^α-high iff it is maximal with respect to the property that $X \cap p^\alpha Y = \{0\}$). In fact, we will mainly be concerned with the case where $n < \omega$, $\alpha = \omega + n$ and $\mathbf{C}$ is the class of $p^{\omega+n}$-*projective groups*; recall that G is $p^{\omega+n}$-projective if $p^{\omega+n}\mathrm{Ext}(G, X) = 0$ for all X, or equivalently, if there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic (see, e.g., [97]). So, a group is p^ω-projective iff it is Σ-cyclic. It follows easily that the class of $p^{\omega+n}$-projectives is closed under arbitrary subgroups. In addition, if G_1 and G_2 are $p^{\omega+n}$-projectives, then G_1 and G_2 are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric (i.e., there is an isomorphism that preserves the height functions on the two groups; see [46]). So, if $\mathbf{C}$ is the class of $p^{\omega+n}$-projective groups and $\alpha = \omega + n$, we have that a group G is $\omega + n$-totally $p^{\omega+n}$-projective iff every $p^{\omega+n}$-bounded subgroup X of G is $p^{\omega+n}$-projective. And since a group is p^{ω}-projective iff it is Σ-cyclic, a group is ω-totally p^{ω}-projective iff it is ω-totally Σ-cyclic.

Note that if $p^{\omega+n}G = \{0\}$, then G is $\omega + n$-totally $p^{\omega+n}$-projective iff it is $p^{\omega+n}$-projective. It is also straightforward to verify that the class of $\omega + n$-totally $p^{\omega+n}$-projectives contains the class of ω-totally Σ-cyclic groups (Corollary **??**). We will say an $\emptyset + n$-totally $p^{\omega+n}$-projective group G is *proper* if it does not belong to either of these two classes; i.e., iff it is not $p^{\omega+n}$-projective and not ω-totally Σ-cyclic. In particular, there are no proper ω-totally p^{ω}-projectives. For $0 < n < \omega$ we study the question of whether there are, in fact, any proper $\emptyset + n$-totally $p^{\omega+n}$-projective groups. In fact, we show that this question is equivalent to a natural construction expressible using *valuated vector spaces* (see, for example, [101] and [45]).

If V is a group, then a *valuation* on V is a function $v : V \to \mathbf{O}_\infty$ (where $\mathbf{O}_\infty$ is the class of all ordinals plus the symbol ∞), such that for all $x, y \in V$, $v(x \pm y) \geq \min\{v(x), v(y)\}$ and $v(px) > v(x)$. It follows that for every $\alpha \in \mathbf{O}_\infty$, $V(\alpha) = \{x \in V : v(x) \geq \alpha\}$ is a subgroup of V. If V and W are valuated groups, then a homomorphism $\phi : V \to W$ will be said to be *valuated* if $v(x) \leq v(\phi(x))$ for all $x \in V$, and an *isometry* if it is bijective and preserves all values. Note that if G is any group and H is a subgroup of G, then the height function on G restricts to a valuation on H. The category of valuated groups clearly has direct sums.

Naturally, a valuated group V is a valuated vector space if $pV = \{0\}$. In particular, the socle of a group will always be a valuated vector space. The valuated vector space V will be said to be *separable* if $V(\omega) = \{x \in V : v(x) \geq \omega\} = \{0\}$ and *free* if it is isometric to the valuated direct sum of valuated vector spaces of rank one. If W is a subspace of V, then the *corank* of W is the dimension of V/W. A subspace E of V will be called *cofree* if there is a valuated decomposition $V = E \oplus F$, where F is free [in other words, V is algebraically the internal direct sum of E and F, and $v(x + y) = \min\{v(x), v(y)\}$ for all $x \in E$ and $y \in F$].

If κ is an infinite cardinal, then a valuated vector space V will be said to be *κ-coseparable* if it is separable and every subspace W of corank strictly less than κ contains a subspace $E \subseteq W$ that is cofree in V. We will really only be concerned with the cases where $\kappa = \aleph_0$ or $\aleph_1$. A κ-coseparable valuated vector space will be said to be *proper* if it is not free. In [38] the existence of a proper $\aleph_1$-coseparable valuated vector space was shown to be equivalent to a question involving the structure of abelian groups, and to be independent of ZFC. We conclude this subsection by showing that for $0 < n < \omega$, the existence of a proper $\aleph_0$-coseparable valuated vector space is equivalent to the existence of a proper $\omega+n$-totally $p^{\omega+n}$-projective group, and we prove that both of these propositions are independent of ZFC (Theorem 5.4).

