SOME CLASSES OF NONCOMMUTATIVE RINGS AND ABELIAN GROUPS

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ABSTRACT and MAIN PURPOSES

The dissertation deals with some aspects of ring theory and commutative group theory as, specifically, it will be focussed on some structural and characterization results concerning the specific ring and group structures. In the machinery that we will use and develop in order to prove the established results, the following directions in modern algebra will be implemented in some of the proofs:

(i) *matrix theory and computations*

(ii) *homological algebra*

(iii) *set theory and formal logic*

(iv) *graph theory*

(v) *number theory*

A brief outline of some of the main results includes the following:

Result 1. *A ring R is invo-clean $\iff$ R decomposes as $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or R_1 is a nil-clean ring of characteristic ≤ 8, and either $R_2 = \{0\}$ or R_2 is embedding in a direct product (i.e., it is a subdirect product) of a family of copies of the field $\mathbb{F}_3$. In particular, if R is strongly invo-clean, then $R_1/J(R_1)$ is Boolean with nil $J(R_1)$ whenever R_1 is non-zero.*

Result 2. *A ring R is uniquely weakly nil-clean $\iff$ R is decomposable as $R \cong R_1 \times R_2$, where either $R_1 = \{0\}$ or $R_1/J(R_1)$ is Boolean with nil $J(R_1)$, and either $R_2 = \{0\}$ or $R_2/J(R_2) \cong \mathbb{Z}_3$ with nil $J(R_2)$.*

Result 3. *The weakly nil-clean index of $\mathbb{T}_2(\mathbb{Z}_p)$ is equal to p, while for $\mathbb{T}_3(\mathbb{Z}_p)$ it is p^2, whenever p is a prime number; the weakly nil-clean index for $\mathbb{M}_2(\mathbb{Z}_3)$ equals to* 5.

Result 4. *A ring R is strongly n-torsion clean for some $n \in \mathbb{N}$ $\iff$ R is strongly clean and $U(R)$ is of finite exponent. In particular, if n is odd, then R is a clean ring in which orders of all units are odd, bounded by n and there exists a unit of order n $\iff$ R is a subdirect product of copies of the fields $\mathbb{F}_{2^{k_i}}$, where $i \in [1, t]$ for some integer $t \geq 1$ such that there exist integers $k_1, \cdots, k_t \geq 1$ with $n = LCM(2^{k_1} - 1, \ldots, 2^{k_t} - 1)$.*

Result 5. *If G is a locally finite group and R is an arbitrary ring, then the group ring $R[G]$ is UU $\iff$ R is UU and G is* 2*-torsion.*

Result 6. *Let $G = A \oplus B$ be a group. Then*

(1) G is socle-regular $\iff$ A is socle-regular, provided B is separable.

(2) A is socle-regular, provided G is socle-regular, that is, a direct summand of a socle-regular group is again a socle-regular group.

(3) Krylov transitive groups are themselves socle-regular with irreversible implication.

(4) There is a weakly transitive group which is not socle-regular.

(5) Any totally projective group of length $\leq \omega^2$ is strongly projectively fully transitive.

(6) If G is a group such that the first Ulm subgroup $p^\omega G$ is elementary, then G is fully transitive $\iff$ the square $G \oplus G$ is strongly projectively fully transitive $\iff$ the square $G \oplus G$ is strongly commutator fully transitive.

(7) Any totally projective group of length $< \omega^2$ is commutator socle-regular.

(8) A direct summand of a commutator socle-regular group is not necessarily commutator socle-regular; a direct summand of a commutator fully transitive group need not be commutator fully transitive too.

(9) Both projective socle-regularity and commutator socle-regularity notions are independent to transitivity and full transitivity.

(10) Commutator fully transitive groups are always commutator socle-regular.

(11) A direct summand of a fully transitive torsion-free IFI-group is again a fully transitive IFI-group.

(12) If G is an IFI-group, then $G \oplus G$ is also an IFI-group.

(13) Any strongly irreducible group G such that $|G/pG| \leq p$ for each prime p is an IFI-group.

Result 7. *Suppose that G is a group such that the factor-group $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective. If $p^{\omega+1}G$ is countable, then G is the direct sum of a $p^{\omega+1}$-projective group and a countable group. Moreover, there is a group G for which $G/p^{\omega+2}G$ is $p^{\omega+2}$-projective and $p^{\omega+2}G$ is countable, but G is not a direct sum of a $p^{\omega+2}$-projective group and a countable group. In particular, if $0 < n < \omega$, then the class of $\omega + n$-totally $p^{\omega+n}$-projective groups is not closed under (finite) direct sums.*

Result 8. *Suppose G is a group, n is an arbitrary natural and λ is an arbitrary ordinal. Then G is n-simply presented $\iff$ both $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented.*

Result 9. *For every $n \in \mathbb{N}$ a direct summand of an n-simply presented group is again an n-simply presented group, provided that the complement is a countable group.*

Result 10. *Let $n \in \mathbb{N}$. Then the following two points hold:*

(a) Nicely ω_1-n-simply presented groups of length $< \omega^2$ are n-simply presented.

(b) Suppose G is a group whose quotient $G/p^\lambda G$ is n-simply presented for some ordinal λ. Then G is nicely ω_1-n-simply presented $\iff$ $p^\lambda G$ is nicely ω_1-n-simply presented.

So, the main purpose of this dissertation is to promote some new ideas in certain contemporary subjects of algebra as well as to demonstrate a new insight of ideas and methods in some branches which could be of further interest for future developments. This will be subsequently achieved in the next sections and their subsections. Our strategically point of view is in developing of a modern technology which will be approachable in many cases in both ring theory and commutative group theory.

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Chapter I. Introduction and Fundamentals

Everywhere in the text of this dissertation, although it is concretely specified in each section, all rings into consideration will be associative unital (sometimes called *unitary*) and all groups unless it is explicitly stated something else (e.g., the unit groups of rings and the groups which form the group rings) will be assumed additive Abelian.

The motivation in writing up this dissertation is to illustrate the study of two different at first glance topics in the modern algebra, which topics actually possess a few close relationships each to other. In fact, one evidence for the existence of such a transversal is the endomorphism ring of Abelian groups. The key approach is that the ring structure unambiguously helps us to decide how the investigated groups are situated into some well-behaved classes of groups.

Specifically, these two subjects are relevant to the following two omnibuses:

(1) *Weakly Exchange Rings with Applications to Group Rings*

We here demonstrate the role of weakly exchange rings and weakly clean as being a common expansion of the classical exchange rings and clean rings to the general theory of rings and modules having numerous applications in the area of (not necessarily commutative) group rings. The new moments are in proving up that some complete descriptions of these ring classes do exist, including also some new dealings with the long-known classes of nil-clean rings, weakly nil-clean rings, invo-clean rings and some their modifications.

(2) *Generalizations of (Fully) Transitive and Simply Presented Abelian Groups*

We here show the majority of some new classes of commutative groups (e.g., the classes of (strongly) socle-regular p-groups and (strongly) n-simply presented p-groups) to the general point of view in the theory of Abelian groups. The new moments are in showing up that some complete descriptions of these group classes do exist, including also some new treatments of the well-known classes of Krylov transitive groups, weakly transitive groups, IFI-groups, n-balanced projective groups, n-simply presented groups, ω_1-n-simply presented groups and some their variations.

To be more concrete, the principally known more important results pertaining to the comments alluded to above, on which results we will somewhat do improvements below, are these:

On **(1)** we have that:

• There exist various characterization results on clean, exchange, nil-clean, weakly nil-clean and some other closely related sorts of rings (see, for more account, [1], [2], [11], [12], [39], [40], [41], [64], [68], [77], [80], [93], [94], [95], [96], [110], etc. some other sources listed below in the literature).

Indeed, invoking the classical source [93], where the pivotal concepts of *clean* and *exchange* rings were defined, what can be more importantly mentioned is that *a ring R is clean (resp., exchange) iff the quotient $R/J(R)$ is clean (resp., exchange) and all idempotents lift modulo $J(R)$ (that is, given $r \in R$ with $r^2 - r \in J(R)$, there is $e \in Id(R)$ having the property that $e - r \in J(R)$).* This was somewhat strengthened in [12] (see also [40] for the commutative case) for the class of weakly nil-clean rings (in fact, it was proven there that any weakly nil-clean ring is necessarily clean, and clean rings are exchange). Same type of results appeared for nil-clean rings in [41] (let us remember that nil-clean rings are always weakly nil-clean). Likewise, further refinements were definitely obtained in [94], [64], [80] and [110], respectively, where under new points of view a new insight in the global structure of clean and exchange rings, arising from some new conditions, was established.

• There exist certain matrix computations regarding to what extent the structure of matrix ring will heavily depend on the structure of the former ring (see, for more information, [4], [11], [81]).

Indeed, inspired by the definition of the notion *clean index* of a ring, which somewhat reaches one of the best knowledge for the class of clean rings, in [4] was defined the concept of *nil-clean index* of a ring. Some interesting results in that matter were proved. Furthermore, in regard to [4], we define in the corresponding subsection below the more general setting of *weakly nil-clean index* of a ring which is of merit investigation being a successful instrument for the full characterization of *uniquely weakly nil-clean* rings (compare also with [D5]) – in that way, some concrete computations were done for certain special full and triangular matrix rings. Also, being involved with certain extremely difficult matrix questions in ring theory, some recent progress was made in [32].

Being closely familiar with the general *theory of matrices* and its computational aspects, we shall try to give a comprehensive presentation of its use in the contemporary directions of the associative rings, especially in their structural characterizations – see, e.g., [99] and [107].

• There exist results focussed on the isomorphic structure of group rings which entirely relies on the group structure of the basis and on the ring structure of the initial ring (see, for more concrete news, [40], [73], [84], [85], [92], [103]).

Indeed, May gave in [84] a complete description of the nil radical of an arbitrary group ring in terms of special elements, whereas Karpilovsky somewhat enlarged that to the Jacobson radical of such a ring. On the other vein, Nicholson explored in [92] local group rings, while in [85] McGovern et al. found a necessary and sufficient condition for a commutative group ring to be nil-clean (for a general necessary and sufficient condition in that way, we refer to [103]). This was substantially strengthened in [40] by the present author of the dissertation along with McGovern to the larger class of weakly nil-clean rings (see [77] too).

On **(2)** we have that:

• There are a series of results dealing with the characterization of both classical classes of transitive and fully transitive groups and their non-trivial extensions.

In fact, the classical properties of transitivity and full transitivity for Abelian groups were firstly defined by Kaplansky in [71] as a common extension of some well-studied classes of primary Abelian groups. Both the definitions entirely rely on the manner how two arbitrary elements of the group are situated, by mapping one to other via an existing group endomorphism, depending on their Ulm sequences in the full group. Likewise, the independence of these two notions was firstly showed by Corner in [29]. Namely, he exhibited an Abelian p-group which is fully transitive but not transitive as well as an Abelian 2-group which is transitive but not fully transitive – note the remarkable fact that every transitive group which is not fully transitive is necessarily a 2-group, a fact first shown by Kaplansky in [71, Theorem 26]. Despite this Corner's result, there is a connection between the two concepts: in fact, Files and Goldsmith showed in [43] that an Abelian p-group G is fully transitive if, and only if, the square's Abelian p-group $G \oplus G$ is transitive. This critical fact will somewhat be refined in one of our subsections. Furthermore, major works on transitive and fully transitive groups were produced in [52] and [61], respectively.

Further very general notions of transitivity were introduced by Goldsmith and Strüngmann in their seminal papers [50] and [51], namely they defined the so-called *Krylov transitive* and *weakly transitive* Abelian p-groups. They proved there that these two concepts are independent each to other as well.

Some recent advantage in the topic was done in [10, Theorem 2.5] by showing that there is a Krylov transitive 2-group that is neither transitive nor fully transitive nor weakly transitive, thus answering a question posed by Danchev-Goldsmith in [D9]. In proving that, they establish the surprising fact stated in [10, Proposition 2.4] that if the Abelian p-group $G \oplus G$ is Krylov transitive, then the Abelian p-group G is fully transitive. Some other effective results could be found in [87, 88] as well.

Nevertheless, among the existing unsettled things of the problematic in the corresponding literature, stated in the reference list of the bibliography, left-open were the questions of what can be said for the structure of the former group, provided its endomorphism ring is (additively) generated by commutators. In other words, all endomorphism are representable as a finite sum of products of commutators. We will be trying to give in the current study some satisfactory affirmative answer in this subject. Our solution will depend heavily on the structure of the first Ulm subgroup of the whole group, determined by the action of the full endomorphism ring on this subgroup (see [D12], [D13] and [D14]).

Our major goals here are to promote a new insight in the structure of the afore-defined (projective, commutator) transitive-like groups and to demonstrate their capability for the classical concepts of transitivity and full transitivity due to Kaplansky in his famous red-book leading to the publication of the monograph [71].

The methods we have developed in order to establish these results are certain innovations in the representation of projective and commutator endomorphisms in terms of matrices, by strengthening the methodology utilized in [D14]. They are concerned with ingenious computations involving number theory and some other not too classical instruments and tricks.

• There are a series of results which deal with the relationships between characteristic, fully invariant and projection-invariant subgroups of Abelian groups (see, e.g., [54], [55, 56], [57], [88]).

In fact, Grinshpon in [54] and Grinshpon et al. in [55],[56] consider those groups (namely, torsion, torsion-free and mixed groups) whose fully invariant subgroups have finite Ulm-Kaplansky invariants and are also endowed with some additional properties. We shall extend this by examining the groups for which all fully invariant subgroups are isomorphic (see [D15], too) as well as we shall consider some other relations and combinations between appropriate group classes.

• There are a series of results pertaining to the generalization of totally projective and simply presented Abelian p-groups in various aspects by considering their purely algebraic structure as well as their homological behavior (see, cf. [44, 47], [62, 63], [66], [74, 75], [76], [97, 98]).

In fact, giving a brief outline of the most important of them, it is a Nunke's achievement in [98] proving the reduction criterion that a group G is totally projective (resp., simply presented) iff so are both the groups $p^\alpha G$ and $G/p^\alpha G$. We considerably supersede that in this dissertation to the class of n-simply presented groups as our proof is rather difficult and long equipping more than ten pages (compare with [D11] as well). On the other hand, concerning their homological shape, totally projective groups are known to be balanced projective with respect to all short-exact sequences (cf. [44]). This will also be improved here for the class of n-balanced projectives, whenever $n \geq 1$ (see [D11] and [76]). Further generalizations are given in [D16]. Some numerical invariants involving set theory machinery were given in [3].

Being closely familiar with the theories of *homological algebra* and *set theory*, we shall try to give a detailed presentation of their usage in the contemporary directions of the commutative groups – see, for instance, [97, 98] and [100]. The theory of valuated groups also plays a crucial role in the structural aspects of Abelian groups – see, for example, [101].

Further details in both points (1) and (2) are stated in each of the subsequent sections and their subsections separately.

As for the fundamental notions, notations and terminology, we will follow mainly those from the classical monograph series of [79], [102] as well as of [44, 47], [71] and [78]. Nevertheless, for readers' convenience and for the sake of completeness, they will be stated in details in the duration.

Chapter II. Background and Conventions

For the present dissertation, a ring R will be assumed to be an associative ring with identity 1 which differs from the zero element 0. We shall use in the sequel the notation $Id(R)$ to denote the set of all idempotents of R, $Nil(R)$ to denote the set of all nilpotents of R, and $U(R)$ to denote the set of all units of R. We also shall use $\mathbb{M}_n(R)$ to denote the ring of all $n \times n$ matrices with entries in R (also called the full matrix ring) and $\mathbb{T}_n(R)$ to denote the ring of all $n \times n$ upper triangular matrices with elements from R (also called the upper triangular matrix ring), whenever $n \in \mathbb{N}$, the set of all positive integers (also termed naturals). Almost all other ring-theoretical notions and terminologies with which we have played will be in agreement with those from [79] as the more profit ones will be explicitly stated and formulated in each separate section and subsection from the corresponding chapters. About the conventions in writing up the text, we shall use "**wnc**" to denote the "**weakly nil-clean**" index of a ring as well as "**a JU-ring**" will mean "**a ring with Jacobson units**".

Likewise, all our groups in Section 2 titled "Applications to Group Rings" of Chapter III "Noncommutative Rings", where group rings are considered, will be written multiplicatively – surely, same appears and for the unit group of an arbitrary ring.

Concerning Abelian group theory, all our groups with which we will play are assumed to be additively written. The notion and notation will follow in general those established in [44, 47] with some little exceptions which will be specified and clarified when needed in the text. About the conventions in writing, used throughout the dissertation, we shall abbreviate "**a dsc-group**" for "**a direct sum of countable groups**" as well as "**projectives**" for "**projective groups**". Besides, abbreviating "**a cft-group**" means "**a commutator fully transitive group**" as well as "**a scft-group**" means "**a strongly commutator fully transitive group**".

Also, we will henceforth use somewhere in the text, where it is possible and better for usage, the widely accepted shorthand abbreviation "**iff**" for the standard phrase "**if and only if**". As for the latter, we shall somewhere write "**if, and only if,**" whenever the text is more specific in the sense that it needs more specifications in the meaning.

Chapter III. Noncommutative Rings

Our main results of this branch are distributed into two sections as follows:

1. Weakly Exchange Rings

Here, for the sake of completeness and for the convenience of the readers, we shall consider below a few more subsections like these:

1.1. **On weakly exchange rings.** The following fundamental notion was defined in [93].

Definition 1.1. A ring R is called *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$.

Likewise, in [93] it was pointed out the fundamental fact that R is clean iff $R/J(R)$ is clean and all idempotents lift modulo $J(R)$.

The "clean" concept was generalized there to the following one:

Definition 1.2. A ring R is said to be *exchange* if, for every $a \in R$, there exists an idempotent $e \in aR$ such that $1 - e \in (1 - a)R$.

It was obtained in [93] that R is an exchange ring iff $R/J(R)$ is an exchange ring and all idempotents lift modulo $J(R)$. Also, it was established there that Definitions 1.1 and 1.2 are equivalent for abelian rings (that are rings for which each idempotent lies in the center of the former ring). However, there is an exchange ring that is not clean.

On the other hand, it was introduced in [2] the notion of *weakly clean* rings but only in a commutative version. We shall do that in the general way as follows:

Definition 1.3. A ring R is called *weakly clean* if each $r \in R$ can be expressed as either $r = u + e$ or $r = u - e$, where $u \in U(R)$ and $e \in Id(R)$.

Evidently, all clean rings are weakly clean, whereas the converse does not hold even in the commutative aspect (see, e.g., [2]). However, every weakly clean ring of characteristic 2 is clean, and vice versa. One of our goals here is to improve this observation by requiring that 2 lies in $J(R)$, which supersedes the condition $2 = 0$.

Definition 1.4. A ring R is called *weakly exchange* if, for any $x \in R$, there exists $e \in Id(R)$ such that $e \in xR$ and either $1-e \in (1-x)R$ or $1-e \in (1+x)R$.

It was established in [24] that the notions of being weakly exchange and weakly clean do coincide for abelian rings, thus extending the aforementioned facts from [93] (see also [109]).

Apparently, all exchange rings are weakly exchange, while the converse does not hold even in the commutative variant as some simple examples demonstrably show. However, every weakly exchange ring of characteristic 2 is exchange, and visa versa. One of our aims here is to enlarge this observation by requiring that 2 lies in $J(R)$, which is weaker than the condition $2 = 0$.

The following technical claim, which considerably extends [93, Proposition 1.1], was stated in [109] without a proof and with two identical misprints in points (3) and (4), which are actually points (c) and (d) below, respectively. We, however, formulate it correctly and provide a transparent proof for the sake of completeness and for the convenience of the reader.

Lemma 1.5. *Let R be a ring. Then the following points are equivalent:*

(a) R is weakly exchange;

(b) For any $x \in R$ there exists $e \in Id(R)$ such that $e - x \in (x - x^2)R$ or $e + x \in (x + x^2)R$;

(c) For any $x \in R$ there exist $e \in Id(R) \cap xR$ and $c \in R$ such that $1 - e - (1 - x)c \in J(R)$ or $1 - e - (1 + x)c \in J(R)$;

(d) For any $x \in R$ there exists $e \in Id(R) \cap xR$ such that $eR + (1 - x)R = R$ or $eR + (1 + x)R = R$.

Proof. (a) $\Rightarrow$ (b). Letting for any $x \in R$ there is $e \in Id(R)$ such that $e \in xR$ and either $1-e \in (1-x)R$ or $1-e \in (1+x)R$. Furthermore, for some $r, a \in R$, one sees that $e-x = (1-x)e-x(1-e) = (1-x)xr-x(1-x)a = (x-x^2)r-(x-x^2)a \in (x - x^2)R$. Moreover, $e + x = (1 + x)e + x(1 - e) = (1 + x)xr + x(1 + x)a = (x + x^2)r + (x + x^2)a \in (x + x^2)R$, as stated.

(b) $\Rightarrow$ (c). If $e - x = (x - x^2)r$ for some $r \in R$, then $e = x(1 + r - xr) \in xR$ and $1 - e = (1 - x)(1 - xr)$ where we may take $c = 1 - xr$. Analogously, if $e + x = (x + x^2)a$ for some $a \in R$, then $1 - e = (1 + x)(1 - xa)$ where we may again choose $c = 1 - xa$.

(c) $\Rightarrow$ (d). Observe that either $u = e + (1 - x)c$ or $v = e + (1 + x)c$ is a unit. Consequently, $1 = eu^{-1} + (1 - x)cu^{-1}$ or $1 = eu^{-1} + (1 + x)cu^{-1}$ and hence, for any $r \in R$, we deduce that $r = eu^{-1}r + (1-x)cu^{-1}r \in eR + (1-x)R$ or $r = eu^{-1}r + (1+x)cu^{-1}r \in eR + (1+x)R$, as required.

(d) $\Rightarrow$ (a). Writing $1 = et + (1-x)s$, we define $f = e + et(1-e)$. It is not too hard to check that $f^2 = f \in xR$ and $1-f = (1-e) - et(1-e) = (1-et)(1-e) = (1-x)s(1-e) \in (1-x)R$.

By symmetry, writing $1 = et + (1+x)s$, we as above set $f = e + et(1-e)$. So, $f \in Id(R) \cap xR$ and $1 - f = (1-et)(1-e) = (1+x)s(1-e) \in (1+x)R$, as expected. $\square$

Recall that the idempotents of a ring R can be *lifted modulo the ideal* L if, given $x \in R$ with $x - x^2 \in L$, there exists $e \in Id(R)$ such that $e - x \in L$. Replacing x by $-x$ this condition is equivalent to the following: if $x + x^2 \in L$, there exists $e \in Id(R)$ such that $e + x \in L$. Especially, we take into account that $x = -(-x) = -y$ and so $x - x^2 \in L \iff y + y^2 \in L$.

So, we come to our first basic result in the current subsection.

Theorem 1.6. *A ring R is weakly exchange if $R/J(R)$ is weakly exchange and all idempotents in R lift modulo $J(R)$. In addition, if $2 \in J(R)$, then the converse is true.*

Proof. Given an arbitrary $x \in R$, we have $x + J(R) \in R/J(R) = \overline{R}$ and thus $\overline{1} - \overline{a} = (\overline{1} - \overline{x})\overline{c}$ or $\overline{1} - \overline{a} = (\overline{1} + \overline{x})\overline{d}$ for some $\overline{a}^2 = \overline{a} \in \overline{x}\overline{R}$ and $\overline{c}, \overline{d} \in \overline{R}$. Writing $\overline{a} = \overline{xr}$, we derive $a = xr + j = a' + j$ where $a' \in xR$ and $j \in J(R)$. Furthermore, either $1 - a' \in (1-x)c + J(R)$ or $1 - a' \in (1+x)d + J(R)$ for some $c, d \in R$. But $a^2 - a \in J(R)$, hence by assumption there is $f \in Id(R)$ such that $a - f \in J(R)$, so that $a' - f \in J(R)$ and $u = 1 - f + a' \in U(R)$. Writing $a' = f + j_1$ with $j_1 \in J(R)$, one sees that $a^2 - a \in J(R)$ is tantamount to $a'^2 - a' \in J(R)$. Define $e = ufu^{-1}$. It is clear that $e = a'fu^{-1} \in Id(R) \cap xR$. We therefore obtain that either $1 - e - (1-x)c \in J(R)$ or $1 - e - (1+x)d \in J(R)$. In fact, what we need to prove is that $e - a' \in J(R)$. To show this, consider $e - a' = a'fu^{-1} - a' = a'(fu^{-1} - 1) = a'(f - u)u^{-1} = a'(2f - 1 - a')u^{-1} = (f+j_1)(2f-1-f-j_1)u^{-1} = (f+j_1)(f-j_1-1)u^{-1} = [(f+j_1)(f-j_1)-f-j_1]u^{-1} = [f^2 - j_1^2 - f - j_1]u^{-1} = [-j_1^2 - j_1]u^{-1} \in J(R)$, as wanted. Now Lemma 1.5 (c) applies to deduce the desired claim that R is weakly exchange.

For the second part-half, we observe that a homomorphic image of a weakly exchange ring is also a weakly exchange ring. Next, as for the lifting property, we observe that $x - x^2 = (x + x^2) - 2x^2$, so that $x - x^2 \in J(R)$ is equivalent to $x + x^2 \in J(R)$, provided $2 \in J(R)$. By virtue of Lemma 1.5 (b), there is $e \in J(R)$ such that $e - x \in (x - x^2)R$ or $e + x \in (x + x^2)R$. In the first case, it follows at once that $e - x \in J(R)$. In the second one, it also follows immediately that $e + x \in J(R)$, as required. $\square$

Similarly, one can derive the following assertion. Before doing that, we need the following technicality.

Lemma 1.7. *For any ring R the following equality holds:*

$$U(R) + J(R) = U(R).$$

Proof. It is self-evident that the left hand-side contains the right one. To treat the converse, given $x \in J(R)+U(R)$, we may write $x = a+u$ where $a \in J(R)$ and $u \in U(R)$. But it is well known that $J(R) = \{x \in R \mid 1-rxs \in U(R), \forall r, s \in R\}$. Hence $a + u = u(1 + u^{-1}a) \in U(R)$ taking $r = -u^{-1}$ and $s = 1$, as required. $\square$

We are now able to proceed by proving the following:

Theorem 1.8. *A ring R is weakly clean if $R/J(R)$ is weakly clean and all idempotents in R lift modulo $J(R)$. In addition, if $2 \in J(R)$, then the converse is true.*

Proof. Choosing an arbitrary $x \in R$, we have $x + J(R) \in R/J(R)$ and so write either $x + J(R) = (u + J(R)) + (e + J(R)) = u + e + J(R)$ or $x + J(R) = (u + J(R)) - (e + J(R)) = u + J(R) - e + J(R) = u - e + J(R)$, where $u + J(R)$ is a unit in $R/J(R)$ and $e + J(R)$ is an idempotent in $R/J(R)$. Consequently, $1 - uv \in J(R)$ and $1 - vu \in J(R)$ for some $v \in R$ as well as $e - e^2 \in J(R)$. Since $1-(1-uv) = uv \in U(R)$ and $1-(1-vu) = vu \in U(R)$, we deduce that $u \in U(R)$. Moreover, $e - f \in J(R)$ for some $f \in Id(R)$. Therefore, $x - (u + f) \in J(R)$ and $x - (u - f) \in J(R)$. We next refer to Lemma 1.7 to infer that $x = w + f$ or $x = w - f$ where $w \in U(R)$, as required.

As for the second part-half, it is obvious that a homomorphic image of a weakly clean ring is again a weakly clean ring. Since weakly clean rings are necessarily weakly exchange, we just apply Theorem 1.6. $\square$

So, we are now ready to extend one of the aforementioned classical results.

Proposition 1.9. *Suppose that R is a ring with $2 \in J(R)$. Then R is weakly exchange iff R is exchange.*

Proof. One direction being trivial, we consider the other one. So, letting R be weakly exchange, we employ Theorem 1.6 to get that $R/J(R)$ is weakly exchange and all idempotents of R are lifted modulo $J(R)$. Since 2 lies in $J(R)$, the factor-ring $R/J(R)$ is exchange possessing characteristic 2. We, further, appeal to [93] and conclude that R is exchange, as required. $\square$

Same type result appears for weakly clean rings. Specifically, the following is valid:

Proposition 1.10. *Suppose that R is a ring with $2 \in J(R)$. Then R is weakly clean iff R is clean.*

Proof. One direction being elementary, we consider the other one. So, given R is weakly clean, we apply Theorem 1.8 to obtain that $R/J(R)$ is weakly clean and all idempotents of R are lifted modulo $J(R)$. Since 2 is in $J(R)$, the quotient ring $R/J(R)$ is clean having characteristic 2. We, furthermore, appeal to [93] and infer that R is clean, as expected. □

1.2. **Rings with Jacobson units.** It is well known that the inclusion $1+J(R) \subseteq U(R)$ or, equivalently, $J(R) \subseteq 1+U(R)$ holds. However, these containments could be strict, so that it is rather natural to state the following:

Definition 1.11. A ring R is called a *JU ring* or a ring with *Jacobson units* if the equality $U(R) = 1 + J(R)$ holds.

Obviously, this is tantamount to the equality $J(R) = 1+U(R)$. In an equivalent form, since one can show

$$U(R)/(1 + J(R)) \cong U(R/J(R)),$$

we observe in the presence of this isomorphism that all JU rings are just those rings R for which $U(R/J(R)) = \{1\}$.

Moreover, note that nil ideals are always contained in the Jacobson radical, so, if $R/J(R)$ is commutative, then the commutator subgroup of $U(R)$ is contained in $1 + J(R)$. Furthermore, if $J(R)$ is nilpotent as an ideal, then $1 + J(R)$ is nilpotent as a group and thus $U(R)$ is solvable. In particular, if $(J(R))^2 = 0$, then $U(R)$ is a metabelian group.

We start with the following technicality.

Lemma 1.12. *For any ring R the following equality is true:*

$$U(R) + J(R) = U(R).$$

Proof. It is self-evident that the left hand-side contains the right hand-side. To treat the converse, given $x \in J(R) + U(R)$, we may write $x = a + u$, where $a \in J(R)$ and $u \in U(R)$. With the aid of the well-known characterization for $J(R) = \{a \in R \mid 1 + RaR \subseteq U(R)\}$, we easily check that $a + u = u(1 + u^{-1}a) \in U(R)$, as required. □

Imitating [96], recall that a ring is said to be *semi-boolean* if each its element is semi-boolean, i.e., it is the sum of an element from $J(R)$ and an element from $Id(R)$. This is equivalent to the conditions that $R/J(R)$ is boolean and idempotents lift modulo $J(R)$. These rings are also termed *J-clean rings*.

So, a non-trivial example of JU rings is the next one.

Example 1.13. J-clean rings are JU.

Proof. If u is a unit in the J-clean ring R, then $u = j + e$, where j is in $J(R)$ and e is in $Id(R)$. So, by Lemma 1.12, $e = u - j \in U(R)$ and hence $e = 1$. This means that $u = j + 1$, as required. $\square$

Referring to [39], a ring R is called a *UU* ring if $U(R) = 1 + Nil(R)$. In accordance with [39] one may observe that a JU ring is a UU ring if, and only if, $J(R)$ is nil. In particular, when R is commutative with $J(R) = Nil(R)$, note that the classes of JU rings and UU rings do coincide. For instance, this holds for *Hilbert rings*, that are commutative rings for which every prime ideal is an intersection of maximal ideals; e.g., the polynomial ring $R[X]$ is Hilbert, whenever R is a commutative ring. This also happens when R is both commutative and finitely generated as an algebra over either a field or the ring of integers $\mathbb{Z}$. In addition, as a special case, the ring $\mathbb{Z}(m)$ is JU (or respectively, UU) if, and only if, $m = 2^k$ for some positive integer k.

Likewise, we emphasize that the Jacobson radical of any artinian ring (in particular, of any finite ring) is nilpotent and thus it is nil. This can be subsumed by the following assertion.

Proposition 1.14. *Finite UU rings are JU, and finite JU rings are UU.*

Proof. Let R be a finite ring. As aforementioned, $J(R) \subseteq Nil(R)$.

To show the first implication, consider an injective function $f : U(R) \to 1 + U(R)$, defined by $f(u) = 1 + u$, which is obviously a surjection and thus it is a bijection. Hence $|U(R)| = |1 + U(R)|$. Notice that $1 + Nil(R) = U(R)$ and so we deduce that $|U(R)| = |Nil(R)|$. We also have that $|U(R)| = |J(R)|$, which follows from the two facts that $|J(R)| \leq |U(R)|$ and that $R/J(R)$ being finite UU must be reduced giving that $Nil(R) \subseteq J(R)$ whence $|Nil(R)| \leq |J(R)|$, thus substantiating our claim. This finally assures that $1 + U(R) = J(R)$, as required.

Treating the second implication, $U(R) = 1 + J(R)$ obviously yields that $U(R) = 1 + Nil(R)$, as expected. $\square$

In the spirit of [93], we recollect that a ring is said to be *clean* if each its element is a sum of an idempotent and a unit. Moreover, recall that a ring R

is called *exchange* provided for any a in R there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$. Notice that clean rings are always exchange, whereas the converse is not true in general; however for abelian rings (that are rings with central idempotents) these two ring classes do coincide.

It is worthwhile noticing that in [39] was established that if R is an exchange UU ring, then $R/J(R)$ is boolean and thus reduced; in particular the same holds for finite UU rings (compare with the proof of Proposition 1.14 quoted above). Therefore, exchange UU rings are always JU. In addition, for finite commutative rings, it is well known that $J(R) = Nil(R)$, so that a finite commutative ring is a UU ring if, and only if, it is a JU ring. Note that finite rings are always clean but not always UU or JU.

Recollect that rings R for which $J(R) = \{0\}$ are called *semiprimitive* rings; for example, any field, any von Neumann regular ring and any left or right primitive ring are semiprimitive. Also, so is the ring of integers $\mathbb{Z}$. Therefore, a semiprimitive ring is JU if, and only if, $U(R) = \{1\}$, and so $\mathbb{Z}$ and any field with at least 3 elements (e.g., $\mathbb{Q}$) is definitely *not* JU.

On the other vein, if K is a field and R is the ring $\mathbb{T}_n(K)$ of all upper triangular $n \times n$ matrices with entries in K, then $J(R)$ consists of all upper triangular matrices with zeros on the main diagonal. Hence, excepting the case when $K = \mathbb{Z}_2$ is the field consisting of two elements, such rings are surely *not* JU.

Moreover, it is clear that any ring R in which the identity is a sum of two units (in particular, if $2 \in U(R)$) is not JU. Also, it is well known that the identity in the full matrix $n \times n$ ring $\mathbb{M}_n(R)$ with $n > 1$ is a sum of two units. For example, we consult with [60, Lemma 1], where it is shown that every diagonal matrix in $\mathbb{M}_n(R)$ with $n \geq 2$ is a sum of two units. Thus, we have accumulated all the information in order to come to the following conclusion.

Theorem 1.15. *No matrix ring over a ring with identity is JU.*

The following constructions illustrate that the JU and UU concepts are independent each to other.

- There is a UU ring which is *not* a JU ring.

If one takes Bergman's ring R as showed in [39], then R is UU as pointed out there. But since $J(R) = \{0\}$, the only J-unit is 1, while $U(R) \neq \{1\}$. This shows that R is not JU.

- There is a JU ring which is *not* a UU ring.

If R is a local ring with residue field $\mathbb{F}_2$, then R is JU, but it is not UU unless its Jacobson radical is nil.

The next properties are also helpful:

(1) The local ring $(R, \mathbf{m})$ is JU if, and only if, $R/\mathbf{m} \cong \mathbb{Z}_2$.

In fact, in both directions the factor-ring $R/\mathbf{m}$ has trivial unit group and simultaneously it is a division ring, whence it must be isomorphic to the field of two elements, as asserted.

Following the standard terminology in the existing literature, a ring R is said to be *J-reduced* if $Nil(R) \subseteq J(R)$ and *reduced* if $Nil(R) = \{0\}$. Clearly, reduced rings are J-reduced but, however, the reverse is manifestly untrue.

(2) If R is a JU ring, then $R/J(R)$ is reduced. In particular, R is J-reduced.

In addition, J-reduced UU rings are JU rings.

Indeed, as we have seen above, $1 + Nil(R/J(R)) \subseteq U(R/J(R)) = \{1\}$, so that $Nil(R/J(R)) = \{0\}$. Thus $R/J(R)$ does not contain non-trivial nilpotents, and it follows at once that R is J-reduced. However, this can be derived directly by observing that $1 + Nil(R) \subseteq U(R) = 1 + J(R)$ and hence $Nil(R) \subseteq J(R)$. The last observation follows immediately.

(3) If R is a JU ring, then 2 lies in $J(R)$.

Indeed, -1 being a unit can be written as $-1 \in 1 + J(R)$ which gives the wanted claim.

(4) If R is a JU ring such that $p \in J(R)$ (in particular, if R is of prime characteristic p), then $p = 2$.

In fact, by assumption, $R/J(R)$ has characteristic p. On the other hand, in view of the preceding point, $R/J(R)$ must have characteristic 2, whence $p = 2$ as stated.

(5) if I is an ideal of a ring R such that R/I has no nontrivial units, then $I \supseteq J(R)$.

In addition, if I is nil and R is JU, then $I = J(R)$.

To prove this, given $u \in U(R)$, it must be that $u + I$ is a unit in R/I and hence $1 - u \in I$. Thus $U(R) \subseteq 1 + I$. But $1 + J(R) \subseteq U(R)$ whence $J(R) \subseteq I$, as asserted. To show the additional part, since $1 + I \subseteq U(R) = 1 + J(R)$, it follows that $I \subseteq J(R)$ which is tantamount to the equality $I = J(R)$, as claimed.

(6) For an ideal I of R the implication R/I is JU $\Rightarrow$ R is JU generally fails.

Indeed, $\mathbb{Z}/2\mathbb{Z}$ is JU but as demonstrated above $\mathbb{Z}$ is not.

Reciprocally, if R is JU, then R/I may not be JU even if I is a nil ideal.

Indeed, appealing to standard arguments and tricks, the abelian 2-group $G = \mathbb{Z}(2^{k_1}) \oplus \mathbb{Z}(2^{k_2}) \oplus \cdots \oplus \mathbb{Z}(2^{k_n})$ with $k_1 < k_2 < \cdots < k_n$, $n \in \mathbb{N}$, has JU endomorphism ring $E(G)$, but $E(G)/2E(G) \cong \mathbb{M}_n(\mathbb{F}_2)$ is not JU by consulting with Theorem 1.15. Notice that here $2E(G)$ is a nil ideal.

The leitmotif of the next chief result listed below is to describe explicitly exchange JU rings. The intersection between these two classes, however, gives nothing new. Specifically, the following is valid:

Theorem 1.16. *A ring R is an exchange JU ring if, and only if, it is J-clean.*

Proof. The sufficiency follows from Example 1.13 and from the obvious fact that J-clean rings being always clean are thereby exchange (see [96] and [93]).

Dealing now with the necessity, since R is exchange, by [93] we deduce that all idempotents in R can be lifted modulo $J(R)$. Moreover, combining again results from [93] with one of the equivalencies above for a ring to be JU, we obtain that the factor-ring $R/J(R)$ is exchange with trivial unit group. Therefore, [39, Corollary 4.2] applies to show that $R/J(R)$ is boolean. We finally apply [96] to get the desired claim. □

Remark 1.17. This result can also be deduced from [80, Theorem 13 (3)] by showing that in JU rings each non-zero idempotent cannot be written as the sum of two units. To show this, in a way of contradiction, given a non-zero idempotent e in R which is a sum of two units, say $e = u_1 + u_2$. Thus, one writes that $e = (j_1 + 1) + (j_2 + 1)$, where both j_1, j_2 are from $J(R)$. Hence $1 - e = j - 1 = -(1 + (-j))$, where $j = -j_1 - j_2$ is in $J(R)$, whence $1 - e$ must be a unit. Consequently, $1 - e = 1$ and hence $e = 0$, a contradiction.

Actually, it is worthwhile noticing that Theorem 13 (3) from [80] is equivalent to our Theorem 1.16 by using the methods for proof developed in [39].

The following assertion gives an element-wise description in parallel to Theorem 1.16.

Proposition 1.18. *Any semi-boolean element is clean. For JU rings, the converse holds as well.*

Proof. Write $r = j + e = (1 - e) + (j + 2e - 1)$, where $j \in J(R)$ and $e \in Id(R)$. Since $1 - e$ is an idempotent and $(2e - 1)^2 = 1$, with Lemma 1.12 at hand we derive that $j + 2e - 1$ belongs to $U(R)$ and hence this is a clean decomposition.

Conversely, write $r = u + e = 1 + j + e = (1 - e) + (j + 2e)$, where $u \in U(R)$ and $e \in Id(R)$. Since in view of property (*) the element 2 lies in $J(R)$, and so $j + 2e$ is in $J(R)$, the claim follows. □

The following two technical statements can be seen in [79], and thus we omit the proofs leaving them to the interested reader.