Our frontier here is the following:

Theorem 5.1. *If G is a group, then the following are equivalent:*

(a) G is a totally Σ-group;
(b) G is ω-totally Σ-cyclic;
(c) G is a Σ-group and $p^\omega G$ is countable;
(d) $G/p^\omega G$ is Σ-cyclic and $p^\omega G$ is countable;
(e) $G \cong C \oplus M$, where C is countable and M is Σ-cyclic;
(f) G is ω-totally pure-complete;
(g) For all $n < \omega$, G is an $\omega + n$-totally dsc-group;
(h) For some $n < \omega$, G is an $\omega + n$-totally dsc-group.

The following result is our main tool in analyzing proper $\omega + n$-totally $p^{\omega+n}$-projective groups. Since non-free separable valuated vector spaces are usually not $\aleph_0$-coseparable, it puts a serious limitation on the structure of proper $\omega+n$-totally $p^{\omega+n}$-projectives, showing that they are relatively rare phenomena.

Theorem 5.2. *Suppose $n < \omega$ and G is a proper $\omega + n$-totally $p^{\omega+n}$-projective group. If V is a separable valuated vector space for which there is an injective valuated homomorphism $V \to G[p]$, then V is $\aleph_0$-coseparable.*

A crucial fact is also the following one:

Theorem 5.3. *The following hold:*

(a) A group G is special iff $p^\omega G$ is finite, $G/p^\omega G$ is $p^{\omega+1}$-projective and $K(G)$ is $\aleph_0$-coseparable.

(b) The class of special groups is closed under arbitrary subgroups.

(c) Any special group is $\omega + n$-totally $p^{\omega+n}$-projective for all $0 < n < \omega$.

We come now to our main theorem on proper $\omega + n$-totally $p^{\omega+n}$-projectives.

Theorem 5.4. *The equivalence of the following three statements is a theorem in ZFC:*

(a) There is a proper $\omega + n$-totally $p^{\omega+n}$-projective group for some $0 < n < \omega$.
(b) There is a proper $\aleph_0$-coseparable valuated vector space.

(c) There is a separable $p^{\omega+1}$-projective group A which is not Σ-cyclic such that whenever G is a group with $p^{\omega}G \cong \mathbb{Z}_p$ and $G/p^{\omega}G \cong A$, then G must also be $p^{\omega+1}$-projective.

On the other hand, all three are undecidable in ZFC; in particular, they all hold in a model of MA+¬CH, whereas they all fail in a model of V=L.

5.2. **On n-simply presented abelian p-groups.** Throughout, by the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration of the subsection. Our terminology and notation will be based upon [44] and [47]. For example, if α is an ordinal, then a group G will be said to be p^{α}-*projective* if $p^{\alpha}\mathrm{Ext}(G, X) = \{0\}$ for all groups X. We will denote the height of an element $x \in G$ by $|x|_G$. We will say G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups.

The *totally projective* groups have a central position in the study of abelian p-groups (see Chapter XII of [44] or Chapter VI of [53]). One reason for their importance is the number of different ways they can be characterized; recall that a group G is totally projective if any one of the following equivalent conditions is satisfied:

(1) G is *simply presented*;
(2) G is *balanced projective*, i.e., $\mathrm{Bext}(G, X) = \{0\}$ for all groups X;
(3) $G/p^{\alpha}G$ is p^{α}-projective for every ordinal α;
(4) G has a *nice system*;
(5) G has a *nice composition series*.

It is worth pointing out that, unlike the treatment in [44], we do not require a simply presented group to be reduced.

In a somewhat different direction, if n is a non-negative integer (that will be fixed for the remainder of this section), then the group G is $p^{\omega+n}$-projective iff there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic (see, e.g., [97]). So, a group is p^{ω}-projective iff it is Σ-cyclic. It follows easily that the class of $p^{\omega+n}$-projectives is closed under arbitrary subgroups. In addition, if G_1 and G_2 are $p^{\omega+n}$-projectives, then G_1 and G_2 are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric (i.e., there exists an isomorphism that preserves the height functions on the two subgroups as computed in the whole groups; see [46]).

A number of papers have been written over the years that combine elements of these two important components of the study of abelian p-groups (see, for example, [48], [49] and [74]). In this and a subsequent section, we will consider several other interesting ways to combine them.

Generalizing (1), a group G will be said to be *n-simply presented* if there is a subgroup $P \subseteq G[p^n]$ such that G/P is simply presented. Such a subgroup will be called *n-simply representing.* It follows, therefore, that the class of n-simply presented groups includes both the simply presented groups and the $p^{\omega+n}$-projective groups.

In terms of homological algebra, we say a short exact sequence $0 \to X \to Y \to G \to 0$ is *n-balanced exact* if it represents an element of $p^n\mathrm{Bext}(G, X)$. Generalizing (2), we say G is *n-balanced projective* if every such n-balanced exact sequence splits. We show that G is n-balanced projective iff it is a summand of a group that is n-simply presented, and that there are enough n-balanced projectives (Theorem 5.5). We also show that a separable group G is n-simply presented iff it is n-balanced projective iff it is $p^{\omega+n}$-projective (Proposition **??**).

If G is $p^{\omega+n}$-projective and P is a subgroup of $G[p^n]$ such that G/P is Σ-cyclic, then P will, in fact, be *nice* in G (i.e., every coset $x+P$ will contain an element of maximal height). This leads to a further generalization of (1): We say the group G is *strongly n-simply presented* if it has an n-simply representing subgroup which is nice.

Continuing in the language of homological algebra, we say a short exact sequence $0 \to X \to Y \xrightarrow{\phi} G \to 0$ is *strongly n-balanced exact* if it is balanced and there is a height-preserving homomorphism $\nu : G[p^n] \to Y[p^n]$ such that $\phi \circ \nu$ is the identity on $G[p^n]$ (note that if $n \geq 1$, then the latter condition already implies that the sequence is balanced - see, for instance, [44, Proposition 80.2]). In other words, we are requiring that the induced exact sequence, $0 \to X[p^n] \to Y[p^n] \to G[p^n] \to 0$, is split in the category of valuated groups. We can, therefore, consider the class of *strongly n-balanced projectives.*

In parallel with the above, we next show that a group G is strongly n-balanced projective iff it is a summand of a group that is strongly n-simply presented, and that there are enough strongly n-balanced projectives (Theorem 5.6). We also show that a $p^{\omega+n}$-bounded group G is strongly n-simply presented iff it is strongly n-balanced projective iff it is $p^{\omega+n}$-projective (Proposition **??**).

One of the most useful and important results in the study of totally projective groups is a theorem of Nunke from [98] which states that if λ is an ordinal, then a group G is totally projective iff $p^\lambda G$ and $G/p^\lambda G$ are both totally projective (see, for example, [53, Theorem 74]). The same property was independently proved by Crawley-Hales for simply presented groups (see [30] and [31]). It is not hard to see that if G is (strongly) n-simply presented or (strongly) n-balanced projective, then $p^\lambda G$ and $G/p^\lambda G$ must share the corresponding property (Theorem 5.7(a) and Proposition **??**(a)). The converse is rather more complicated. We show that if $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are strongly n-simply presented or strongly n-balanced projective, then so is G (Theorem 5.7(b) and Proposition **??**(b)). On the other hand, for ordinals not of the form $\lambda + n$ (e.g., limit ordinals), we show that this can fail for strongly n-simply presented groups (Example **??**).

The next part of the subsection is devoted to showing that for an arbitrary ordinal λ, if $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented or n-balanced projective, then the same can be said of G (Theorem 5.8 and Corollary **??**). This surprisingly very difficult proof requires a detailed examination of the behavior of bounded subgroups P of G for which G/P is simply presented.

These properties allow us to conclude that for any group G of length strictly less than ω^2, that G is (strongly) n-simply presented iff it is (strongly) n-balanced projective (Corollaries **??** and **??**). In other words, the (strongly) n-simply presented groups of length less than ω^2 are closed under taking direct summands. Later, we will establish some further statements of this sort.