Lemma 1.19. *Let A, B be subsets of a ring R with A a subgroup of R^+. If $a \in A$, then $a + (A \cap B) = A \cap (a + B)$.*

Lemma 1.20. *Let $0 \neq e = e^2$ in a ring R. Then*

(a) $J(eRe) = (eRe) \cap J(R) = eJ(R)e$.

(b) $U(eRe) = (eRe) \cap (1 - e + U(R))$.

And so, we are in a position to proceed by proving of the following.

Proposition 1.21. *Any JU ring passes to corners, that is, each corner ring of a JU ring is again a JU ring.*

Proof. According to the Definition 1.11, and with Lemmas 1.19 and 1.20 at hand, we deduce that $e + U(eRe) = e + [(eRe) \cap (1 - e + U(R))] = (eRe) \cap (1 - e + e + U(R)) = (eRe) \cap (1 + U(R)) = (eRe) \cap J(R) = J(eRe)$, as required. $\square$

1.3. **On exchange π-UU unital rings.** We begin here with recalling some useful concepts as follows:

Definition 1.22. A ring R is said to be *UU* if $U(R) = 1 + Nil(R)$.

Definition 1.23. A ring R is said to be *exchange* if, for each $r \in R$, there is an idempotent $e \in rR$ such that $1 - e \in (1 - r)R$.

It was proved in [39] that a ring R is an exchange UU ring iff $J(R)$ is nil and $R/J(R)$ is Boolean.

Before proceed by proving our chief result, we need a few more technicalities. Generalizing Definition 1.22, one can state the following.

Definition 1.24. Let $n \in \mathbb{N}$. A ring R is called *n-UU* if the inclusion $U^n(R) \subseteq 1 + Nil(R)$ holds.

Clearly, UU rings just coincide with 1-UU rings.
This can be substantially expanded to the following:

Definition 1.25. A ring R is called *π-UU* if, for any $u \in U(R)$, there exists $i \in \mathbb{N}$ such that $u^i \subseteq 1 + Nil(R)$.

These rings play a key, if not (at least) facilitate, role in developing a new modern theory of *periodic rings* (see, e.g., [32]).

The next statement considerably supersedes [1, Lemma 4.4] by dropping off the unnecessary limitation on the ring to be "**exchange**".

Proposition 1.26. *Let R be a 2-UU ring. Then $J(R)$ is nil.*

Proof. Given $x \in J(R)$, it follows that $(1+x)^2 = 1+2x+x^2 \in 1+Nil(R)$ which amounts to $2x+x^2 \in Nil(R)$. Similarly, replacing x by $-x$, we derive that $-2x+x^2 \in Nil(R)$. Since these two sums commute, it follows immediately that $2x^2 \in Nil(R)$. Finally, using the above trick for x^2, we deduce that $2x^2+x^4 \in Nil(R)$. Since $2x^2 \in Nil(R)$, we conclude that $x^4 \in Nil(R)$, i.e., $x \in Nil(R)$, as required. $\square$

A difficult question which still eludes us is of whether or not $J(R)$ is nil whenever R is a π-UU ring.

Concentrating now on the 2-UU case, the next consequence could be useful for further applications.

Corollary 1.27. *A ring R is 2-UU iff $J(R)$ is nil and $R/J(R)$ is 2-UU.*

Proof. According to Proposition 1.26, the argument follows in the same manner as [39, Theorem 2.4 (2)]. $\square$

We continue with

Lemma 1.28. *Let R be a ring. Then the following two points hold:*
(i) If R is n-UU for some $n \in \mathbb{N}$, then eRe is also n-UU for any $e \in Id(R)$.
(ii) If R is π-UU, then eRe is also π-UU for any $e \in Id(R)$.

Proof. We shall show the validity only of (ii). The proof of (i) is analogous and so it will be omitted. As in [39], letting $w \in U(eRe)$ with inverse v, it follows that $w+1-e \in U(R)$ with inverse $v+1-e$. Therefore, there exists $i \in \mathbb{N}$ such that $(w+1-e)^i = w^i+1-e \in 1+Nil(R)$, that is, $w^i - e = q \in Nil(R)$. But $q \in Nil(R) \cap (eRe) = Nil(eRe)$ which leads to $w^i = e+q \in 1_{eRe}+Nil(eRe)$, as expected. $\square$

Lemma 1.29. *For any $n \in \mathbb{N}$ and any non-zero ring R the full matrix ring $\mathbb{M}_n(R)$ is not 2-UU.*

Proof. Since $\mathbb{M}_2(R)$ is isomorphic to a corner ring of $\mathbb{M}_n(R)$ for $n \geq 2$, in view of Lemma 1.28 it suffices to establish the claim for $n=2$. To that goal, as in [39], let us consider the invertible matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with the inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, we infer that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ which is the same invertible with the inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ and thus certainly not a nilpotent, as wanted. $\square$

Let us recall that a ring is said to be *tripotent* if every its element satisfies (is a solution of) the equation $x^3 = x$, and is said to be *invo-clean* if each its element is represented as a sum of an involution (= a unit of order at most 2) and an idempotent.

We shall now restate and reproof the main result from [1] by giving a more convenient form and more transparent proof arising from well-known recent results in [39] and [D4] (compare with the subsequent subsection, too), respectively. Actually, a new substantial achievement, including new points with more strategic estimations, arises as follows:

Theorem 1.30. *Suppose that R is a ring. Then the following five items are equivalent:*

(a) R is exchange 2-UU.

(b) $J(R)$ is nil and $R/J(R)$ is commutative invo-clean.

(c) $J(R)$ is nil and $R/J(R) \cong B \times C$, where $B \subseteq \prod_\lambda \mathbb{Z}_2$ and $C \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

(d) $J(R)$ is nil and $R/J(R)$ is tripotent.

(e) $J(R)$ is nil and $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals λ and μ.

Proof. The equivalence (b) $\iff$ (c) is exactly Corollary 2.17 from [D4], whereas the equivalence (d) $\iff$ (e) is obvious.

We shall show that (a) $\iff$ (b) is valid. To prove the left-to-right implication, we first consider the semi-primitive case when $J(R) = \{0\}$. Imitating the basic idea from the proof of [39, Theorem 4.1], we arrive at the case when $eRe \cong \mathbb{M}_2(T)$ for some idempotent $e \in R$ and some non-zero ring T depending on R, provided $Nil(R) \neq \{0\}$. However, with Lemma 1.28 at hand we deduce that eRe is 2-UU, while with the aid of Lemma 1.29 this property does not hold for $\mathbb{M}_2(T)$. This contradiction substantiates that R is reduced, i.e., $Nil(R) = \{0\}$ and thus abelian. Hence R is clean with $U^2(R) = \{1\}$ which allows us to conclude with an appeal to [D4] that R abelian invo-clean and so commutative invo-clean. Suppose now that $J(R) \neq \{0\}$. The fact that $J(R)$ is nil follows directly from Proposition 1.26. Owing to [97] and Corollary 1.27, one sees that $R/J(R)$ is exchange 2-UU, and so by what we have just already shown so far, the factor-ring $R/J(R)$ has to be commutative invo-clean, as asserted.

As for the right-to-left implication, it follows immediately by virtue of [93] that R is exchange. That R is 2-UU follows like this: Using the isomorphism $U(R)/(1 + J(R)) \cong U(R/J(R)) \cong U(B) \times \prod_\mu U(\mathbb{Z}_3)$ and so $U^2(R/J(R)) = \{1\}$.

Furthermore, for any $u \in U(R)$ it must be that $u + J(R) \in U(R/J(R))$ and hence $(u + J(R))^2 = u^2 + J(R) = 1 + J(R)$ which means that $u^2 - 1 \in J(R) \subseteq Nil(R)$, as required.

The implication (c) $\Rightarrow$ (d) is elementary. What remains to illustrate is the truthfulness of the implication (d) $\Rightarrow$ (a). Since tripotent rings are always exchange, the application of [93] shows that R is exchange. On the other side, since $U(R/J(R)) \cong U(R)/(1 + J(R))$ and $U^2(R/J(R)) = \{1\}$, as shown above it follows that R is 2-UU, thus completing the proof after all. $\square$

The next construction manifestly demonstrates that the theorem is *no* longer true for n-UU rings when $n > 2$.

Example 1.31. Consider the full matrix 2×2 ring $R = \mathbb{M}_2(\mathbb{Z}_2)$. It was proved in [11] that R is nil-clean and hence exchange. Moreover, R is a 3-UU ring. However, it is easily checked that $J(R) = \{0\}$ and that R is even not tripotent (whence not Boolean). In fact, $U(R)$ has 6 elements satisfying the following identities:

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

We finish off our work with the following question of some interest and importance. Recall that a ring R is termed π-*Boolean* if, for any $r \in R$, there is $i \in \mathbb{N}$ with $r^i = r^{2i}$.

Problem 1.32. Does it follow that R is an exchange π-UU ring iff $J(R)$ is nil and $R/J(R)$ is π-Boolean?

1.4. **Invo-clean unital rings.** The following concept appeared in [93].

Definition 1.33. A ring R is called *clean* if each $r \in R$ can be expressed as $r = u+e$, where $u \in U(R)$ and $e \in Id(R)$. If, in addition, the existing idempotent e is unique, then R is said to be *uniquely clean*.

A clean ring R with $ue = eu$ is said to be *strongly clean*.

In particular, in [41] was introduced the following concept:

Definition 1.34. A ring R is called *nil-clean* if each $r \in R$ can be written as $r = q + e$, where $q \in Nil(R)$ and $e \in Id(R)$.

A nil-clean ring R with $qe = eq$ is said to be *strongly nil-clean*.

On the other hand, the latter concept of nil-cleanness was extended in [40] and [12] respectively by defining the notion of *weak nil-cleanness* as follows:

Definition 1.35. A ring R is called *weakly nil-clean* if every $r \in R$ can be presented as either $r = q + e$ or $r = q - e$, where $q \in Nil(R)$ and $e \in Id(R)$.

A weakly nil-clean ring with $qe = eq$ is said to be *strongly weakly nil-clean*.

It was established in [12] and [40] that weakly nil-clean rings are themselves clean. Likewise, in [12] was established a complete characterization of abelian weakly nil-clean rings as those abelian rings R for which $J(R)$ *is nil and* $R/J(R)$ *is isomorphic to a Boolean ring* B*, or to* $\mathbb{Z}_3$*, or to* $B \times \mathbb{Z}_3$. We notice also that abelian weakly nil-clean rings were classified in [D5] in another direction (compare with the results in the next subsection).

The next notion is our basic tool here.

Definition 1.36. A ring R is said to be *invo-clean* if every $r \in R$ can be written as $r = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$. If, in addition, the existing idempotent e is unique, then R is said to be *uniquely invo-clean*.

An invo-clean ring with $ve = ev$ is called *strongly invo-clean*.

Interestingly, any idempotent is an invo-clean element due to the record $e = (2e - 1) + (1 - e)$, because $(2e - 1)^2 = 1$ and $(1 - e)^2 = 1 - e$.

Moreover, simple examples of invo-clean rings that could be plainly verified are these: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_8$. Oppositely, both $\mathbb{Z}_5$ and $\mathbb{Z}_7$ are not invo-clean, but however they are clean.

The objective of the current subsection is to explore invo-clean rings by giving a complete description of their algebraic structure. As an application, we will characterize some related classes of rings.

We start now with the mathematical part.

Lemma 1.37. *Homomorphic images of invo-clean rings are again invo-clean.*

Proof. Since homomorphic images of involutions and idempotents are again involutions and idempotents, the statement follows easily. ☐

Lemma 1.38. *If R is an invo-clean ring, then $24 = 0$. In particular, $6 \in Nil(R)$.*

Proof. Write $3 = v + e$, where v is an involution and e is an idempotent. Thus $(3-v)^2 = 3-v$ implies that $5v = 7$, whence $24 = 0$ by squaring both sides of the equality. In addition, $6^3 = 216 = 24.9 = 0$, hence $6 \in Nil(R)$, as asserted. ☐

The following technical assertion is our critical instrument for the further attack of the main result.

Lemma 1.39. *Suppose R is a ring with $u \in U(R)$ and $e \in Id(R)$ such that $u^2e = eu^2 = e$ and $u = e + q$, where $q \in Nil(R)$. Then $e = 1$.*

Proof. Letting $u = e + q$ for some $e \in Id(R)$ and $q \in Nil(R)$ with $q^t = 0, t \in \mathbb{N}$ say, we obtain that $u^2 = e+eq+qe+q^2$ and hence $u^2e = e = e+eqe+qe+q^2e$ which forces that $(q + q^2)e = -eqe$, Similarly, $eu^2 = e$ insures that $e(q + q^2) = -eqe$. Thus e commutes with the nilpotent $(q + q^2)^n = [q(1 + q)]^n = q^n(1 + q)^n$ for all $n \in \mathbb{N}$, and therefore the same is valid for u. Furthermore, $u - (q + q^2) = e - q^2$ with $u - (q + q^2) = u_2 = e - q^2$ being a unit, one sees that $u_2 - (2q^3 + q^4) = e-(q^2+2q^3+q^4) = e-(q+q^2)^2$. Putting $u_3 = u_2+(q+q^2)^2$, we observe that u_3 is a unit since u_2 commutes with $(q+q^2)^2$ and that $u_3 = e+q^3(2+q)$. Repeating the same procedure t-times, we will find a unit u_t such that $u_t = e + q^t.a = e$ for some element a depending on q; $a = -1 = -q^0$ provided $t = 2$. This yields that $e = 1$, which exhausts our claim. ☐

The following consequence is immediate.

Corollary 1.40. *If R is a ring with $u \in Inv(R)$ such that $u = e+q$ for $e \in Id(R)$ and $q \in Nil(R)$, then $e = 1$.*

Proof. Just take $u \in Inv(R)$ in Lemma 1.39 and this allows us to infer that $e = 1$, as promised. ☐

One further continues with the following.

Proposition 1.41. *If R is an invo-clean ring with $2 \in U(R)$, then $Nil(R) = J(R) = \{0\}$.*

Proof. If $q \in Nil(R)$, write $q = v + e$ where $v \in Inv(R)$ and $e \in Id(R)$. Thus $-v = -q+e$, where $-v \in Inv(R)$ and $-q \in Nil(R)$. Appealing to Corollary 1.40, we conclude that $e = 1$. Therefore, $q = v+1$ and hence $q^2 = 2+2v = 2(1+v) = 2q$. This leads to $q(2-q) = 0$. Since $2 - q \in U(R)$, we finally infer that $q = 0$, as expected.

Concerning the second part, given $z \in J(R)$ we have $z = v+e$ for v, e as above. Consequently, $z - v = e \in U(R) \cap Id(R) = \{1\}$ means that $z = v + 1$ and since $2 - z \in U(R)$ the same trick as above works to get that $z = 0$, as promised. □

Proposition 1.42. *If R is an invo-clean ring with $Id(R) = \{0, 1\}$ and $2 \in U(R)$, then $R \cong \mathbb{Z}_3$.*

Proof. Each element r of R can be written as either $r = v + 1$ or $r = v$, where $v \in Inv(R)$. However, $\frac{1-v}{2}$ is always an idempotent, whence $\frac{1-v}{2} = 0$ or $\frac{1-v}{2} = 1$. In the first case $v = 1$, while in the second one $v = -1$. Consequently, all the elements of R are $\{0, -1, 1, 2\}$. But it must be that $2 = -1$, because only $2.(-1) = 1$ is possible. So, $3 = 0$ and $R = \{0, 1, 2\}$, as needed. □

Proposition 1.43. *If R is an invo-clean ring with $2 \in Nil(R)$, then R is nil-clean with bounded index of nilpotence. In particular, an invo-clean ring is nil-clean iff 2 is a nilpotent.*

Proof. Given $r \in R$, we write $r = v + e$, where $v^2 = 1$ and $e^2 = e$. But $(1+v)^2 = 2+2v = 2(1+v)$ and hence $(1+v)^3 = 2(1+v)^2 = 2^2(1+v)$, etc. by induction we derive that $(1+v)^{n+1} = 2^n(1+v)$ for all $n \in \mathbb{N}$. Thus $(1+v)^t = 0$ for some appropriate natural t, that is, $1 + v \in Nil(R)$. Furthermore, one may write that $r = (v+1) - (1-e)$, whence R is nil-clean, as claimed.

For the second part, given $q \in Nil(R)$, we write that $q = i + e$ for some $i \in Inv(R)$ and $e \in Id(R)$. Thus $-i = (-q) + e$ and, since $-i \in Inv(R)$ and $(-q \in Nil(R)$, Corollary 1.40 is applicable to infer that $q = i + 1$. Furthermore, one verifies that $q^2 = 2q$ and hence, by induction, $q^{n+1} = 2^n q$ for all $n \in \mathbb{N}$. Thus $q^k = 0$ for some fixed $k \in \mathbb{N}$, as required. □

Under certain additional circumstances the converse is true; even more a criterion when a nil-clean ring is invo-clean is deducible.

Proposition 1.44. *Suppose that R is a nil-clean ring. Then R is invo-clean iff any $q \in Nil(R)$ satisfies the equation $q^2 + 2q = 0$.*

Proof. "$\Rightarrow$". As in Proposition 1.43, we derive that $q^2 = 2q$. Substituting q by $-q$, we are set.

"$\Leftarrow$". Writing $r = q + e = (1 + q) - (1 - e)$ for any $r \in R$ with $q \in Nil(R)$ and $e \in Id(R)$, one checks that $(1 + q)^2 = 1 + 2q + q^2 = 1$ and $(1 - e)^2 = 1 - e$, as required. □

As an interesting consequence, we obtain the following one.

Corollary 1.45. *Suppose R is a nil-clean ring of characteristic 2. Then R is invo-clean iff the index of nilpotence of R is 2.*

Remark 1.46. In regard to the above statement, it is worth noticing that $\mathbb{Z}_8$ is both invo-clean and nil-clean containing the element 2 of nilpotence index 3. However, it is readily seen that 2 satisfies the equality $q^2 + 2q = 0$, because $2^2 + 2.2 = 8 = 0$.

Likewise, $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$ is a nil-clean ring which is not necessarily invo-clean (compare with Corollary 1.45 above). In fact, $\mathbb{Z}_{16}$ is indecomposable, that is, the only idempotents are 0 and 1 as well as all involutions are 1, 7, 9 and 15. So, the unit 5 cannot be represented as a sum of an involution and an idempotent, as expected.

Proposition 1.47. *The (finite or infinite) direct product of invo-clean rings is again an invo-clean ring.*

Proof. This fact has routinely technical check, so we leave it to the reader. □

So, consulting with [40, Proposition 1.9 (ii)], we come to the following:

Example 1.48. The ring $\mathbb{Z}_3 \times \mathbb{Z}_3$ is invo-clean but not weakly nil-clean. Also, referring to [12], $\mathbb{Z}_9$ is a weakly nil-clean ring but an easy computation shows that it is not invo-clean. Thereby, these two notions are independent each to other.

We come now to our main result in which we give a satisfactorily complete description of invo-clean rings.

Theorem 1.49. *A ring R is invo-clean iff $R \cong R_1 \times R_2$, where R_1 is an invo-clean ring of characteristic at most 8 which is nil-clean, and R_2 is either $\{0\}$ or a commutative semiprimitive (and hence reduced) invo-clean ring of characteristic 3 such that each its element is the sum of two idempotents (respectively, of two involutions). In addition, R_2 can be embedded as an isomorphic copy in the direct product of copies of $\mathbb{Z}_3$.*

Proof. Treating the necessity, by virtue of Lemma 1.38 we know that $6^n = 0$ for some $n \in \mathbb{N}$. Since $(2^n, 3^n) = 1$, i.e., there exist non-zero integers k, l such that

$2^n k+3^n l=1$, it follows that $R=2^nR\oplus 3^nR$ because $2^nR\cap 3^nR=\{0\}$. In fact, to show that this intersection is zero, given $x=2^na=3^nb$ for some $a,b\in R$, we have $2^nak=3^nbk$. However, $a(1-3^nl)=3^nbk$ whence $3^n(al+bk)=a$. Multiplying both sides by 2^n, we derive that $0=2^na=x$, as required. Furthermore, $3^nR\cong R/2^nR$ as well as $2^nR\cong R/3^nR$, so that $R\cong R_1\times R_2$, where we put $R_1=R/2^nR$ and $R_2=R/3^nR$. Certainly, using the same trick, one can also decompose R as $R\cong (R/8R)\times(R/3R)$ because $(8,3)=1$. Next, since $R\to R/2^nR=R_1$ and $R\to 3^nR=R_2$ are epimorphisms, it follows from Lemma 1.37 that both R_1 and R_2 are invo-clean. Hence, in view of Lemma 1.38, $6\in Nil(R_1)$ and $6\in Nil(R_2)$. But it is obviously true that $2\in J(R_1)$ whence $3\in U(R_1)$ which assures that $2\in Nil(R_1)$ and even employing the second part of Lemma 1.38 we will have $2^3=8=0$ in R_1. In accordance with Proposition 1.43, the ring R_1 has to be nil-clean.

Regarding the second direct factor, $3\in J(R_2)$ ensures that $2\in U(R_2)$ and thus owing to Proposition 1.41 we obtain $3\in Nil(R_2)=J(R_2)=\{0\}$ which amounts to $3=0$ in R_2. Next, given arbitrary $a\in R_2$, we write $2a=v+e$ where $v\in Inv(R_2)$ and $e\in Id(R_2)$ whence $a=\frac{v+1}{2}+\frac{e+2}{2}$. It is readily verified that both $\frac{v+1}{2}$ and $\frac{e+2}{2}$ are idempotents, as asserted. But R_2 being reduced is necessarily abelian whence commutative. On the other side, we can write $a-1=v+e$ with v,e as above, which means that $a=v+(1+e)$. Since $(1+e)^2=1+3e=1$, we are done. That is why, with the Chinese Remainder Theorem at hand, we deduce that $R_2\cong R_2/J(R_2)$ can be embedded in the direct product of invo-clean domains of characteristic 3 which, in conjunction with Proposition 1.42, are isomorphic to $\mathbb{Z}_3$, as claimed.

The sufficiency follows immediately from Proposition 1.47. □

As a nontrivial immediate consequence, which seems not to have a direct proof, is the following one:

Corollary 1.50. *If R is an invo-clean ring, then $J(R)$ is nil with index of nilpotence not exceeding* 3.

Proof. According to Theorem 1.49, one can decompose the invo-clean ring R as $R\cong R_1\times R_2$, where R_1 is a nil-clean ring and R_2 is a ring with zero Jacobson radical. Since $J(R)\cong J(R_1)$, we next just apply [41] to get the desired claim.

Concerning the second part that the index of nilpotence of R is at most 3, writing $j=v+e$ for any $j\in J(R)$ with $v\in Inv(R)$ and $e\in Id(R)$, we have $j-v=e\in U(R)\cap Id(R)=\{1\}$. Hence $j=v+1$ and so $j^2=2j$ which yields $j^3=4j$ and $4j^2=8j$. Since $2j\in J(R)$, repeating the same procedure for this element, we deduce that $(2j)^2 = 2(2j)$, that is, $4j^2 = 4j$. Thus $8j = 4j$, i.e., $4j = 0 = j^3$, as required. $\square$

Another consequence gives a comprehensive description of invo-clean rings having the strongly property in the following manner:

Corollary 1.51. *A ring R is strongly invo-clean iff $R \cong R_1 \times R_2$, where R_1 is a strongly invo-clean ring of characteristic less than or equal to* 8 *which is strongly nil-clean, and R_2 is either $\{0\}$ or a commutative semiprimitive (and hence reduced) invo-clean ring of characteristic* 3 *which can be embedded as an isomorphic copy in the direct product of copies of $\mathbb{Z}_3$.*

Proof. To treat the necessity, the first part concerning the full classification of the subring R_1 as well as some facts for the subring R_2 follow directly from Theorem 1.49. As for the more concrete classification of the subring R_2 having characteristic 3, one observes that since any involution v and any idempotent e satisfy $v^3 = v$ and $e^3 = e$ and they also commute, every element $y = v + e$ in R_2 satisfies the equation $y^3 = y$. Therefore, applying [68], the ring R_2 must be commutative. (Note that it is trivially seen that R_2 is also a von Neumann regular ring and this also yields that $J(R_2) = \{0\}$.) Further on, the description of R_2 follows repeating the same trick as that from Theorem 1.49.

The sufficiency follows directly from Proposition 1.47. $\square$

Remark 1.52. In regard to Theorem 1.49 and its Corollary 1.51, does it follow that (strongly) nil-clean rings of characteristic ≤ 8 are (strongly) invo-clean? It is also worthwhile noticing that it was somewhat a curiosity that R_2 is a commutative ring.

It was established in [95] that a ring R is *uniquely clean* precisely when R is abelian, $R/J(R)$ is boolean and idempotents of R lift modulo $J(R)$. Thus, one can expect that any *uniquely invo-clean* ring is strongly nil-clean, and hence by an appeal to [39] it will follows that $R/J(R)$ is boolean and $J(R)$ is nil.

1.5. **Weakly nil-clean index and uniquely weakly nil-clean rings.** In [41] a ring R is said to be *nil-clean* if each element $a \in R$ can be represented as $a = b + e$, where $b \in Nil(R)$ and $e \in Id(R)$; note that this is equivalent to the representation that, for every $a \in R$, we have $a = b - e$. If this presentation is unique, the ring R is called *uniquely nil-clean*. It is not too hard to check that this is tantamount to the requirement that the existing idempotent e is unique (see, e.g., [23] and [41]).

On the other vein, in [40] and [12] was stated the definition of a *weakly nil-clean* ring as such a ring R for which any element $a \in R$ is of the form $a = b + e$ or

$a = b - e$, where $b \in Nil(R)$ and $e \in Id(R)$. Moreover, a ring R is said to be *uniquely weakly nil-clean* if the existing idempotent e is unique.

Our further work is motivated by the notions of *unique nil-cleanness* and *weak nil-cleanness* as we will combine them into a new concept. So, the aim here is to explore some variations of unique weak nil-cleanness in order to enlarge the principal known results on unique nil-cleanness from [41] and [23]. In doing that, we set and explore in details the weakly nil-clean index of rings and discuss the original notion of uniquely weakly nil-clean rings stated in Problem 3 of [40]. We sshall also investigate here some other aspects of unique weak nil-cleanness which arise from its specific definition.

For any $a \in R$, let $\mathcal{E}(a) = \{e \in R \mid e^2 = e,\ a - e \in U(R)\}$ and then the *clean index* of R, denoted as $c(R)$, is defined in [81] by $c(R) = sup\{|\mathcal{E}(a)| : a \in R\}$. For any $a \in R$, set $\eta(a) = \{e \in R \mid e^2 = e$ and $a - e \in Nil(R)\}$ and then the *nil-clean index* of R, denoted as $Nin(R)$, is defined in [4] by $sup\{|\eta(a)| : a \in R\}$. In this way, for a more comprehensive investigation of these two notions and, especially, as a natural generalization of the nil-clean index, we also define the concept of *weakly nil-clean index* of a ring. Thereby, as it will be showed below, a ring is uniquely weakly nil-clean if and only if it is weakly nil-clean of weakly nil-clean index 1.

In [81] the clean index $c(R)$ of a ring R was defined and studied. Imitating this, in [4] was introduced the *nil-clean index* $Nin(R)$ of R and some detailed study was given.

In parallel to these two notions, we proceed by stating the following concepts.

Definition 1.53. Let R be a ring and $a \in R$. We define the set

$$\alpha(a) = \{e \in R : e^2 = e \text{ and } a - e \text{ or } a + e \text{ is a nilpotent}\}.$$

Definition 1.54. For an element $a \in R$ the *weakly nil-clean index* of a, abbreviated as $wnc(a)$, is defined to be the cardinality of the set $\alpha(a)$.

Definition 1.55. We define the *weakly nil-clean index* of a ring R as follows:

$$wnc(R) = \sup\{|\alpha(a)| :\ a \in R\}.$$

We foremost start with a series of elementary but useful basic properties of the operator $wnc(R)$ which extend the analogous ones in [4].

Lemma 1.56. *For any ring R the inequality $wnc(R) \geq 1$ holds. In addition, if R is a ring which has at most n idempotents or at most n nilpotents, then $wnc(R) \leq n$.*

Proof. Straightforward. □

Example 1.57. A direct check shows that $wnc(\mathbb{Z}_3) = 1$.

Lemma 1.58. *If R is a ring with a subring S, then $wnc(R) \geq wnc(S)$.*

Proof. Follows in the same manner as [4, Lemma 2.2]. □

Lemma 1.59. *If R is a ring with a nil-ideal I, then $wnc(R/I) \leq wnc(R)$.*

Proof. Letting $a \in R$ be an arbitrary element, then for any idempotent $b + I \in \alpha(a + I)$, so $b^2 - b \in I$ and there exists $e \in Id(R)$ with $b + I = e + I$, one may derive that $(a + I) - (b + I) = (a - e) + I$ with $(a - e)^t \in I$ or that $(a + I) + (b + I) = (a + e) + I$ with $(a + e)^t \in I$ for some $t \in \mathbb{N}$. Since I is nil, it follows that either $a - e \in Nil(R)$ or $a + e \in Nil(R)$. Consequently, $e \in \alpha(a)$ and thus $|\alpha(a)| \geq |\alpha(a + I)|$, as needed. □

Remark 1.60. In [4, Lemma 2.4 (1)] the condition "*If idempotents lift modulo I*" is absolutely redundant, because I is a nil-ideal. Moreover, the inequality $Nin(R/I) \geq Nin(R)$ is *not* true and the purported there proof is erroneous. This can be subsumed via the following construction: set $R = \mathbb{Z}_p$ and $I = \{(a_{ij}) \in \mathbb{T}_n(R) \ : \ \forall a_{ii} = 0\}$. It is readily seen that this is a nil-ideal of $\mathbb{T}_n(R)$ with the property that $\mathbb{T}_n(R)/I \cong R \times \cdots \times R$, where the product is taken n times.

Next, choosing $n = 2 = p$, we detect that $\mathbb{T}_2(\mathbb{Z}_2)/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, whence with the aid of [4, Lemma 2.3] we derive $Nin(\mathbb{T}_2(\mathbb{Z}_2)/I) = Nin(\mathbb{Z}_2 \times \mathbb{Z}_2) = Nin(\mathbb{Z}_2)Nin(\mathbb{Z}_2) = 1 \cdot 1 = 1$. On the other hand, [41, Theorem 4.1] is a guarantor that $\mathbb{T}_2(\mathbb{Z}_2)$ is nil-clean, so that $Nin(\mathbb{T}_2(\mathbb{Z}_2)) = wnc(\mathbb{T}_2(\mathbb{Z}_2)) = 2$, owing to Example 1.74 listed below. Thus this contradiction demonstrates $Nin(R/I) < Nin(R)$.

If now we choose $n = 3 = p$, then the same trick successfully works to manifestly illustrate with the help of Example 1.75 quoted below that $wnc(R) > wnc(R/I)$.

The next assertion improves [4, Lemma 2.8].

Lemma 1.61. *For any ring R the inequality $wnc(R) \geq Nin(R)$ holds.*

Proof. It is trivial, so we will omit the details. □

Remark 1.62. Note the simple fact, which was already used above, that if R is a nil-clean ring, then $wnc(R) = Nin(R)$.

About the truthfulness of the inequality $c(R) \geq wnc(R)$, one can say the following: It is rather logically to expect that it is always true. Nevertheless, if we define $wnc'(R)$ in the same meaning as $wnc(R)$ but containing only the sign "+", whereas $wnc''(R)$ to contain only the sign "-", we see that these are certainly two different numbers possessing the property that $c(R) \geq \max(wnc'(R), wnc''(R))$ as well as $wnc(R) \geq \max(wnc'(R), wnc''(R))$, as supposed.

The next assertion extends [4, Theorem 3.2].

Proposition 1.63. *Suppose R is a ring. Then $wnc(R) = 1$ iff R is abelian.*

Proof. First of all we will prove that $wnc(R) = 1$ iff R is abelian and for any non-zero idempotent $e \in R$, the relation $e \neq m + n$ holds for all $m, n \in Nil(R)$.

Since with Lemma 1.61 at hand we have $1 = wnc(R) \geq Nin(R) \geq 1$, it follows that $Nin(R) = 1$ and by [4, Lemma 3.1] we get that R is abelian and for any idempotent $0 \neq e \in R$, the ratio $e \neq m + n$ is valid for all $m, n \in Nil(R)$.

Now let R be abelian and, for any idempotent $e \in R \setminus \{0\}$, the inequality $e \neq m + n$ is true for all $m, n \in Nil(R)$. Suppose, for concreteness, $a \in R$ has two weakly nil-clean decompositions. We have three possible cases:

(1) $a = e_1 + n_1 = e_2 + n_2$, with e_1 and e_2 idempotents and $n_1, n_2 \in Nil(R)$. In this case the decompositions are actually nil-clean, so this situation was handled in [4, Lemma 3.1] and leaded to $e_1 = e_2$. It follows now that $wnc(R) = 1$.

(2) $a = -e_1 + n_1 = -e_2 + n_2$, with e_1 and e_2 idempotents and $n_1, n_2 \in Nil(R)$. Then $-e_1(1 - e_1) + n_1(1 - e_1) = -e_2(1 - e_1) + n_2(1 - e_1)$, so $e_2(1 - e_1) = n_2(1 - e_1) - n_1(1 - e_1)$. Since R is abelian, the element $e_2(1 - e_1)$ is an idempotent and both $n_2(1 - e_1)$, $n_1(1 - e_1)$ are nilpotents. So, by hypothesis, we get $e_2(1 - e_1) = 0$, that is $e_2 = e_1e_2$. Consequently, $n_1 - n_2 = e_1 - e_2 = e_1(1 - e_2)$, and hence by hypothesis we derive that $e_1(1 - e_2) = 0$. Thus $e_1 = e_1e_2$ and $e_1 = e_2$. It again follows that $wnc(R) = 1$.

(3) $a = -e_1 + n_1 = e_2 + n_2$, with e_1 and e_2 idempotents and $n_1, n_2 \in Nil(R)$. Then $e_2(1 - e_2) + n_2(1 - e_2) = -e_1(1 - e_2) + n_1(1 - e_2)$ and so $e_1(1 - e_2) = n_1(1 - e_2) - n_2(1 - e_2)$ Thus $e_1(1 - e_2) = 0$, i.e., $e_1 = e_1e_2$.

Let $n_1^m = 0$ and $f = e_1 + e_2$. Now lifting $f + n_2 = n_1$ to the $m - th$ power, we obtain that $\sum_{k=0}^{m} \binom{m}{k} f^{m-k} n_2^k = 0$. But $f^k = (e_1 + e_2)^k = \sum_{l=0}^{m} \binom{m}{k} e_1^{k-l} e_2^k = e_1 + e_2 + e_1e_2(2^k - 2) = e_1 + e_2 + e_1(2^k - 2) = (2^k - 1)e_1 + e_2$, so $\sum_{k=0}^{m} \binom{m}{k}((2^{m-k} - 1)e_1 + e_2)n_2^k = 0$, which gives $e_1 \sum_{k=0}^{m} \binom{m}{k} 2^{m-k} n_2^k + (e_2 - e_1) \sum_{k=0}^{m} \binom{m}{k} n_2^k = 0$. This is equivalent to $e_1(2 + n_2)^k + (e_2 - e_1)(n_2 + 1)^k = 0$. Hence $e_2(n_2 + 1)^k + e_1((2 + n_2)^k - (1 + n_2)^k) = 0$. Multiplying by $(1 - e_1)$ we get $(1 - e_1)e_2(n_2 + 1)^k = 0$, but $n_2 + 1$ is a unit, so $e_2 = e_1e_2$ and from $e_1 = e_1e_2$ we have $e_1 = e_2$. It follows once again that $wnc(R) = 1$.

Knowing that $wnc(R) = 1$ iff R is abelian and for any non-zero idempotent $e \in R$, the relation $e \neq m + n$ holds for all $m, n \in Nil(R)$ and by Lemma 3.1 from [4], we infer that $wnc(R) = 1$ iff $Nin(R) = 1$ and now using Theorem 3.2 from [4] we get the desired result. □

It is worthwhile noticing that indecomposable rings, and hence local rings, always have weakly nil-clean index one. In that aspect, we will now consider in general this special case of rings having the weakly nil-clean index one and shall completely characterize them. So, we come now to one of our basic statements.

Theorem 1.64. *The following are equivalent for a ring R:*

(1) *R is uniquely weakly nil-clean;*
(2) *R is abelian weakly nil-clean;*
(3) *$R \cong R_1 \times R_2$, where R_1 is either $\{0\}$ or an abelian nil-clean ring and R_2 is either $\{0\}$ or a local weakly nil-clean ring such that $J(R_2)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$.*

Proof. (1)⇒(2). We will show that R is abelian and so [12] will apply to get the desired claim. To that goal, let $e^2 = e \in R$. Then, for any $r \in R$, one writes that $e = e + 0 = (e - er(1-e)) + er(1-e)$ are two decompositions into the sum of an idempotent and a nilpotent. Thus $e = e - er(1-e)$, i.e., $er(1-e) = 0$. Similarly, $(1-e)re = 0$. Hence $er = re = ere$, and so all idempotents in R are central, that is, R is abelian, as wanted.

(2)⇒(1). By virtue of [12], R is weakly nil-clean. As $J(R) = J(R_1) \times J(R_2)$ is nil and $R/J(R) \cong [R_1/J(R_1)] \times [R_2/J(R_2)]$ is reduced, we derive that $J(R) = Nil(R)$. Assume that, for $a \in R$, there exist idempotents e and f and nilpotents b and c such that $a = b + e$ or $a = b - e$ and that $a = c + f$ or $a = c - f$. We must show that $e = f$. There are four cases that we have to consider:

(i) $a = b + e = c + f$;
(ii) $a = b + e = c - f$;
(iii) $a = b - e = c + f$;
(iv) $a = b - e = c - f$.

For case (i) or (iv), we have $e - f \in J(R)$. For case (ii) or (iii), we have $e + f \in J(R)$. Thus, in any case, we have $e - f = (e - f)(e + f) \in J(R)$. It follows that $(1 - e)f = -(1 - e)(e - f) \in J(R)$ and $e(1 - f) = (e - f)(1 - f) \in J(R)$. As both $(1 - e)f$ and $e(1 - f)$ are idempotents, we conclude that $(1 - e)f = 0$ and $e(1 - f) = 0$. So $f = ef = e$, as required.

(1) $\iff$ (2). This is a direct consequence of Proposition 1.63.

(2) $\iff$ (3). It follows directly from [12]. □

We recall from [23, Theorem 5.4] that a ring R is uniquely nil-clean iff R is abelian nil-clean. So, with Theorem 1.64 at hand, one can deduce the following:

Corollary 1.65. *A ring R is uniquely nil-clean iff R is uniquely weakly nil-clean and $2 \in J(R)$.*

As a connection to strongly π-regular rings, one may state the following strengthening of results on unique nil-cleanness of rings from [23] and [41].

Corollary 1.66. *A ring R is uniquely weakly nil-clean iff R is abelian strongly π-regular such that $R/J(R)$ is isomorphic to either a Boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings.*

Proof. It is well known that strongly π-regular rings R have nil $J(R)$. We therefore employ [12] and Theorem 1.64 to get what we asserted. $\square$

Remark 1.67. We shall now explore two various notions of unique weak nil-cleanness. At the beginning, if we use the *"weak unicity"* for a ring R, i.e., every element $r \in R$ can be written down in at most one way as a nil-clean element or $-r$ in at most one way as a nil-clean element, then we just obtain uniquely weakly nil-clean rings and vice versa.

However, if we use the *"strong unicity"* for a ring R, i.e., every element $r \in R$ can be written down in a unique way as $n + f$, with n a nilpotent and f or $-f$ an idempotent, then such a ring is either uniquely nil-clean of characteristic 2 or uniquely weakly nil-clean but not nil-clean. This follows because we can write $-1 = 0 + (-1) = (-2) + 1$, so if $2 \neq 0$ we have that 2 is not a nilpotent.

The next comments are also useful shedding some light on the newly defined notion.

Remark 1.68. For any ring R and any $s \in R$, we set $P_s = es(1-e)$ and $P'_s = (1-e)se$.

Let us now R be a ring and $r \in R$. We then have the following weakly nil-clean decompositions for each idempotent e

$$e = e + 0 = (e - P_r) + P_r = (e - P'_r) + P'_r = (e + P_r) - P_r = (e + P'_r) - P'_r.$$

We now continue with

Proposition 1.69. *Let R be a ring with $wnc(R) \leq 2$. Then, for any $s \in R$ and for any $e \in Id(R)$, we have $2es(1-e) = 0$.*

Proof. Let $e \in R$ be an idempotent and let $s \in R$.

If e is central, then $R = C(e)$, so for every $s \in R$ we obtain $es = se = ese$ and, therefore, $es(1-e) = 0$, hence $2es(1-e) = 0$. If e is not central, then there is

$s \notin C(e)$ and so $P_s \neq 0$ or $P'_s \neq 0$. We have $e = e+0 = (e-P_s)+P_s = (e-P_{2s})+P_{2s}$ and by $wnc(R) \leq 2$ and $P_s \neq 0$ we get $P_{2s} = 0$ or $P_{2s} = P_s$. If $P_{2s} = P_s$, it follows $e2s(1-e) = es(1-e)$ and thus $es(1-e) = 0$, which is a contradiction because $P_s \neq 0$. Consequently, $P_{2s} = 0$, so $2es(1-e) = 0$. $\square$

Remark 1.70. Another proof for Proposition 1.69 is as follows:

Let e be idempotent. We have

$$e = e + 0 = (e + er(1-e)) - er(1-e) = (e - er(1-e)) + er(1-e),$$

thus we get three weakly nil-clean decompositions of e. Therefore,

$$e = e \pm er(1-e) \text{ or } e + er(1-e) = e - er(1-e),$$

which is equivalent to

$$er(1-e) = 0 \text{ or } 2er(1-e) = 0.$$

Corollary 1.71. *Let R be a ring with $wnc(R) \leq 2$. Then, for any $s \in R$ and for any $e \in Id(R)$, we have $2(es - se) = 0$.*

Proof. Utilizing P_s as in Proposition 1.69, we obtain that $2es(1-e) = 0$. Now, considering P'_s, we have $2(1-e)se = 0$ and, therefore, $2es = 2ese = 2se$, so $2(es - se) = 0$. $\square$

Proposition 1.72. *Let R be a ring with $wnc(R) \leq 2$ and $e \in Id(R)$. Then $|R/C(e)| \leq 2$.*

Proof. If we assume the contrary, $|R/C(e)| > 2$, then there are two different elements, say $s, t \notin C(e)$, such that $s - t \notin C(e)$. By using Remark 1.70 and $wnc(R) \leq 2$, we differ the following cases:

- $P_s = P_t$ and $P'_s = P'_t$, then $es(1-e) = et(1-e)$ and $(1-e)se = (1-e)te$, hence $e(s-t) = e(s-t)e$ and $(s-t)e = e(s-t)e$, so $e(s-t) = (s-t)e$, which is a contradiction.
- $P_s = 0$ and $P'_s = 0$, then $es = ese$ and $ese = se$, so $es = se$, which is a contradiction.
- $P_s = 0$ and $P'_t = 0$, then since s and t are not in $C(e)$, it follows $P'_s \neq 0$ and $P_t \neq 0$ and $P'_s = P_t$ and by this we get $e(1-e)se = eet(1-e)$, so $et(1-e) = 0$, which is a contradiction.
- $P_s = P_t$ and $P'_s = 0$, then $P'_t \neq 0$, so $P_s = P_t = P'_t$, which is a contradiction.

$\square$

Proposition 1.73. *Let R be a ring and $e \in Id(R)$. Then*

$$|R/A(e)| \leq |\alpha(e)|,$$

where $A(e) = \{r \in R \mid er(1-e) = 0\}$.

Proof. Letting $n+1 \leq |R/A(e)|$, then we can find an inclusion

$$\{A(e), r_1 + A(e), \dots, r_n + A(e)\} \subseteq R/A(e).$$

So, for any r_i, r_j such that $i, j \in \{1, 2, \dots, n\}$, we have $r_i + A(e) \neq r_j + A(e)$ and, therefore, $r_i - r_j \notin A(e)$. It follows that $P_{r_i - r_j} \neq 0$. Thus $P_{r_i} \neq P_{r_j}$. Also, for any r_i, we have $r_i \notin A(e)$. Hence $P_{r_i} \neq 0$. So the set $\{0\} \cup \{P_{r_i} | i \in \{1, 2, \dots, n\}\}$ has $n+1$ elements and since for an idempotent e we get $e = e + 0 = (e - P_{r_i}) + P_{r_i}$ for any $i \in \{1, 2, \dots, n\}$, the desired inequality $|R/A(e)| \leq |\alpha(e)|$ follows, as asserted. $\square$

We will now compute $wnc(R)$ for some concrete rings R. Specifically, we will show that the following equalities hold:

Example 1.74. $wnc(\mathbb{T}_2(\mathbb{Z}_p)) = p$, where p is a prime number.

Proof. It is a well-known fact that a matrix in $\mathbb{T}_2(\mathbb{Z}_p)$ is a nilpotent if and only if it has a zero principal diagonal. We are looking now for idempotents. In fact, $\begin{pmatrix} x_1 & a \\ \overline{0} & x_2 \end{pmatrix} \begin{pmatrix} x_1 & a \\ \overline{0} & x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 & a(x_1 + x_2) \\ \overline{0} & x_2^2 \end{pmatrix}$, hence $\begin{pmatrix} x_1 & a \\ \overline{0} & x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 & a(x_1 + x_2) \\ \overline{0} & x_2^2 \end{pmatrix}$, and thus $x_1, x_2 \in \{\overline{0}, \overline{1}\}$ and $a(x_1 + x_2 - \overline{1}) = \overline{0}$ Each pair (x_1, x_2) will give a set of solutions for the problem of idempotent matrices.

- case I : $x_1 = \overline{0}, x_2 = \overline{0}$, then $S_1 = \{\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}\}$;
- case II : $x_1 = \overline{0}, x_2 = \overline{1}$, then $S_2 = \{\begin{pmatrix} \overline{0} & \alpha \\ \overline{0} & \overline{1} \end{pmatrix}, \alpha \in \mathbb{Z}_p\}$;
- case III : $x_1 = \overline{0}, x_2 = \overline{1}$, then $S_3 = \{\begin{pmatrix} \overline{1} & \alpha \\ \overline{0} & \overline{0} \end{pmatrix}, \alpha \in \mathbb{Z}_p\}$;
- case IV : $x_1 = \overline{1}, x_2 = \overline{1}$, then $S_4 = \{\begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}\}$.

Let $A \in \mathbb{T}_2(\mathbb{Z}_p)$. Letting $A - E$ be a nilpotent, where E is an idempotent, then A has the main diagonal of the form of an idempotent diagonal (so it has $\overline{0}$ and/or $\overline{1}$). If $A + E$ is a nilpotent, with E an idempotent, then A has in the main diagonal an element from $\{\overline{0}, \overline{-1}\}$. Therefore, except of A with main zero diagonal, only one of the following can hold: $A+E$ or $A-E$ a nilpotent, with E an idempotent.

Let A be with $\overline{0}$ or $-\overline{1}$ in the main diagonal. We look for m as big as possible such that $A+E_1,\cdots,A+E_m$ are nilpotents. Thus $E_1,\cdots,E_m$ share the same main diagonal, that is, they are in the same S_i. Hence the problem is reduced to finding the maximum cardinality of $S_i, i\in\{1,2,3,4\}$. Also, trying to find out the maximum r such that $A-E_1,\cdots,A-E_r$ are nilpotents, with A having $\overline{0}$ and $\overline{1}$ in the main diagonal and $E_1,\cdots,E_r$ being idempotents leads to the same problem, finding the maximum cardinality of $S_i, i\in\{1,2,3,4\}$. We finally conclude that $|S_1|=|S_4|=1$ and $|S_2|=|S_3|=p$, because the free variable α can take exactly the p values $\overline{0},\overline{1},\ldots,\overline{p-1}$. So, $wnc(\mathbb{T}_2(\mathbb{Z}_p))=p$, as promised. $\square$

Example 1.75. $wnc(\mathbb{T}_3(\mathbb{Z}_p))=p^2$, where p is a prime number.

Proof. It is a well-known fact that a matrix in $\mathbb{T}_3(\mathbb{Z}_p)$ is a nilpotent if and only if it has a zero main diagonal. We are looking now for idempotents. In fact,
$\begin{pmatrix} x_1 & a & b\\ \overline{0} & x_2 & c\\ \overline{0} & \overline{0} & x_3\end{pmatrix}\begin{pmatrix} x_1 & a & b\\ \overline{0} & x_2 & c\\ \overline{0} & \overline{0} & x_3\end{pmatrix}=\begin{pmatrix} x_1 & a & b\\ \overline{0} & x_2 & c\\ \overline{0} & \overline{0} & x_3\end{pmatrix}$, which is equivalent to
$\begin{pmatrix} x_1^2 & a(x_1+x_2) & b(x_1+x_3)+ac\\ \overline{0} & x_2^2 & c(x_2+x_3)\\ \overline{0} & \overline{0} & x_3^2\end{pmatrix}=\begin{pmatrix} x_1 & a & b\\ \overline{0} & x_2 & c\\ \overline{0} & \overline{0} & x_3\end{pmatrix}$, which is equivalent to

$$\begin{cases} x_1\in\{\overline{0},\overline{1}\}\\ x_2\in\{\overline{0},\overline{1}\}\\ x_3\in\{\overline{0},\overline{1}\}\\ a(x_1+x_2-\overline{1})=\overline{0}\\ b(x_1+x_3-\overline{1})=-ac\\ c(x_2+x_3-\overline{1})=\overline{0}\end{cases}$$

For $x_1=\overline{0},x_2=\overline{0},x_3=\overline{0}$, we have $S_1=\{O_3\}$.

For $x_1=\overline{0},x_2=\overline{0},x_3=\overline{1}$, we have $S_2=\{\begin{pmatrix} \overline{0} & \overline{0} & \alpha\\ \overline{0} & \overline{0} & \gamma\\ \overline{0} & \overline{0} & \overline{1}\end{pmatrix}\mid \alpha,\gamma\in\mathbb{Z}_p\}$.

For $x_1=\overline{0},x_2=\overline{1},x_3=\overline{0}$, we have $S_3=\{\begin{pmatrix} \overline{0} & \alpha & \alpha\gamma\\ \overline{0} & \overline{1} & \gamma\\ \overline{0} & \overline{0} & \overline{0}\end{pmatrix}\mid \alpha,\gamma\in\mathbb{Z}_p\}$.

For $x_1 = \overline{0}, x_2 = \overline{1}, x_3 = \overline{1}$, we have $S_4 = \{\begin{pmatrix} \overline{0} & \alpha & \beta \\ \overline{0} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{0}, x_3 = \overline{0}$, we have $S_5 = \{\begin{pmatrix} \overline{0} & \alpha & \beta \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{pmatrix} \mid \alpha, \beta \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{0}, x_3 = \overline{1}$, we have $S_6 = \{\begin{pmatrix} \overline{1} & \alpha & -\alpha\gamma \\ \overline{0} & \overline{0} & \gamma \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix} \mid \alpha, \gamma \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{1}, x_3 = \overline{0}$, we have $S_7 = \{\begin{pmatrix} \overline{1} & \overline{0} & \alpha \\ \overline{0} & \overline{1} & \gamma \\ \overline{0} & \overline{0} & \overline{0} \end{pmatrix} \mid \alpha, \gamma \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{1}, x_3 = \overline{1}$, we have $S_8 = \{I_3\}$. Following the same argument as in Example 1.74, we derive that $wnc(\mathbb{T}_3(\mathbb{Z}_3))$ is the maximum cardinality of S_i, $i \in \{1, 2, \dots, 8\}$. Since $|S_1| = |S_8| = 1$ and $|S_2| = |S_3| = \dots = |S_7| = p^2$ (2 free variables and $|\mathbb{Z}_p| = p$), it finally follows that $wnc(\mathbb{T}_3(\mathbb{Z}_p)) = p^2$, as stated. $\square$

Remark 1.76. When studying weakly nil-clean matrices, it is not enough to study companion matrices which are (or are not) blocks of other companion matrices. In fact, note that *not* all matrices are similar to a companion matrix (see the proof of the main result in [11] or [13]).

Let A be a matrix with $A = E + N$ for an idempotent E and a nilpotent N. It is a well-known fact that there is a matrix C such that $C^{-1}AC = B$, where B is a companion matrix. Then $B = C^{-1}EC + C^{-1}NC$, so B is the sum of a matrix similar to a nilpotent and a matrix similar to an idempotent, thus the sum of a nilpotent and an idempotent. The same argument works for $A = N - E$, too.

Example 1.77. $wnc(\mathbb{M}_2(\mathbb{Z}_3)) = 5$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_3)$. Then $A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$. We claim $A^2 = A$ in order to find the wanted idempotents. They are the following:

$$\begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{1} & \overline{0} \\ s & 0 \end{pmatrix}, \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix},$$

where $s \in \mathbb{Z}_3$ and also

$$\begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}.$$

Next, we claim $A^2 = O_2$ to find out nilpotents. They are the following:

$$\begin{pmatrix} \overline{1} & \overline{1} \\ \overline{2} & \overline{2} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{2} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{1} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$$

and also

$$\begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{0} \end{pmatrix}, \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{0} \end{pmatrix},$$

where $s \in \mathbb{Z}_3$.

If $A + E$, with E an idempotent, is nilpotent, then $tr(A + E) = 0$, whence $trA = -trE$. If $A - E$, with E an idempotent, is nilpotent, then $tr(A - E) = 0$, whence $trA = trE$.

For an idempotent E, we deduce:

- $trE = \overline{1}$ if and only if $E \neq O_2$, $E \neq I_2$;
- $trE = \overline{2}$ if and only if $E = I_2$;
- $trE = \overline{0}$ if and only if $E = O_2$.

Let $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{0} \end{pmatrix}$. Then, for an idempotent E, if $A + E$ is a nilpotent, then $tr(E) = 0$, and thus $E = O_2$. Also, if $A - E$ is a nilpotent, then $tr(E) = 0$ and hence $E = O_2$. Therefore, if $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{0} \end{pmatrix}$, we have $\alpha(A) = \{O_2\}$, and it follows that

$$wnc(\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{0} \end{pmatrix}) \leq 1$$

such that $wnc(\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{pmatrix}) = 1$ and $wnc(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix}) = wnc(\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{0} \end{pmatrix}) = 0$.

Let $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix}$. Furthermore, for an idempotent E, if $A+E$ is a nilpotent, then $tr(E) = \overline{2}$, and so $E = I_2$. But $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{1} & y \\ \overline{1} & \overline{2} \end{pmatrix}$ is a nilpotent if and only if $y = 2$. Also, if $A - E$ is a nilpotent, then $tr(E) = 1$ and hence $E \neq O_2, I_2$.

We infer that

- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & y - s \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = y$;
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{2} & y \\ \overline{1} - s & \overline{1} \end{pmatrix}$ is a nilpotent if and only if ($y = \overline{2}$ and $s = \overline{0}$) or ($y = \overline{1}$ and $s = \overline{2}$);

- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & y-s \\ \overline{1} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & y \\ \overline{1}-s & \overline{1} \end{pmatrix}$, is a nilpotent if and only if ($y = \overline{0}$ and $s \in \mathbb{Z}_3$) or ($y \in \mathbb{Z}_3$ and $s = \overline{1}$);
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{1} & y-\overline{2} \\ \overline{2} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $y = \overline{0}$;
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & y-\overline{1} \\ \overline{0} & \overline{2} \end{pmatrix}$, which is not a nilpotent.

By virtue of the above results, we get the following:

For $A = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$ we have $E_1 = \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{2} & \overline{1} \end{pmatrix}$, $E_5 = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix}$ such that $A - E_i$ is a nilpotent ($i \in \{1,2,3,4,5\}$) and there are no idempotents E such that $A + E$ is a nilpotent. So

$$\text{wnc } (\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}) = 5.$$

For $A = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{1} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{2} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$ such that $A - E_i$ is a nilpotent, $i \in \{1,2,3\}$ and there are no idempotents E such that $A + E$ is a nilpotent. So

$$\text{wnc } (\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{1} \end{pmatrix}) = 3.$$

For $A = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{1} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$ such that $A - E_i$ is a nilpotent, $i \in \{1,2,3\}$ and there is one idempotent, namely $E = I_2$ such that $A + E$ is a nilpotent. So

$$\text{wnc } (\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{1} \end{pmatrix}) = 4.$$

Let $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{2} \end{pmatrix}$. Furthermore, for an idempotent E, if $A+E$ is a nilpotent, then $tr(E) = \overline{1}$, and thus $E \neq I_2, O_2$. Also, if $A-E$ is a nilpotent, then $tr(E) = \overline{2}$ and hence $E = I_2$.

We derive
$\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{2} \end{pmatrix} - \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{2} & y \\ \overline{1} & \overline{1} \end{pmatrix}$, which is a nilpotent if and only if $y = \overline{2}$

- Let $A = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix}$. Then $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & s \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = 0$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{1}+s & \overline{2} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & s \\ \overline{1} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1}+s & \overline{0} \end{pmatrix}$, is a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{0} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{2} & \overline{1} \end{pmatrix}$, which is a nilpotent.

For $A = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix}$ we have the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}$ such that $A + E_i$ is a nilpotent, $i \in \{1,2,3,4\}$ and there are no idempotents E such that $A - E$ is a nilpotent. So

$$\operatorname{wnc}\left(\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix}\right) = 4.$$

Let $A = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}$.

- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & s+1 \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{1}+s & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{1}$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & s+1 \\ \overline{1} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{1}$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & \overline{1} \\ s+1 & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1}+s & \overline{0} \end{pmatrix}$, is a nilpotent if and only if $s = \overline{2}$;

- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{1} \end{pmatrix}$, which is not a nilpotent.

For $A = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{1} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{2} & \overline{1} \end{pmatrix}$ such that $A + E_i$ is a nilpotent, $i \in \{1, 2, 3, 4\}$ and there are no idempotents E such that $A - E$ is a nilpotent. So

$$\text{wnc}\,\left(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}\right) = 4.$$

Let $A = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$. We have

- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & s+\overline{2} \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = \overline{1}$;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{1}+s & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{1}$.
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & s+\overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{0}$;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & \overline{1} \\ s+1 & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1}+s & \overline{0} \end{pmatrix}$, is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{2} & \overline{1} \end{pmatrix}$, which is not a nilpotent.

For $A = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{2} & \overline{2} \end{pmatrix}$ such that $A + E_i$ is a nilpotent, $i \in \{1, 2, 3, 4\}$ and there is one idempotent $E = I_2$ such that $A - E$ is a nilpotent. So

$$\text{wnc}\,\left(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}\right) = 5.$$

In conclusion, $\text{wnc}(\mathbb{M}_2(\mathbb{Z}_3)) = 5$, as expected. $\square$

For rings A and B and for a bimodule ${}_AM_B$, we denote by $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ the formal triangular matrix ring.

The next statement strengthens [4, Theorem 4.1].

Proposition 1.78. *Let R be a ring. The following statements are equivalent:*

(1) *$wnc(R)=2$;*

(2) *$R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where A and B are abelian rings, and ${}_AM_B$ is a bimodule with $|M| = 2$.*

Proof. *(1) implies (2)*:
If $wnc(R) = 2$, since $wnc(R) \geq Nin(R)$, then $Nin(R) = 1$ or $Nin(R) = 2$.

- If $Nin(R) = 1$, then R is abelian and so $wnc(R) = 1$, which is a contradiction.
- If $Nin(R) = 2$, then by Theorem 4.1 in [4] we get the desired form of R.

(2) implies (1):
Nilpotent elements in R are $\begin{pmatrix} n_A & w \\ 0 & n_B \end{pmatrix}$, where n_A is a nilpotent in A, n_B is a nilpotent in B and w is any element in $M = \{0, x\}$.
Idempotent elements in R are $\begin{pmatrix} e_A & w \\ 0 & e_B \end{pmatrix}$, where e_A is an idempotent in A, e_B is an idempotent in B and $w \in M$ which satisfies the condition $e_Aw + we_B = w$.
Since $wnc(A) = wnc(B) = 1$ and $x = x + 0 = 0 + x = x - 0 = 0 - x$ are the only decompositions of x, we have at most four weakly nil clean decompositions for $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$ as follows:

$$\begin{pmatrix} a & w \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & x \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & 0 \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_Ax + xe_B = 0;$$

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & x \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & 0 \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_Ax + xe_B = x.$$

Hence we get at most two idempotents in $\alpha(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix})$.

Since $wnc(A) = wnc(B) = 1$ and $0 = 0 + 0 = x + x = 0 - 0 = x - x$ are the only decompositions of x, we have at most four weakly nil clean decompositions for $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ as follows:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & 0 \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & x \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_Ax + xe_B = x;$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & 0 \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & x \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_Ax + xe_B = x.$$

Hence we got at most 2 idempotents in $\alpha(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix})$.

Therefore, $wnc(R) \leq 2$, and so if we find q in R such that we can get two idempotents in $\alpha(q)$, then $wnc(R) = 2$. Thus q is $\begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ and the idempotents are $\begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ and $\begin{pmatrix} 0 & x \\ 0 & 1_B \end{pmatrix}$. □

We continue by showing that the next assertion is *not* an analogue of [4, Proposition 4.2].

Example 1.79. If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $wnc(A) = wnc(B) = 1$ and ${}_AM_B$ is a bimodule with $|M| = 3$, then $wnc(R) = 3$ cannot be happen in general. In fact, in accordance with Example 1.77, $R = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_3 & 0 \end{pmatrix}$ is a ring with $wnc(R) = 5 > 3$.

Note that if $P = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix}$, then $P/J(P) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

We now proceed by extending [4, Proposition 4.4] in the following manner:

Proposition 1.80. *Let R be a ring and let $n \geq 1$ be an integer. Then*
(i) $wnc(\mathbb{M}_n(R)) \geq 3$, provided $n \geq 2$.
(ii) $wnc(\mathbb{M}_n(R)) = 3$ if and only if $n = 2$ and $R \cong \mathbb{Z}_2$.

Proof. (i) Applying Lemma 1.61, it follows that $wnc(\mathbb{M}_n(R)) \geq Nin(\mathbb{M}_n(R))$. Furthermore, [4, Proposition 4.4 (1)] applies to get the wanted inequality.

(ii) Referring again to Lemma 1.61, $Nin(\mathbb{M}_n(R)) \leq wnc(\mathbb{M}_n(R))$ so that either $Nin(\mathbb{M}_n(R)) = 1$, or $Nin(\mathbb{M}_n(R)) = 2$, or $Nin(\mathbb{M}_n(R)) = 3$. The first two cases are impossible appealing to [4, Theorem 3.2] or to [4, Theorem 4.1], respectively. The third case is handled in [4, Proposition 4.4 (2)], which gives our claim. □

Remark 1.81. It is noteworthy that, by virtue of [11], the ring $\mathbb{M}_2(R) \cong \mathbb{M}_2(\mathbb{Z}_2)$ is nil-clean and, consequently, $wnc(\mathbb{M}_2(\mathbb{Z}_2)) = Nin(\mathbb{M}_2(\mathbb{Z}_2))$.

1.6. **n-Torsion clean rings.** Our notations and notions here are in agreement with those from [79]. For instance, for such a ring R, the symbol $U(R)$ denotes the group of units, $\mathrm{Id}(R)$ the set of idempotents and $J(R)$ the Jacobson radical of R, respectively. Besides, the finite field with m elements will be denoted by $\mathbb{F}_m$, and $\mathbb{M}_k(R)$ will stand for the $k \times k$ matrix ring over R; $k \in \mathbb{N}$. For an element u of a group G, the letter $o(u)$ will denote the order of u. And finally, the symbol $LCM(n_1, \ldots, n_k)$ will be reserved for the least common multiple of $n_1, \ldots, n_k \in \mathbb{N}$.

We will say a nil ideal I of R is nil of index k if, for any $r \in I$, we have $r^k = 0$ and k is the minimal natural number with this property. Likewise, we will say that I is nil of bounded index if it is nil of index k, for some fixed k.

Let us recall that a ring R is said to be *clean* if, for every $r \in R$, there are $u \in U(R)$ and $e \in \mathrm{Id}(R)$ with $r = e + u$. If, in addition, the commutativity condition $ue = eu$ is satisfied, the clean ring R is called *strongly clean*. These rings were introduced by Nicholson in [93] and [94]. Both clean rings and their various specializations or generalizations are intensively studied since then (see, for example, [12], [23], [D4], [40], [41] and references within).

A decomposition $r = e + u$ of an element r in a ring R will be called *n-torsion clean decomposition* of r if $e \in \mathrm{Id}(R)$ and $u \in U(R)$ is n-torsion, i.e. $u^n = 1$. We will say that such a decomposition of r is *strongly n-torsion clean*, if additionally e and u commute.

The aim of this article is to investigate in detail the following proper subclasses of (strongly) clean rings:

Definition 1.82. A ring R is said to be (*strongly*) *n-torsion clean* if there is $n \in \mathbb{N}$ such that every element of R has a (strongly) n-torsion clean decomposition and n is the smallest possible natural number with the above property.

It is easy to see that boolean rings are precisely the rings which are (strongly) 1-torsion clean. Thus the classes introduced above can be treated as natural generalizations of boolean rings.

Let us notice that in [D4] the class of *(strongly) invo-clean* rings was investigated. In our terminology, (strongly) invo-clean rings are precisely rings which are either (strongly) 1-torsion clean or (strongly) 2-torsion clean.

It is clear that every clean ring having the unit group of bounded exponent s is n-torsion clean for some n with $1 \leq n \leq s$. We will see below that n has to divide s, but does not have to be equal to s. Let us also observe that a homomorphic image of an n-torsion clean ring is always m-torsion clean, for some $m \leq n$. Hoverer, it is not clear whether n is a multiple of m. Notice that finite rings are always clean, so they are n-torsion clean for suitable n and it would be of interest to compute n for some classes of finite rings; for instance, for matrix rings over finite fields.

In the present subsection we mainly concentrate on the case of strongly n-torsion clean rings. Our work is organized as follows: We first state some facts of introductory character containing some basic observations and examples. Strongly n-torsion clean rings are next examined. In particular, it is shown in Theorem 1.93 that such rings have to satisfy a polynomial identity of degree $2n$ and that their Jacobson radical is nil of bounded index. Likewise, Theorem 1.102 offers a description of such rings which are abelian. Surprisingly, when n is odd, strongly n-torsion clean rings appeared to be commutative. Their precise description is given in the subsequent Theorem 1.104. We finish off with some locally-open queries of some interest and importance.

We begin with the following simple but useful observation. Its proof is provided for the sake of completeness.

Lemma 1.83. *Let R be a (strongly) n-torsion clean ring. Then there exist a finite number of elements $r_1, \ldots, r_k \in R$ with (strongly) clean decompositions $r_i = e_i + u_i$, $1 \leq i \leq k$, such that $n = LCM(o(u_1), \ldots, o(u_k))$. In particular:*

1. *When the group $U(R)$ has finite exponent s, then n divides s.*
2. *When R is commutative, then $U(R)$ contains an element of order n.*

Proof. For $r \in R$, let us set

$$r_{min} = \min\{o(u) \mid r = e + u \text{ is a (strongly) } n\text{-torsion clean decomposition of } r \\ \text{and } o(u) \text{ divides } n\}.$$

Then each r_{min} divides n. Thus $LCM(r_{min} \mid r \in R)$ exists and also divides n. Moreover, we can pick elements $r_1, \ldots, r_k \in R$ such that $LCM(r_{min} \mid r \in R) = LCM(r_{1min}, \ldots, r_{kmin})$. The minimality of n gives $LCM(r_{min} \mid r \in R) = n$. This completes the proof of the main statement. Subsequently, (1) and (2) follow. $\square$

It is well known that $1 + J(R) \subseteq U(R)$. In the class of rings for which the equality holds, the notation of n-torsion clean rings boils down to rings R for which the unit group $U(R)$ is of finite exponent n. Indeed, we have:

Proposition 1.84. *Let R be a ring and $n \in \mathbb{N}$. Then:*

(1) *If $r \in J(R)$, then the unit $1 + r$ has exactly one clean decomposition.*
(2) *Suppose $U(R) = 1 + J(R)$. Then the following two conditions are equivalent:*
   (a) *R is (strongly) n-torsion clean.*
   (b) *R is (strongly) clean and the group $U(R)$ is of finite exponent n.*

Moreover, if one of the equivalent conditions holds, then $R/J(R)$ is a boolean ring.

Proof. (1) Let $r \in J(R)$. Observe that if $1 + r = e + u$ is a clean decomposition of $1 + r$, then $1 - e = u - r \in \mathrm{Id}(R) \cap U(R) = \{1\}$, that is, $e = 0$. This implies that $1 + r$ has the unique clean decomposition $r + 1 = 0 + (1 + r)$.

(2) Suppose R is (strongly) n-torsion clean and $u \in U(R) = 1 + J(R)$. Then, by (1), $u = 0 + u$ is the only clean decomposition of u and $u^n = 1$ follows, i.e. $U(R)$ is of finite exponent $s \leq n$.

Conversely suppose that R is (strongly) clean and $U(R)$ is a group of finite exponent s. Then it is clear that R is n-torsion clean ring, for some $n \leq s$. This yields the equivalence $(a) \Leftrightarrow (b)$.

Since units always lift modulo the Jacobson radical, we have $U(R/J(R)) = \{1\}$. If R is strongly n-torsion clean, then $R/J(R)$ is m-torsion clean for some $m \leq n$. The above yields that $m = 1$, i.e. $R/J(R)$ is a boolean ring. □

Notice that the ring $T_m(\mathbb{F}_2)$ of all upper triangular $m \times m$ matrices over the field $\mathbb{F}_2$ is clean, its Jacobson radical J consists of all strictly upper triangular matrices and $U(T_m(\mathbb{F}_2)) = 1 + J$. Thus, with Proposition 1.84 at hand, we deduce:

Example 1.85. Let $m \in \mathbb{N}$ and let k be the smallest nonnegative integer such that $m \leq 2^k$. Then the ring $T_m(\mathbb{F}_2)$ is (strongly) 2^k-torsion clean.

Recall that a ring R is uniquely clean if every element of R has a unique clean presentation. Such rings were characterized in [95] as those abelian rings R for which $R/J(R)$ is boolean (whence $U(R) = 1 + J(R)$) and idempotents lift modulo $J(R)$. Notice that, as idempotents always lift modulo nil ideals, every ring R such that $R/J(R)$ is boolean and $J(R)$ is nil must be clean. Therefore, the above proposition also gives the following corollary. Its second statement generalizes Example 1.85.

Corollary 1.86. (1) *Let R be a uniquely clean ring. Then R is n-torsion clean iff $U(R)$ is of exponent n;*

(2) *Let R be a ring such that $R/J(R)$ is boolean and $J(R)$ is nil of bounded index. Then R is n-torsion clean, where n is the exponent of $U(R)$. Moreover, n is a power of 2.*

Proof. (1) being an immediate consequence of the preceding discussion, let the ring R be as in (2). Then R is a UU ring (i.e. all units are unipotent) and so [39, Theorem 3.4 (2)] implies that $U(R)$ is a 2-group. Now the thesis is a simple consequence of Proposition 1.84. □

Likewise, Proposition 1.84 demonstrates that, from the point of view of n-torsion clean property, rings with $U(R) = 1 + J(R)$ are, in some sense, not too interesting. The situation when the ring is Jacobson semisimple and has non-trivial group of units is much more interesting. The next example is of such nature and it shows that a ring can be n-torsion clean with n strictly smaller than the exponent of the group $U(R)$.

Example 1.87. Let $R = \mathbb{M}_2(\mathbb{F}_2)$. Then R is 2-torsion clean and strongly 6-torsion clean.

Proof. The ring R is nil clean by virtue of [11]. Since the index of nilpotence of elements of R is at most 2, Corollary 2.11 from [D4] implies that R invo-clean which is not boolean, so that it is 2-torsion clean. The above can be also checked by direct computations. For instance, the unit $r = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ of order 3 has a 2-torsion clean decomposition $r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (but it does not have strongly 2-torsion clean decomposition). It is also easy to make elementary computations showing that R is strongly 6-torsion clean. □

Let us notice that the unit group of $\mathbb{M}_2(\mathbb{F}_2)$ is isomorphic to the symmetric group $\mathcal{S}_3$.

Proposition 1.88. *Let $m, k \in \mathbb{N}$ be such that $m \leq 2^k$. Then $\mathbb{M}_m(\mathbb{F}_2)$ is n-torsion clean for some natural $n \leq 2^k$.*

Proof. Set $R = \mathbb{M}_m(\mathbb{F}_2)$. It is known (cf. [11, Theorem 3]) that R is a nil-clean ring. Thus every element x of R can be written as $x = e + z$ with $e = e^2$ and $z^m = 0$, as the index of nilpotence of elements in R is bounded by m. Now we can write $x = (1+e)+(1+z)$, where $1+e$ is an idempotent and $(1+z)^{2^k} = 1+z^{2^k} = 1$, as $m \leq 2^k$. This enables us to conclude that every element of R has 2^k-torsion clean decomposition. In particular, R is n-torsion clean for some $n \leq 2^k$. $\square$

With the help of this proposition, we derive:

Example 1.89. The rings $\mathbb{M}_3(\mathbb{F}_2)$ and $\mathbb{M}_4(\mathbb{F}_2)$, in view of Proposition 1.88, are n-torsion clean for some $n \leq 2^2$. The rings are, however, not invo-clean by virtue of Corollary 2.11 in [D4], and so $n \in \{3, 4\}$.

The linear group $GL(3, \mathbb{F}_2)$ is the unit group of $\mathbb{M}_3(\mathbb{F}_2)$. The group is known to be simple of order 168 and exponent 84.

The following technical lemma is crucial for our further considerations.

Lemma 1.90. *Suppose that R is a ring and the element $a \in R$ possesses strongly n-torsion clean decomposition. Then the equality $(a^n - 1)((a-1)^n - 1) = 0$ holds.*

Proof. Let $a = e + v$ be a strongly n-torsion clean decomposition of a. Since e, v commute and $v^n = 1$, we deduce:

$$a^n - 1 = (e+v)^n - 1 = \sum_{i=0}^{n} \binom{n}{i} e^i v^{n-i} - 1 = \sum_{i=1}^{n} \binom{n}{i} e^i v^{n-i} \in Re.$$

This implies that

$$(a^n - 1)(1 - e) = 0 \quad \text{and, consequently,} \quad a^n - 1 = (a^n - 1)e. \tag{1.1}$$

Using $ve = (a-1)e$, we also have $1 = v^n = (a-e)^n = (a-1)^n e - a^n e + a^n$. This yields

$$a^n - 1 = (a^n - (a-1)^n)e. \tag{1.2}$$

Applying (1.2) and the second equation of (1.1), we get $((a-1)^n - 1)e = 0$. This equality and the first equation of (1.1) now give together that $(a^n - 1)((a-1)^n - 1) = (a^n - 1)((a-1)^n - 1)(e + (1-e)) = 0$, as desired. $\square$

The following assertion is central.

Lemma 1.91. *Let $n \in \mathbb{N}$ and let R be a ring satisfying the identity $(x^n - 1)((x-1)^n - 1) = 0$. Then:*

(1) $\mathrm{char}(R) := |1 \cdot \mathbb{Z}\,|$ *is finite and $J(R)$ is a nil ideal;*
(2) *If n is odd, then R is a reduced ring of characteristic 2 and $J(R) = 0$;*
(3) *If R is an algebra over a field F, then either R is abelian (i.e., all idempotents of R are central) or* $\mathrm{char}(F)$ *divides n.*

Proof. Set $\phi(x) = (x^n - 1)((x-1)^n - 1) \in \mathbb{Z}[x]$.

(1) Substituting $x = 3 \cdot 1$ in the identity $\phi(x) = 0$, we see that there exists $0 \neq m \in \mathbb{Z}$ such that $1 \cdot m = 0$ in R. This shows that characteristic $\mathrm{char}(R)$ of R is finite.

Let $r \in J(R)$. Assume that r is not nilpotent. Then the multiplicatively closed set $S = \{r^k \mid k \in \mathbb{N}\}$ does not contain 0. Let P be a maximal ideal of R in the class of all ideals having empty intersection with S. So, P is a prime ideal of R, the ring $\bar{R} = R/P$ satisfies the same identity as R does and $\bar{r} = r + P \in J(\bar{R})$. Moreover $\bar{r}$ is not nilpotent, as $S \cap P = \emptyset$. Thus, eventually replacing R by $\bar{R}$, we may additionally assume that the ring R is prime. Then, its subring $1 \cdot \mathbb{Z} = F$ of R is a domain. By the first part of the proof, $1 \cdot \mathbb{Z}$ is finite, so it is a field. This means that the element $r \in J(R)$ is algebraic over the field F and, as such, has to be nilpotent (cf. [79, Proposition 4.18]). This contradicts the choice of r and shows that every element of $J(R)$ is nilpotent.

(2) Suppose n is odd. Then, substituting $x = 0$ in the identity $\phi(x) = 0$, we obtain $2 = 0$, i.e. $\mathrm{char}(R) = 2$.

Let $r \in R$ be such that $r^2 = 0$. If $n = 1$, then the identity $\phi(x) = 0$ shows that $(r-1)r = 0$ and $r = 0$ follows immediately, as $r - 1$ is invertible.

Suppose now that $n \geq 3$. Notice that $(r-1)^2 = 1$. Thus, as n is odd, $(r-1)^n = (r-1)$. Therefore, $r = \phi(r) = 0$. This shows that R has no nonzero nilpotent elements, i.e. R is reduced. Then also $J(R) = 0$ as, by (1), $J(R)$ is a nil ideal.

(3) Suppose R is an algebra over a field F. By (1), $\mathrm{char}(F) = p \neq 0$. If n is odd then, using (2), R is a reduced ring, so it is abelian. Suppose now that n is even and p does not divide n. Thus $1 \cdot n$ is invertible in R. Let $e = e^2, r \in R$. Substituting $x := er(1-e)$ in the identity $\phi(x) = 0$ and using the fact that n is even, we obtain $0 = ((er(1-e))^n - 1)((er(1-e) - 1)^n - 1) = ner(1-e)$ and thus the equality $er(1-e) = 0$ follows. Similarly $(1-e)re = 0$. The above shows that every idempotent e of R is central, provided that $\mathrm{char}(R)$ does not divide n. This completes the proof of the lemma. $\square$

Remark 1.92. In regard to point (2) stated above, a routine argument demonstrates that when R is an n-torsion clean ring and n is odd, then $J(R) = 0$ and $\mathrm{char}(R) = 2$. Indeed, let $0 = f + v$ be an n-torsion clean decomposition of 0. Then $-f = (-f)^n = v^n = 1$ and $\mathrm{char}(R) = 2$ follows. Now, Proposition 1.84(1) yields that $1 = (r-1)^n = \sum_{i=0}^{n} \binom{n}{i}(-1)^{n-i} r^i$, for any $r \in J(R)$. As n is odd, this equation gives $0 = rw$, where $w = \sum_{i=1}^{n} \binom{n}{i}(-1)^{n-i} r^{i-1} \in 1 + J(R)$ is invertible in R, i.e. $r = 0$, as required.

Now we are ready to establish the following theorem.

Theorem 1.93. *Let $n \in \mathbb{N}$. Suppose R is a strongly n-torsion clean ring. Then:*

(1) *R is a PI-ring satisfying the polynomial identity $(x^n-1)((x-1)^n-1)=0$;*
(2) *R has finite characteristic $\mathrm{char}(R) = |1 \cdot \mathbb{Z}\,|$;*
(3) *$J(R)$ is a nil ideal of index smaller than $(\mathrm{char}(R))^n$;*
(4) *When n is odd, then R is a reduced ring of characteristic 2 and $J(R) = 0$;*
(5) *If R is an algebra over a field F, then:*
(i) $J(R)$ is a nil ideal of index bounded by n;
(ii) either R is abelian (i.e., all idempotents of R are central) or $\mathrm{char}(F)$ divides n.

Proof. The first statement is a direct consequence of Lemma 1.90. Notice that, in virtue of Lemma 1.91, for completing the proof it remains only to show that $J(R)$ is nil of index bounded as indicated in the theorem.

Let $r \in J(R)$. We claim that $r^{(\mathrm{char}(R))^n} = 0$. By Lemma 1.91(1), r is nilpotent. Furthermore, Proposition 1.84 (1) shows that the unit $1+r$ has exactly one clean presentation. Thus $(1+r)^n = 1$ follows, as R is n-torsion clean. Therefore $r^n \in S = (1 \cdot \mathbb{Z})[r] = (1 \cdot \mathbb{Z})r^{n-1} + \ldots + (1 \cdot \mathbb{Z})$. By (1), $1 \cdot \mathbb{Z}$ is a finite ring with $c := \mathrm{char}(R)$ elements. Hence the ring S is finite and has at most c^n elements. As $r \in S$ is nilpotent, its index of nilpotence has to be smaller than $|S| \leq c^n$ (to argue this, just consider the set $A \subseteq S$ of all powers of the element r and show that $|A|$ is the nilpotence index of r). This gives (3).

Suppose now that R is an algebra over a field F. Then S defined as above is, in this case, a finite dimensional algebra over $\mathbb{F}_p = 1 \cdot \mathbb{Z} \subseteq F$ of dimension not bigger than n. The dimension argument applied to the sequence of subspaces $S \supseteq Sr \supseteq Sr^2 \supseteq \ldots$ shows that $r^n = 0$, when $r \in S$ is nilpotent. This yields (5)(i) and completes the proof of the theorem. $\square$

It is an important open question (see [94, Question 2] and [14]) whether strongly clean rings are Dedekind finite. Since PI rings are Dedekind finite, the above theorem gives immediately the following consequence:

Corollary 1.94. *Strongly n-torsion clean rings are Dedekind finite.*

We also state the following consequence.