A group G is $p^{\omega+n}$-projective iff there is a Σ-cyclic group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$ (see, e.g., [46]). The proof of this property depends solely on the fact that T is Σ-cyclic iff p^nT is Σ-cyclic. Similarly, we say G is *n-co-simply presented* if there is a simply presented group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$. Since T is also simply presented iff p^nT is simply presented, the same proof shows that G is n-simply presented iff it is n-co-simply presented.

We begin by describing the summands of the n-simply presented groups.

Theorem 5.5. *The group G is n-balanced projective iff it is a summand of a group that is n-simply presented. There are enough n-balanced projectives.*

We also have the following analogue of Theorem 5.5.

Theorem 5.6. *The group G is strongly n-balanced projective iff it is a summand of a group that is strongly n-simply presented. There are enough strongly n-balanced projectives.*

Nunke's-like theorems for our point of view are of the type:

Theorem 5.7. *Suppose λ is an ordinal and G is a group.*

(a) If G is (strongly) n-simply presented, then both $p^\lambda G$ and $G/p^\lambda G$ are (strongly) n-simply presented.

(b) If both $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are (strongly) n-simply presented, then G is (strongly) n-simply presented.

By a *graded vector space*, we will mean a collection of vector spaces indexed by the ordinals, $U = [U_\alpha]_{\alpha<\infty}$, such that there is an ordinal λ with $U_\alpha = \{0\}$ for all $\alpha \geq \lambda$; the smallest such ordinal λ we call the *length* of U. The definition of a graded homomorphism or isomorphism follows naturally and the resulting category of graded vector spaces clearly has direct sums. We say $x \in U$ if there is an α such that $x \in U_\alpha$ and if $x \neq 0$ we write $|x|_U = \alpha$. We say U is *admissible* if its *Ulm function* $f_U(\alpha) = r(U_\alpha)$ is admissible in the usual sense. Let $R(U) = \sum_{\alpha<\infty} r(U_\alpha)$, and if β is an ordinal, let $R_\beta(U) = \sum_{\beta \leq \alpha < \beta+\omega} r(U_\alpha)$.

Our motivating example is where V is a valuated vector space (e.g., the socle of some group) and $U(V)$ is the graded vector space $[U_\alpha(V)]_{\alpha<\infty} = [V(\alpha)/V(\alpha+1)]_{\alpha<\infty}$. We let $R(V) = R(U(V))$ and $R_\beta(V) = R_\beta(U(V))$. If $\mathcal{L}$ is a subset of a valuated vector space V, then for each ordinal α we let $\mathcal{L}_\alpha = \{x \in \mathcal{L} : |x|_V = \alpha\}$ and we let $\mathrm{span}(\mathcal{L})$ be the graded vector space $[\mathrm{span}(\mathcal{L}_\alpha)]_{\alpha<\infty}$.

This brings us to the objective of this section.

Theorem 5.8. *Suppose G is a group and λ is any ordinal. Then G is n-simply presented iff $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented.*

Theorem 5.9. *If G is a group and λ is a limit ordinal such that $p^\lambda G$ is bounded and $G/p^\lambda G$ is n-simply presented, then G is n-simply presented.*

5.3. **On ω_1-n-simply presented abelian p-groups.** Throughout the present subsection, let all groups into examination be p-torsion abelian written additively as is the custom when discussing such groups. Also, let $n \geq 0$ be a non-negative integer. Most of the used notions and notations are standard and can be seen in the classical sources [44, 47] and [53]. For the more specific terminology the interested reader can read [37, 38] and [D11] (actually, representing the statements from the previous subsection). For instance, we will abbreviate G as a *dsc-group* if it is a *direct sum of countable groups*. Besides, imitating [D11] (compare with the preceding subsection, too) a group G is called *n-simply presented* if there is $P \leq G[p^n]$ such that G/P is simply presented. When P is nice in G, such groups are said to be *strongly n-simply presented* or *nicely n-simply presented*. The last is a common generalization of the well-known concept of $p^{\omega+n}$-projectivity due to Nunke where G is *$p^{\omega+n}$-projective* whenever there exists a p^n-bounded subgroup $P \leq G$ such that G/P is *Σ-cyclic* (= a direct sum of cyclics). Later on, Keef enlarged in [75] that notion to the so-called *ω_1-$p^{\omega+n}$-projective* groups that are groups G for which there exist countable (nice) subgroups C such that G/C are $p^{\omega+n}$-projective.