Corollary 1.95. *Let R be a strongly n-torsion clean ring. If R is a finitely generated algebra over a central noetherian subring, then $J(R)$ is nilpotent.*

Proof. By Theorem 1.93(1), R satisfies a monic polynomial identity and $J(R)$ is a nil ideal. Now the thesis is a direct consequence of [9, Theorem 2.5]. $\square$

The following example shows that, in general, Jacobson radical of strongly n-torsion clean rings does not have to be nilpotent.

Example 1.96. Let $F = \mathbb{F}_{p^k}$ and $F[X]$ be the polynomial ring in infinitely many commuting indeterminacies from the set X. Set $R = F[X]/I$, where I is the ideal of $F[X]$ generated by all elements x^p, $x \in X$. Then R is a local ring and its Jacobson radical is not nilpotent. Making use of Propositions 1.84(1) and 1.97(1), one can easily check that R is $p(p^k-1)$-torsion clean.

In Theorem 1.104, stated in the sequel, we will present a complete characterization of strongly n-torsion clean rings in the case when n is odd. For doing so, the following proposition, which gives a characterization of strongly n-torsion clean rings which are subdirect products of fields, is needed.

Proposition 1.97. (1) *Let F be a field. Then F is n-torsion clean iff F is finite and $n = |F| - 1$.*

(2) *A product of fields $\mathbb{F}_{p_1^{k_1}} \times \ldots \times \mathbb{F}_{p_t^{k_t}}$ is n-torsion clean, where n is equal to $LCM(p_1^{k_1}-1, \ldots, p_t^{k_t}-1)$;*

(3) *A product $\prod_{i\in I} F_i$ of fields is n-torsion clean iff all fields F_i, $i \in I$, are finite, $LCM(|F_i| - 1 \mid i \in I)$ exists and is equal to n;*

(4) *Let R be a subdirect product of fields F_i, $i \in I$. Then R is n-torsion clean iff $\prod_{i\in I} F_i$ is n-torsion clean.*

Proof. (1) Notice that any finite field F is n-torsion clean for some divisor n of $|F|-1$. On the other hand, if F is any field which is n-torsion clean then, by Theorem 1.93, every element of F is a root of the polynomial $(x^n-1)((x-1)^n - 1) \in F[x]$, so $|F|$ is finite and $|F| \le 2n$. Suppose that F is a finite n-torsion clean field and let $|F| - 1 = l \cdot n$. By what we have just shown it follows that $l = \frac{|F|-1}{n} \le 2 - \frac{1}{n} < 2$ and so $l = 1$ holds, i.e. $s = |F| - 1$, as required.

(2) Let $T = \mathbb{F}_{p_1^{k_1}} \times \ldots \times \mathbb{F}_{p_t^{k_t}}$ and let n be as defined in (2). Notice that $n = \max\{o(u) \mid u \in U(T)\}$ and the order of any $u \in U(T)$ divides n. Therefore, T is m-torsion clean for some $m \le n$.

For showing that $n = m$, it suffices to show that $n_i = p_i^{k_i} - 1$ divides n, for any $1 \le i \le t$. Note that, by (1), $F_i = \mathbb{F}_{p_i^{k_i}}$ is n_i-torsion clean. Furthermore, using Lemma 1.83, we can pick elements $r_1, \ldots, r_s \in T$ and their clean decompositions $r_j = e_j + u_j$, $1 \le j \le s$, such that $m = LCM(u_1, \ldots, u_s)$. For a fixed $1 \le i \le t$ consider the set $\{\pi_i(r_1), \ldots, \pi_i(r_s)\} \subseteq F_i$, where π_i denotes the canonical projection of R onto F_i. Then, for every $a \in F_i$, a can be presented as $e + u$ with $u^{z_i} = 1$, where $z_i = LCM(o(\pi_i(u_1)), \ldots, o(\pi_i(u_s)))$. Thus $n_i \le z_i$. As z_i is a LCM of orders of elements in a cyclic group $U(F_i)$ of order n_i, we also deduce that $z_i \leq n_i$, i.e. $z_i = n_i$. This implies that n_i divides m, for any $1 \leq i \leq t$, as desired.

(3) Suppose the product $\prod_{i\in I} F_i$ is n-torsion clean. Then every field F_i is a homomorphic image of $\prod_{i\in I} F_i$. Thus, owing to (1), each F_i is a finite field. If $LCM(|F_i|-1 \mid i \in I)$ would not exist, then there would exist indexes $i_1, \ldots, i_k \in I$ such that $m = LCM(|F_{i_1}| - 1, \ldots, |F_{i_k}| - 1) > n$. However, in virtue of (2), $T = F_{i_1} \times \cdots \times F_{i_k}$ is m-torsion clean and $m \leq n$, as T is a homomorphic image of R. Thus $LCM(|F_i| - 1 \mid i \in I)$ do exist and we can assume that $LCM(|F_i| - 1 \mid i \in I) = LCM(|F_{i_1}| - 1, \ldots, |F_{i_k}| - 1) = m$. Then it is clear that $n \leq m$. Notice also that $m \leq n$, as T is a homomorphic image of R, i.e. $n = m$. This gives (3).

(4) Let R be a subdirect product of fields F_i, $i \in I$. Suppose R is m-torsion clean. For every $i \in I$, F_i is a homomorphic image of R so, with the aid of (1), the field F_i is n_i-torsion clean, where $n_i = |F_i| - 1$. We also have $n_i \leq m$. Therefore, $LCM(|F_i|-1 \mid i \in I)$ exists and the statement (3) shows that $\prod_{i\in I} F_i$ is n-torsion clean, where $n = LCM(|F_i| - 1 \mid i \in I)$. In particular, the order of any unit of R divides n and thus $m \leq n$ follows.

Let us fix $i \in I$ and let $F = F_i$ with $s = n_i$. Then, any $a \in F$ can be presented as $a = e + u$ with $u^m = 1$. Let $k_1, \ldots k_s$ be orders of units in such presentations of all elements of F. Then, by construction, $k = LCM(k_1, \ldots, k_s)$ divides m and also divides $s = |F| - 1$ (as s is equal to the order of the group $U(F)$). In particular $k \leq s$. On the other hand, appealing to (1), F is s-torsion clean and this forces that $s \leq k$. However, this shows that $k = s = n_i$ divides m. This means that, for any $i \in I$, $n_i = s$ divides n. Consequently, $n = LCN(n_i \mid i \in I)$ divides m. By the first part of the proof $m \leq n$, so $n = m$ really follows.

Suppose now that $\prod_{i\in I} F_i$ is n-torsion clean and R is a subdirect products of fields F_i, $i \in I$. To complete the proof, it is enough to show R is m-torsion clean for some m. The statement (3) implies that the group $U(R)$ is of finite exponent, say k is the exponent. Then, for any $r \in R$, $e = r^k$ is an idempotent, and $r = (1-e)+((e-1)+r)$ is a clean decomposition of a with $((e-1)+r)^k = 1$. This allows us to conclude that R is m-torsion clean, for some $m \leq k$, as required. □

The following result, which is needed later in the text, is also of some independent interest. Before stating it, let us recall that idempotents lift modulo an ideal J of R if, for any $a \in R$ such that $a^2 - a \in J$, there exists an idempotent $e \in R$ such that $e - a \in J$. If the idempotent e is uniquely determined by the element a, then we say that idempotents lift uniquely modulo I.

It is known that idempotents lift modulo nil ideals, thus the following lemma applies when J is a nil ideal of a ring R.

Lemma 1.98. *Let $J \subseteq J(R)$ be an ideal of R. Suppose that idempotents lift modulo J. Then the following conditions are equivalent:*

(1) *R is an abelian ring;*
(2) *R/J is an abelian ring and idempotents lift uniquely modulo J.*

Proof. Let $\pi\colon R \to R/J$ denotes the canonical homomorphism.

(1) $\Rightarrow$ (2). Suppose the ring R is abelian. Since idempotents lift modulo J, $\mathrm{Id}(R/J) = \pi(\mathrm{Id}(R))$. Thus the ring R/J is abelian, as R is such. Let $e, f \in \mathrm{Id}(R)$ be such that $e - f \in J \subseteq J(R)$. Then, by [69, Corollary 11], e and f are conjugate in R, i.e., there exists $u \in U(R)$ such that $e = ufu^{-1}$. However, all idempotents of R are central, so $e = f$. This, together with the assumption that idempotents lift modulo J yield that idempotents lift uniquely modulo J.

(2) $\Rightarrow$ (1). The commutator of elements $a, b \in R$ will be denoted by $[a, b] := ab - ba$. Suppose (2) holds and let $e \in \mathrm{Id}(R)$, $r \in R$. Then $f = e + er(1 - e)$ is also an idempotent and $[f, e] = er(1 - e)$. By assumption R/J is abelian, so $\pi([f, e]) = 0$. This shows that $er(1 - e) \in J$. Since $\pi(e) = \pi(f)$ and, by assumption, idempotents lift uniquely modulo J, we obtain $e = f$, i.e. $er(1-e) = 0$. Now, replacing e by $1 - e$, we also have $(1 - e)re = 0$, for any $r \in R$. This means that every idempotent e of R is central, i.e. R is abelian, as required. □

We will need in the sequel the following direct application of [39, Theorem 3.2.].

Lemma 1.99. *Let R be a ring and $u \in R$. Suppose that $m := \mathrm{char}(R)$ is finite and $J(R)$ is a nil ideal of index $s + 1$, where $s \geq 0$. If $u^t - 1 \in J(R)$, then $u^{tm^s} = 1$.*

Proof. [39, Theorem 3.2.] states that if R is a ring satisfying assumptions of the lemma, then $(1 - r)^{m^s} = 1$, for any $r \in J(R)$. Now, if $u^t - 1 \in J(R)$, then there exists $r \in J(R)$ such that $u^t = 1 - r$ and $u^{tm^s} = 1$ follows. □

The above lemma gives immediately the following corollary:

Corollary 1.100. *Let R be a ring of such that $\mathrm{char}(R)$ is finite and $J(R)$ is nil of bounded index. If the group $U(R/J(R))$ is of finite exponent, then so is $U(R)$. If additionally $R/J(R)$ is clean (so R is also clean, as units and idempotents lift modulo nil ideals), then R is n-torsion clean, for some $n \in \mathbb{N}$.*

Corollary 1.101. *Let R be a ring of finite characteristic and J a nil ideal of R of bounded index. Then the following conditions are equivalent:*

(1) *R is an n-torsion clean ring, for some $n \in \mathbb{N}$.*
(2) *R/J is an t-torsion clean ring, for some $t \in \mathbb{N}$.*

Proof. Suppose R/J is an m-torsion clean ring, for some $m \in \mathbb{N}$. Let $r \in R$. Since units and idempotents lift modulo J we can find $e \in \mathrm{Id}(R)$ and $u \in U(R)$ such that $\bar{r} = \bar{e} + \bar{u}$ is an t-torsion clean decomposition of $\bar{r}$ in R/J, where $\bar{r}$ denotes the natural image of r in R/J. By Lemma 1.99, $u = 1$, where $m = \mathrm{char}(R)$ and $s+1$ is the nil index of the ideal J. This implies that R is n torsion clean, for some $n \leq tm^s$.

The reverse implication is clear. □

The following theorem offers a characterization of strongly n-torsion clean abelian rings (compare with Theorem 1.93).

Theorem 1.102. *For a ring R, the following conditions are equivalent:*

(1) *There exists $n \in \mathbb{N}$ such that R is an n-torsion clean abelian ring.*
(2) (a) $\mathrm{char}(R)$ *is finite;*
 (b) *The Jacobson radical $J(R)$ is nil of bounded index;*
 (c) *Idempotents lift uniquely modulo $J(R)$;*
 (d) *$R/J(R)$ is a subdirect product of finite fields F_i, where i ranges over some index set I, such that $LCM(|F_i| - 1 \mid i \in I)$ exists.*
(3) *R is an abelian clean ring such that the unit group $U(R)$ is of finite exponent.*

Proof. (1) $\Rightarrow$ (2). Suppose R is an n-torsion clean abelian ring. Then, Theorem 1.93 guarantees that $\mathrm{char}(R)$ is finite and $J(R)$ is nil of finite index. In particular, R has properties (a) and (b). Since $J(R)$ is a nil ideal, idempotents lift modulo $J(R)$ and, by Lemma 1.98, they lift uniquely, so (c) holds.

Finally, by Theorem 1.93(1), $R/J(R)$ satisfies the polynomial identity $\phi(x) = 0$, where $\phi(x) = (x^n - 1)((x-1)^n - 1) \in \mathbb{Z}[x]$. Therefore, $R/J(R)$ is a subdirect product of primitive PI-rings, say $R/J(R)$ is a subdirect product of primitive rings $\{R_i\}_{i \in I}$, for some index set I. Let us fix $i \in I$. Then R_i, as a homomorphic image of R, also satisfies the identity $\phi(x) = 0$. Consequently, by the classical Kaplansky's theorem (cf. [102]), each R_i has to be a central simple algebra, finite dimensional over its center C. Notice that, as $\mathrm{char}(R)$ is finite, C is a field of nonzero characteristic, say $\mathbb{F}_p \subseteq C$. Observe also that, by Lemma 1.98, R_i is an abelian ring. This implies that R_i has to be a division algebra over $\mathbb{F}_p$. It is known (cf. [42, Corollary from page 48]) that every division algebra which is algebraic over a finite field is necessarily commutative. In particular, R_i has to be a field. In fact, it is a finite field, as R_i is contained in the spitting field of $\phi(x) \in \mathbb{F}_p[x]$.

The above shows that $R/J(R)$ is a subdirect product of finite fields. Moreover, $R/J(R)$ is also, as a homomorphic image of R, strongly n'-torsion clean, for some $n' \leq n$. Therefore, making use of Proposition 1.97, we see that R satisfies the property (d). This completes the proof of the implication.

(2) $\Rightarrow$ (3). Suppose (2) holds. We know, by (d) and Proposition 1.97, that $R/J(R)$ is a clean ring with the unit group $U(R)$ of finite exponent. The property (b) guarantee that $J(R)$ is a nil ideal and Corollary 1.100 yields that R is a clean ring with the unit group $U(R)$ is of finite exponent. Finally, properties (d), (c) together with Lemma 1.98 imply that R is an abelian ring.

The implication (3) $\Rightarrow$ (1) is obvious. □

In parallel to Theorem 1.102, one can state the following:

Theorem 1.103. *For a ring R, the following conditions are equivalent:*

(1) *R is strongly n-torsion clean, for some $n \in \mathbb{N}$.*
(2) *R is strongly clean and $U(R)$ is of finite exponent.*

Proof. (1) $\Rightarrow$ (2). Suppose R is strongly n-torsion. Then clearly R is strongly clean. Next, observe that Theorem 1.93 implies that R is a PI-ring satisfying an identity of degree $2n$ and $J(R)$ is a nil ideal of bounded index. Using similar arguments as in the proof of Theorem 1.102, one can see that the quotient $R/J(R)$ is a subdirect product of a matrix rings, say $R_i = M_{m_i}(F_i)$, over finite fields F_i. Notice that, as $\mathrm{char}(R)$ is finite, the set of characteristics of fields from the set $\mathcal{F} = \{F_i \mid i \in I\}$ is finite and also the number of fields of a given characteristic p is finite, as every such field is contained in the splitting field of a given polynomial of degree $2n$. Thus there are only finitely many classes of isomorphic fields in the set $\mathcal{F}$. Moreover, by the classical Amitsur-Levitzki's theorem (cf. [102]), each m_i is not grater than n, as every R_i satisfies a polynomial identity of degree $2n$. Therefore, the unit group of the product $\prod_{i \in I} R_i$ is a group of finite exponent. By Theorem 1.93, $\mathrm{char}(R)$ is finite and $J(R)$ is a nil ideal of bounded index. Now, we can apply Lemma 1.99 to obtain that the group $U(R)$ is of finite exponent.

The implication (2) $\Rightarrow$ (1) is clear. □

We now have at our disposal all the necessary information to present a satisfactory structural characterization of strongly n-torsion clean rings, for all odd n.

Theorem 1.104. *Suppose $n \in \mathbb{N}$ is odd. For a ring R, the following conditions are equivalent:*

(1) *R is a strongly n-torsion clean ring;*

(2) *There exist integers* $k_1, \dots, k_t \geq 1$ *such that* $n = LCM(2^{k_1}-1, \dots, 2^{k_t}-1)$ *and* R *is a subdirect product of copies of fields* $\mathbb{F}_{2^{k_i}}$, $1 \leq i \leq t$*;*

(3) R *is a clean ring in which orders of all units are odd, bounded by* n *and there exists a unit of order* n*.*

Proof. (1) $\Rightarrow$ (2). Suppose R is a strongly n-torsion clean ring. Then, by Theorem 1.93 (4), R is a reduced ring of characteristic 2 and $J(R) = 0$. Thus, as every reduced ring is abelian, we can apply Theorem 1.102 to obtain that R is a subdirect product of finite fields F_i of characteristic 2, where $i \in I$, for some index set I. Now, Proposition 1.97 completes the proof of the implication.

The reverse implication (2) $\Rightarrow$ (1) is a direct consequence of Proposition 1.97. The implication (2) $\Rightarrow$ (3) is a tautology.

(3) $\Rightarrow$ (1). Let R be as in (3). Then, as $(-1)^2 = 1$ and R has no units of even order, $-1 = 1$, i.e., $\mathrm{char}(R) = 2$. Let us observe that R has to be reduced. Indeed, if $r^2 = 0$ for some $r \in R$, then $(1+r)^2 = 1+r^2 = 1$. Using again the fact that R has no units of even order, we get $r = 0$. It is known that in a reduced ring all idempotents are central. Moreover, by assumption, R is a clean ring and, as every unit of R is of finite order bounded by n, the ring must be strongly m-torsion clean, for some $m \leq n!$. Now, because orders of units are odd, m has to be odd (as $u^{2k} = 1$ yields $u^k = 1$, when $o(u)$ is odd). Furthermore, bearing in mind the equivalence of statements (1) and (2), we conclude that $n = m$, as required. $\square$

It is worth to mention certain slightly unexpected, non-trivial consequences of the above theorem. Namely, not every odd natural number n can serve as torsion degree of strongly n-torsion clean rings and, for odd, n-torsion clean rings are always commutative.

Notice that, because every finite ring is clean, Theorem 1.104 forces the following:

Corollary 1.105. *For a finite ring* R *the following conditions are equivalent:*

(1) R *is strongly* n*-torsion clean for some odd* n*;*

(2) R *has no units of even order;*

(3) R *is isomorphic to a finite direct product of fields of characteristic* 2.

Proof. By the Chinese Remainder Theorem, any subdirect product of finite number of fields is isomorphic to a direct product of fields. Now, the corollary is a straightforward consequence Theorem 1.104. $\square$

We close the work with some problems of interest.

Question 1.106. The matrix ring $\mathbb{M}_n(\mathbb{F}_{2^k})$ is always m-torsion clean for some m. Compute m in terms of n and k; is $m = n$ if $k = 1$?

Recall that some basic observations related to the above problem can be found in Proposition 1.88 and Examples 1.87 and 1.89. In particular $M_2(\mathbb{F}_2)$ is 2-torsion clean and, when $n \in \{3, 4\}$ then $M_n(\mathbb{F}_2)$ is m-torsion clean, where $2 < m \leq 4$. It could be checked, with the help of SageMath, that $n = m$ in the above cases.

We have seen in Theorem 1.103 that strongly n-torsion clean rings have units group $U(R)$ of finite exponent. For odd n, by Theorem 1.104, $n = \exp(U(R))$. Example 1.87 shows also such equality in the case of the ring $M_2(\mathbb{F}_2)$. We were kindly informed by Pace Nielsen, that such equality also holds for $M_3(\mathbb{F}_2)$, i.e. $M_3(\mathbb{F}_2)$ is strongly 84-torsion clean. Notice also that Example 1.96 offers yet another instance of equality $n = \exp(U(R))$.

Thus we pose the following two questions.

Question 1.107. Let R be a strongly n-torsion clean ring. Is it true that $n = \exp(U(R))$?

Question 1.108. Let R be an n-torsion clean ring. Is then necessary $U(R)$ of finite exponent?

For odd n, strongly n-torsion clean rings were characterized in Theorem 1.104. Besides, Theorem 1.102 offers a description of strongly n-torsion clean rings with extra assumption that the considered rings are abelian. So, we come to

Question 1.109. Characterize strongly n-torsion clean rings, for even $n \in \mathbb{N}$.

If R is not abelian, then Theorem 1.93 (5) and arguments used in the proof of Theorem 1.103 show that, modulo the Jacobson radical (which is nil of bounded index), Question 1.109 essentially reduces to the investigation of matrix rings over finite fields of characteristic dividing n.

It is also worthwhile noticing that (strongly) 2-torsion clean rings were classified in [D4] under the name (strongly) invo-clean rings by using another approach. In fact, R is strongly invo-clean iff $R \cong R_1 \times R_2$, where R_1 is a ring for which $R_1/J(R_1)$ is boolean with $z^2 = 2z$ for every $z \in J(R_1)$, and R_2 is a ring which can be embedded in a direct product of copies of the field $\mathbb{F}_3$.

We now arrive at our other, final, section of applicable character.

2. Applications to Group Rings

Here, as usual, the symbol $R[G]$ stands for the group ring of an arbitrary multiplicative group G over an arbitrary unital ring R, and $\omega(R[G])$ is its standard augmentation ideal, generated by the elements $1-g$, where g runs over G.

Imitating [39], we state the following:

Definition 2.1. A ring R is said to be *UU* if its unit group $U(R)$ satisfies the equality $U(R) = 1 + Nil(R)$, where $Nil(R)$ is the set of all nilpotent elements of R.

However, this definition is rather clumsy for applications and so the next necessary and sufficient condition from [39] will be useful in the sequel.

Proposition 2.2. *A ring R is UU if, and only if, $2 \in Nil(R)$ and $U(R)$ is a 2-group.*

On the other hand, mimicking [41], a ring R is called *nil-clean* if $R = Nil(R) + Id(R)$, that is, for every $r \in R$ there exist $q \in Nil(R)$ and $e \in Id(R)$ such that $r = q + e$. If, in addition, $qe = eq$ holds, the nil-clean ring is called *strongly nil-clean*.

The following criterion, which was used in the proof of [103, Theorem 2.12], was independently proved in [39] and [77], respectively: *A ring R is strongly nil-clean if, and only if, the Jacobson radical $J(R)$ is nil and $R/J(R)$ is boolean.*

Even much more, in [39] was showed that *a ring is strongly nil-clean exactly when it is nil-clean UU* which amounts to *a ring is strongly nil-clean uniquely when it is nil-clean and its unit group is a 2-group.* In order to simplify the proof of Theorem 2.12 from [103], we shall use in what follows this key assertion (e.g., in Corollary 2.6) without any concrete referring.

A brief history of the best known principal achievements on group rings over such rings is like this: In [85] was found a complete description when the group ring $R[G]$ is nil-clean. This was further expanded in the non-commutative case in both [77] and [103] to the classes of strongly nil-clean and nil-clean rings, respectively. For local group rings, the interested reader can be consulted with [92].

The leitmotif of the brief result stated below is to generalize the aforementioned results to the large class of UU rings as well as to give a more elementary and direct proof of a theorem from [77] (see [103, Theorem 2.12], too). It is worthwhile noticing that some partial statements on commutative group rings of UU rings are given in [D2, Section 5] in terms of divisions of the ring R and the group G.

We recall that a (possibly non-commutative) group is said to be *locally finite* if each its finite subset generates a finite subgroup, that is, each its finitely generated subgroup is finite. These groups are necessarily torsion. A type of such groups are the so-called *locally normal* groups, that are groups for which every finite subset can be embedded in a finite normal subgroup. For torsion abelian groups this property is always fulfilled, whereas in the non-abelian case the situation is more delicate being the classical Burnside's problem solved in the negative.

Before proceed by proving our major assertion, we need the next pivotal instrument from [39].

• Let I be a nil-ideal of a ring R. Then R is UU precisely when R/I is UU.

Our basic statement is the following one:

Theorem 2.3. *Let G be a group and R a ring.*

(i) If $R[G]$ is UU, then R is UU and G is a 2-group.

(ii) If G is locally finite, then $R[G]$ is UU if, and only if, R is UU and G is a 2-group.

(iii) If H is a normal subgroup of G such that H is locally normal and if $R[G]$ is UU, then $R[G/H]$ is UU.

Proof. (i) According to Proposition 2.2, we know that 2 is nilpotent in $R[G]$ and $U(R[G])$ is a 2-group. It now follows immediately that 2 is nilpotent in R and that $U(R) \leq U(R[G])$ and $G \leq U(R[G])$ are both 2-groups. Again Proposition 2.2 applies to get that R is UU, as wanted.

(ii) In view of point (i), we need to show only the "if" part. To that goal, since G is locally finite, choosing $x \in \omega(R[G])$, we deduce that $x \in \omega(R[H])$ for some finite subgroup H of G. But it is well known that the ideal $\omega(R[H])$ is nilpotent, and thus it is necessarily nil (see, e.g., [25, Theorem 9]). Hence the element x is nilpotent, so that the ideal $\omega(R[G])$ is nil. Taking into account that $R[G]/\omega(R[G]) \cong R$ along with the truthfulness of the bullet above, we are now done.

(iii) First of all, we observe that the following isomorphism of group rings

$$R[G]/(\omega(R[H]) \cdot R[G]) \cong R[G/H]$$

is fulfilled. We claim that the relative augmentation ideal $\omega(R[H]) \cdot R[G]$ of $R[G]$ is nil. In fact, since H is locally normal, each element z of this ideal is contained in the ideal $\omega(R[F]) \cdot R[G]$, where F is a finite normal subgroup of H and so it is normal in G as well. As above, it follows from [25, Theorem 9] that $\omega(R[F])$ is nilpotent whence so is $\omega(R[F]) \cdot R[G]$, because for any natural i the formula $(\omega(R[F]) \cdot R[G])^i = (\omega(R[F]))^i \cdot R[G]$ holds by taking into account that F is a normal subgroup of G. That is why, z is a nilpotent and so the claim sustained. Furthermore, the bullet alluded to above allows us to deduce in turn that $R[G/H]$ is a UU ring, as pursued. □

It is worth to noticing that the claim in point (i) that G is a 2-group contrasts the comments before Proposition 2.9 in [103].

The next affirmation is an immediate consequence of point (ii) from the preceding theorem.

Corollary 2.4. *Suppose R is a ring and G is an abelian group. Then $R[G]$ is a UU ring if, and only if, R is a UU ring and G is a 2-group.*

We now come to the promised above generalization of the major result in [85].

Corollary 2.5. *A group ring $R[G]$ is a commutative UU ring if, and only if, R is a commutative UU ring and G is an abelian 2-group.*

So, we are also ready to provide a more transparent proof of the following fact from [77] commented above:

Corollary 2.6. *Suppose that R is a ring and G is a locally finite group. Then $R[G]$ is strongly nil-clean if, and only if, R is strongly nil-clean and G is a 2-group.*

Proof. The necessity is well-known and trivial, so we omit its proof. As for the sufficiency, it follows directly from Theorem 2.3 and [103, Theorem 2.3]. □

We close the work with the following question of interest: *Does it follow that Theorem 2.3 remains true without the assumption that G is locally finite*? This query will definitely hold in the affirmative, provided that the next implication is valid: If R is a ring having $\mathrm{char}(R) = 2$ and G is a finitely generated 2-group, then the augmentation ideal $\omega(R[G])$ of the group ring $R[G]$ is nil.

Chapter IV. Abelian Groups

Our chief results of this branch are distributed into two sections as follows:

3. Generalizations of transitive and fully transitive Abelian groups

We shall distinguish here six subsections as follows:

3.1. **On the socles of fully invariant subgroups of Abelian p-groups.** The classification of all the fully invariant subgroups of a reduced Abelian p-group is a difficult and long-standing problem, not withstanding the progress made by Kaplansky in the 1950s utilizing the notion of a fully transitive group - see $\Sigma 18$ in [71]. Further progress was made for the special class of so-called *large* subgroups by Pierce in [100, Theorem 2.7]. A somewhat less ambitious programme is to try to characterize the socles of fully invariant subgroups and this is the subject of our discussions here. Despite the seeming simplification engendered by restricting attention to socles, the situation is still complicated once one moves away from fully transitive groups. We will show by means of examples that full transitivity is not the real core of the problem. We remark at the outset that the consideration of reduced groups only, is not a serious restriction; see the Note after Lemma 3.4 below. *Hence, in the sequel, we shall assume that our groups are always reduced p-groups for some arbitrary but a fixed prime p.*

Our notation is standard and follows [44, 71], an exception being that maps are written on the right. Finally we recall the notion of a U-sequence from [71]: a U-sequence relative to a p-group G is a monotone increasing sequence of ordinals $\{\alpha_i\}(i \geq 0)$ (each less than the length of the group G) except that it is permitted that the sequence be ∞ from some point on but that if a gap occurs between α_n and α_{n+1}, the α_n^{th} Ulm invariant of G is non-zero.

We introduce two additional concepts, the first of which shall be the primary focus our interest:

(i) A group G is said to be *socle-regular* if for all fully invariant subgroups F of G, there exists an ordinal α (depending on F) such that $F[p] = (p^\alpha G)[p]$.

(ii) Suppose that H is an arbitrary subgroup of the group G. Set $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min(H[p])$; clearly $H[p] \leq (p^\alpha G)[p]$.

If K is also a subgroup of G containing H, then of course there may be two different values of min associated to H, depending on where the heights of elements are calculated. We will distinguish these if necessary by writing $\min^G(H[p])$ and

$\min^K(H[p])$; note that if K is an isotype subgroup of G then the respective values of min coincide. However if K is not an isotype subgroup of G then all that one can say is that $\min^K(H[p]) \leq \min^G(H[p])$. Our first result collects some elementary facts about the function min being our major instrument for further attacks on the explored groups.

Proposition 3.1. (1) *If F is a subgroup of the group G and $(p^nG)[p] \leq F[p]$ for some integer n, then $\min(F[p])$ is finite.*

(2) *If F is a fully invariant subgroup of the group G and $\min(F[p]) = n$, a finite integer, then $F[p] = (p^nG)[p]$.*

Proof. (i) Suppose that $\alpha = \min(F[p])$, so that $\alpha \leq \min\{h_G(x) : x \in (p^nG)[p]\}$. Now if $\alpha \geq \omega$, then $(p^nG)[p] \leq p^\omega G = p^\omega(p^nG)$, so that writing $X = p^nG$, one has $X[p] \leq p^\omega X$, which forces X to be divisible contrary to the assumption that G is reduced. Hence $\min(F[p])$ is finite as required.

(ii) As observed above, one inclusion holds always. Conversely, suppose that $x \in F[p]$ and $h_G(x) = n$. Then $x = p^ny$ and the subgroup generated by y is a direct summand of G - see, e.g., [44, Corollary 27.2]. Thus $G = \langle y\rangle \oplus G_1$ for some subgroup G_1. Now if $0 \neq z$ is an arbitrary element of $(p^nG)[p]$, then $z = p^nw$ for some $w \in G$. Since the elements y, w are both of order p^{n+1} we may define a homomorphism $\phi : G \to G$ by sending $y \mapsto w$ and mapping G_1 to zero; note that $x\phi = z$. Since $F[p]$ is fully invariant in G, it follows that $z \in F[p]$ and so $(p^nG)[p] \leq F[p]$. □

Corollary 3.2. *If G is a separable group, then G is socle-regular.*

Proof. This is immediate since the hypothesis of separability implies that for any fully invariant subgroup F of G, $\min(F[p])$ is finite. □

Let us notice that Corollary 3.2 could have been deduced directly from our next result but we preferred to give the more elementary proof as an introduction to the type of arguments needed.

Theorem 3.3. *If G is a fully transitive group, then G is socle-regular.*

Proof. Since G is, by hypothesis, fully transitive, one may make use of Kaplansky's classification of fully invariant subgroups - see Theorem 25 in [71]. Thus, the fully invariant subgroup F has the form $F = \{x \in G : U_G(x) \geq U\}$, where $U = \{\alpha_i\}$ is a U-sequence relative to G. Now, if $x \in F[p]$, then $U_G(x) = \{\beta, \infty, \dots\}$ for some ordinal $\beta \geq \alpha_0$. Clearly, $x \in (p^{\alpha_0}G)[p]$ and so $F[p] \leq (p^{\alpha_0}G)[p]$.

Conversely, if $y \in (p^{\alpha_0}G)[p]$, then $U_G(y) = \{\gamma, \infty, \dots\}$, where $\gamma \geq \alpha_0$. But now it is immediate that $y \in \{x \in G : U_G(x) \geq U\} = F$, so that $(p^{\alpha_0}G)[p] \leq F[p]$, as required. $\square$

It follows, of course, that the class of socle-regular groups is large since the class of fully transitive groups is known to contain the λ-separable groups for all limit ordinals λ, the totally projective groups and Crawley's generalized torsion-complete groups; for further details of the latter see [51]. It is perhaps worth remarking that, as observed in [51], for $p \neq 2$, the concept of full transitivity coincides with Krylov's notion of transitivity, i.e. there exists an endomorphism mapping any element of the group to any other element which has the same Ulm sequence.

In this section, we explore some of the properties of the class of socle-regular groups. We shall have need of the following result, which is a slight variation of a well-known result.

Lemma 3.4. *Suppose that* $A = \bigoplus_{i \in I} G_i$ *and that* F *is fully invariant in* A. *Then*

(1) $F = \bigoplus_{i \in I}(G_i \cap F)$
(2) *each* $G_i \cap F$ *is fully invariant in* G_i.

Proof. Let $\pi_i : A \twoheadrightarrow G_i$ denote the canonical projections onto G_i. It is easy to see that $F = \bigoplus_{i \in I} F\pi_i$. Since F is fully invariant in A, $F\pi_i \leq F$ and it follows easily that $F\pi_i = G_i \cap F$, establishing (i). Suppose now that ϕ_i is an arbitrary endomorphism of G_i. Then $(G_i \cap F)\phi_i = F\pi_i\phi_i \leq F$ since F is fully invariant in A and $\pi_i\phi_i$ can be identified with an endomorphism of A. Since $(G_i \cap F)\phi_i \leq G_i$ also, the result follows. $\square$

Note: This Lemma allows one to justify the restriction of consideration to reduced groups. For if $G = D \oplus R$ is a group with maximal divisible subgroup D, then for any fully invariant subgroup F of G, one has $F = (F \cap D) \oplus (F \cap R)$ and $F \cap D$, $F \cap R$ are fully invariant in D, R respectively. However it is well known that the socle $(F \cap D)[p]$ must be either 0 or $D[p]$ and so the determination of $F[p]$ reduces to the determination of the socle of the fully invariant subgroup $F \cap R$ of the reduced group R.

Given that the class of fully transitive groups is closed under the addition of separable summands – see, e.g., [15, Proposition 2.6] – it is reasonable to ask whether the class of socle-regular groups has a similar property. A strong positive answer is given by the following.

Theorem 3.5. *Suppose that $A = G \oplus H$ where H is separable, then A is socle-regular if, and only if, G is socle-regular.*

Proof. Suppose that G is socle-regular and that F is fully invariant in A, so that, by Lemma 3.4, $F = (F \cap G) \oplus (F \cap H)$ and $(F \cap G), (F \cap H)$ are fully invariant in G, H respectively. If $F \cap H \neq 0$ then, since H is separable, it follows that $\min^H((F \cap H)[p])$ is finite. But $F[p] = (F \cap G)[p] \oplus (F \cap H)[p]$ and so

$$\min{}^A(F[p]) \leq \min{}^A((F \cap H)[p]) = \min{}^H((F \cap H)[p]),$$

the last equality following since H is pure in A. Thus it follows that $\min^A(F[p])$ is also finite, and so by Proposition 3.1, $F[p] = (p^n A)[p]$ for some integer n.

If $F \cap H = 0$, then F is a fully invariant subgroup of the socle-regular group G. Hence $F[p] = (p^\alpha G)[p]$ for some ordinal α. If $\alpha \geq \o$, then $p^\alpha A = p^\alpha G$ since H is separable and so $F[p] = (p^\alpha A)[p]$. Otherwise $F[p] = (p^n G)[p]$ and F is a fully invariant subgroup of G. It follows from Proposition 3.1(i) that $\min^G(F[p])$ is finite, and since G is pure in A, we also have that $\min^A(F[p])$ is finite. Now an appeal to Proposition 3.1(ii) yields the desired result.

Conversely suppose that A is socle-regular and assume for a contradiction that G is not. Then there exists a fully invariant subgroup K of G such that $K[p] \neq (p^\alpha G)[p]$ for any α. Note that $\min(K[p])$ must be infinite, for if it were finite, then by Proposition 3.1(ii), $K[p] = (p^n G)[p]$ for some finite n – contradiction. So $\min(K[p])$ is infinite and thus $K[p] \leq p^\omega G$. Furthermore $K[p]$ is fully invariant in G since K is. It follows from Lemma 3.6 below that $K[p]$ is fully invariant in the socle-regular group A. Thus $K[p] = (p^\alpha A)[p]$ for some α. Since $K[p] \leq p^\omega G$, α must be infinite. But then $p^\alpha H = 0$ and so $K[p] = (p^\alpha G)[p] \oplus (p^\alpha H)[p] = (p^\alpha G)[p]$ – contradiction. Thus, G is socle-regular as required. $\square$

We remark that the last possibility examined in the proof above never actually occurs: $\min^G(F[p])$ finite implies that there is an $x \in F[p]$ which can be embedded in a cyclic summand of G and then this element x can be mapped outside of F contrary to full invariance of F.

The proof of Theorem 3.5 is completed by the following:

Lemma 3.6. *A subgroup F of G is fully invariant in $A = G \oplus H$, where H is separable, if F is fully invariant in G and $F \leq p^\omega G$.*

Proof. Suppose that $F \leq p^\omega G$ and that F is fully invariant in G. Let $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ be any endomorphism of A. Then $(F \oplus 0)\Phi \leq (F\alpha \oplus F\gamma) \leq (F \oplus F\gamma)$ since F is fully invariant in G. Moreover, γ is a homomorphism $\gamma : G \to H$, and since H is separable, $p^\omega G$ must be mapped to zero by γ. Since $F \leq p^\omega G$, one must have that $F\gamma = 0$ and so $(F \oplus 0)\Phi \leq (F \oplus 0)$ and F is fully invariant in A as required. $\square$

We can also show that direct powers of a single socle-regular group are again socle-regular. In fact we have the stronger:

Theorem 3.7. *The group G is socle-regular if, and only if, the direct sum $G^{(\kappa)}$ is socle-regular for any cardinal κ.*

Proof. Suppose that F is fully invariant in $G^{(\kappa)}$, so that in view of Lemma 3.4, $F = \bigoplus_{i<\kappa}(G_i \cap F)$ where each G_i is isomorphic to G. Then the socle $F[p] = \bigoplus_{i<\kappa}(G_i \cap F)[p]$ and each $G_i \cap F$ is fully invariant in G_i. Since G is socle-regular, each $(G_i \cap F)[p]$ can be expressed as $(p^{\alpha_i}G_i)[p]$ for ordinals α_i. However if the α_i are not all equal, the subgroup $\bigoplus_{i<\kappa}(p^{\alpha_i}G_i)[p]$ is not fully invariant. It follows immediately that $F[p] = (p^{\alpha}G^{(\kappa)})[p]$, where α is the common value of the α_i, as required.

Conversely suppose that $G^{(\kappa)}$is socle-regular and that F is an arbitrary fully invariant subgroup of G. Since the endomorphism ring of $G^{(\kappa)}$ may be construed as the ring of row-finite matrices over $\mathrm{End}(G)$, it is easy to see that the subgroup $F^{(\kappa)}$ is fully invariant in $G^{(\kappa)}$. Since the latter is socle-regular, we have $(F^{(\kappa)})[p]$ $= (p^{\alpha}G^{(\kappa)})[p]$ for some ordinal α. It follows immediately that $F[p] = (p^{\alpha}G)[p]$ and thus G is socle-regular. $\square$

Recall that a fully invariant subgroup L of a group G is said to be large if $G = L + B$ for every basic subgroup B of G. Our next result shows that socle-regularity is inherited by large subgroups.

Proposition 3.8. *If A is a socle-regular group and L is a fully invariant subgroup of A such that $p^{\omega}L = p^{\omega}A$, then L is socle-regular. In particular, large subgroups of socle-regular groups are again socle-regular.*

Proof. Let F be a fully invariant subgroup of L. Then F is also fully invariant in A and hence, as A is socle-regular, $F[p] = (p^{\alpha}A)[p]$ for some ordinal α. Since $p^{\omega}A = p^{\omega}L$ by hypothesis, it follows from a simple transfinite induction argument that $p^{\alpha}A = p^{\alpha}L$ for all ordinals $\alpha \geq \omega$. Thus, if $\alpha \geq \omega$, $F[p] = (p^{\alpha}A)[p] = (p^{\alpha}L)[p]$. If α is finite, then $F[p] = (p^{n}A)[p] \geq (p^{n}L)[p]$ and so it follows from Proposition 3.1(i) that $\min^{L}(F[p])$ is finite. Applying the second part of the same Proposition gives that $F[p] = (p^{m}L)[p]$ for some integer m. The final claim in relation to large subgroups follows from the fact that if L is a large subgroup of A, then $p^{\omega}A = p^{\omega}L$ – see, e.g., [44] or [47]. $\square$

Once we drop the hypothesis of full transitivity, it is possible to exhibit groups of varying levels of complexity which are not socle-regular. Our first result shows that this failure can happen at the next stage beyond separability. We give two examples, the first based on the well-known realization theorem of Corner in [27], while the second is essentially due to Megibben in [87] – compare also with Chapter III above for some similar results on noncommutative rings pertaining to the endomorphism ring of groups of the present type.

Theorem 3.9. *There exist groups of length $\omega + 1$ which are not socle-regular.*

Proof. For the first class of examples let $H = \langle a\rangle \oplus \langle b\rangle$ where a, b are of order p and set $K = \langle a\rangle$ and $L = \langle b\rangle$. The endomorphism ring of H contains a subring Φ consisting of the diagonal matrices with entries from $\mathrm{End}(K)$ and $\mathrm{End}(L)$. Now apply Corner's realization result [27, Theorem 6.1] to obtain a group G such that $p^\omega G = H$ and $\mathrm{End}(G) \restriction H = \Phi$. (Note that G is neither transitive nor fully transitive since K, L are both fully invariant subgroups of G but the elements a, b have the same Ulm sequence $(\omega, \infty, \dots)$.)

In particular K is fully invariant in G and $K[p] = K$. However $(p^\omega G)[p] = K \oplus L$, $p^{\omega+1}G = 0$ and $p^n G$ is unbounded for all positive integers n, so that $K[p] \neq (p^\alpha G)[p]$ for any α. Hence G is not socle-regular as desired.

For the second class of examples let $A = G \oplus H$, where $p^\omega G \cong p^\omega H \cong \mathbb{Z}(p)$, $G/p^\omega G$ is a direct sum of cyclic groups and $H/p^\omega H$ is torsion-complete. It follows easily – e.g., see [87, Theorem 2.4] – that $p^\omega H$ is fully invariant in A. We claim that A is not socle-regular. If it were, then there is an ordinal $\alpha \geq 0$ such that $p^\omega H = (p^\omega H)[p] = (p^\alpha A)[p] = (p^\alpha G)[p] \oplus (p^\alpha H)[p]$. Therefore, $(p^\alpha G)[p] = 0$, i.e., $p^\alpha G = 0$ and hence $\alpha = \omega + 1$. Thus, $p^\omega H = (p^{\omega+1} H)[p] = 0$ - a contradiction. $\square$

The next comments could be useful in order to build a new strategy and to attack the general case.

Note: (i) The first class of examples shows that elongations of socle-regular groups by socle-regular groups need not be socle-regular: $p^\omega G$ and $G/p^\omega G$ are clearly both socle-regular while G is not. Notice, however, that it is easy to show that for any ordinal α and any socle-regular group A, the subgroup $p^\alpha A$ is always socle-regular.

(ii) These same examples show that Kaplansky's classification of fully invariant subgroups fails if we drop the full transitivity hypothesis: the subgroup K above is fully invariant but it cannot have the form $M(\{\alpha_i\})$ for any U-sequence $\{\alpha_i\}$. To see this observe that $U_G(a) = (\omega, \infty, \dots)$ and so if $K = M(\{\alpha_i\})$ for some U-sequence $\{\alpha_i\}$, then $\alpha_0 \leq \omega$. But it follows immediately that b, which has Ulm sequence $U_G(b) = (\omega, \infty, \dots)$, must also belong to $M(\{\alpha_i\})$, implying that $b \in K$ – contradiction. A similar observation has been made by Megibben in [87].

(iii) The second class of examples shows that one cannot drop the separability condition from Theorem 3.5: since $p^\omega G \cong p^\omega H \cong \mathbb{Z}(p)$, it is easy to see that G, H are both fully transitive and hence socle-regular by Theorem 3.3. However $A = G \oplus H$ is not socle-regular and so direct sums of socle-regular groups need not be socle-regular.

As noted above, elongations of socle-regular groups by socle-regular groups need not be socle-regular. We can however obtain some additional information in the special situation where the quotient $G/p^\omega G$ is a direct sum of cyclic groups.

Theorem 3.10. *Let G be a group such that $G/p^\omega G$ is a direct sum of cyclic groups. Then G is socle-regular if, and only if, $p^\omega G$ is socle-regular.*

Proof. We have already noted that G socle-regular implies that $p^\alpha G$ is socle-regular for any ordinal α, so it suffices to handle the sufficiency. Let F be an arbitrary fully invariant subgroup of G. Consider the socle $F[p]$. If $F[p] \nleq (p^\omega G)[p]$, then $\min(F[p])$ is finite and it follows from Proposition 3.1 that $F[p] = (p^n G)[p]$ for some finite integer n. If, however $F[p] \leq (p^\omega G)[p]$ we claim that $F[p]$ is fully invariant in $p^\omega G$. Assuming that this is true, it then follows immediately that $F[p] = (p^\alpha(p^\omega G))[p]$ since $p^\omega G$ is socle-regular by hypothesis. Thus $F[p] = (p^{\omega+\alpha} G)[p]$ and we are finished. Thus it remains to show that $F[p]$ is fully invariant in $p^\omega G$.

If ϕ is an arbitrary endomorphism of $p^\omega G$, then it follows from Hill's work on totally projective groups – see [61, Theorem 2] – that every endomorphism of $p^\omega G$ is induced from an endomorphism of G in this situation. The desired result follows immediately. $\square$

We have seen in Theorem 3.7 that direct powers of socle-regular groups must be socle-regular, but we have been unable to determine whether or not summands of socle-regular groups are, in general, socle-regular. The best we can offer is the rather weak:

Proposition 3.11. *Let $G = A \oplus B$ be a socle-regular group such that every homomorphism from A to B is small, then A is socle-regular.*

Proof. Let F be a fully invariant subgroup of A. If $\min(F[p])$ is finite then it follows from Proposition 3.1 that $F[p] = (p^n A)[p]$ for some finite integer n. If $\min(F[p])$ is infinite, then $F[p] \leq p^\omega A$. Claim that $F[p] \oplus 0$ is fully invariant in G: the argument is similar to that used in Lemma 3.6 with smallness replacing the argument using separability. If Φ is any endomorphism of G then Φ may be written as a matrix $\left(\begin{smallmatrix}\alpha & \gamma\\ \delta & \beta\end{smallmatrix}\right)$, where $\gamma \in \mathrm{Hom}(A, B)$, so that γ is small. Then $(F[p] \oplus 0)\Phi \leq (F[p]\alpha \oplus F[p]\gamma)$. However $F[p] \leq p^{\omega}A$ implies that $F[p]\gamma = 0$ as γ is small. So $F[p] \oplus 0$ is fully invariant in G and hence $F[p] \oplus 0 = (p^{\nu}G)[p]$ for some ordinal ν. Hence $F[p] = (p^{\nu}A)[p]$ as required. $\square$

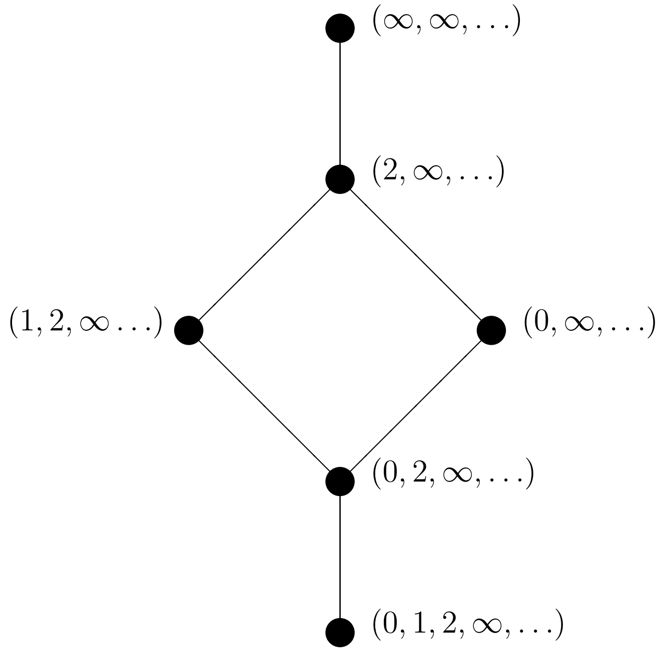
It is, however, possible to construct a group which is not fully transitive but is transitive (and hence is a 2-group) and has the property that it is socle-regular.

Example 3.12. There is a socle-regular 2-group which is transitive but *not* fully transitive.

Proof. Let G be the transitive, non fully transitive 2-group constructed by Corner in [28]. The group G has the property that $2^{\omega}G = H$, $\mathrm{Aut}(G) \restriction 2^{\omega}G = \mathrm{Aut}(H)$, $\mathrm{End}(G) \restriction 2^{\omega}G = \Phi$, where Φ is the subring of $\mathrm{End}(H)$ generated by $\mathrm{Aut}(H)$ and the group $H = \langle a\rangle \oplus \langle b\rangle$, where a has order 2 and b has order 8. Note that H has six different associated Ulm sequences:

$$(\infty, \infty, \dots); (2, \infty, \dots); (0, \infty, \dots); (1, 2, \infty, \dots); (0, 2, \infty, \dots); (0, 1, 2, \infty, \dots).$$

Fuller details of this group, relevant for our present purposes, may be found in [51, Example 3.16]. In particular, the associated lattice has just one pair of incomparable Ulm types and it is easy to check, using the calculations and discussions of Example 3.16 in [51], that the only fully invariant subgroups of G contained in $2^{\omega}G$ are $F_1 = \{0, 4b, a - 2b, a + 2b\}$, $F_2 = \{0, a, 4b, a + 4b\}$ and $F_3 = \{0, 4b\}$. (This is essentially because it is possible to map from any vertex of the lattice, other than the vertex labelled $(0, 2, \infty, \dots)$, to any other one above it.)



Now if F is an arbitrary fully invariant subgroup of G and $\min(F[2])$ is finite, then $F[2] = (2^n G)[2]$ for $n = \min(F[2])$ by Proposition 3.1. If $\min(F[2]) \geq \omega$ then $F[2]$ is one of $F_i[2]$, $i = 1, 2, 3$. However, a simple check shows that $F_1[2] = (2^{\omega+1}G)[2]$, $F_2[2] = (2^{\omega}G)[2]$ while $F_3[2] = (2^{\omega+2}G)[2]$. Thus the socle of each fully invariant subgroup of G is of the form $(2^{\alpha}G)[2]$ for some α and G is socle-regular. $\square$

The following commentaries are somewhat helpful to shed some light on the general situation.

Note: It is now rather easy to show that neither transitivity nor full transitivity is the core concept in determining whether or not a group is socle-regular. For if G is the group in the example above, it follows from Theorem 3.7 that $A = G \oplus G$ is socle-regular. However A is neither transitive nor fully transitive; it cannot be fully transitive since direct summands of such groups are again fully transitive [15, Theorem 3.4] and it cannot be transitive since if it were, it would follow from [43, Corollary 3] that G was fully transitive which it is not.

We finish off our discussion by posing three questions of interest:

(1) Does there exist a transitive group which is not socle-regular? Such a group would, of course, necessarily be a 2-group.

(2) Does Theorem 3.10 generalize to arbitrary infinite ordinals α, if $G/p^{\alpha}G$ is assumed to be totally projective?

(3) Is a summand of a socle-regular group again socle-regular?

3.2. **On socle-regularity and some notions of transitivity for Abelian p-groups.** Early work in the theory of infinite Abelian p-groups focused on issues such as classification by cardinal invariants. This led initially to the rich theory known now as Ulm's theorem and, in some sense, culminated in deep classification of the class of groups known variously as *simply presented, totally projective* or *Axiom 3* groups. Such groups are, of necessity, somewhat special. On the other hand, there was also interest in properties of groups that were held by "the majority" of Abelian p-groups. Within this latter category, the extensive classes of transitive and fully transitive groups were prominent. Recently, the present authors introduced two new classes of p-groups which, respectively, properly contained the corresponding classes of transitive and fully transitive groups: these are the socle-regular and strongly socle-regular groups developed in [D8] and [35] – see also Chapter III for some related results on ring theory which could be interpreted on the endomorphism ring of abelian groups of these special kinds. The present subsection looks further at the interconnections between these classes and some other recent notions of transitivity.

Throughout, all groups will be additively written, reduced Abelian p-groups; standard concepts relating to such groups may be found in [44, 47] or [71]. We follow the notation of these texts but write mappings on the right. To avoid subsequent need for definitions of fundamental ideas, we mention that the *height* of an element x in the group G (written like $h_G(x)$) is the ordinal α if $x \in p^\alpha G \setminus p^{\alpha+1}G$ with the usual convention that $h(0) = \infty$. The *Ulm sequence* of x with respect to G is the sequence of ordinals or symbols ∞ given by $U_G(x) = (h_G(x), h_G(px), h_G(p^2x), \dots)$; the collection of such sequences may be partially ordered pointwise. Finally we recall an *ad hoc* notion introduced in [D8] which continues to be useful here: suppose that H is an arbitrary subgroup of the group G. Set $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min(H[p])$; clearly $H[p] \leq (p^\alpha G)[p]$.

We will now explore some various notions of transitivity. The notions of transitivity and full transitivity for Abelian p-groups were introduced by Kaplansky in [70] and became a topic of ongoing interest in Abelian group theory with the publication of Kaplansky's famous "little red book" [71]. Recall that a p-group G is said to be *transitive* (resp., *fully transitive*) if for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$ (resp., $U_G(x) \leq U_G(y)$) there is an automorphism (endomorphism) ϕ of G with $x\phi = y$. In recent times two addition notions of transitivity have been introduced: in [51] a group G is said to be *Krylov transitive* if, for each pair of elements $x, y \in G$ with $U_G(x) = U_G(y)$, there is an endomorphism ϕ of G with $x\phi = y$. Finally, a group G was said in [51] to be *weakly transitive* if, given

$x, y \in G$ and endomorphisms ϕ, ψ of G with $x\phi = y,\ y\psi = x$, there is an automorphism θ of G with $x\theta = y$. Notice in this last concept that although there is no explicit reference to Ulm sequences, the existence of the endomorphisms ϕ, ψ ensures that $U_G(x) = U_G(y)$.

To avoid a great deal of repetition, we find it convenient to use the expression *G is *-transitive* to mean that G has a fixed one of the the four transitivity properties discussed above.

In [27], Antony Corner showed that transitivity and full transitivity of a group G are determined by the action of the endomorphism ring on the first Ulm subgroup $p^\omega G$. Following his example, we say that if Φ is a unital subring of the endomorphism ring $\mathrm{End}(G)$ of G and if H is a Φ-invariant subgroup of G, then

(i) Φ is transitive on H if, for any x, y in H with $U_G(x) = U_G(y)$, there is a unit $\phi \in \Phi$ with $x\phi = y$;

(ii) Φ is Krylov transitive on H if, for any x, y in H with $U_G(x) = U_G(y)$, there is an element $\phi \in \Phi$ with $x\phi = y$;

(iii) Φ is fully transitive on H if, for any x, y in H with $U_G(x) \leq U_G(y)$, there is an element $\phi \in \Phi$ with $x\phi = y$;

(iv) Φ is weakly transitive on H if, for any x, y in H and elements $\phi, \psi \in \Phi$ with $x\phi = y$ and $y\psi = x$, there is a unit $\theta \in \Phi$ with $x\theta = y$.

Our first result follows exactly as in [27, Lemma 2.1] or [51, Proposition 3.8], so we state it without proof:

Proposition 3.13. *The p-group G is *-transitive if, and only if,* $\mathrm{End}(G)$ *acts *-transitively on $p^\omega G$.*

An immediate consequence of Proposition 3.13 is the fact that addition of a separable summand has no influence on the transitivity properties.

Corollary 3.14. *If G is *-transitive and H is separable, then $K = G \oplus H$ is *-transitive.*

Proof. The proof for transitivity, full transitivity and Krylov transitivity follows by an identical argument to that given in [15, Proposition 2.6]. Suppose then that G is weakly transitive. It suffices, by Proposition 3.13, to show that $\mathrm{End}(K)$ acts weakly transitively on $p^\omega K = p^\omega G \oplus 0$. Suppose $(g_0, 0),\ (g_1, 0) \in p^\omega K$ and there are endomorphisms ϕ, ψ of K with $(g_0, 0)\phi = (g_1, 0),\ (g_1, 0)\psi = (g_0, 0)$. Representing ϕ, ψ as matrices in the standard way, $\phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ and $\psi = \left(\begin{smallmatrix} \alpha_1 & \gamma_1 \\ \delta_1 & \beta_1 \end{smallmatrix}\right)$, we conclude that $g_0\alpha = g_1$ and $g_1\alpha_1 = g_0$ for endomorphisms α, α_1 of G. Since G is weakly transitive, there is an automorphism θ of G with $g_0\theta = g_1$ and $g_1\theta^{-1} = g_0$. The matrix $\Delta = \left(\begin{smallmatrix} \theta & 0 \\ 0 & 1_H \end{smallmatrix}\right)$ then represents an automorphism of K and it is easy to check that $(g_0, 0)\Delta = (g_1, 0)$. $\square$

There are, of course, many interrelations between the various notions of transitivity: for example, it is immediate that either transitivity or full transitivity implies Krylov transitivity. We list a representative sample of these connections:

Proposition 3.15. *(i) A group G is fully transitive if, and only if, its square $G \oplus G$ is transitive;*

(ii) If $p \neq 2$ and G is transitive, then G is fully transitive;

(iii) A direct summand of a transitive group is Krylov transitive;

(iv) If $p \neq 2$, then G is fully transitive if, and only if, G is Krylov transitive if, and only if, G is a summand of a transitive group;

(v) If G is Krylov transitive and weakly transitive, then G is transitive and vice versa;

(vi) If G is fully transitive and weakly transitive, then G is transitive.

Proof. A proof of (i) may be found as Corollary 3 in [43]; (ii) is a fundamental observation of Kaplansky [71, Theorem 26]. For (iii) assume $G = H \oplus K$ and that $x, y \in H$ with $U_H(x) = U_H(y)$. But then $U_G((x,0)) = U_G((y,0))$ and so there is an automorphism $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ with $(x,0)\Phi = (y,0)$. Hence, $y = x\alpha$ for the endomorphism α of H and H is Krylov transitive.

The equivalence of the first two parts of (iv) may be found in [51], while the final equivalence is Corollary 5 in [43].

Observe that (v) follows easily: if $U_G(x) = U_G(y)$, then Krylov transitivity implies that there are endomorphisms α, β of G with $x\alpha = y,\ y\beta = x$. By weak transitivity, there is the required automorphism ψ of G with $x\psi = y$. Conversely, G being transitive directly ensures that G is both Krylov transitive and weakly transitive. Finally (vi) follows immediately since full transitivity implies Krylov transitivity. □

Our next result is an analogue for Krylov transitivity of part of a well-known result of Kaplansky [71, Theorem 26], the other part being contained in (ii) and (iii) above.

Theorem 3.16. *Suppose G is a Krylov transitive reduced p-group and that G has at most two Ulm invariants equal to 1, and if it has exactly two, they correspond to successive ordinals, then G is fully transitive.*

Before embarking on the proof of Theorem 3.16, we establish two simple lemmas; recall from [71] that a group is said to have *property $P(\alpha)$*, for an ordinal α, if for any element $x \in (p^\alpha G)[p] \setminus p^{\alpha+1}G$ there is an element y such that both y and $x + y$ are also of order p and height α.

Lemma 3.17. *Suppose that $x \in G$ with $U_G(x) = (\alpha_0, \alpha_1, \dots)$ and $y \in G[p]$ with $U_G(y) = (\alpha_0, \infty, \dots)$. Then if G is Krylov transitive and has property P(α_0), there is an endomorphism ϕ of G with $x\phi = y$.*

Proof. If $h(x + y) = \alpha_0$, then $U_G(x + y) = U_G(x)$ and so, by Krylov transitivity, there is an endomorphism ψ of G with $x\psi = x + y$. The mapping $\phi = \psi - 1$ then has the desired property. Suppose then that $h(x + y) > \alpha_0$. Since we are assuming P(α_0), there is an element z of height α_0 and order p such that $y - z$ also has height α_0 and order p. Now $(x + z) = (x + y) - (y - z)$ has height exactly α_0 since $h(x + y) > \alpha_0$, while $h(y - z) = \alpha_0$. It follows that $U_G(x + z) = (\alpha_0, \alpha_1, \alpha_2, \dots) = U_G(x)$. Thus, by Krylov transitivity, there is an endomorphism of G mapping x to $x + z$ and so, of course, there is a mapping $\psi : x \mapsto z$. Moreover, $U_G(z) = U_G(y)$ and so there is an endomorphism θ with $z\theta = y$. The composite $\phi = \psi\theta$ then maps $x \mapsto y$, as required. □

Our second lemma has been used previously in [51, Lemma 2.2]; its elementary proof may be found there.

Lemma 3.18. *If G is a p-group such that for all $x, y \in G$ with $y \in G[p]$ and $U_G(x) \leq U_G(y)$, there is an endomorphism φ of G mapping x onto y, then G is fully transitive.*

Proof of Theorem 3.16. It suffices to show that the conditions of Lemma 3.18 above are satisfied. So assume that y is a fixed but arbitrary element of $G[p]$ and $x \in G$ with $U_G(x) \leq U_G(y)$; clearly we may assume $y \neq 0$. The proof is by induction on the order of the element x. Denote $U_G(x)$ by $(\alpha_0, \alpha_1, \dots)$.

If $o(x) = p$, then $\alpha_1 = \infty$ and we have $U_G(y) = (\beta_0, \infty, \dots)$ with $\beta_0 \geq \alpha_0$. If $\beta_0 = \alpha_0$, then x, y will have equal Ulm sequences and so, by Krylov transitivity, there is a map $\phi : x \mapsto y$. If $\beta_0 > \alpha_0$, then the Ulm sequences of x and $x + y$ will be equal and Krylov transitivity yields a map $\psi : x \mapsto x + y$. The mapping $\phi = \psi - 1_G$ will then have the desired property.

So now assume that x is of order p^n and that for all elements t with $o(t) < p^n$, if $U_G(t) \leq U_G(s)$ with $s \in G[p]$, there is an endomorphism : $t \mapsto s$. Now the Ulm sequence of x has the form $(\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \infty, \dots)$; note that by an identical argument to that used in the previous paragraph, we may assume $U_G(y) = (\alpha_0, \infty, \dots)$. It follows from the existence of the gaps in the Ulm sequences for x, y that the Ulm invariants $f_G(\alpha_0), f_G(\alpha_{n-1})$ are both non-zero. Moreover if $h(x + y) = \alpha_0$, we are finished since $U_G(x + y)$ would then be $(\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \infty, \dots) = U_G(x)$ and Krylov transitivity would yield the required mapping.

Suppose then that $h(x+y) = \delta_0 > \alpha_0$. Then $U_G(x+y) = (\delta_0, \alpha_1, \ldots, \alpha_{n-1}, \infty, \ldots)$ and so $\alpha_0 < \delta_0 < \alpha_1$ i.e. there is a gap between α_0 and α_1. In particular $\alpha_{n-1} > \alpha_0 + 1$, so that α_0 and α_{n-1} are not successive ordinals. By our assumption on the Ulm invariants, one of the non-zero cardinals $f_G(\alpha_0), f_G(\alpha_{n-1})$ is not equal to 1. If $f_G(\alpha_0) \neq 1$, then G has the property $\mathrm{P}(\alpha_0)$ and so by Lemma 3.17, there is the desired mapping $x \mapsto y$.

If $f_G(\alpha_{n-1}) \neq 1$, then G has property $\mathrm{P}(\alpha_{n-1})$. Furthermore, by [71, Lemma 29], we may write $x = v + w$ in such a way that $o(v) < o(x)$ and w is normal relative to x, has order p^n and $h(p^{n-1}w) = \alpha_{n-1}$. By induction there is an endomorphism mapping $v \mapsto y$. The proof is completed by an appeal to [71, Lemma 31]. This completes the proof of the theorem.

By making use of ideas from [43], we can derive more information about the inter-relationships of the various transitivity properties. Recall that for a reduced group of length τ, the *Ulm support* $\mathrm{supp}(G)$ of G is the set of all ordinals $\sigma < \tau$ for which $f_G(\sigma)$ is non-zero.

Theorem 3.19. *If $G = G_1 \oplus G_2$ and supp($p^\omega G_1$) = supp($p^\omega G_2$), then the following are equivalent:*
(i) G is Krylov transitive;
(ii) G is fully transitive;
(iii) G is transitive.

Proof. Under the hypothesis of the theorem, the equivalence of (ii) and (iii) follows from [43, Theorem 1]. The implication (ii) $\Rightarrow$(i) holds even without the additional hypothesis. Thus it remains only to establish that (i) $\Rightarrow$ (ii).

Suppose then that G is Krylov transitive. Let B denote the standard basic group and set $H = G \oplus B \oplus B$; note that it follows from Corollary 3.14 that H is again Krylov transitive. Moreover a straightforward check shows that no Ulm invariant of H is equal to 1. It follows then from Theorem 3.16 that H is fully transitive. But then G, as a summand of a fully transitive group, is also fully transitive. □

Thus we can immediately deduce:

Corollary 3.20. *If $G = A \oplus A$ for some group A, then G is Krylov transitive if, and only if, it is fully transitive if, and only if, it is transitive.*

In [51] it was shown that full transitivity (Krylov transitivity) and weak transitivity are independent notions and Corner's original examples of non-transitive but fully transitive groups, and vice versa, show that Krylov transitivity is independent of the notions of transitivity and full transitivity. It would be interesting to know (see also Question 2.1 from [D9]):

Question 2.1. Does there exist a Krylov transitive group which is neither transitive nor fully transitive? Such a group would necessarily be a 2-group.

However, this was answered in the affirmative by Theorem 2.5 in the very recent article [10].

The four notions of transitivity also share the property that subgroups of the form $p^\beta G$ are, in some circumstances, the key to determining the *-transitivity of the whole group G. The following generalizes [61, Theorems 3 and 4].

Proposition 3.21. *Suppose that $G/p^\beta G$ is totally projective for some ordinal β. Then G is *-transitive if, and only if, $p^\beta G$ is *-transitive.*

Proof. Let $H = p^\beta G$ and observe that if $h_H(x) = \alpha$, then $h_G(x) = \beta + \alpha$. Consequently, if $x, y \in H$ and $U_H(x) = U_H(y)(U_H(x) \leq U_H(y))$, then $U_G(x) = U_G(y)(U_G(x) \leq U_G(y))$. Thus if G is transitive, Krylov transitive or fully transitive, it follows easily that $p^\beta G$ has the same property. If G is weakly transitive and $x, y \in H$ are such that there are endomorphisms ϕ, ψ of H with $x\phi = y, y\psi = x$, then since ϕ, ψ do not decrease heights computed in G and G/H is totally projective, it follows from a well-known result of Hill (see, e.g., [61]) that ϕ, ψ extend to endomorphisms ϕ', ψ' of G and, of course, $x\phi' = y,\ y\psi' = x$. Since G is, by assumption, weakly transitive, there is an automorphism, θ' say, of G with $x\theta' = y$. Then $\theta = \theta' \restriction H$ is an automorphism of H with $x\theta = y$, as required.

Conversely suppose that $p^\beta G$ is *-transitive and let $x, y \in G$ be elements such that $U_G(x) = U_G(y)(U_G(x) \leq U_G(y))$ [there exist endomorphisms ϕ, ψ of G with $x\phi = y,\ y\psi = x$]; note that in the third case one also has that $U_G(x) = U_G(y)$. Let n, m be the smallest integers such that $p^n x \in p^\beta G, p^m y \in p^\beta G$; observe that $m \leq n$ with equality in the first and third cases. In the case of transitivity or Krylov transitivity of $p^\beta G$, we have an automorphism (endomorphism) ϕ of H with $p^n x\phi = p^n y$ and this extends to an isomorphism (endomorphism) of $\langle H, x\rangle \to \langle H, y\rangle$ by mapping $x \mapsto y$. Since this is height-preserving (not height-decreasing) in G, the aforementioned Hill's result again yields an extension of ϕ to G with the required property.

In the case of weak transitivity, we have a pair of endomorphisms ϕ, ψ of G and their restrictions to H also interchange x and y. Hence there is an automorphism of H mapping x to y and again, by the total-projectivity of G/H, we get the desired automorphism of G sending x to y.

Finally consider the case where H is assumed to be fully transitive. As noted above, $p^n x, p^n y$ both belong to $p^\beta G$ and $U_H(p^n x) \leq U_H(p^n y)$. So there exists an endomorphism of H mapping $p^n x \mapsto p^n y$. This mapping extends to a mapping from $\langle H, x\rangle \to \langle H, y\rangle$ by mapping $x \mapsto y$. Since heights in G are not decreased and the quotient G/H is totally projective, there exists the desired endomorphism of G mapping $x \mapsto y$. □

Remark 2.1. In the cases of transitivity, Krylov transitivity and full transitivity, it is not necessary to assume that $G/p^\beta G$ is totally projective in order to deduce that $p^\beta G$ inherits the corresponding transitivity property.

The various notions of transitivity behave somewhat differently in relation to the formation of direct summands: a summand of a fully transitive group is again fully transitive, but this is not true in general for transitive or weakly transitive groups – see, for example, [15] and [51].

Proposition 3.22. *Let $G = H \oplus K$, then if G is Krylov-transitive, H is also Krylov-transitive.*

Proof. Suppose that G is Krylov transitive and let x, y be elements of H with $U_H(x) = U_H(y)$. Then the elements $(x, 0), (y, 0)$ of G have equal Ulm sequences in G and consequently there is an endomorphism of G mapping $(x, 0)$ to $(y, 0)$; this, of course, necessitates the existence of an endomorphism of H mapping x to y. □

There are, however, some situations in which summands of transitive (weakly transitive) groups inherit the transitivity property. Recall that a homomorphism $\phi : G \to H$ is said to be *small* if for every natural number k, there is a natural number n such that $(p^n G)[p^k]\phi = 0$. Weakening this definition, we shall say that a homomorphism $\varphi : G \to H$ is *almost small* if $p^\omega G \subseteq ker\varphi$. Clearly, every small homomorphism is almost small, whereas the converse does not hold always. Also, if H is separable, then each homomorphism between G and H is almost small.

Recall - see [15] - that a group G is said to be of *type A* if $U(\mathrm{End}(G) \restriction p^\omega G) = \mathrm{Aut}(G) \restriction p^\omega G$. Before stating our result on summands, we derive the following assertion:

Lemma 3.23. *Suppose that $K = G \oplus H$ and every homomorphism from G to H is almost small. Then if G is of type A and $\Phi = \left(\begin{smallmatrix}\alpha & \gamma \\ \delta & \beta\end{smallmatrix}\right)$ is an automorphism of K, there is an automorphism ϕ of G with $\phi \restriction p^\omega G = \alpha \restriction p^\omega G$.*

Proof. Since Φ is an automorphism of K, its restriction to $p^\omega K$ is an automorphism of $p^\omega K = p^\omega G \oplus p^\omega H$. Letting bars denote restrictions to first Ulm subgroups, we get $\bar{\Phi} = \left(\begin{smallmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\delta} & \bar{\beta} \end{smallmatrix}\right)$ and the assumption of almost smallness forces $\bar{\gamma} = 0$. Since every endomorphism of $p^\omega K$ must have a matrix representation which is lower triangular, we deduce that $\bar{\alpha}$ is a unit of $\mathrm{End}(p^\omega G)$. Since G is of type A, there is an automorphism ϕ of G with $\phi \restriction p^\omega G = \bar{\alpha} = \alpha \restriction p^\omega G$, as required. □

Proposition 3.24. *If $K = G \oplus H$ and every homomorphism from G to H is almost small, then if K is transitive (weakly transitive) and G is of type A, then G is also transitive (weakly transitive).*

Proof. We consider first the situation where K is transitive. Suppose that $x, y \in p^\omega G$ with $U_G(x) = U_G(y)$. Then $(x, 0)$ and $(y, 0)$ are elements of $p^\omega K$ having the same Ulm sequences in K. Since K is transitive, there is an automorphism $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ of K with $(x, 0)\Phi = (y, 0)$. By the previous Lemma 3.23, there is an automorphism ϕ of G with $\phi \restriction p^\omega G = \alpha \restriction p^\omega G$, so that $x\phi = x\alpha = y$. Hence $\mathrm{End}(G)$ acts transitively on $p^\omega G$ and by Proposition 3.13 we have that G is transitive, as required.

Finally suppose that K is weakly transitive and x, y are as above with endomorphisms θ, ψ of G such that $x\theta = y, y\psi = x$. Then the endomorphisms of K, given by the matrix representations $\Theta = \left(\begin{smallmatrix} \theta & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\Psi = \left(\begin{smallmatrix} \psi & 0 \\ 0 & 0 \end{smallmatrix}\right)$, have the property that $(x, 0)\Theta = (y, 0)$ and $(y, 0)\Psi = (x, 0)$. Since K is weakly transitive, there is an automorphism $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ of K with $(x, 0)\Phi = (y, 0)$. An appeal again to the preceding Lemma 3.23 yields an automorphism ϕ of G with $\phi \restriction p^\omega G = \alpha \restriction p^\omega G$. Thus $x\phi = x\alpha = y$ and $\mathrm{End}(G)$ acts weakly transitively on $p^\omega G$, as required. □

Our next result shows that Krylov transitive groups behave nicely when "squared", provided that the lattice of Ulm sequences of the first Ulm subgroup is a chain.

Theorem 3.25. *Suppose G is a group such that all elements of $p^\omega G$ have comparable Ulm sequences. Then $G \oplus G$ is Krylov transitive if, and only if, G is Krylov transitive. This property may fail if there are elements of $p^\omega G$ with incomparable Ulm sequences.*

Proof. The necessity follows directly from Proposition 3.22; in fact, there is no need for any assumption about $p^\omega G$ for this implication. For the sufficiency, let $H = G \oplus G$ and suppose $\underline{x}, \underline{y} \in p^\omega H$ with $U_H(\underline{x}) = U_H(\underline{y})$, where $\underline{x} = (x_1, x_2)$ and $\underline{y} = (y_1, y_2)$. By assumption, the Ulm sequences of elements of $p^\omega G$ are comparable, so there is no loss in generality in assuming that $U_H(\underline{x}) = U_G(x_1)$, $U_H(\underline{y}) = U_G(y_1)$. Since G is Krylov transitive, there is an endomorphism

$\theta : G \to G$ with $x_1\theta = y_1$. Appealing to the comparability hypothesis again, either $U_G(y_1) = U_G(y_2)$ or $U_G(y_1) < U_G(y_2)$.

If the first possibility arises, $U_G(x_1) = U_G(y_2)$ and so there is an endomorphism ψ of G with $x_1\psi = y_2$. If $\Delta = \left(\begin{smallmatrix} \theta & \psi \\ 0 & 0 \end{smallmatrix}\right)$, then $\Delta \in \mathrm{End}(H)$ and $\underline{x}\Delta = \underline{y}$.

In the second situation $U_G(y_1) < U_G(y_2)$ and so we have $U_G(y_1 + y_2) = U_G(y_1)$; again there is an endomorphism of G, ϕ say, with $x_1\phi = y_1 + y_2$. Now set $\Delta = \left(\begin{smallmatrix} \theta & \theta+\phi \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{End}(H)$ and $\underline{x}\Delta = \underline{y}$.

For the second part of the theorem, recall that Corner [27, Section 4] has constructed a transitive, but non fully transitive 2-group C with $2^\omega C = \langle a\rangle \oplus \langle b\rangle$, where a and b have orders 2 and 8 respectively. Note that C is, of course, Krylov transitive since it is even transitive and that the elements $a, 2b$ of $2^\omega C$ have incomparable Ulm sequences. It is shown in [51, Example 3.16] that $C \oplus C$ is weakly transitive. However $C \oplus C$ is not Krylov transitive: if it were, it would follow from Proposition 3.15 (v) that $C \oplus C$ is transitive, which in turn implies by (i) of the same proposition that C is fully transitive – contradiction. $\square$

Question 2.2. Does there exist a non fully transitive Krylov transitive group which satisfies the above theorem? Owing to [51, Proposition 2.3] such a group must necessarily be a 2-group.

Remark 2.2. In the second part of of the proof of Theorem 3.25 it is possible to show directly, arguing as in the proof of [51, Example 3.16], that there is no endomorphism of $C \oplus C$ mapping the element $(a + 2b, 0)$ to $(a + 2b, a)$ although both elements have Ulm sequence $(\omega, \omega + 2, \infty, \dots)$. Moreover, by what we have shown above, if G is transitive, then $G \oplus G$ need not be Krylov transitive. So, is it true that G is Krylov transitive non transitive if, and only if, $G \oplus G$ is Krylov transitive non transitive?

Our next result is simply a reworking of Corner's Proposition 2.2 in [27]: observe that in the proof there, it suffices to have Krylov transitivity at each of the key stages.

Proposition 3.26. *Let G be a Krylov transitive group such that $p^\omega G$ is a homocyclic group of finite exponent. Then*

(i) G is fully transitive.

(ii) If there is a direct decomposition $G = G_1 \oplus G_2$ with $p^\omega G_i \neq 0$ $(i = 1, 2)$, then G is transitive.

It is possible to improve somewhat on Theorem 3.19 by using the methods of [43] and replacing full transitivity by Krylov transitivity. Rather than adopt either extreme of leaving the task to the reader or re-writing the proofs in their entirety, we point out the key argument needed to replace the use of full transitivity.

Lemma 3.27. *Suppose that $G = G_1 \oplus G_2$ and $x \in G_1, y \in G_2$ with $U_{G_1}(x) \geq U_{G_2}(y)$. Then if G is Krylov transitive, there is a homomorphism $\delta : G_2 \to G_1$ with $y\delta = x$.*

Proof. Consider the elements (x, y) and $(0, y)$ of G. Since their Ulm sequences are, respectively, the infima $U_{G_1}(x) \wedge U_{G_2}(y)$ and $U_{G_2}(y) \wedge U_{G_1}(0)$, it follows that they are both equal to $U_{G_2}(y)$. By Krylov transitivity, there is a matrix $\Phi = \left(\begin{smallmatrix} \theta & \psi \\ \delta & \beta \end{smallmatrix}\right)$ with $(0, y)\Phi = (x, y)$ which gives $y\delta = x$ as required. □

We continue with the rather curious statement like this:

Proposition 3.28. *If $G = G_1 \oplus G_2$ is Krylov transitive, G_2 is transitive and $supp(p^\omega G_1) \subseteq supp(p^\omega G_2)$, then G is transitive.*

Proof. The proof is based on Lemma 2 in [43]. In the proof of Lemma 2, two applications of full transitivity are made. The first such is actually based on elements a_1, b_2 with equal Ulm sequences and it is immediate that Krylov transitivity will suffice for the argument there. The second application of full transitivity occurs at the bottom of [43, p.1607] but it is easily seen to involve the set-up invoked in Lemma 3.27 above. Consequently this too will carry over to the Krylov transitivity situation.

The final stage of the proof is carried out in an identical fashion to Proposition 2 in [43]. However the appeals to full transitivity can be replaced by the argument of Lemma 3.27. □

We now give an example that indicates that Question 2.1 may be rather difficult.

Example 3.29. If $G = C_1 \oplus C_2$ where C_1 (respectively C_2) is a non transitive, fully transitive 2-group (is a transitive, non fully transitive 2-group) as constructed by Corner in [27], then G is not fully transitive and it is Krylov transitive if, and only if, it is transitive.

Proof. That G is not fully transitive is immediate since the summand C_2 is not fully transitive. One implication is trivial. Note then that the group $2^\omega C_1$ is elementary while $2^\omega C_2 \cong \mathbb{Z}(2) \oplus \mathbb{Z}(8)$ and so $\mathrm{supp}(2^\omega C_1) \subseteq \mathrm{supp}(2^\omega C_2)$. If G is Krylov transitive, then since C_2 is transitive, it would follow from Proposition 3.28 that G is transitive. □

We remark that it can be shown directly that the group G above is not transitive.

We close this section with a generalization of a problem due to Corner [27].

Question 2.3. Are Krylov transitive groups with finite first Ulm subgroup weakly transitive?

Notice that it follows from [51, Corollary 3.11] this is true for groups of type A (even without the assumption of Krylov transitivity); by a simple modification of the argument in [51, Corollary 3.13], one can show that the converse does not hold.

We now arrive at some critical properties of socle-regularity and strong socle-regularity. In [D8] and [35] the notions of socle-regularity and strong socle-regularity were introduced; the question of whether or not a summand of a socle-regular group is again socle-regular, was left unanswered in [D8]. We can now answer this in the affirmative. Notice that the same problem was settled in the negative for strongly socle-regular groups in [35]. Recall the definitions: a group G is said to be *socle-regular* (*strongly socle-regular*) if for all fully invariant (characteristic) subgroups F of G, there exists an ordinal α (depending on F) such that $F[p] = (p^\alpha G)[p]$. It is self-evident that strongly socle-regular groups are themselves socle-regular, whereas the converse is not valid (see [35]).