This article is an extension of n-simply presented groups in the spirit of (the previous generalizations of) ω_1-$p^{\omega+n}$-projective groups. It is organized as follows: In the first part, i.e. here, we put the main definitions. In the second one, we prove some useful preliminary assertions and state some background material, and in the third one we state with proofs the major results in the subject. Next, in the final stage, we prove a series of statements concerning the important full Nunke-esque property, and we close in the remaining part with some unsettled challenging questions.

Definition 1. The group G is called *ω_1-n-simply presented* if there is a countable subgroup K of G such that G/K is n-simply presented. In addition, if K is finite, G is said to be *ω-n-simply presented*.

When $n = 0$, and as a result G/K is simply presented, we will just say that G is *ω_1-simply presented*. But if K is nice in G, G is just simply presented (see [37] or [38]). Likewise, ω-n-simply presented groups are precisely n-simply presented.

When K is a priory chosen nice in G, one may state:

Definition 2. The group G is called *nicely ω_1-n-simply presented* if there is a countable nice subgroup N of G such that G/N is n-simply presented.

When $n = 0$, and hence G/N is simply presented, we observe with the aid of [37, 38] that G must be simply presented, too.

Definition 3. The group G is said to be *strongly ω_1-n-simply presented* if there exists a countable subgroup C of G such that G/C is strongly n-simply presented. In addition, if C is finite, we will say that G is *strongly ω-n-simply presented*.

In case that C is taken a priory nice in G, one can state:

Definition 4. The group G is said to be *strongly nice ω_1-n-simply presented* if there exists a countable nice subgroup M of G such that G/M is strongly n-simply presented.

Apparently, because $p^{\omega+n}$-projective groups are strongly n-simply presented, the ω_1-$p^{\omega+n}$-projectives, defined as in [75], are themselves strongly nice ω_1-n-simply presented. Moreover, strongly ω-n-simply presented groups are strongly nice ω_1-n-simply presented, because finite subgroups are always nice. As indicated in [D11], strongly ω-n-simply presented groups need not be strongly n-simply presented.

Also, it is clear that Definition 4 yields Definition 2 and Definition 3 implies Definition 1. Likewise, some enlargements of this kind for the n-totally projective groups from [76] can be given as well.

On the other vein, Hill and Megibben gave in [63] the definition of *a c.c. group* as a group G such that $p^{\omega}(G/C)$ is countable whenever $C \leq G$ is a countable subgroup. For our applicable purposes we shall now enlarge this concept to the so-called *α-countably groups* where α is an arbitrary ordinal. This is necessary because the approach used in [D11] does not work here because $p^{\alpha+n}(G/C)$ is not always contained in $(p^{\alpha}G + C)/C$ if $p^nC \neq \{0\}$.

Definition 5. We will say that the group G is *α-countably* if for any its countable subgroup C the factor-group $p^{\alpha}(G/C)/(p^{\alpha}G + C)/C$ is always countable.

Note also that if either C is a nice subgroup or $\alpha \in \mathbb{N}$, the factor-group $p^{\alpha}(G/C)/(p^{\alpha}G + C)/C$ equals to zero, and so Definition 5 is satisfied in both situations.

When $\alpha = \omega$, the posed condition is equivalent to the countability of the quotient $[\cap_{i<\omega}(p^iG+C)]/[\cap_{i<\omega}p^iG]$ which in turn is tantamount to the countability of the quotient $[\cap_{i<\omega}(p^iG + C)]/(\cap_{i<\omega}p^iG + C)$. Apparently, c.c. groups are always ω-countably. To treat the converse relationship, one sees that if $p^{\omega}G$ is countable, then every ω-countably group is a c.c. group, and thus these two notions do coincide. In particular, weakly ω_1-separable groups (which are of necessity separable), are ω-countably as well as ω-countably separable groups are weakly ω_1-separable.

Definition 6. We will say that the group G is *α-boundary* if for any its countable subgroup C the factor-group $p^{\alpha}(G/C)/[(p^{\alpha}G + C)/C]$ is always bounded.