Proposition 3.30. *A summand of a socle-regular group is again socle-regular.*

Proof. Let $G = A \oplus B$ be a socle-regular group; we show A is also socle-regular.

Let F be an arbitrary fully invariant subgroup of A and set $C = \langle x\gamma : x \in F[p],\ \gamma \in \mathrm{Hom}(A, B)\rangle$. Note that C is an elementary group. We claim that (i) $C\delta \leq F[p]$ for all $\delta : B \to A$ and (ii) $C\beta \leq C$ for all $\beta \in \mathrm{End}(B)$.

Assuming for the moment that we have established these claims, consider the subgroup $F[p] \oplus C$ of G. If $\Delta = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ is an arbitrary endomorphism of G (with the usual conventions), then $(F[p] \oplus C)\Delta \leq (F[p]\alpha + C\delta, F[p]\gamma + C\beta)$. Clearly $F[p]\alpha \leq F[p]$ by the full invariance of F in A and $F[p]\gamma \leq C$ by definition, so that the claims above yield $(F[p] \oplus C)\Delta \leq F[p] \oplus C$, i.e., $F[p] \oplus C$ is fully invariant in G. Now G is socle-regular, so there is an ordinal τ such that $F[p] \oplus C = (p^\tau G)[p] = (p^\tau A)[p] \oplus (p^\tau B)[p]$. It follows that $F[p] = (p^\tau A)[p]$ and since F was an arbitrary fully invariant subgroup of A, we have that A is socle-regular, as required.

To establish the first claim, note that if $c \in C$, then $c = \sum x_i\gamma_i$ for some $x_i \in F[p]$, $\gamma_i : A \to B$. But then $c\delta = \sum(x_i\gamma_i)\delta = \sum x_i(\gamma_i\delta)$ and $\gamma_i\delta \in \mathrm{End}(A)$. Thus $x_i(\gamma_i\delta) \in F[p]$ since the latter is fully invariant in A.

For the second claim, it suffices to note, using the same notation as above, that $\gamma_i\beta \in \mathrm{Hom}(A, B)$ so that $c\beta = \sum x_i(\gamma_i\beta) \in C$ by definition. □

As noted above, it was shown in [35] that summands of strongly-socle-regular groups need not be strongly socle-regular. However, we do have the following elementary classification showing that socle-regular groups are precisely the summands of strongly socle-regular groups:

Corollary 3.31. *The following are equivalent for a group G:*
(i) G is a summand of a strongly socle-regular group;
(ii) G is a socle-regular group;
(iii) the square $G \oplus G$ is strongly socle-regular;
(iv) the square $G \oplus G$ is socle-regular.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Proposition 3.30, while $(ii) \Rightarrow (iii)$ was established in [35, Theorem 3.6]. The final implication $(iii) \Rightarrow (i)$ is immediate. The equivalence $(ii) \iff (iv)$ was obtained in Theorem 1.4 of [D8]. □

The following extends [35, Proposition 3.3 (i)].

Corollary 3.32. *Any summand A of a strongly socle-regular group is strongly socle-regular if* $\mathrm{End}(A)$ *is additively generated by* $\mathrm{Aut}(A)$.

Proof. Suppose $G = A \oplus B$ is a strongly socle-regular group. Applying Proposition 3.30, we deduce that A is socle-regular. Since $\mathrm{Aut}(A)$ generates $\mathrm{End}(A)$, in view of [35, Proposition 2.5] every characteristic subgroup C of A is fully invariant in A and hence has the required form $C[p] = (p^\alpha A)[p]$. □

Under certain circumstances, socle-regularity and strong socle-regularity do coincide.

Proposition 3.33. *Suppose that $G = A \oplus B$ with $p^n A \cong p^n B$, for some non-negative n. Then G is socle-regular if, and only if, G is strongly socle-regular.*

Proof. One implication is clear and does not depend on the additional hypothesis on G. Conversely suppose that G is socle-regular; note that it is immediate that $p^n G$ is also socle-regular. Then, $p^n G$ is isomorphic to the square of a fixed group, $p^n A$, and consequently its endomorphism ring is additively generated by its automorphism group. Furthermore, it follows from Corollary 3.31 above that $p^n G$ is strongly socle-regular. That G itself is strongly socle regular follows from [35, Proposition 2.6 (iii)]. □

It was shown in Theorem 0.3 of [D8] that fully transitive groups were socle-regular and in [35, Theorem 2.4] that transitive groups were socle-regular (indeed they are even strongly socle-regular). It is, perhaps, not surprising then that Krylov transitive groups share the same property.

Proposition 3.34. *If the group G is Krylov transitive, then G is socle-regular.*

Proof. Let F be a fully invariant subgroup of G and let $\alpha_0 = \min(F[p])$, so that $h(y) \geq \alpha_0$ for all $y \in F[p]$. Clearly $F[p] \leq (p^{\alpha_0}G)[p]$.

Conversely, suppose that $x \in (p^{\alpha_0}G)[p]$, so that $U_G(x) = (\alpha, \infty, \dots)$ for some $\alpha \geq \alpha_0$. Choose a fixed, but arbitrary, $z \in F[p]$ such that $h(z) = \alpha_0$. If $\alpha = \alpha_0$, then $U_G(x) = U_G(z)$ and so, by Krylov transitivity, there is an endomorphism ϕ of G with $z\phi = x$. Hence $x \in (F[p])\phi \leq F[p]$, since F is fully invariant in G. If $\alpha > \alpha_0$, then $h(x + z) = \alpha_0$ and so $U_G(x + z) = (\alpha_0, \infty, \dots) = U_G(z)$. Again, by Krylov transitivity, there is an endomorphism ψ with $z\psi = x + z$. But then $\psi - 1_G$ is an endomorphism of G and $z(\psi - 1_G) = (x+z) - z = x$, forcing $x \in F[p]$ since the latter is fully invariant in G. Thus in either case $(p^{\alpha_0}G)[p] \leq F[p]$, as required. □

Remark 3.1. There is, however, no possibility of extending the above proposition to strong socle-regularity: in [35, Example 3.5] a fully transitive, and hence Krylov transitive, group is exhibited which is not strongly socle-regular. Moreover, the above proposition cannot be reversed. In fact, there even exists a strongly socle-regular group which is not Krylov transitive. Indeed, the group C, discussed in the second part of the proof of Theorem 3.25, has been shown in Example of [D8] to be socle-regular. Hence its square $C \oplus C$ is also socle-regular, whence by Corollary 3.31 (iii) it is strongly socle-regular, but is not Krylov transitive.

The class of weakly transitive groups is not, however, contained in the class of socle-regular groups:

Proposition 3.35. *There exists a weakly transitive group X which is not socle-regular.*

Proof. Let T be a separable p-group such that $\mathrm{End}(T) = J_p \oplus E_s(T)$ where J_p is the ring of p-adic integers and $E_s(T)$ is the ideal consisting of all small endomorphisms of T; such groups are easy to find, the first example being due to Pierce [100]. Let B be a basic subgroup of T, so that T/B is divisible of rank $\lambda > 1$ say. Now construct a p-group X such that $X/p^{\omega}X \cong T$ and $p^{\omega}X$ is elementary of rank λ – for instance, use the pullback construction of Lemma 44.1 in [104].

Then $\mathrm{End}(X/p^\omega X) = J_p \oplus E_s(X/p^\omega X)$ and hence $\mathrm{End}(X) = J_p \oplus E_\theta(X)$, where $E_\theta(X)$ is the ideal of thin endomorphisms of X – see [27]. In this situation every thin endomorphism is actually small – see [27, Lemma 7.2]. Note that if φ is small, $(p^\omega X)\varphi = 0$.

We claim X is weakly transitive (by Proposition 3.13 it is enough to check this on $p^\omega X$): if $x, y \in p^\omega X$ with $x\phi = y$ and $y\psi = x$ for endomorphisms ϕ, ψ, then $\phi = r + \varphi_1$ and $\psi = s + \varphi_2$, where each φ_i is small. This forces r, s to be mutually inverse p-adic integers with $xr = y,\ ys = x$, so X is certainly weakly transitive.

Finally we claim that X is not socle-regular. Consider any proper subgroup F of $p^\omega X$. Since $p^{\omega+1}X = 0$, $F = F[p] \neq (p^\alpha X)[p]$ for any α. However F is fully invariant since endomorphisms of X act on $p^\omega X$ as scalar multiplications. Thus X is not socle-regular, as claimed. □

In virtue of Proposition 3.34 and [51] there is an abundance of socle-regular groups that are not weakly transitive. However, it would be interesting to know whether or not there exists a strongly socle-regular group which is not weakly transitive.

In light of Proposition 3.24, the following is not too surprising.

Proposition 3.36. *If $G = K \oplus H$ is strongly socle-regular, where K is of type A and each homomorphism between K and H is almost small (in particular, either H is separable or $Hom(K, H) = Small(K, H)$), then K is strongly socle-regular.*

Proof. Suppose that C is an arbitrary characteristic subgroup of K. If $C[p] \not\subseteq p^\omega K$, then applying Proposition 1.1 (ii) from [35] we get that $C[p] = (p^t K)[p]$ for some natural t, and we are done. So, we may assume that $C[p] \leq p^\omega K$. Assume that we have shown that $C[p] \oplus \{0\}$ is characteristic in G. Then, by strong socle-regularity, we will have $C[p] \oplus \{0\} = (p^\delta G)[p] = (p^\delta K)[p] \oplus (p^\delta H)[p]$ for some $\delta \geq \emptyset$, insuring that $C[p] = (p^\delta K)[p]$ as required.

It suffices then to show that $C[p] \oplus \{0\}$ is characteristic in G. If $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ is an arbitrary automorphism of G, then $C[p]\gamma = 0$ since γ is, by assumption, almost small. Thus $(C[p] \oplus \{0\})\Phi = (C[p]\alpha \oplus \{0\})$. It follows from Lemma 3.23 that there is an automorphism ϕ of K with $\phi \restriction p^\omega K = \alpha \restriction p^\omega K$ and this clearly yields the desired result since $C[p]$ is a characteristic subgroup of K. □

It was shown in [D8] (resp., in [35]) that socle-regularity (resp., strong socle-regularity) of a group G is inherited by the subgroups $p^\alpha G$ for all α. It is clear that the converse cannot hold in general, but it was shown in [35, Proposition 2.6] that strong socle-regularity "lifts" from a subgroup $p^\alpha G$ to G provided that $G/p^\alpha G$ is totally projective and $\alpha < \omega^2$; we do not know if the ordinal restriction is necessary. It is, of course, possible to modify the argument in [35, Proposition 2.6] to show directly that an analogous result holds for socle-regularity. However with our classification of socle-regularity in terms of strong socle-regularity, we can easily deduce the result. Our first observation has no ordinal restriction.

Proposition 3.37. *Let G be a group such that the quotient $G/p^\alpha G$ is totally projective for some ordinal α. Then the implication (b)$\Rightarrow$(a) holds, where*

(a) G is socle-regular if $p^\alpha G$ is socle-regular;

(b) G is strongly socle-regular if $p^\alpha G$ is strongly socle-regular.

Proof. Suppose $p^\alpha G$ is socle-regular. Then, in view of [35, Theorem 3.6], $p^\alpha(G \oplus G) = p^\alpha G \oplus p^\alpha G$ is strongly socle-regular. Observe that $(G \oplus G)/p^\alpha(G \oplus G) = (G \oplus G)/(p^\alpha G \oplus p^\alpha G) \cong (G/p^\alpha G) \oplus (G/p^\alpha G)$ is totally projective, whence by hypothesis $G \oplus G$ is strongly socle-regular. It follows again from [35, Theorem 3.6] that G is socle-regular, as asserted. □

As an immediate consequence of Proposition 3.37 and Proposition 2.6 (v) from [35], we have the following strengthening of Theorem 1.7 from [D8].

Corollary 3.38. *Suppose G is a group such that the factor-group $G/p^\alpha G$ is totally projective for some ordinal $\alpha < \omega^2$. Then G is socle-regular if, and only if, $p^\alpha G$ is socle-regular.*

Next, we show that the converse of Proposition 1.8 from [D8] does not hold.

Example 3.1. There are socle-regular groups A and B, with each homomorphism between them being small, such that $A \oplus B$ is not socle-regular.

Proof. Let A, B be 2-groups with $2^\omega A \cong 2^\omega B \cong \mathbb{Z}(2) \oplus \mathbb{Z}(8)$ as in Corner's construction [27] of transitive but not fully transitive groups. It is easy to arrange that the groups A, B have the additional property that $\mathrm{Hom}(A, B) = \mathrm{Hom}_s(A, B)$ and $\mathrm{Hom}(B, A) = \mathrm{Hom}_s(B, A)$. Now consider the group $G = A \oplus B$ and its subgroup $H = (2^\omega A)[2] \oplus (2^{\omega+1} B)[2]$. The latter is fully invariant in G because any endomorphism of G acts diagonally on H since the cross homomorphisms, being small, act trivially on the components of H. However, an easy check shows that H cannot be of the form $(2^\alpha G)[2]$ for any α. □

3.3. **On projectively fully transitive Abelian p-groups.** In 1952 Kaplansky, [70], began his investigations into the fully invariant and characteristic subgroups of an Abelian p-group. He followed this up in his now famous "*little red book*", Infinite Abelian Groups, [71], and introduced the notions of transitive and fully transitive p-groups in a natural way arising from his investigations in [70]; these notions have been of interest in Abelian group theory ever since. There is another notion, closely related to full invariance, which has also been studied: projection invariance. Recall that a subgroup H of the group G is said to be *projection-invariant in* G if $\pi(H) \leq H$ for all idempotent endomorphisms π of G. Significant work on this topic was produced by Hausen [57] and Megibben [88], concentrating in the main in establishing when projection-invariant subgroups are actually fully invariant; the socles of such subgroups have been investigated by the present authors in [36]. In this work we follow a somewhat different path and explore a new notion of transitivity which we shall call projective full transitivity. Recall that a group G is said to be fully transitive if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there is an endomorphism ϕ of G with $\phi(x) = y$. Our modification is to say that G is *projectively fully transitive* if the endomorphism ϕ can be chosen to be in the subring of the full endomorphism ring generated by the idempotent endomorphisms; clearly a projectively fully transitive group is always fully transitive.

We shall establish a number of basic properties of projectively fully transitive groups; in particular we shall show that this class of groups is properly contained in the class of fully transitive groups. Moreover, the class is large but is not closed under the taking of direct summands, unlike the situation which pertains for fully transitive groups. Recent work on various types of transitivity - see, for example, [D9] - has revealed the role played by 'squares' of a group in this connection and similar properties re-appear here (compare also with Chapter III concerning some ring-theoretic results which might be translated for the endomorphism ring of abelian groups of the mentioned above sorts).

To simplify the notation and to avoid risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism **ring** of G and $\mathrm{End}(G)$ for the endomorphism **group** of G. We shall denote by $\mathrm{Proj}(G)$ the **subring** of $\mathrm{E}(G)$ generated by the idempotents of $\mathrm{E}(G)$; thus an element $\phi \in \mathrm{Proj}(G)$ will have the form $\phi = \sum_{\text{finite}} \pm\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$, where each π_{i_j} is an idempotent in $\mathrm{E}(G)$.

In the final part of the subsection, we shall examine briefly an apparently stronger notion. Following Hausen, [57], we let $\Pi(G)$ denote the **subgroup** of the endomorphism group $\mathrm{End}(G)$ generated by the idempotent endomorphisms; so $\phi \in \Pi(G)$ has the form $\phi = \sum_{i=1}^{n} \pm\pi_i$ for some finite n, where each π_i is an idempotent endomorphism. Then a group G is said to be *strongly projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there exists $\phi \in \Pi(G)$ with $\phi(x) = y$; clearly a strongly projectively fully transitive group is projectively fully transitive. Our results here are somewhat sketchier.

Throughout, the word group will denote an additively written Abelian p-group. In this context our notation is standard and follows Fuchs [44, 47] and Kaplansky [71, 72]; mappings are written on the left.

Since it is clear that a fully transitive group G is projectively fully transitive if $\mathrm{E}(G) = \mathrm{Proj}(G)$ (and similarly it is strongly projectively fully transitive if $\mathrm{End}(G) = \Pi(G)$), we consider firstly this situation. To simplify our terminology we shall say that a group G is an *idempotent-generated* group (or IG-group) if $\mathrm{E}(G) = \mathrm{Proj}(G)$; we say that G is an *idempotent-sum* group (or IS-group) if $\mathrm{End}(G) = \Pi(G)$. If $\mathrm{E}(G)$ is commutative, then it is obvious that $\mathrm{Proj}(G) = \Pi(G)$ so that the IG-groups are then precisely the IS-groups; in general an IS-group is always an IG-group. However, this situation is rather rare for a primary group: it follows from results of Szele and Szendrei - see Exercise 6, p. 227 in [44] - that groups with commutative endomorphism ring are precisely subgroups of $\mathbb{Z}(p^\infty)$ and it is easy to see that any cyclic group is an IS-group, while the quasi-cyclic group $\mathbb{Z}(p^\infty)$ is not even an IG-group.

We begin with an elementary but useful observation:

Proposition 3.39. *If $G = A_1 \oplus \cdots \oplus A_n$ where the A_i are IG (resp. IS)-groups, then G is an IG (resp. IS)-group. In particular,*

(i) if A is an IG (resp. IS)-group, then so also is $A^{(n)}$ for each finite n;

(ii) if F is a finite group, then it is an IS-group.

Proof. By induction it suffices to show the result for the direct sum of two groups, so suppose that $G = A \oplus B$ where each A, B is an IG (resp. IS)-group. If $\phi \in \mathrm{E}(G)$, then we can write ϕ in the form $\phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$, where $\alpha \in \mathrm{E}(A)$, $\beta \in \mathrm{E}(B)$, $\gamma \in \mathrm{Hom}(B, A)$ and $\delta \in \mathrm{Hom}(A, B)$. But then we have

$$\phi = \left(\begin{smallmatrix} \alpha - 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & \beta - 1_B \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1_A & \gamma \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1_B \end{smallmatrix}\right).$$

The latter two matrices represent idempotent endomorphisms of G. Since $\alpha - 1_A \in \mathrm{E}(A), \beta - 1_B \in \mathrm{E}(B)$ and A, B are IG (resp. IS)-groups, the endomorphisms $\alpha - 1_A, \beta - 1_B$ may be expressed as sums of products of idempotents (resp. sums of idempotents) and hence the matrices

$$\left(\begin{smallmatrix} \alpha - 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & \beta - 1_B \end{smallmatrix}\right)$$

have the same properties since the embeddings of $\mathrm{E}(A), \mathrm{E}(B)$ into $\mathrm{E}(G)$ are ring homomorphisms. The particular cases are immediate. □

Corollary 3.40. *If $G = A \oplus B$, where A is a fully invariant subgroup of G, then G is an IG (resp. IS)-group if, and only if, both A, B are IG (resp. IS)-groups.*

In particular, if $G = D \oplus R$, where D is divisible and R is reduced, then G is an IG (resp. IS)-group if, and only if, both D, R are IG (resp. IS)-groups.

Proof. If A, B are IG (resp. IS)-groups, then it follows immediately from Proposition 3.39 that G is an IG (resp. IS)-group; the full invariance of A is not needed here.

Conversely suppose that G ia an IG (resp. IS)-group and that A is fully invariant in G. Let χ denote the restriction map $\chi : \mathrm{E}(G) \to \mathrm{E}(A)$ with $\phi \mapsto \phi \restriction A$ for each $\phi \in \mathrm{E}(G)$; the full invariance of A ensures that χ is a ring homomorphism $\mathrm{E}(G) \twoheadrightarrow \mathrm{E}(A)$. Consequently, $\chi(\mathrm{Proj}(G)) \leq \mathrm{Proj}(A)$, $\chi(\Pi(G)) \leq \Pi(A)$ and hence $\mathrm{E}(A) = \chi(\mathrm{E}(G)) \leq \mathrm{Proj}(A) \leq \mathrm{E}(A)$ if G is an IG-group. Similarly, $\mathrm{E}(A) = \Pi(A)$ if G is an IS-group. Thus if G is an IG (resp. IS)-group, then so also is A. □

It follows immediately from Corollary 3.40 that the study of IG (resp. IS)-groups may be reduced to the separate study of divisible and reduced IG (resp. IS)-groups.

In fact we can say a great deal more about IG-groups, thus generalizing Proposition 3.39 (i).

Proposition 3.41. *If A is an arbitrary group and $\kappa \geq 2$ is any cardinal, then $G = A^{(\kappa)}$ is an IG-group.*

Proof. If κ is finite then the result is immediate from Proposition 2.1 in [36]. If κ is infinite, then write $G = X \oplus X$, where $X \cong A^{(\kappa)}$, so that $\mathrm{E}(G)$ is isomorphic to the ring of 2×2 matrices over $S = \mathrm{E}(X)$. However, it follows again from Proposition 2.1 in [36] that $\mathrm{E}(G) = \mathrm{Proj}(G)$, so that G is an IG-group. □

Notice that it follows from Proposition 3.41 that a summand of an IG-group need not be an IG-group: in fact, if G is any group which is not an IG-group (for example, $\mathbb{Z}(p^\infty)$), then its square is an IG-group.

We also have the simple consequence:

Corollary 3.42. *A divisible group D is an IG-group if, and only if,* $\mathrm{rk}(D) \geq 2$.

Proof. The sufficiency follows immediately from Proposition 3.41. However, if $\mathrm{rk}(D) = 1$, then $\mathrm{E}(D) \cong J_p$, the ring of p-adic integers which has only 0 and 1 as idempotents and consequently the endomorphism ring of D is not generated by idempotents. □

Note also that Proposition 3.41 does *not* generalize to IS-groups as the next example shows.

Example 3.43. The group $G = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ is not an IS-group.

Proof. The endomorphism ring of G is, of course, isomorphic to the ring of 2×2 matrices over J_p. We claim that the trace of an idempotent 2×2 p-adic matrix is one of $\{0, 1, 2\}$. To see this suppose that $\Delta = \left(\begin{smallmatrix} x & a \\ b & y \end{smallmatrix}\right)$ is an idempotent p-adic matrix. Direct calculation gives

(i) $xa + ay = a$ and $bx + yb = b$, so that $a(x + y - 1) = 0 = b(x + y - 1)$.

(ii) $x^2 + ab = x$ and $y^2 + ab = y$, so that $(x - y)(x + y - 1) = 0$.

Since J_p is a domain, we have that either $x + y - 1 = 0$ or $x = y$. In the first case, the trace of Δ is 1, so we may restrict our attention to the case where $x + y - 1 \neq 0$ and $x = y$. From the observation in (i) above, we conclude that $a = 0 = b$ and this in turn forces $x^2 = x, y^2 = y$. Thus x, y take the values 0 or 1 and hence the trace of Δ is either 0, 1 or 2, as required. In particular, we see that the trace of any idempotent matrix must be a non-zero integer in this situation.

Suppose now, for a contradiction, that $\mathrm{End}(G) = \Pi(G)$. Choose a p-adic integer u which is not a rational integer and consider the matrix $\phi = \left(\begin{smallmatrix} u & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By assumption then ϕ is a linear combination of idempotent matrices and hence the trace of ϕ, u, is a finite sum of terms from the set $\{0, 1, 2\}$; in particular it is a rational integer – contradiction. □

Returning to the reduced situation, it seems rather difficult to give a satisfactory description of the class of IG (resp. IS)-groups. Since a bounded group can be expressed in the form $G = A \oplus F$ where F is finite and each homocyclic component of A is of rank at least 2, it follows easily from Proposition 3.39 and Proposition 3.41 that a bounded group is an IG-group. In fact Hausen has shown, [57, Corollary 6], that every bounded group is even an IS-group. On the other hand, it is relatively easy to exhibit reduced groups which are not IG-groups. A simple, but useful, observation here is that if the ring $\mathrm{E}(G)$ is generated as a ring (resp. additively) by idempotents, then the same is true of $\mathrm{E}(G)/I$ for any two-sided ideal I. This, combined with Corner's realization theorems [27, 28, 29], gives the following:

Proposition 3.44. *If A is a ring whose additive group is the completion of a free p-adic module of countable rank and A is not generated as a ring (resp. additively) by its idempotents, then there exists an unbounded separable group G_A which is not an IG (resp. IS)-group.*

Proof. From Theorem 1.1 in [27], we conclude that there is an unbounded separable group G_A with $\mathrm{E}(G_A) = A \oplus E_s(G_A)$, where $E_s(G_A)$ is the ideal of small endomorphisms of G_A. Since $\mathrm{E}(G_A)/E_s(G_A) \cong A$ is not generated as a ring (resp. additively) by its idempotents, we conclude that G_A is not an IG (resp. IS)-group. $\square$

Rings of the type required for Proposition 3.44 are easy to construct: for example the ring A which is the completion (in the p-adic topology) of the polynomial ring $J_p[X]$ has the property and so also does the ring direct product $A = J_p \times \cdots \times J_p = J_p^{(n)}$ for a finite n. In fact, if A is a ring whose additive group is the completion of a free p-adic module of countable rank and A is commutative, then A is not generated as a ring by its idempotents: if it were, it would follow from Bergman's lemma [44, Lemma 97.2] that the additive group of A would be free - a contradiction.

Corollary 3.45. *If G is an unbounded essentially indecomposable group, then G is not an IG-group.*

Proof. If G is an unbounded essentially indecomposable p-group, then it follows from a result of Monk [91, Corollary to Theorem 1] that the only idempotents in $\mathrm{E}(G)/E_s(G)$ are 0 and 1 but these cannot generate this quotient as a ring since the quotient must always contain a copy of the center of $\mathrm{E}(G)$ which is isomorphic to J_p as G is unbounded. $\square$

It is also easy to construct non-separable groups which are not IG (resp. IS)-groups; to do this we make use of Corner's second realization theorem [28, Theorem 10.2]. Hence, we have:

Proposition 3.46. *If A is a reduced separable group with basic subgroups of rank ≥ 2 and of cardinal $< 2^{\aleph_0}$, then for any infinite ordinal $\alpha < \omega^2$, there is a group G with $p^\alpha G = A$ and G is not an IG-group.*

Proof. Using Corner's theorem we construct a group G with $p^\alpha G = A$ and $\mathrm{E}(G)_A = \{\phi \restriction A : \phi \in \mathrm{E}(G)\} = \Phi$, where Φ is any complete separable p-adic subalgebra of $\mathrm{E}(A)$. Consider firstly the case where A is unbounded. Then the choice $\Phi = J_p$ is possible and so we have $\mathrm{E}(G))_A = J_p$. Since J_p is not generated as a ring by its idempotents, the ring $\mathrm{E}(G)$ cannot be generated by its idempotents since $\mathrm{E}(G)_A$ is a ring homomorphic image of $\mathrm{E}(G)$.

Now suppose that A is bounded and write $A = B \oplus C$, where $B = \mathbb{Z}(p^{n_1}) \oplus \mathbb{Z}(p^{n_2})$ and $n_1 \leq n_2$; this is possible since by assumption the rank of a basic subgroup of A is at least 2. Let Φ be the set of matrices of the form $\left(\begin{smallmatrix} \phi & 0 \\ 0 & \psi \end{smallmatrix}\right)$, where $\phi \in \Phi_1, \psi = r1_C$ $(0 \leq r < p^{e(C)})$ and Φ_1 is a subring of $\mathrm{E}(B)$, which we will define. Since $\mathrm{E}(B)$ is finite, Φ will be complete and separable, so we may apply Corner's theorem to find a group G with $p^\alpha G = A$ and $\mathrm{E}(G)_A = \Phi$. Now choose

Φ_1 to be the subring of $\mathrm{E}(B)$ consisting of lower triangular matrices of the form $\left(\begin{smallmatrix} r & 0 \\ s & r \end{smallmatrix}\right)$, where $0 \leq r < p^{n_2}$ and $s \in \mathrm{Hom}(\mathbb{Z}(p^{n_1}), \mathbb{Z}(p^{n_2}))$.

We claim that Φ is not generated as a ring by its idempotents - notice that this suffices to show that G is not an IG-group. For if it were, then the same would be true of any ring homomorphic image of Φ, in particular Φ_1 would be generated as a ring by its idempotents. However by direct calculation we can see that any idempotent matrix in Φ_1 must have entries satisfying $r^2 = r, 2rs = s$. Hence $r = 0$ or 1, which in turn implies that $s = 0$. Hence, the only idempotents in Φ_1 are $0, 1$ and these clearly do not generate all of Φ_1. This completes the proof. □

We have already seen that a summand of an IG-group need not be an IG-group, however we do have:

Proposition 3.47. *If G is an IG (resp. IS)-group , then so also is p^nG for any finite n.*

Proof. The mapping $\chi : \mathrm{E}(G) \to \mathrm{E}(p^nG)$ given by $\chi(\phi) = \phi \restriction p^nG$ is a ring homomorphism. However, if $\theta \in \mathrm{E}(p^nG)$ then it follows easily from Proposition 113.3 in [44] that there is an endomorphism $\phi \in \mathrm{E}(G)$ with $\phi \restriction p^nG = \theta$. Hence the mapping χ is onto and $\mathrm{E}(p^nG)$ is a ring epimorphic image of $\mathrm{E}(G)$. The result follows now immediately. □

This assertion may be extended to subgroups $p^\alpha G$ provided the quotient $G/p^\alpha G$ is totally projective: the proof is essentially identical but the ontoness property of the mapping χ now comes from Hill's result on totally projective groups [61]. So, we have:

Proposition 3.48. *If G is an IG (resp. IS)-group and the quotient $G/p^\alpha G$ is totally projective, then $p^\alpha G$ is an IG (resp. IS)-group.*

We close all the work done above with the following question:

Problem 1. If A is an IS-group and $\kappa \geq 1$ is any cardinal, does it follow that $A^{(\kappa)}$ is also an IS-group?

We will now carefully consider the class of projectively fully transitive groups as follows:

In the classical theory of transitive and fully transitive groups, it is usual to restrict consideration to reduced groups. However, it is not difficult to extend the theory to non-reduced groups. This is normally achieved by modifying the definition of an Ulm sequence for an element of a divisible group - see [71, p.57] - so that if D is divisible and $x \in D$, then $U_D(x) = (0, \dots 0, \infty, \dots)$ where the symbol ∞ occurs at precisely the $(n+1)^{st}$ place if x has order p^n; with this understanding it is easy to show that divisible groups are fully transitive – see, for example, [71, Exercise 71] or [15, Proposition 2.1]. In fact, we can show even that any divisible group is necessarily a projectively fully transitive group. Recall once again from the introduction that a group G is said to be *projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there exists $\phi \in \mathrm{Proj}(G)$ with $\phi(x) = y$; clearly a projectively fully transitive group is fully transitive.

Theorem 3.49. *If D is a divisible group, then D is a projectively fully transitive group.*

Proof. Since a divisible group is fully transitive and any divisible group of rank ≥ 2 is an IG-group (by Proposition 3.41), we deduce immediately that the result is true provided $\mathrm{rk}(D) \geq 2$. Suppose then that D is divisible with $\mathrm{rk}(D) = 1$. Note that, in this situation, we have for $x, y \in D$ that $o(x) \geq o(y)$ if, and only if, $U_D(x) \leq U_D(y)$.

Clearly, if $o(x) = p^n \geq o(y) = p^m$ we can write $x = ua, y = p^{n-m}va$ where a is the generator of the $\mathbb{Z}(p^\infty)[p^n]$ and $(u, p) = 1 = (v, p)$. Let $\lambda p^n + \mu u = 1$ so that $a = \mu ua$ and then observe that the mapping $\phi = \mu v p^{n-m} : x \mapsto v p^{n-m} \mu ua = y$. Since ϕ is an integer multiple of the identity map, it certainly belongs to $\mathrm{Proj}(\mathbb{Z}(p^\infty))$. $\square$

Recall [43, Definition 1] that the groups G_1, G_2 form a *fully transitive pair* if, for every $x \in G_i, y \in G_j (i, j \in \{1, 2\})$ with $U_G(x) \leq U_G(y)$, there exists $\alpha \in \mathrm{Hom}(G_i, G_j)$ with $\alpha(x) = y$. Note that $\{G_1, G_2\}$ is a fully transitive pair if, and only if, if $G_1 \oplus G_2$ is fully transitive – see [43, Proposition 1].

Proposition 3.50. *If A, B are projectively fully transitive groups and $\{A, B\}$ is a fully transitive pair, then $A \oplus B$ is a projectively fully transitive group.*

Proof. Let $G = A \oplus B$ and suppose that $x, y \in G$ with $U_G(x) \leq U_G(y)$. We proceed by induction on the order of y. In fact, we claim that if there is always an endomorphism $\phi \in \mathrm{Proj}(G)$ mapping x to y when $o(y) = p$, then the proposition follows: for suppose we have shown the result for $o(y) = p^n$ and consider the situation where $o(y) = p^{n+1}$. Then, arguing exactly as in [51, Lemma 2.2], $U_G(px) \leq U_G(py)$ and $o(py) \leq p^n$. So there is a $\theta \in \mathrm{Proj}(G)$ with $\theta(px) = py$. Set $y' = y - \theta(x)$ so that $y' \in G[p]$ and clearly $U_G(x) \leq U_G(y')$. Hence there is $\varphi \in \mathrm{Proj}(G)$ with $\varphi(x) = y'$. But then $\theta + \varphi \in \mathrm{Proj}(G)$ and $(\theta + \varphi)(x) = \theta(x) + (y - \theta(x)) = y$. So the claim is established and it remains only to verify the result when $o(y) = p$.

Let $x = (a, b), y = (a_1, b_1) \in A \oplus B$ with $py = 0$. By re-labelling if necessary, we may assume $ht_G(x) = ht_A(a)$. Now we have $U_A(a) \leq U_G(y) \leq U_B(b_1)$ In particular, since $\{A, B\}$ is a fully transitive pair, there is a homomorphism $\gamma : A \to B$ with $\gamma(a) = b_1$. However, $U_A(a) \leq U_A((a_1 - a)$ since $p(a_1 - a) = -pa$. Since by hypothesis A is a projectively fully transitive group, there is an endomorphism $\varphi \in \mathrm{Proj}(A)$ with $\varphi(a) = a_1 - a$. Now if $\Delta = \left(\begin{smallmatrix} \varphi & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Gamma = \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 0 \end{smallmatrix}\right)$, then Δ and Γ are endomorphisms of G and $\Delta + \Gamma$ maps (a, b) to (a_1, b_1) as required. It remains to show that $\Delta, \Gamma \in \mathrm{Proj}(G)$. Since the embedding of $\mathrm{E}(A)$ into $\mathrm{E}(G)$ is a ring homomorphism, it is clear that $\Delta \in \mathrm{Proj}(G)$, but a direct calculation shows that Γ is idempotent also and so we have the result. $\square$

We note for later use that the proof of Proposition 3.50 carries over to the situation where the groups A, B are strongly projectively fully transitive groups.

We have a partial converse in the situation where one of the groups is divisible.

Proposition 3.51. *Suppose that D is a divisible group and R is reduced, then if $G = D \oplus R$ is projectively fully transitive, then so also is R.*

Proof. Suppose that $x, y \in R$ and that $U_R(x) \leq U_R(y)$. Then $U_G((0, x)) \leq U_G((0, y))$ and so there is a $\varphi \in \mathrm{Proj}(G)$ with $\varphi(0, x) = (0, y)$, say $\varphi = \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$ is the matrix representation. Now an idempotent matrix in $\mathrm{E}(G)$ necessarily has diagonal entries which are idempotents in $\mathrm{E}(D)$ and $\mathrm{E}(R)$ respectively. Consequently any product of idempotent matrices must also have idempotent diagonal entries. Since φ can be expressed as a sum (or difference) of such products, it follows that its diagonal entry β has the same property, i.e. $\beta \in \mathrm{Proj}(R)$. Since $\beta(x) = y$, R is projectively fully transitive. $\square$

Summarizing these results we have:

Theorem 3.52. *If $G = D \oplus R$, where D is divisible and R is reduced, then G is projectively fully transitive if, and only if, R is projectively fully transitive.*

Proof. The necessity follows from Proposition 3.51 and the sufficiency follows from Proposition 3.50 since R, being projectively fully transitive, is certainly fully transitive and then the divisibility of D ensures that $\{D, R\}$ is a fully transitive pair. $\square$

It follows immediately that we may restrict our attention to reduced groups when we are considering projective full transitivity. Hence for the remainder of this section we shall assume that all groups discussed are *reduced*.

Until this point we have not given an example of a fully transitive group which is not projectively fully transitive. We remedy this in the next result:

Proposition 3.53. *The class of projectively fully transitive groups is strictly contained in the class of fully transitive groups.*

Proof. In fact we will exhibit three examples of groups with this phenomenon; the first having an infinite elementary first Ulm subgroup, the second a finite elementary first Ulm subgroup and the third a finite non-elementary first Ulm subgroup.

(i) Suppose that G is a group in which $p^\omega G$ is elementary of infinite rank and G is fully transitive but not transitive – such groups exist, for example, the groups constructed by Corner in Section 3 of [29]. Then, as shown in Lemma 1.6 of [36], every subgroup of $p^\omega G$ is a projection-invariant subgroup of G. We claim that G is not projectively fully transitive: choose basis elements a, b of $p^\omega G$ and note that $U_G(a) = (\omega, \infty, \dots) = U_G(b)$. However, if $\phi \in \mathrm{Proj}(G)$, then $\phi(\langle a\rangle) \leq \langle a\rangle$ since $\langle a\rangle$ is projection invariant in G; in particular there cannot be a $\phi \in \mathrm{Proj}(G)$ with $\phi(a) = b$, so that G is not projectively fully transitive, as claimed.

Our construction of the second and third examples is based on Corner's Theorem 6.1 in [28] and we make use of Lemma 3.60 below.

(ii) Let $H = \mathbb{Z}(p) \oplus \mathbb{Z}(p) = \langle a\rangle \oplus \langle b\rangle$ and define the endomorphism ϕ by $\phi : a \mapsto b,\ b \mapsto a + b$; note that $\phi^2 = I + \phi$, where I is the identity on H. Let Φ be the subring generated by I, ϕ. Then Φ consists of the elements $\{rI + s\phi : 0 \leq r, s \leq p - 1\}$.

Suppose that $p \neq 2$ and we make the additional assumption that p is a prime of the form $p = 5n + 2$; note that it follows from Dirichlet's theorem on primes in arithmetic progression that there are infinitely many primes of this form. Consider an idempotent $rI + s\phi \in \Phi$. Then it follows immediately that $(r^2 + s^2 - r)I + (2rs + s^2 - s)\phi = 0$. Applying this expression to the element a, we deduce that

$$r^2 + s^2 - r \equiv 0 \mod p \qquad (1)$$

and that $2rs + s^2 - s \equiv 0 \mod p$.

Consider now the situation where $s \neq 0$; the last congruence may then be simplified to

$$2r + s - 1 \equiv 0 \mod p \qquad (2).$$

Now multiply the relation (1) by 4 and substitute for $2r$, to obtain $(1 - s)^2 + 4s^2 - 2(1 - s) \equiv 0 \mod p$. Simplifying, we get

$$5s^2 \equiv 1 \mod p \qquad (3).$$

Since $s \neq 0$, the congruence (3) has a solution if, and only if, 5 is a quadratic residue mod p, i.e., employing the standard Legendre symbol notation, if, and only if, $(\frac{5}{p}) = -1$. Now it follows from the Quadratic Reciprocity theorem that $(\frac{5}{p})(\frac{p}{5}) = (-1)^{(5-1)(p-1)/4} = 1$ and hence we conclude that $(\frac{5}{p}) = (\frac{p}{5})$, since the Legendre symbol is ± 1 in each case. We claim that $(\frac{p}{5}) = -1$; for suppose not, then we have $p \equiv x^2 \mod 5$ for some x and from this it follows that $x^2 \equiv 2 \mod 5$. This latter is impossible since the only squares mod 5 are $0, 1, 4$. Hence the only idempotents in Φ must have $s = 0$ and it follows from a straightforward calculation that then $r = 0, 1$ and so the only idempotents in Φ are $0, I$.

Note that as a consequence of $(\frac{5}{p}) = -1$, we have that the expression $t^2 + t - 1 \not\equiv 0 \mod p$ for any $0 \leq t \leq p - 1$: for if $t^2 + t - 1 \equiv 0 \mod p$ then $4t^2 + 4t - 4 = (2t + 1)^2 - 5 \equiv 0 \mod p$, contradicting $(\frac{5}{p}) = -1$.

Now construct, using once again the discussed above Corner's realization theorem, a group G such that $p^\omega G = H$ and $\mathrm{E}(G) \restriction H = \Phi$. Since $\mathrm{Proj}(\Phi)$ consists only of the multiples of the identity, it is clear that $\mathrm{E}(G)$ does not act projectively fully transitively on $p^\omega G$ and so G is certainly not projectively fully transitive by Lemma 3.60 below. However, Φ acts fully transitively on $p^\omega G$: to see this observe that the Ulm sequences of H are only of two types, *viz.*, $(\infty, \infty, \dots)$ and $(0, \infty, \dots)$ and these correspond respectively to the sets of elements $\{0\}, \{ra + sb : 0 \leq r, s \leq p - 1; r, s \text{ not both } 0\}$. Since it is trivial to find a map in Φ taking a to $ra + sb$, $(0 \leq r, s \leq p - 1)$, it will suffice to show that for an arbitrary element $ra + sb$, with not both of $r, s = 0$, that there is a mapping in $yI + z\phi$ taking $ra + sb$ to a.

We consider a number of cases:

(a) if $s = 0$, choose $y = r^{-1}$ and $x = 0$;

(b) if $s \neq 0$, let $t = rs^{-1}$ and note that multiplication by s^{-1} maps $ra+sb \mapsto ta+b$. Thus it will suffice to show that we can map an arbitrary element of the form $(ta+b)$ to a. Applying the map $yI+x\phi$ to $(ta+b)$ we get $(yt+x)a+(y+xt+x)b$, so we need to choose y, x in order that $(yt + x) \equiv 1 \mod p$ and simultaneously that $(y + xt + x) \equiv 0 \mod p$. If we set $x = 1 - yt$ then certainly the first congruence is satisfied. Substituting we see that the second congruence reduces to $y(1-t-t^2)+(1+t) \equiv 0 \mod p$. As noted above, our choice of p ensures that $(1 - t - t^2) \not\equiv 0 \mod p$ and hence the choice $y = (1 + t)/(t^2 + t - 1), x = 1 - yt$ ensures that $yI + x\phi$ maps $(ta + b) \mapsto a$, as required.

It follows from Lemma 2.1 in [29] that G is fully transitive.

(iii) The proof of the final part is similar to that of (ii). Let $H = \mathbb{Z}(2) \oplus \mathbb{Z}(4) = \langle a \rangle \oplus \langle b \rangle$ and define the endomorphism ϕ by $\phi : a \mapsto a + 2b,\ b \mapsto a + b$; note that $\phi^2 = 3I$ and $2\phi = 2I$, where I is the identity on H. Let Φ be the subring generated by I, ϕ. A routine check using the identities noted above shows that Φ has order 8 and consists of the elements $\{0, I, 2I, 3I, \phi, I + \phi, 2I + \phi, 3I + \phi\}$;

observe that the only idempotents in Φ are $0, I$. Now construct, using Corner's theorem, a group G such that $2^\omega G = H$ and $\mathrm{E}(G) \restriction H = \Phi$. Since $\mathrm{Proj}(\Phi)$ consists only of the multiples of the identity, it is clear that $\mathrm{E}(G)$ does not act projectively fully transitively on $2^\omega G$ and so G is certainly not projectively fully transitive by Lemma 3.60 below. However, Φ acts fully transitively on $2^\omega G$: to see this observe that the Ulm sequences of H form a chain with four nodes consisting of the elements with Ulm sequences $(0, 1, \infty, \dots), (0, \infty, \dots), (1, \infty, \dots)$ and $(\infty, \infty, \dots)$. The four types consist respectively of the sets of elements $\{b, 3b, a+b, a+3b\}, \{a, a+2b\}, \{2b\}, \{0\}$. A straightforward check shows that $3I$ interchanges $b, 3b$ and also $a+3b, a+b$ while $\phi : b \mapsto a+b, 2I+\phi : a+b \mapsto b$; thus the elements of Ulm sequence $(0, 1, \infty, \dots)$ lie in a single orbit under the action of Φ. Since $2I+\phi : b \mapsto a$, $3I+\phi : a \mapsto 2b$ and $2I : 2b \mapsto 0$, we can establish that Φ acts fully transitively on H if we can show that $a, a+2b$ are in the same orbit of Φ; but this is immediate since a simple calculation shows that ϕ interchanges $a, a+2b$. It follows from Lemma 2.1 in [29] that G is fully transitive. □

Remark 3.54. The choice of the prime $p = 5n+2$ in Proposition 3.53 (ii) was made purely to simplify the calculations; it is not a necessary condition. For example, it is straightforward to show that in the case $p = 2$, the only idempotents in the subring Φ are $0, I$ and that Φ acts fully transitively on H.

Our next result shows the close connection between the various types of transitivity that we have discussed:

Theorem 3.55. *Suppose $\kappa > 1$. Then the following are equivalent:*
(i) G is fully transitive;
(ii) $G^{(\kappa)}$ is fully transitive;
(iii) $G^{(\kappa)}$ is transitive;
(iv) $G^{(\kappa)}$ is projectively fully transitive.

Proof. The equivalence of (i) and (ii) follows from Corollary 1 in [43], while the equivalent of (ii) and (iii) follows from Corollary 4 of the same paper. We show the equivalence of (ii) and (iv). Since projectively fully transitive groups are always fully transitive, it is immediate that (iv) $\Rightarrow$ (ii). Conversely, if $G^{(\kappa)}$ is fully transitive, then, since we are assuming that $\kappa > 1$, it follows from Proposition 3.41 that $G^{(\kappa)}$ is a fully transitive IG-group and so necessarily is a projectively fully transitive group. □

Corollary 3.56. *If G is projectively fully transitive, then for every cardinal κ, $G^{(\kappa)}$ is projectively fully transitive.*

Corollary 3.57. *A direct summand of a projectively fully transitive group is not necessarily a projectively fully transitive group.*

Proof. Choose a group G as in Proposition 3.53 above, so that G is fully transitive but not projectively fully transitive. If $H = G \oplus G$, then it follows from Theorem 3.55 that H is projectively fully transitive while its summand G is not. $\square$

Remark 3.58. The role of transitivity in this connection is not clear. In Theorem 3.55 it is not possible to replace condition (i) with the statement "G is projectively fully transitive": for if G is chosen as in the proof of Corollary 3.57, then G is not projectively fully transitive but its square $G \oplus G$ is transitive, since G is fully transitive – see Corollary 3 in [43]. It would be interesting to know if a transitive, fully transitive group is necessarily projectively fully transitive. In fact, it was shown in [36] that C_λ-groups of length λ are both transitive and fully transitive; however whether or not they are projectively fully transitive is not obvious (compare also with Corollary 3.67 (ii) listed below). Moreover, note that it is well known that there exist transitive 2-groups which are not fully transitive (see, for example, Section 4 in [29]) and hence, *a fortiori*, not projectively fully transitive.

In the classical notions of transitivity a key observation due to Corner [29] is that the transitivity property depends on the action of the endomorphism ring on the first Ulm subgroup. A similar phenomenon occurs here; let us say that a subring Φ of $\mathrm{E}(G)$ acts *projectively fully transitively* on a subgroup X of G if, given $x, y \in X$ with $U_G(x) \leq U_G(y)$, there is an endomorphism $\phi \in \Phi$ such that $\phi(x) = y$ and ϕ belongs to the subring of Φ generated by the idempotents in Φ.

The following extremely simple assertion has been used previously by both Hausen [57] and Megibben [88]; we include the short proof for the sake of completeness and readers' convenience.

Lemma 3.59. *If $G = A \oplus H$ and $\psi(A) \leq H, \psi(H) = 0$, then $\psi \in \Pi(G)$.*

Proof. The standard matrix representation for ψ is $\psi = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1_H \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_H \end{smallmatrix}\right)$. Since there last two matrices are clearly idempotent, $\psi \in \Pi(G)$. $\square$

Our next result is a careful re-working of Lemma 2.1 in [28]. We were faced here with the dilemma of whether to simply tell the reader that the necessary changes can be made to Corner's original proof or of reworking the proof in detail. We have chosen the latter since it enables us to point out the more substantial changes needed to extend the proof to strongly projectively fully transitive groups, which we will discuss in the next section.

Lemma 3.60. *A group G is projectively fully transitive if, and only if,* $\mathrm{E}(G)$ *acts projectively fully transitively on* $p^\omega G$.

Proof. The sufficiency is trivial; so assume that $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$ and consider $x, y \in G$ with $U_G(x) \leq U_G(y)$. Let r, s be the least natural numbers such that $p^r x, p^s y \in p^\omega G$; if $r = 0$, then both $x, y \in p^\omega G$ and the result follows immediately, so we may assume that $r \neq 0$. Note that $s \leq r$, so that in any case $p^r y \in p^\omega G$ and $U_G(p^r x) \leq U_G(p^r y)$. By hypothesis there is an endomorphism $\phi_0 \in \mathrm{Proj}(G)$ with $\phi_0(p^r x) = p^r y$. Since $r \neq 0$, we may choose an integer $m > \max\{ht_G(p^{r-1}x), ht_G(p^{s-1}y)\}$ – if $s-1$ is negative we simply omit the final term $ht_G(p^{s-1}y)$.

So we can choose $x_0 \in G$ with $p^r x = p^{r+m}x_0$; then $p^r y = \phi_0(p^r x) = p^{r+m}y_0$ where $y_0 = \phi_0(x_0)$. Thus $x = x_1 + p^m x_0, y = y_1 + p^m y_0$ where $p^r x_1 = p^r y_1 = 0$. Note that $o(x_1) = p^r$ for otherwise $p^t x = p^{t+m}x_0$ for some $t < r$, contradicting the choice of m. Also $ht_G(p^{r-1}x_1) = ht_G(p^{r-1}x)$ since $x_1 = x - p^m x_0$ and so $ht_G(p^{r-1}x_1) < m$. Thus $\langle x_1 \rangle \cap p^m G = 0$ and so there is a $p^m G$-high subgroup A such that $x_1 \in A$. It follows that A is necessarily a bounded pure subgroup and so we may write $G = A \oplus H$ for some complement H; note that $p^m G \leq H$. Let π denote the projection of G onto H with kernel A.

Let $y_1 = a_1 + h_1$, where $a_1 \in A, h_1 \in H$. Since the decomposition is direct, $U_G(y_1) = U_G(a_1) \wedge U_G(h_1)$ and so $U_G(x_1) \leq U_G(a_1)$ and $U_G(x_1) \leq U_G(h_1)$. Now $x_1, a_1 \in A$, a bounded group and $U_G(x_1) = U_A(x_1)$ and $U_G(a_1) = U_A(a_1)$. Since bounded groups are fully transitive there is an endomorphism θ of A with $\theta(x_1) = a_1$. But, as observed earlier, a bounded group is an IG-group and so we can extend θ to an endomorphism θ_1 of G with $\theta_1 \in \mathrm{Proj}(G)$. (It is even possible to choose a suitable $\theta_1 \in \Pi(G)$ by using Corollary 6 in [57].)

Since A is a bounded summand we can certainly find an endomorphism ϕ' of G with $\phi'(x_1) = h_1$. Set $\psi = \pi\phi'(1-\pi)$ and observe that $\psi(x_1) = h_1$. Moreover, $\psi(A) \leq \pi(G) = H$ while $\psi(H) = 0$, so by Lemma 3.59, ψ is in $\Pi(G)$; in particular $\psi \in \mathrm{Proj}(G)$. Set $\phi_1 = \theta_1 + \psi$ and note that $\phi_1(x_1) = \theta_1(x_1) + \psi(x_1) = a_1 + h_1 = y_1$; also observe that $\phi_1 \in \mathrm{Proj}(G)$. (In fact it is even in $\Pi(G)$.)

Finally set $\phi = \phi_0\pi + \phi_1(1-\pi)$ so that $\phi \in \mathrm{Proj}(G)$. (Note that this construction is not possible if one wants to stay within $\Pi(G)$.) Now $\phi(x) = \phi_0\pi(x) + \phi_1(1-\pi)(x)$ and, since $x = x_1 + p^m x_0$, we have $\phi_0\pi(x) = \phi_0\pi(p^m x_0)$ as $x_1 \in A$. So $\phi_0\pi(x) = \phi_0(p^m x_0)$, since $p^m G \leq H$, as noted above. Thus $\phi_0\pi(x) = p^m\phi_0(x_0) = p^m y_0$. We also have $\phi_1(1-\pi)(x) = \phi_1(1-\pi)(x_1 + p^m x_0) = \phi_1(1-\pi)(x_1)$ since $p^m x_0 \in H$, and this gives $\phi_1(1-\pi)(x) = \phi_1(x_1) = y_1$. Therefore, $\phi(x) = y_1 + p^m y_0 = y$ with $\phi \in \mathrm{Proj}(G)$, as required. □

Corollary 3.61. *(i) A separable group is projectively fully transitive;*

(ii) if $p^\omega G \cong \mathbb{Z}(p^n)$ for some finite n, then G is projectively fully transitive;

(iii) if A is projectively fully transitive and B is separable, then $A \oplus B$ is projectively fully transitive.

Proof. Part (i) follows immediately from Lemma 3.60. For (ii) observe that if $x, y \in p^\omega G$ with $U_G(x) \leq U_G(y)$, then it is easy to see that an integer multiple of the identity (and hence an endomorphism which is even in $\Pi(G)$) maps x to y; the result then follows from Lemma 3.60. For the final part, note that both A, B are fully transitive and the direct sum $A \oplus B$ is also fully transitive by [15, Proposition 2.6]. Hence $\{A, B\}$ is a fully transitive pair and the result follows from Proposition 3.50. $\square$

Note that it is not possible to extend part (ii) of Corollary 3.61 even to the situation where $p^\omega G = \mathbb{Z}(p) \oplus \mathbb{Z}(p)$; indeed, Megibben [87] has constructed an example of such a group which is not even fully transitive.

Recall that the notions of socle-regularity and strong socle-regularity have been introduced in [D8] and [35], respectively; these concepts were generalizations of full transitivity and transitivity respectively. In [36], a p-group G was said to be *projectively socle-regular* if for all projection-invariant subgroups P of G, there is an ordinal α (depending on P) such that $P[p] = (p^\alpha G)[p]$. Our next result shows that projective socle-regularity is likewise a generalization of projective full transitivity.

Proposition 3.62. *If G is projectively fully transitive, then G is projectively socle-regular; if $p \neq 2$, then G is strongly socle-regular.*

Proof. Suppose P is an arbitrary projection-invariant subgroup of G and $\alpha = \min\{ht_G(z) : z \in P[p]\}$, so that $P[p] \leq (p^\alpha G)[p]$. Choose $x \in P[p]$ of height exactly α so that $U_G(x) = (\alpha, \infty, \dots)$. Let $y \in (p^\alpha G)[p]$ be arbitrary; then $U_G(y) = (\beta, \infty, \dots)$ where $\beta \geq \alpha$. Since G is projectively fully transitive there is a $\phi \in \mathrm{Proj}(G)$ such that $\phi(x) = y$. But because ϕ is a linear combination of products of idempotents and P is projection invariant, we have that $y = \phi(x) \in P[p]$. Since y was arbitrary in $(p^\alpha G)[p]$, we deduce that $(p^\alpha G)[p] \leq P[p]$ and hence we have the desired equality. The final conclusion follows immediately from Proposition 1.5 in [36] once we have that G is projectively socle-regular. $\square$

We now consider subgroups of projectively fully transitive groups. We begin with the elementary:

Proposition 3.63. *If G is projectively fully transitive, then $p^\beta G$ is projectively fully transitive for all ordinals β.*

Proof. Let $H = p^\beta G$ and observe that if $x, y \in H$ with $U_H(x) \leq U_H(y)$, then $U_G(x) \leq U_G(y)$. So there is a $\phi \in \mathrm{Proj}(G)$ with $\phi(x) = y$. However, as H is fully invariant in G, it is easy to see that if $\phi \in \mathrm{Proj}(G)$, then $\phi \restriction H \in \mathrm{Proj}(H)$. $\square$

For *finite* ordinals β it is easy to establish the converse:

Proposition 3.64. *If $p^n G$ is projectively fully transitive for some finite n, then G is projectively fully transitive.*

Proof. By induction it suffices to establish the result for pG, so let $H = pG$. Furthermore, by Lemma 3.60 it suffices to show that $ER(G)$ acts projectively fully transitively on $p^\omega G$. So let $x, y \in p^\omega G$ with $U_G(x) \leq U_G(y)$. Note that $x, y \in p^\omega G = p^\omega H$ since $p^\omega H = p^{1+\omega} G = p^\omega G$. Consider $U_H(x) = (\alpha_0, \alpha_1, \dots)$, say. Since $x \in p^\omega H$, each $\alpha_i \geq \omega$ and then $p_i^\alpha H = p_i^\alpha G$, so that $U_H(x) \leq U_H(y)$. Since H is, by assumption, projectively fully transitive there is a $\phi \in \mathrm{Proj}(H)$ with $\phi(x) = y$. It follows from Theorem 1.11 in [36] that every idempotent in $\mathrm{E}(H)$ lifts to an idempotent in $\mathrm{E}(G)$ and so every element of $\mathrm{Proj}(H)$ lifts to an element of $\mathrm{Proj}(G)$. In particular, ϕ lifts to an element $\psi \in \mathrm{Proj}(G)$ with $\psi(x) = y$. $\square$

If we wish to extend Proposition 3.64 to ordinals $\beta \geq \omega$, it seems inevitable that we must introduce some restriction on the quotient $G/p^\beta G$: we know from the proof of Proposition 3.53 that there is a group G such that $p^\omega G$ is an elementary group of infinite rank (and hence projectively fully transitive) but G is not projectively fully transitive. An obvious restriction is to assume that the quotient $G/p^\beta G$ is totally projective. We begin by examining the situation when $\beta = \omega$.

Lemma 3.65. *If $G/p^\omega G$ is a direct sum of cyclic groups and $p^\omega G$ is projectively fully transitive, then G is projectively fully transitive.*

Proof. We show that $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$. Let $x, y \in p^\omega G$ with $U_G(x) \leq U_G(y)$. Since for any $g \in p^\omega G, ht_G(g) = \omega + ht_{p^\omega G}(g)$, we have $U_{p^\omega G}(x) \leq U_{p^\omega G}(y)$. By assumption there is a $\phi \in \mathrm{Proj}(p^\omega G)$ with $\phi(x) = y$. It follows from Theorem 11 in [62] that every idempotent in $\mathrm{E}(p^\omega G)$ lifts to an idempotent in $\mathrm{E}(G)$, so the mapping ϕ lifts to a mapping $\psi \in \mathrm{Proj}(G)$ and $\psi(x) = y$. Thus $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$ and, in virtue of Lemma 3.60, G is projectively fully transitive as required. $\square$

Theorem 3.66. *Suppose that α is an ordinal strictly less than ω^2 and $G/p^\alpha G$ is totally projective. If $p^\alpha G$ is projectively fully transitive, then so also is G.*

Proof. The proof is by induction; if $\alpha \leq \omega$ we have already established the result in Proposition 3.64 and Lemma 3.65. So suppose that the result is true for all ordinals $< \alpha$. There are two possibilities: either α is a limit of cofinality ω or $\alpha = \beta + 1$ for some β.

Consider firstly the case $\alpha = \beta + 1$ for some β. Set $X = p^\beta G$ and note that $pX = p^\alpha G$ is projectively fully transitive. It follows from Proposition 3.64 that X is projectively fully transitive. Moreover, $G/p^\beta G \cong (G/p^\alpha G)/(p^\beta G/p^\alpha G) \cong (G/p^\alpha G)/p^\beta(G/p^\alpha G)$ and hence $G/p^\beta G$ is totally projective by a well-known result of Nunke – see, e.g., [44, Exercise 82.3]. So, by our induction hypothesis, we conclude that G is projectively fully transitive.

In the limit case $\alpha = \beta + \omega$ for some β. Set $X = p^\beta G$ so that $p^\omega X = p^\alpha G$ is projectively fully transitive. Now $X/p^\omega X \cong p^\beta G/p^\alpha G$ is totally projective again by the aforementioned Nunke's result. It follows from Lemma 3.65 that $X = p^\beta G$ is projectively fully transitive. Since $G/p^\beta G$ is totally projective and $\beta < \alpha$, the induction hypothesis gives us that G is projectively fully transitive. □

Corollary 3.67. *(i) If G is totally projective of length $\leq \omega^2$, then G is projectively fully transitive;*

(ii) if λ is cofinal with ω and G is a C_λ-group of length $\lambda \leq \omega^2$, then G is projectively fully transitive.

Proof. (i) If G is totally projective of length $< \omega^2$, then the result follows immediately from Theorem 3.66 above. If G has length ω^2, then G is actually a direct sum of totally projective groups of length $< \omega^2$, say $G = \bigoplus_{i \in I} G_i$ where $l(G_i) < \omega^2$ for each $i \in I$. If $x, y \in G$ and $U_G(x) \leq U_G(y)$, then there is a finite set $\{i_1, \dots, i_n\} \subseteq I$ such that $x, y \in H = \bigoplus_{j=1}^{n} G_{i_n}$; moreover, $U_H(x) = U_G(x) \leq U_G(y) = U_H(y)$. If we can show that H is projectively fully transitive, then we have a mapping $\phi \in \mathrm{Proj}(H)$ with $\phi(x) = y$. If $G = H \oplus K$ and we set $\psi = \phi \oplus 0_K$, then it is easy to see that $\psi \in \mathrm{Proj}(G)$ and $\psi(x) = y$. Thus to establish part (i) it suffices to show that H is projectively fully transitive.

Now each G_{i_j} is totally projective of length $< \omega^2$, so each G_{i_j} is projectively fully transitive. Moreover, given any i_1, i_2 the sum $G_{i_1} \oplus G_{i_2}$ is totally projective and hence fully transitive, i.e. $\{G_{i_1}, G_{i_2}\}$ is a fully transitive pair and hence it follows from Proposition 3.50 that $G_{i_1} \oplus G_{i_2}$ is projectively fully transitive. A simple induction now yields the desired result that H is projectively fully transitive. (This argument is presented in a more formalized way in Corollary 3.74 below.)

(ii) If G is a C_λ-group of length λ cofinal with ω and $x, y \in G$ with $U_G(x) \leq U_G(y)$, let $H = \langle x, y \rangle$. Since H is a finite group and λ is a limit ordinal, there is an ordinal $\alpha < \lambda$ such that $H \cap p^\alpha G = \{0\}$. Then it follows from [82, Proposition

4] that G decomposes as $G = A \oplus K$ where A is totally projective of length $< \lambda$ and $x, y \in A$. Since $U_A(x) = U_G(x) \leq U_G(y) = U_A(y)$ and A is projectively fully transitive by part (i), we have an endomorphism $\phi \in \mathrm{Proj}(A)$ with $\phi(x) = y$. But then an identical argument to that in the proof of part (i) gives a mapping $\psi \in \mathrm{Proj}(G)$ with $\psi(x) = y$. Thus G is projectively fully transitive as required. □

We are now in a position to consider the class of strongly projectively fully transitive groups.

Recall from the introduction that a group G is said to be *strongly projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there exists $\phi \in \Pi(G)$ with $\phi(x) = y$; clearly a strongly projectively fully transitive group is projectively fully transitive. We pointed out in the final paragraph of the proof of Lemma 3.60 the difficulty in extending that result to strongly projectively fully transitive groups. We can, however, obtain the corresponding result by taking a little more care. The notion of acting strongly projectively fully transitively on $p^\omega G$ is analogous to that acting projectively fully transitively: specifically, a subgroup Φ of $\mathrm{End}(G)$ acts *strongly projectively fully transitively* on a subgroup X of G if, given $x, y \in X$ with $U_G(x) \leq U_G(y)$, there is an endomorphism $\phi \in \Phi$ such that $\phi(x) = y$ and ϕ belongs to the subgroup of Φ additively generated by the idempotents in Φ.

Lemma 3.68. *A reduced group G is strongly projectively fully transitive if, and only if,* $\mathrm{End}(G)$ *acts strongly projectively fully transitively on $p^\omega G$.*

Proof. The sufficiency is trivial, so assume that $\mathrm{End}(G)$ acts strongly projectively fully transitively on $p^\omega G$. Our arguments follow exactly those described in the proof of Lemma 3.60 and we use the same notation as in that lemma. Observe firstly that the hypothesis that $\mathrm{End}(G)$ acts strongly projectively fully transitively on $p^\omega G$ means that ϕ_0 can be chosen to be in $\Pi(G)$. Now consider the endomorphism θ of the bounded group A. As noted in the proof of Lemma 3.60, we may use Hausen's result [57, Corollary 6] to choose $\theta \in \Pi(A)$. We now extend θ to an endomorphism of G taking a little more care than in the previous proof.

If ε is an idempotent endomorphism of the direct summand A, where $G = A \oplus H$, then we can extend ε to an endomorphism ε^* by setting $\varepsilon^* = \varepsilon \oplus 0_H$. Note that ε^* is then an idempotent endomorphism of G and, if π is the canonical projection of G onto H along A, we have $\varepsilon^*(1 - \pi)(H) = 0 = \varepsilon^*(H)$, while $(\varepsilon^*(1 - \pi))(a) = \varepsilon^*(a)$ for all $a \in A$. Consequently $\varepsilon^*(1 - \pi) = \varepsilon^*$. Applying this method of extension to the map $\theta \in \Pi(A)$ we get an endomorphism $\theta_1 \in \Pi(G)$ and $\theta_1(1 - \pi) = \theta_1$.

Returning to the proof of Lemma 3.60, we note that the mapping ψ, where $\psi(x_1) = h_1$, belongs to $\Pi(G)$ and by construction it satisfies $\psi = \psi(1 - \pi)$.

Consequently the map $\phi_1 = \theta_1 + \psi$ also belongs to $\Pi(G)$ and satisfies $\phi_1 = \phi_1(1-\pi)$.

In the final paragraph of the proof of Lemma 3.60 it is shown that the map $\phi = \phi_0\pi + \phi_1(1-\pi)$ has the desired property that $\phi(x) = y$. However, if we now define a new map $\phi^* = \phi_0 + \phi_1$, then certainly $\phi^* \in \Pi(G)$ since both $\phi_0, \phi_1 \in \Pi(G)$. But $G = A \oplus H$ and A is bounded, so $p^\omega G \leq H$ and hence $\phi_0\pi(x) = \phi_0(x)$ as $x \in p^\omega G$.

Moreover, as we noted above, $\phi_1 = \phi_1(1-\pi)$ and so $\phi^*(x) = \phi(x) = y$, as required. $\square$

Corollary 3.69. *(i) If B is a separable group, then B is strongly projectively fully transitive;*

(ii) if A is strongly projectively fully transitive and B is separable, then $A \oplus B$ is strongly projectively fully transitive;

(iii) if $p^\omega G \cong \mathbb{Z}(p^n)$ for some finite n, then G is strongly projectively fully transitive.

Proof. Point (i) is immediate from the previous result, and (ii) follows immediately from part (i) and Proposition 3.50 – recall our observation at the end of the proof of Proposition 3.50. The final part follows by an identical argument to that used in Corollary 3.61 (ii). $\square$

We remark that it is possible to prove directly (i.e. without invoking Lemma 3.68) that a separable group is strongly projectively fully transitive: the argument utilizes Lemma 65.5 in [44].

Although a separable group is necessarily strongly projectively fully transitive, it does not follow that it is an IS-group; recall from Corollary 3.45 that a separable essentially indecomposable group need not be even an IG-group.

Since Proposition 3.50 and Proposition 3.51 carry over unchanged to strongly projectively fully transitive groups, we see that a group $G = D \oplus R$, with D divisible and R reduced, is strongly projectively fully transitive if, and only if, D, R are both strongly projectively fully transitive.

In fact, we derive:

Theorem 3.70. *A group $G = D \oplus R$, where D is divisible and R is reduced, is strongly projectively fully transitive if, and only if, R is strongly projectively fully transitive.*

Proof. By the preceding observation it is clearly enough to show that any divisible group is strongly projectively fully transitive. If D is of rank one then the result follows from the proof of Theorem 3.49: just observe that the mapping used to send the element x to y was an integer multiple of the identity. If D is of finite rank then the result follows from Proposition 3.39 (ii) and the fact that a divisible group is always fully transitive. Finally, if D has infinite rank, the result follows from Proposition 3.73 below. $\square$

Corollary 3.71. *A divisible group is strongly projectively fully transitive.*

The following somewhat combines Corollaries 3.69 (iii) and 3.71 into a more general case.

Proposition 3.72. *Let G be a group such that $p^\omega G$ is the direct sum of a divisible group and a cyclic group of order p^n for some $n \in \mathbb{N}$. Then G is strongly projectively fully transitive.*

Proof. One may decompose $G = D \oplus C$ where D is divisible and $p^\omega C \cong \mathbb{Z}(p^n)$. In fact, $p^{\omega+n}G$ is the maximal divisible part in $p^\omega G$, so that $p^\omega G = p^{\omega+n}G \oplus R$ where $R \cong \mathbb{Z}(p^n)$. But $G = p^{\omega+n}G \oplus C$ for some group C, and hence $C \cong G/p^{\omega+n}G$ and $p^\omega C \cong p^\omega(G/p^{\omega+n}G) = p^\omega G/p^{\omega+n}G \cong R$. This substantiates our claim. Furthermore, we apply a combination of Theorem 3.70 and Corollary 3.69 (iii) to deduce that G is strongly projectively fully transitive, as asserted. $\square$

Proposition 3.73. *If the group $G^{(n)}$ is strongly projectively fully transitive for every finite n, then $H = G^{(\kappa)}$ is strongly projectively fully transitive for any infinite cardinal κ.*

Proof. If $x, y \in H$ with $U_H(x) \leq U_H(y)$, then there exists a finite integer m such that $x, y \in H_m = G^{(m)}$. Now $U_{H_m}(x) = U_H(x)$ and similarly for y, so there is a mapping $\phi \in \Pi(H_m)$ with $\phi(x) = y$. However ϕ can be expressed as a linear combination of idempotents in $\mathrm{E}(H_m)$ and each of these may be extended trivially to an idempotent of H by acting as the zero map on the canonical complement. The resulting sum is a map $\psi \in \Pi(H)$ with $\psi(x) = y$. Thus H is strongly projectively fully transitive, as required. $\square$

In fact, the argument in Proposition 3.73 easily generalizes to give:

Corollary 3.74. *If $G_i (i \in I)$ is a collection of groups with the property that $\bigoplus_{i \in J} G_i$ is strongly projectively fully transitive (respectively, projectively fully transitive) for every finite subset $J \subseteq I$, then we have that $\bigoplus_{i \in I} G_i$ is strongly projectively fully transitive (respectively, projectively fully transitive).*

Next, we record some crucial properties of strongly projectively fully transitive groups; the proofs of these results follow by identical arguments to those used for the corresponding results on projectively fully transitive groups; the proof of part (v) follows from part (iv) and Proposition 3.73.

Theorem 3.75. *(i) If G is strongly projectively fully transitive, then $p^\beta G$ is strongly projectively fully transitive for all ordinals β;*

(ii) if $p^n G$ is strongly projectively fully transitive for some finite n, then G is strongly projectively fully transitive;

(iii) if α is an ordinal strictly less than ω^2 and $G/p^\alpha G$ is totally projective, then if $p^\alpha G$ is strongly projectively fully transitive, so also is G;

(iv) if A, B are strongly projectively fully transitive and $\{A, B\}$ is a fully transitive pair, then $A \oplus B$ is strongly projectively fully transitive;

(v) if G is strongly projectively fully transitive, then $G^{(\kappa)}$ is strongly projectively fully transitive for any cardinal κ;

(vi) if G is totally projective of length $\leq \omega^2$, then G is strongly projectively fully transitive;

(vii) if λ is cofinal with ω and G is a C_λ-group of length $\lambda \leq \omega^2$, then G is strongly projectively fully transitive.

An easy consequence of Corollary 3.74 is the following result which generalizes Proposition 3.50:

Proposition 3.76. *If G is a fully transitive group which is an arbitrary direct sum of (strongly) projectively fully transitive groups, then G is (strongly) projectively fully transitive.*

In light of Theorem 3.55, one might expect a similar result with strongly projectively fully transitive replacing projectively fully transitive. This seems to be difficult and the best we can offer is the following.

Proposition 3.77. *If $p^\omega G$ is an elementary group, then G is fully transitive if, and only if, $G \oplus G$ is strongly projectively fully transitive.*

Proof. Sufficiency is immediate since summands of fully transitive groups are fully transitive; in fact there is no need for the additional hypothesis on $p^\omega G$ for this argument. Conversely, suppose that G is fully transitive and $p^\omega G$ is elementary. Let $H = G \oplus G$ and consider any elements $(a, b), (c, d)$ in $p^\omega H$. Consider firstly the situation where $a, b \neq 0$. Since all the elements of $p^\omega G$ have the same Ulm sequence $(\omega, \infty, \dots)$ in G, there are endomorphisms $\gamma : b \mapsto c$ and $\delta : a \mapsto d$. The matrix $\Delta = \left(\begin{smallmatrix} 0 & \gamma \\ \delta & 0 \end{smallmatrix}\right)$ represents an endomorphism of H which maps (a, b) to

(c, d), but $\Delta = \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and each of these matrices is idempotent, so that $\Delta \in \Pi(H)$.

If one of $a, b = 0$ (the situation where both are zero is trivial), then we may assume without loss that $a \neq 0, b = 0$. As before, we have the endomorphisms of G that are $\alpha : a \mapsto c$, $\delta : a \mapsto d$. Now the matrix $\Delta = \left(\begin{smallmatrix} \alpha & \alpha \\ \delta & 1-\alpha \end{smallmatrix}\right)$ represents an endomorphism of H and maps $(a, 0)$ to (c, d). However, $\Delta = \left(\begin{smallmatrix} \alpha & \alpha \\ 1-\alpha & 1-\alpha \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 0 \\ \delta+\alpha-1 & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and direct calculation gives that each of these matrices is idempotent. Thus $\Delta \in \Pi(H)$.

It follows immediately from Lemma 3.68 that $H = G \oplus G$ is strongly projectively fully transitive. $\square$

Remark 3.78. The condition that $p^\omega G$ be elementary in Proposition 3.77 is far from necessary. For instance, if C is a bounded group and G is a group with $p^\omega G = C$ constructed via Corner's Theorem 6.1 in [28], with $\mathrm{E}(G)$ acting on $p^\omega G$ in the same manner as the full endomorphism ring $\mathrm{E}(C)$, then G is certainly fully transitive and $H = G \oplus G$ is strongly projectively fully transitive. To see the latter, observe that if $(x, y), (u, v) \in p^\omega H$ with $U_H((x, y)) \leq U_H((u, v))$, then we can assume without loss that $U_G(x) \leq U_G(y)$ and $U_G(u) \leq U_G(v)$. By the full transitivity of G we have endomorphisms $\gamma : x \mapsto u$, $\delta : x \mapsto v$ and $\gamma \restriction p^\omega G \in \mathrm{E}(C)$. However, $\left(\begin{smallmatrix} 0 & 0 \\ \delta & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are both idempotents and so the sum $\Delta_1 = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Pi(H)$. Since $\gamma \restriction p^\omega G$ can be written as a sum of idempotents in $\mathrm{E}(C)$, say $\gamma \restriction p^\omega G = \pi_1 + \cdots + \pi_n$, then we obtain from Corner's construction that there are idempotents $e_1, \ldots, e_n \in \mathrm{E}(G)$ with $e_i \restriction p^\omega G = \pi_i$ and $\gamma = e_1 + \cdots + e_n$. The matrices $\left(\begin{smallmatrix} e_i & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are again idempotents in $\mathrm{E}(H)$ and if $\Delta_2 = \left(\begin{smallmatrix} e_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \cdots + \left(\begin{smallmatrix} e_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then it is immediate that $\Delta = \Delta_1 + \Delta_2 \in \Pi(H)$ and Δ maps $(x, y) \mapsto (u, v)$, as required.

Finally, we note that, similar to the situation for projectively fully transitive groups, a summand of a strongly projectively fully transitive group need not be strongly projectively fully transitive, even when the first Ulm subgroup is elementary.

Proposition 3.79. *There is a non-strongly projectively fully transitive group G, with elementary first Ulm subgroup, such that $G \oplus G$ is strongly projectively fully transitive.*

Proof. Let G be a fully transitive group as constructed in either part (i) or part (ii) of Proposition 3.53 above; note that in either case the first Ulm subgroup of G is an elementary group. It follows immediately from Proposition 3.77 that $G \oplus G$ is strongly projectively fully transitive. However, as pointed out in the proof of Proposition 3.53, neither group G is even projectively fully transitive. $\square$

The reader will have noted that we have not shown that the classes of projectively fully transitive and strongly projectively fully transitive groups are distinct. This seems to be reasonably difficult, so we pose:

Problem 2. Find a projectively fully transitive group which is not strongly projectively fully transitive.

We finish the work with a further question; we believe that an answer to this question will shed further light on the nature of projective and strong projective full transitivity.

Problem 3. Are reduced totally projective p-groups (in particular, reduced countable p-groups) necessarily (strongly) projectively fully transitive?

In closing we also state the following specification:

Note. Question 2.2 from [D9] has obviously a negative solution. In fact, every Krylov transitive group G such that all elements of $p^\omega G$ have comparable Ulm sequences, is fully transitive. To show that, we apply subsequently the first part of Theorem 2.13 and Corollary 2.8 again from [D9]. Thus $G \oplus G$ has to be fully transitive, whence so is G as being a direct summand, as asserted.

3.4. **On commutator socle-regular Abelian p-groups.** Throughout our discussion, we shall focus on additively written Abelian p-groups, where p is a prime fixed for the rest of the present work, although many of the topics we investigate can be considered in a much wider context. The notion of a fully invariant subgroup of a group is, of course, a classical notion in algebra, as is the weaker notion of a characteristic subgroup. Kaplansky devoted a section of his famous "Little Red Book"[71] (see also [70]) to the study of such subgroups and, arising from this, he introduced the much-studied classes of transitive and fully transitive groups – see, for example, [27, 28, 29, 43]. Recall that a group G is said to be *transitive* (respectively, *fully transitive*) if given $x, y \in G$ with Ulm sequences $\mathrm{U}_G(x) = \mathrm{U}_G(y)$ (respectively, $\mathrm{U}_G(x) \leq \mathrm{U}_G(y)$), there exists an automorphism (respectively, an endomorphism) ϕ such that $\phi(x) = y$. But there are several other weaker notions which have been of interest: recall that a subgroup H of a group G is said to be *projection invariant* in G if $\pi(H) \leq H$ for all idempotent endomorphisms π of G – see, for instance, [57, 88, 36] as well as [D9] – while a subgroup H of G is said to be *commutator invariant* in G if $[\phi, \psi](H) \leq H$ for all $\phi, \psi \in \mathrm{E}(G)$, where, as usual, $[\phi, \psi]$ denotes the additive commutator $\phi\psi - \psi\phi$. These two notions are independent of each other; in fact, there is a commutator invariant subgroup that is not projection invariant, and a projection invariant subgroup which is not commutator invariant. For the first case, consider the group $A = \langle a \rangle \oplus \langle b \rangle$ such that $\mathrm{o}(a) = p$ and $\mathrm{o}(b) = p^3$ with a proper subgroup $H = \langle a + pb \rangle$. It was established in [19] that H is commutator invariant in A but not a fully invariant subgroup. With the aid of [88] we also deduce that H is not projection invariant in A because in finite groups full invariance and projection invariance coincide. For the second case, the group G of Example 3.92 below will suffice; see the note immediately following the proof of Example 3.93 as well.

In [D8] and [35] the authors generalized the classes of transitive and fully transitive groups by focusing on the possible socles of characteristic and fully invariant subgroups (see [D9] too). In [36] full invariance was replaced by projection invariance and the current work continues this theme by replacing full invariance with commutator invariance. Our interest in this was sparked by the timely appearance of Chekhlov's interesting paper [19].

We show that in relation to commutator socle-regularity, one can restrict attention to reduced groups: if $A = D \oplus R$, where D is divisible and R is reduced, then A is commutator socle-regular if, and only if, R is commutator socle-regular- Theorem 3.83. Using realization results of Corner, we establish a useful method of constructing groups whose commutator socle-regularity is precisely determined by that of its first Ulm subgroup. We then exploit this result to show, *inter alia*, that for groups G with $G/p^\alpha G$ totally projective and $\alpha < \omega^2$, commutator socle-regularity of G is determined by that of $p^\alpha G$ - Theorem 3.89; on the other hand we construct groups G, K with $p^\omega G = p^\omega K$ but K is commutator socle-regular while G is not - Example 3.90.

Next, we relate the various notions of socle-regularity that have previously been investigated in [D8], [D9] and [35, 36] with commutator socle-regularity. Our principal results show that the notions are equivalent when the group involved is the direct sum of at least two copies of a fixed group - Theorem 3.95 - but we provide examples showing that the notions are, in fact, different in general. It follows easily from this that summands of commutator socle-regular groups need not be commutator socle-regular- Corollary 3.97. However, we also show that the addition of a separable summand to a group does not influence commutator socle-regularity - Theorem 3.98.

Our interest here will focus on the Abelian p-groups involved but we should point out that a ring-theoretic perspective is also possible: Kaplansky in [72] raised the notion of rings in which every element is a sum of additive commutators - the so-called *commutator rings*. These too have been the subject of a great deal of interest; see, e.g., the recent significant work of Mesyan in [90].