In particular, there is a natural number m such that the inclusion $p^{\alpha+m}(G/C) \subseteq (p^{\alpha}G + C)/C$ holds.

Note also that if either C is a nice subgroup or $\alpha \in \mathbb{N}$ then the quotient $p^{\alpha}(G/C)/[(p^{\alpha}G+C)/C]$ equals to zero, as well as if $p^mC = \{0\}$ then the inclusion $p^{\alpha+m}(G/C) \subseteq (p^{\alpha}G + C)/C$ holds appealing to Lemma 3.1 of [D11], and thus in all cases Definition 6 is fulfilled.

Our first result here is the following:

Theorem 5.10. *The following points are equivalent:*

(i) G is ω_1-n-simply presented;

(ii) $G/(C \oplus L)$ is simply presented where C is a countable subgroup of G and L is a p^n-bounded subgroup of G;

(iii) G/L is ω_1-simply presented for some $L \leq G[p^n]$.

Our next claim asserts the following:

Theorem 5.11. *Nicely ω_1-n-simply presented groups of length $< \omega^2$ are n-simply presented.*

We are now ready to state the following central result:

Theorem 5.12. *The class of ω_1-n-simply presented groups is closed under the formation of ω_1-bijections, and is the smallest class containing n-simply presented groups with this property.*

In other words, if $f : G \to A$ is an ω_1-bijective homomorphism and G is an ω_1-n-simply presented group, then A is an ω_1-n-simply presented group, and ω_1-n-simply presented groups form the minimal class of groups possessing that property.

We are now proceed with the following main result:

Theorem 5.13. *The classes of strongly ω_1-n-simply presented groups, nicely ω_1-n-simply presented groups and strongly nice ω_1-n-simply presented groups are closed under taking ω-bijections. Moreover, the class of strongly nice ω_1-n-simply presented groups is the smallest (minimal) class containing strongly n-simply presented groups possessing that property.*

So we come to the following two assertions of Nunke's-esque form.

Theorem 5.14. *Suppose G is a λ-boundary group for some ordinal λ such that $p^\lambda G$ is n-simply presented. Then G is ω_1-n-simply presented if, and only if, $G/p^\lambda G$ is ω_1-n-simply presented.*

Theorem 5.15. *Suppose G is a λ-countably group for some ordinal λ such that $p^\lambda G$ is n-simply presented. Then G is ω_1-n-simply presented if, and only if, $G/p^\lambda G$ is ω_1-n-simply presented.*

Chapter V. Left-Open Problems

We shall here state some still unsettled intriguing questions as well as we shall restate for completeness of the exposition some already putted queries in the corresponding subsections quoted above.

Concerning ring theory (possibly non-commutative), recall that a ring R is said to be π-*regular* if, for each $a \in R$, there is a natural number n (depending on a) such that $a^n \in a^n R a^n$.

Problem 5.16. Does it follow that all weakly exchange (respectively, all exchange) rings whose units are sums of two idempotents are π-regular?

Problem 5.17. Suppose that R is a ring with $R = Id(R) + Id(R)$ such that $U(R) = 1 + Nil(R)$. Is it true that R is (weakly) exchange or even π-regular?

We close the queries on ring theory with

Problem 5.18. Let R be a ring and G a group. Is the group ring $R[G]$ a UU-ring iff R is a UU-ring and G is a 2-group? If *not*, find a necessary and sufficient condition for $R[G]$ to be UU only in terms of R, G and their sections.

Concerning Abelian group theory, we finish off our work with a few challenging problems of certain interest and importance, some of which are also relevantly stated for concreteness in the separate subsections alluded to above. So, we ask for the following:

Problem 5.19. Are reduced simply presented p-groups necessarily projectively fully transitive?

Problem 5.20. Suppose $n \in \mathbb{N}$. If the direct sum $G \oplus H$ is a strongly n-simply presented group for two groups G and H such that H is countable, does the complement G is also strongly n-simply presented?

We end all the work with our final query.

Problem 5.21. In the presence of **ZFC**, if G is a proper $(\omega + n)$-totally $p^{\omega+n}$-projective p-group for some $n \in \mathbb{N}$, does it follow that $p^\omega G$ is necessarily countable?

Chapter VI. References/Bibliography

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