We re-iterate that all groups throughout the current work are additively written Abelian p-groups, where p is an arbitrary but fixed prime. Our notation and terminology not explicitly stated herein are standard and follow mainly those in [44, 47]. As usual, $\mathrm{E}(G)$ denotes the *endomorphism ring* of a group G. We close this introduction by recalling an important result of A.L.S. Corner from [28, Theorem 6.1] which we shall use repeatedly in the sequel: *If H is a countable bounded p-group and Φ is a countable subring of $E(H)$, then H may be imbedded as the subgroup $p^\omega G$ of a p-group G such that $E(G)$ acts on H as Φ and with the property that each $\phi \in \Phi$ extends to an endomorphism ϕ^* of G.* The mapping $\phi \mapsto \phi^*$ may even be taken as a semigroup homomorphism between the respective multiplicative semigroups of the rings; we shall need this semigroup property only in Example 3.93. We shall also exploit the groups constructed by Corner using this imbedding result: there is a fully transitive non-transitive p-group with first Ulm subgroup elementary of countably infinite rank and a transitive 2-group which is not fully transitive having a finite first Ulm subgroup which is the direct sum of cycles of order 2 and 8 - see Sections 3 & 4 in [29] and [51] for further details as well as Chapter III for some related results in ring theory relevant to the endomorphism ring of such abelian groups.

The construction of examples in this area invariably leads one to considerable amounts of reasonably straightforward but somewhat laborious calculations. These calculations have been recorded separately in an Appendix in order not to interfere with the presentation of results.

In the upcoming lenes we investigate some of the fundamental properties of the class of commutator socle-regular groups; we begin with the appropriate definitions.

Definition 1. A subgroup C of a group G is said to be *commutator invariant* if $f(C) \leq C$ for every $f \in \mathrm{E}(G)$ which is of the form $f = [\phi, \psi] = \phi\psi - \psi\phi$, where $\phi, \psi \in \mathrm{E}(G)$.

Clearly each fully invariant subgroup is commutator invariant, whereas the converse fails (see, e.g., [19]). Nevertheless, in some concrete situations, commutator invariant subgroups are fully invariant. Specifically, the following result from [19] holds:

Proposition 3.80. *(Chekhlov) Suppose A is a group such that $A = \oplus_{i\in I} G$ for some group G, where $|I| > 1$. Then in A any commutator invariant subgroup is fully invariant.*

Proof. We outline an alternative approach to that in [19], utilizing Mesyan's result [90] and some standard matrix representation. Let H be an arbitrary commutator invariant subgroup of A. If $|I|$ is infinite, then every element of $\mathrm{E}(A)$ is a sum of commutators – see [90, Theorem 13] – and so if H is commutator invariant, it is then certainly fully invariant.

Suppose then that $A = \bigoplus_{i=1}^{n} G_i$, $n > 1$, where each $G_i \cong G$, say. Let $E_{ij}(s)$ be the $n \times n$ matrix over the ring $S = \mathrm{E}(G)$ with ij^{th}-entry equal to s and all other entries zero. Recall that an arbitrary endomorphism of A can be represented as an $n \times n$ matrix Δ over S, $\Delta = \begin{pmatrix} d_{11} & \dots & d_{1n} \\ \cdot & \dots & \cdot \\ d_{n1} & \dots & d_{nn} \end{pmatrix}$. Now $E_{ij}(d_{ij})E_{jj}(1) = E_{ij}(d_{ij})$ while $E_{jj}(1)E_{ij}(d_{ij}) = 0$ provided $i \neq j$. So, for $i \neq j$, $E_{ij}(d_{ij})$ is a commutator. Hence $\Delta = diag\{d_{11}, \dots, d_{nn}\} + \Delta'$, where Δ' is a sum of commutators. Thus, to establish that H is fully invariant, it suffices to show that H is invariant under the diagonal matrix $diag\{d_{11}, \dots, d_{nn}\}$; in fact, it follows easily that it will suffice to show that H is invariant under the diagonal matrix $diag\{d, 0, \dots, 0\}$, where $d = d_{11}$.

Now $E_{n1}(d)$ is a commutator, so if $(g_1, \dots, g_n)^t \in H$ - we are writing elements of G as column vectors and using $(\ )^t$ to denote transposes - then it follows that the matrix product $E_{n1}(d).(g_1, \dots, g_n)^t = (0, \dots, 0, dg_1)^t$ is also an element of H. However, the matrix obtained by interchanging the first and last columns of the identity matrix and 0 elsewhere is also a commutator: $E_{1n}(1) + E_{n1}(1) = [(E_{1n}(1) + E_{n1}(-1)), E_{nn}(1)]$. It follows immediately now that

$$diag\{d, 0, \dots, 0\}.(g_1, \dots, g_n)^t = (dg_1, 0, \dots, 0)^t \in H$$

and so H has the required invariance property. □

The next result is elementary and we state it without proof for convenience of reference; the content also appears in [19].

Lemma 3.81. *(i) If A is a commutator invariant subgroup of the fully invariant subgroup B of a group C, then A is commutator invariant in C.*

(ii) If A is fully invariant in B and B is a a commutator invariant subgroup of C, then A is commutator invariant in C.

Motivated by similar definitions used previously in [D8] and [35, 36], we introduce the following:

Definition 2. A group G is said to be *commutator socle-regular* if, for each commutator invariant subgroup C of G, there exists an ordinal α (depending on C) such that $C[p] = (p^\alpha G)[p]$.

Our first observation is that the property of being commutator socle-regular is inherited by certain subgroups.

Proposition 3.82. *If G is a commutator socle-regular group, then so is $p^\beta G$ for all ordinals β.*

Proof. Let C be a commutator-invariant subgroup of $p^\beta G$. Since the latter is fully invariant in G, it follows from Lemma 3.81 that C is commutator invariant in G. Consequently, there is an ordinal α such that $C[p] = (p^\alpha G)[p]$. Intersecting both sides of the last equality with $p^\beta G$, we obtain that $C[p] = (p^\gamma G)[p]$ where $\gamma = \max(\alpha, \beta)$. But $\gamma = \beta + \delta$ for some $\delta \geq 0$, so that we can write $C[p] = (p^\delta(p^\beta G))[p]$, as required. □

The next result allows us to restrict our attention hereafter to reduced groups.

Theorem 3.83. *(i) If D is a divisible group, then D is commutator socle-regular.*

(ii) Let $A = D \oplus R$ be a group, where D is a divisible subgroup and R is a reduced subgroup. Then A is commutator socle-regular if, and only if, R is commutator socle-regular.

Proof. (i) If H is a commutator invariant subgroup of D, then it follows from Proposition 3.80 that H is fully invariant in D. Then H has the form $H = D$ or $H = D[p^n]$ for some non-negative integer n – see, for example, Exercise 68 in [71]. Hence, in both situations, we have $H[p] = (D[p^n])[p] = D[p]$, as required.

(ii) **"Necessity".** Suppose that C is an arbitrary commutator invariant subgroup of R. We claim that $D \oplus C$ is then a commutator invariant subgroup of A. Assuming we have established this, it follows that $(D \oplus C)[p] = D[p] \oplus C[p] = (p^\alpha A)[p] = (p^\alpha D)[p] \oplus (p^\alpha R)[p]$ for some ordinal α. Thus it readily follows that $C[p] = (p^\alpha R)[p]$. Hence it remains only to establish the claim.

Since endomorphisms of A have matrix representations as upper triangular matrices, an easy calculation shows that any commutator homomorphism in $\mathrm{E}(A)$ must have the form $\Delta = \begin{pmatrix} [\alpha,\alpha_1] & \delta \\ 0 & [\beta,\beta_1] \end{pmatrix}$ for endomorphisms α, α_1 of D, β, β_1 of R and a homomorphism $\delta : R \to D$. Since C is commutator invariant in R, it follows easily that $\Delta(D \oplus C) \leq D \oplus C$, as required.

"Sufficiency". Given that K is an arbitrary commutator invariant subgroup of A, Theorem 2 from [19] ensures that K has one of the forms $K = D \oplus C$ or $K = D[p^t] \oplus C$ for some $t \in \mathbb{N} \cup \{0\}$, where in both cases C is a commutator invariant subgroup of R. In the first case, $K[p] = D[p] \oplus C[p] = D[p] \oplus (p^\lambda R)[p] = (D \oplus p^\lambda R)[p] = (p^\lambda D \oplus p^\lambda R)[p] = (p^\lambda A)[p]$, as desired. For the second case we have $K[p] = (D[p^t])[p] \oplus C[p] = D[p] \oplus C[p] = (p^\lambda A)[p]$, as required. □

For the remainder of the work, we shall assume that all groups being discussed, unless explicitly stated to the contrary, are reduced.

We shall made use of the following technical lemma in our next result.

Lemma 3.84. *Suppose that $A = \langle a \rangle, B = \langle b \rangle$ are isomorphic cyclic summands of order p^n of the group G. Then there is a commutator f from $E(G)$ such that $f(a) = b$ or $f(a) = b - sa$, where s is a unit* $\mod p^n$.

Proof. Since A is a finite group, it has the exchange property – see, e.g., [44, Theorem 72.1]. Thus if $G = A \oplus N = B \oplus M$, then there exist summands E_1, E_2 of B, M respectively such that $G = A \oplus E_1 \oplus E_2$; let $B = E_1 \oplus E_1', M = E_2 \oplus E_2'$ so that $A \cong E_1' \oplus E_2'$ – see [44, Section 72, (a), (b)]. Since B is cyclic, either $E_1 = \{0\}$ or $E_1 = B$.

Case (1): If $E_1 = \{0\}$, then $E_1' = B$ and so $E_2' = \{0\}$, implying that $M = E_2$. So in this case we have

$$G = A \oplus M = B \oplus M.$$

Case (2): If $E_1 = B$, then

$$G = A \oplus B \oplus E_2.$$

We now consider the cases separately:

Case (1): $G = A \oplus M = B \oplus M$.

Note that if $a = rb+m$ and $b = sa+m_1$ for some $m, m_1 \in M$, with r, s integers mod p^n, then $a = rsa+(rm_1+m)$, whence we deduce that $rs \equiv 1 \mod p^n$. Now define $\phi : G \to G$ by $\phi(a) = sb, f(m) = 0$ for all $m \in M$, and $\psi : G \to G$ by $\psi(b) = a, \psi(m) = 0$ for all $m \in M$. Set $f = \phi\psi - \psi\phi$; a direct calculation shows that $f(a) = b - sa$, as required.

Case (2): $G = A \oplus B \oplus E_2$.

Define $\phi : G \to G$ by $\phi(a) = b, \phi(b) = 0$ and $\phi(e) = 0$ for all $e \in E_2$, and $\psi : G \to G$ by $\psi(b) = b, \psi(a) = 0$ and $\psi(e) = 0$ for all $e \in E_2$. Set $f = \phi\psi - \psi\phi$; a direct computation shows that $f(a) = b$, as required. □

Suppose H is an arbitrary subgroup of the group G. Let $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min_G(H[p])$; the inclusion $H[p] \leq (p^\alpha G)[p]$ clearly holds. Our next result illustrates some elementary but useful properties of the function $\min_G$.

Proposition 3.85. *If C is a commutator-invariant subgroup of the group G and $min_G(C[p]) = n$, a natural number, then $C[p] = (p^n G)[p]$.*

Proof. Suppose that C is an arbitrary commutator-invariant subgroup of G and $\min_G(C[p]) = n$, a finite integer. Therefore, there is an element $x \in C[p]$ such that $h_G(x) = n$ and so $x = p^n y$ where y is the generator of a direct summand of G, say $G = \langle y \rangle \oplus G_1$; see Corollary 27.2 from [44]. Let $z \in (p^n G)[p] \setminus (p^{n+1} G)[p]$, so that we write $z = p^n w$ for some element w of height zero; thus $G = \langle w \rangle \oplus G_2$. Notice that $\langle w \rangle \cong \mathbb{Z}(p^{n+1}) \cong \langle y \rangle$. By Lemma 3.84, there is a commutator endomorphism f of G such that $f(y) = w$ or $f(y) = w - sy$. Thus we have that $f(x) = z$ or $f(x) = z - sx$ for some s. Since $x \in C$ and C is commutator invariant in G, either $z \in C$ or $z - sx \in C$; in either case we can conclude that $z \in C$.

If now z' is an arbitrary element of $(p^{n+1}G)[p]$, then $z+z' \in (p^nG)[p]\setminus(p^{n+1}G)[p]$ and so $z + z' \in C$, whence $z' \in C$. Hence $(p^nG)[p] \leq C$. Since $\min_G(C[p]) = n$, we certainly have $C[p] \leq (p^nG)[p]$ and so we obtain the desired equality $C[p] = (p^nG)[p]$. □

The next result is simple but worthwhile for further applications.

Proposition 3.86. *Any large subgroup of a reduced commutator socle-regular group is also commutator socle-regular.*

Proof. Let C be a commutator invariant subgroup of a large subgroup L of a commutator socle-regular group G. If $\min_L(C[p])$ is finite, n say, then it follows from Proposition 3.85 that $C[p] = (p^nL)[p]$. If $\min_L(C[p])$ is infinite then so also is $\min_G(C[p])$, thus $C[p] \leq (p^\beta G)[p]$ for some infinite ordinal β. However, utilizing Lemma 3.81, C is commutator invariant in G as well, so $C[p] = (p^\alpha G)[p]$ for some ordinal α and it is immediate that $\alpha \geq \beta$ is infinite. It follows from [7] that $p^\alpha G = p^\alpha L$, whence $C[p] = (p^\alpha L)[p]$. Thus L is commutator socle-regular, as claimed. □

An examination of the proof of the proposition above shows that the result holds for any fully invariant subgroup F of a group G having the property that $p^\omega F = p^\omega G$ (compare also the difference with Example 3.90 below).

Our next proposition is somewhat technical but will enable us to deduce some interesting consequences.

Proposition 3.87. *If G is a group with $p^\omega G = H$ and for each $\phi \in E(H)$ there is an endomorphism $\phi^* \in E(G)$ with $\phi^* \restriction H = \phi$, then G is commutator socle-regular if, and only if, H is commutator socle-regular.*

Proof. The necessity follows from Proposition 3.82 above.

Conversely, to treat the sufficiency, suppose that H is commutator socle-regular and let C be an arbitrary commutator invariant subgroup of G. If $\min_G(C[p])$ is finite then it follows from Proposition 3.85 that $C[p] = (p^nG)[p]$ for some finite n. If $\min_G(C[p])$ is infinite, then $C[p] \leq H$. We claim that $C[p]$ is actually a commutator invariant subgroup of H. Assuming this for the moment, we conclude, as H is commutator socle-regular, that $C[p] = (p^\alpha H)[p]$ for some ordinal α and hence $C[p] = (p^\alpha(p^\omega G))[p] = (p^{\omega+\alpha}G)[p]$, as required.

It remains then to establish the claim. If $f = \phi\psi - \psi\phi$ is any commutator in $\mathrm{E}(H)$, then $f^* = \phi^*\psi^* - \psi^*\phi^*$ is a commutator in $\mathrm{E}(G)$. But if $x \in H$, then $(\phi^*\psi^*)(x) = \phi^*(\psi(x))$ since $\psi^* \restriction H - \psi$; note that $y = \psi(x) \in H$ since $\psi \in \mathrm{E}(H)$. Thus $(\phi^*\psi^*)(x) = \phi^*(y) = \phi(y) = \phi(\psi(x)) = (\phi\psi)(x)$ and we have that $(\phi^*\psi^*) \restriction H = \phi\psi$; similarly $(\psi^*\phi^*) \restriction H = \psi\phi$. In particular, if $x \in C[p]$, then $f(x) = f^*(x) \in C[p]$ since C is a commutator invariant subgroup of G which in turn makes $C[p]$ commutator invariant in G. Since f was an arbitrary commutator in $\mathrm{E}(H)$, we conclude that $C[p]$ is a commutator invariant subgroup of H, as claimed. $\square$

In the proof of our next theorem we shall need an easy extension of a well-known result on extending automorphisms from the subgroup p^nG, n an integer, to automorphisms of the whole group G. It is possible to prove this directly using a modification of the argument in [44, Proposition 113.3] but we give here a simple argument which utilizes the result for automorphisms given by Fuchs.

Lemma 3.88. *If n is finite and ϕ is an arbitrary endomorphism of the subgroup p^nG of G, then ϕ extends to an endomorphism ϕ^* of G.*

Proof. Consider the group $H = G \oplus G$ and note that $p^nH = p^nG \oplus p^nG$. Regard endomorphisms of H as 2×2 matrices over $\mathrm{E}(G)$ and endomorphisms of p^nH as 2×2 matrices over $\mathrm{E}(p^nG)$. Let $\phi \in \mathrm{E}(p^nG)$ be arbitrary. Then $\Delta = \begin{pmatrix} \phi & 1_{p^nG} \\ 1_{p^nG} & 0 \end{pmatrix}$ is an endomorphism of p^nH which is easily seen to actually be an automorphism. By [44, Proposition 113.3], Δ extends to an automorphism $\Delta^* = \begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix}$ of H, where $\alpha, \beta, \gamma, \delta \in \mathrm{E}(G)$. Thus $\Delta\binom{x}{0} = \Delta^*\binom{x}{0}$ for all $x \in p^nH$, i.e., $\binom{\phi(x)}{x} = \binom{\alpha(x)}{\delta(x)}$. Set $\phi^* = \alpha$, an endomorphism of G and note that $\phi^* \restriction p^nG = \alpha \restriction p^nG$, as required. $\square$

Our next result indicates, *inter alia*, that the class of commutator socle-regular groups is quite large.

Theorem 3.89. *(i) If G is a group such that either $p^\omega G = \{0\}$ or $p^\omega G \cong \mathbb{Z}(p^n)$ for some finite n, then G is commutator socle-regular;*

(ii) A group G is commutator socle-regular if, and only if, p^nG is commutator socle-regular for some $n \in \mathbb{N}$;

(iii) If G is a group such that $G/p^\alpha G$ is totally projective for some ordinal $\alpha < \omega^2$, then G is commutator socle-regular if, and only if, $p^\alpha G$ is commutator socle-regular;

(iv) Totally projective groups of length $< \omega^2$ are commutator socle-regular.

Proof. (i) follows immediately from Proposition 3.87 and the observation that in either case the endomorphisms of $p^\omega G$ are scalars and hence give rise in a natural way to the desired semigroup homomorphism.

The necessity in (ii) follows directly from Proposition 3.82. The proof of sufficiency is similar to the proof of Proposition 3.87; let K be a commutator-invariant subgroup of G and if $\min_G(K[p])$ is finite, say m, then with the aid of Proposition 3.85 we may write $K[p] = (p^m G)[p]$, as required. Otherwise, if $\min_G(K[p]) \geq \omega$, then clearly $K[p] \leq p^\omega G \leq p^n G$. We assert that $K[p]$ is a commutator-invariant subgroup of $p^n G$. This follows as in Proposition 3.87 using Lemma 3.88 to deduce that endomorphisms of $p^n G$ extend to endomorphisms of G. Since $p^n G$ is commutator socle-regular, we have that $K[p] = (p^\alpha(p^n G))[p]$ for some ordinal α. Consequently, $K[p] = (p^{n+\alpha} G)[p]$ and G is commutator socle-regular, as desired.

We will establish (iii) by first considering the case $\alpha = \omega$. In this special case the proof follows from Proposition 3.87 and the observation that as $G/p^\omega G$ is totally projective, it follows from [61, Theorem 2] that every endomorphism of $p^\omega G$ extends to an endomorphism of G, thereby giving the extension property required to apply Proposition 3.87.

Suppose now α has the form $\alpha = \omega \cdot m$ for some $1 < m < \omega$. Since $p^\alpha G = p^{\omega \cdot m} G = p^\omega(p^{\omega\cdot(m-1)} G)$ is commutator socle-regular and $G/p^\alpha G = G/p^{\omega\cdot m} G$ is totally projective, whence so is $p^{\omega\cdot(m-1)}(G/p^\alpha G) = p^{\omega\cdot(m-1)} G/p^{\omega\cdot m} G$, we apply the preceding case $\alpha = \omega$ for $A = p^{\omega\cdot(m-1)} G$ to derive that $p^{\omega\cdot(m-1)} G$ is commutator socle-regular. Moreover, as $G/p^\alpha G$ is totally projective so also is $G/p^\beta G$ for any $\beta < \alpha$. Thus, after $m-1$ steps, we deduce that $p^\omega G$ is commutator socle-regular and $G/p^\omega G$ is a direct sum of cyclic groups. Again by what we have shown in the previous paragraph, G will be commutator socle-regular, finishing this case.

Finally, consider the case where $\alpha = \omega \cdot m + n$ with $m, n < \omega$. Since $p^\alpha G = p^{\omega\cdot m+n} G = p^n(p^{\omega\cdot m} G)$ is commutator socle-regular, we can conclude from (ii) above that the same holds for $p^{\omega\cdot m} G$. As already observed, if $G/p^\alpha G$ is totally projective, then so also is $G/p^{\omega\cdot m} G$. We therefore may employ the previous step to conclude that G is commutator socle-regular, indeed.

Part (iv) follows immediately from (iii) by choosing α to be the length of G. $\square$

Our next example shows that one cannot extend part (i) of the preceding theorem even to the situation where $p^\omega G$ is an elementary group of rank 2.

Example 3.90. There are groups G, K with $p^\omega G = \mathbb{Z}(p) \oplus \mathbb{Z}(p) = p^\omega K$ where K is commutator socle-regular but G is not.

Proof. Let $H = \langle a \rangle \oplus \langle b \rangle$, where each of a, b is of order p. Let Φ denote the subring of $\mathrm{E}(H)$ consisting (in the usual matrix representation) of the 2×2

upper triangular matrices Δ over the field of p elements. A straightforward calculation gives that any commutator in Φ is strictly upper triangular, i.e., the diagonal entries are also 0. Applying Theorem 6.1 in [28], we find a group G with $p^\omega G = H$ such that $\mathrm{E}(G)$ acts on $p^\omega G$ as Φ. Consequently, if ϕ is any commutator in $\mathrm{E}(G)$, then ϕ acts on $p^\omega G$ as a strictly upper triangular matrix. In particular, any commutator maps the subgroup $\langle a \rangle$ to 0 and so $\langle a \rangle$ is commutator invariant. But clearly $\langle a \rangle = \langle a \rangle[p]$ cannot have the form $(p^\alpha G)[p]$ for any ordinal α and hence G is not commutator socle-regular.

The construction of K is similar, but this time we take Φ to be the full endomorphism ring of H. An application of Theorem 6.1 in [28] yields a group K with $p^\omega K = H$ and a function $()^*$ from $\mathrm{E}(H) \to \mathrm{E}(K)$ with the properties required to apply Proposition 3.87. Since the finite group H is certainly commutator socle-regular, it follows immediately from Proposition 3.87 that K is also commutator socle-regular. $\square$

We remark that it is possible to give a much simpler example than the group G constructed above - for instance, the commutative subring of diagonal matrices would suffice - but, as we shall have need of this particular example later, we have chosen to give this slightly more complicated construction here.

We now arrive at the various classes of socle-regularity as follows: In the previous subsections the authors have considered various notions of socle-regularity. These notions have a great degree of similarity since they may be defined in a common way as follows:

A group G is said to be **-socle-regular* if every *-subgroup P of G has the property that $P[p] = (p^\alpha G)[p]$ for some ordinal α.

When *-subgroup corresponds to fully invariant (characteristic) subgroup, we get the notions that were called socle-regular (strongly socle-regular) groups in [D8] and [35]; when *-subgroup corresponds to projection invariant (commutator invariant) subgroup, we get the notion of projectively socle-regular (commutator socle-regular) groups introduced in [36] and the present work respectively.

It is easy to see that the class of socle-regular groups contains each of the other three classes. In this section we investigate the relationships between these different classes; recall that it follows from examples given in [35, 36] that the strongly socle-regular and projectively socle-regular classes are properly contained in the class of socle-regular groups. It was also established in [D8] that fully transitive groups are socle-regular, while in [35] that transitive groups are strongly socle-regular.

Our first example shows that the class of commutator socle-regular groups is also properly contained in the class of socle-regular groups.

Example 3.91. There is a transitive (and hence strongly socle-regular) group which is neither commutator socle-regular nor projectively socle-regular.

Proof. Let G be the transitive non-fully transitive group constructed as in [29]. Recall that G is a 2-group with $2^\omega G = \langle a\rangle \oplus \langle b\rangle = A$, where $o(a) = 2,\ o(b) = 8$ and the restriction of $\mathrm{E}(G)$ to A, $\mathrm{E}(G) \restriction A = \Phi$, where Φ is the subring generated by the automorphisms of A. This group has been thoroughly investigated in [51]; note that the elements of Φ can be described by two families $\{\theta_{i\lambda}\}$ and $\{\phi_{j\mu}\}$ with the parameters $1 \leq i, j \leq 4$ and $\lambda \in \{\pm 1, \pm 3\},\ \mu \in \{0, \pm 1, 2\}$. The images of the element a under Φ are restricted to the possibilities $0, a, 4b, a + 4b$ and then a straightforward, but somewhat laborious, calculation - see the Appendix for details - shows that every commutator of the form $[\alpha, \beta]$ with $\alpha, \beta \in \Phi$ maps $a \mapsto 0$. We claim that $\langle a\rangle$ is commutator invariant in G. For if $[\gamma, \delta]$ is any commutator in $\mathrm{E}(G)$, then $[\gamma, \delta](a) = [\alpha, \beta](a)$ for some $\alpha, \beta \in \Phi$ and so, by the previous observation, we have $[\gamma, \delta](a) = 0$. So $\langle a\rangle$ is certainly a commutator invariant subgroup of G. However, a direct computation shows that $\langle a\rangle[2] = \langle a\rangle$ is not equal to any of the subgroups $(2^\omega G)[2], (2^{\omega+1}G)[2], (2^{\omega+2}G)[2]$ and since $\langle a\rangle$ cannot be of the form $(2^n G)[2]$ for any finite n, we conclude that $\langle a\rangle[2] \neq (2^\lambda G)[2]$ for any ordinal λ and so G is not commutator socle-regular.

However, G is transitive and hence, by [35, Theorem 4], it is strongly socle-regular; moreover, it follows from [36, Proposition 1.13] that G is not projectively socle-regular. $\square$

Our next two examples demonstrate that the classes of commutator socle-regular, projectively socle-regular groups and strongly socle-regular groups are distinct.

Example 3.92. There exists a fully transitive commutator socle-regular group that is neither projectively socle-regular nor strongly socle-regular.

Proof. Suppose that G is the example constructed by Corner in [29] of a non-transitive fully transitive group with $p^\omega G \cong H = \bigoplus_{\aleph_0} \mathbb{Z}(p)$ and having the property that $\mathrm{E}(G) \restriction p^\omega G = \Phi$ acts as a dense algebra of endomorphisms of H. Claim that G is commutator socle-regular.

To see this, let C be an arbitrary commutator invariant subgroup of G. If $\min C[p]$ is finite, then with Proposition 3.85 at hand we have that $C[p] = (p^n G)[p]$ for some finite integer n; if not, then $C[p] \leq (p^\omega G)[p]$. Now suppose that $0 \neq c \in C[p]$ and let x be an arbitrary element of $(p^\omega G)[p]$ which is linearly independent of

c. It is straightforward to show that there is a commutator $\phi \in \mathrm{E}(\langle c\rangle \oplus \langle x\rangle)$ with $\phi(c) = x$; say $\phi = fg - gf$ for $f, g \in \mathrm{E}(\langle c\rangle \oplus \langle x\rangle)$. Now, as observed by Corner [29, p. 19], the density property of Φ means that every endomorphism of a finite subgroup of $p^\omega G$ extends to an endomorphism of G; in particular f, g extend to mappings f', g' of G and so there is a commutator $\phi' = f'g' - g'f' \in \mathrm{E}(G)$ such that $\phi'(c) = \phi(c) = x$. Since $C[p]$ is obviously commutator invariant in G, it follows that $x \in C[p]$. Consequently, if the socle of an arbitrary commutator invariant subgroup of G is contained in $p^\omega G$, then it must equal $p^\omega G$ itself. It now follows immediately that G is commutator socle-regular.

However, G is not projectively socle-regular - see [36, Proposition 1.7] as well as it is not strongly socle-regular - see [35, Theorem 2.3]. □

Example 3.93. There is a projectively socle-regular group (and hence strongly socle-regular p-group for $p > 2$) which is not commutator socle-regular.

Proof. We utilize the group G constructed previously in Example 3.90 having $p^\omega G = H = \langle a\rangle \oplus \langle b\rangle$, where each of a, b is of order p and where $\mathrm{E}(G)$ acts on H as the subring Φ of $\mathrm{E}(H)$ consisting (in the usual matrix representation) of the 2×2 upper triangular matrices Δ over the field of p elements. We have seen in that example that G is not commutator socle-regular.

We claim, however, that G is projectively socle-regular. Observe firstly that the only idempotent matrices in Φ are the trivial zero and identity matrices along with the four matrices $\Delta_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \Delta_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right), \Delta_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Delta_4 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$; this is easily verified by a simple matrix calculation.

Now suppose that $0 \neq P$ is a projection invariant subgroup of G. If $\min_G P[p]$ is finite, then $P[p] = (p^n G)[p]$ for some finite n by Proposition 1.1 of [36]. If $\min_G P[p]$ is infinite, then $P[p]$ is a projection invariant subgroup of $H = p^\omega G$. It follows from Corner's construction that if π is an idempotent in Φ, then the corresponding extension $\pi^* \in \mathrm{E}(G)$ is also an idempotent, since the mapping $()*$ is actually a semigroup homomorphism from the multiplicative semigroup of $\mathrm{E}(H)$ to that of $\mathrm{E}(G)$, and $\pi^* \restriction H = \pi$. Since $P[p]$ is projection invariant in both $p^\varrho G$ and G, it follows that $\Delta_i(P[p]) \leq P[p]$ for $1 \leq i \leq 4$.

Let $(0, 0) \neq (ua, vb) \in P[p]$, where u, v are integers. If both $u, v \neq 0$, then applying Δ_1 and Δ_3 to the element (ua, vb) gives us that both $(ua, 0)$ and $(0, vb)$ belong to $P[p]$ and it follows readily that $P[p]$ must then be all of H, i.e., $P[p] = (p^\omega G)[p]$. If $u = 0, v \neq 0$, then applying Δ_4 to $(0, vb)$ we get that $(va, 0)$, and hence also $(a, 0)$, belongs to $P[p]$; this again implies that $P[p] = H = H[p]$. If finally $u \neq 0, v = 0$, then an identical argument using Δ_2 yields the same result.

In summary, we deduce that $P[p] = (p^\omega G)[p]$, and G is therefore projectively socle-regular, as required.

By taking $p \neq 2$, we obtain with Proposition 1.5 of [36] at hand that G is strongly socle-regular, as asserted. $\square$

Note that it follows immediately that the group in Example 3.92 has a projection invariant subgroup which is not commutator invariant, while the group in Example 3.93 has a commutator invariant subgroup which is not projection invariant.

Our final example shows that full transitivity is not enough to ensure commutator socle-regularity; our construction is given as a 2-group, but this was merely to simplify calculations and plays no real part.

Example 3.94. There exists a fully transitive (and hence socle-regular) group that is not commutator socle-regular.

Proof. Let H be the finite group $\langle a_1\rangle \oplus \langle a_2\rangle \oplus \langle a_3\rangle$, where a_i has order $2^i (i = 1, 2, 3)$. Let e_{ii} denote the canonical projection of H onto a_i; let $\sigma_{ij}(i < j)$ be the forward shift mapping $a_i \mapsto 2^{j-i}a_j$ and denote by $\tau_{ji}(j > i)$, the backward shift mapping $a_j \mapsto a_i$. Set Φ to be the subring of $\mathrm{E}(H)$ generated by $\{e_{11}, (e_{22} + e_{33}), \sigma_{12}, \sigma_{13}, \tau_{21}, \sigma_{23}, \tau_{31}, \tau_{32}\}$. It is easy to check that the ring generators are linearly independent of additive orders $2, 2^3, 2, 2, 2, 2^2, 2, 2^2$, so that additively they generate a subgroup of order 2^{12}. Hennecke [59] has shown that this subring acts fully transitively on G and has order 2^{13}, so that additively Φ is not the direct sum of the subgroups generated by the elements listed above. However, the product $\tau_{32}\sigma_{23} = 2e_{22}$ is an element of Φ and it follows easily that the enlarged set $S = \{2e_{22}, e_{11}, (e_{22} + e_{33}), \sigma_{12}, \sigma_{13}, \tau_{21}, \sigma_{23}, \tau_{31}, \tau_{32}\}$ of linearly independent elements generates the ring Φ additively since the subgroup generated by S has order 2^{13}. Hence to check the possible actions of commutators from Φ on G, it suffices to consider commutators involving the elements of S. Moreover, since a commutator $[\alpha, \beta] = -[\beta, \alpha]$, we can reduce the calculations by half. On the other hand, a straightforward, but rather tedious, direct calculation – see the Appendix listed below – shows that the commutators of the additive generators of Φ map a_2 to either $0, 2a_2$ or $4a_3$. It follows that the cyclic subgroup $\langle 2a_2\rangle$ is then mapped to 0 by the commutators of Φ.

Now use Corner's realization result to construct a 2-group G such that $2^\omega G = H$ and $\mathrm{E}(G)$ acts on $2^\omega G$ as Φ. It follows immediately that G is fully transitive, whence by Theorem 0.3 of [D8] it is socle-regular. Furthermore, the action of $\mathrm{E}(G)$ assures that the subgroup $\langle 2a_2\rangle$ is commutator invariant in G. However, the socle of $\langle 2a_2\rangle$ is just the subgroup itself since a_2 has order 4 but $(2^\omega G)[2] = \langle a_1\rangle \oplus \langle 2a_2\rangle \oplus \langle 4a_3\rangle$, $(2^{\omega+1}G)[2] = \langle 2a_2\rangle \oplus \langle 4a_3\rangle$, $(2^{\omega+2}G)[2] = \langle 4a_3\rangle$, so that $\langle 2a_2\rangle$ is not a socle of the form $(2^\alpha G)[2]$ for any infinite α; since it is clearly not of the form $(2^n G)[2]$ for any finite n, we conclude that G is not commutator socle-regular, as required. $\square$

Nevertheless, in some specific cases, the concepts do coincide. As usual, for each cardinal $\kappa \geq 0$, the κ-power subgroup $G^{(\kappa)}$ denotes the direct sum $\oplus_\kappa G$ of κ copies of G.

Theorem 3.95. *Let $\kappa > 1$. The following conditions are equivalent:*
(i) G is socle-regular;
(ii) $G^{(\kappa)}$ is socle-regular;
(iii) $G^{(\kappa)}$ is strongly socle-regular;
(iv) $G^{(\kappa)}$ is projectively socle-regular;
(v) $G^{(\kappa)}$ is commutator socle-regular.

Proof. In view of Theorem 2.4 of [36], it suffices to obtain only the implication (ii) $\iff$ (v). The implication (v) $\Rightarrow$ (ii) is trivial, and the reverse implication follows easily from Proposition 3.80 above. $\square$

A direct consequence is the following:

Corollary 3.96. *If G is a commutator socle-regular group, then $G^{(\kappa)}$ is commutator socle-regular for any $\kappa \geq 0$.*

Proof. As we have seen above, every commutator socle-regular group is socle-regular. Thus [D8] applies to show that $G^{(\kappa)}$ is socle-regular. We now employ Theorem 3.95 to conclude that this κ-power group is commutator socle-regular, as desired. $\square$

Another consequence is that summands of commutator socle-regular groups need not be again commutator socle-regular.

Corollary 3.97. *A summand of a commutator socle-regular group is not necessarily commutator socle-regular.*

Proof. Let G be the socle-regular group from Example 3.91 above, which is not commutator socle-regular. However, it follows from Theorem 3.95 that $G \oplus G$ is commutator socle-regular. $\square$

Nevertheless, in a certain specific case the following direct summand property holds:

Theorem 3.98. *Suppose that $A = G \oplus H$ and H is separable. Then A is commutator socle-regular if, and only if, G is commutator socle-regular.*

Proof. Suppose that G is commutator socle-regular and X is a commutator invariant subgroup of A. If $\min_A(X[p])$ is finite then, by Proposition 3.85, $X[p] = (p^n A)[p]$ for some finite n. So, supposing $\min_A(X[p])$ is infinite, then $X[p] \le (p^\omega A)[p] = (p^\omega G)[p]$, as H is separable. However, X is a commutator invariant subgroup of A and so $X[p]$ is a commutator invariant subgroup of A which is actually contained in G. Since endomorphisms of G extend trivially to endomorphisms of A, it is easy to see that $X[p]$ is actually a commutator invariant subgroup of G and so $X[p] = (p^\lambda G)[p]$ for some ordinal λ. Thus $(p^\lambda G)[p] \le (p^\omega G)[p]$ and so $\lambda \ge \omega$. It follows immediately that $X[p] = (p^\lambda G)[p] = (p^\lambda A)[p]$ since $p^\lambda H = 0$.

Conversely, suppose that A is commutator socle-regular and let Y be an arbitrary commutator invariant subgroup of G. As before, if $\min_G(Y[p])$ is finite, then Proposition 3.85 assures that $Y[p] = (p^n G)[p]$ for some positive integer n. Suppose then that $\min_G(Y[p])$ is infinite, so that $Y[p] \le (p^\omega G)[p] = (p^\omega A)[p]$. We claim that $Y[p]$ is a commutator invariant subgroup of A. Assuming for the moment that we have established this claim, it then follows that $Y[p] = (p^\lambda A)[p]$ for some ordinal λ. Hence $Y[p] = (p^\lambda A)[p] \le (p^\omega A)[p]$, yielding $\lambda \ge \omega$. Since $p^\lambda A = p^\lambda G$ for $\lambda \ge \omega$, we get the desired result that $Y[p] = (p^\lambda G)[p]$ for some λ. It remains then only to establish the claim.

Observe firstly that if $\phi = \begin{pmatrix} \alpha & \delta \\ \gamma & \beta \end{pmatrix}$ and $\psi = \begin{pmatrix} \alpha_1 & \delta_1 \\ \gamma_1 & \beta_1 \end{pmatrix}$ are arbitrary endomorphisms of A (in the standard matrix representation), then the commutator $[\phi, \psi]$ can be represented as a matrix $\Delta = \begin{pmatrix} [\alpha,\alpha_1] & f \\ g & [\beta,\beta_1] \end{pmatrix}$, where $f : H \to G,\ g : G \to H$ are homomorphisms. Note, however, that as H is separable and $Y[p] \le (p^\omega G)[p]$, the image under g of each element of $Y[p]$ is necessarily 0. Identifying $Y[p]$ with $Y[p] \oplus 0$, a straightforward calculation shows that $\Delta(Y[p]) = [\alpha, \alpha_1](Y[p])$ and this is clearly contained in $Y[p]$ since Y is, by assumption, a commutator invariant subgroup of G. □

We finish with a question which we have not been able to resolve at this stage.

• Does there exist a commutator socle-regular group of length $\ge \omega^2$; in particular, is the restriction on the ordinal α in Theorem 3.89 (iii) necessary?

We now come to our Appendix containing the requested calculations:

Calculations for Example 3.91 Let A be the finite group defined as $A = \langle a \rangle \oplus \langle b \rangle$, where a has order 2 and b has order 8. Let Φ denote the subring of the full endomorphism ring generated by the automorphisms. It is known from [51] that Φ has order 32 and the elements of Φ can be labeled as $\{\theta_{1\lambda}, \theta_{2\lambda}, \theta_{3\lambda}, \theta_{4\lambda}\}(\lambda = \pm 1, \pm 3)$ and $\{\phi_{1\mu}, \phi_{2\mu}, \phi_{3,\mu}, \phi_{4\mu}\}(\mu = 0, \pm 1, 2)$. These are the mappings given by:

- $\theta_{1\lambda} : a \mapsto a, b \mapsto \lambda b$
- $\theta_{2\lambda} : a \mapsto a + 4b, b \mapsto \lambda b$
- $\theta_{3\lambda} : a \mapsto a, b \mapsto a + \lambda b$
- $\theta_{4\lambda} : a \mapsto a + 4b, b \mapsto a + \lambda b$
- $\phi_{1\mu} : a \mapsto 4b, b \mapsto 2\mu b$
- $\phi_{2\mu} : a \mapsto 0, b \mapsto a + 2\mu b$
- $\phi_{3\mu} : a \mapsto 4b, b \mapsto a + 2\mu b$
- $\phi_{4\mu} : a \mapsto 0, b \mapsto 2\mu b$.

In our calculations we shall frequently make use of the following simply verified statement without comment:

if $\lambda, \sigma \in \{\pm 1, \pm 3\}$ then $\lambda - \sigma$ is even; in particular, if $\lambda \in \{\pm 1, \pm 3\}$ then $\lambda - 1$ is even.

Our objective is to show that for every commutator $[\alpha, \beta]$, where $\alpha, \beta \in \Phi$, we have that $[\alpha, \beta](a) = 0$. Clearly we may reduce the amount of calculation by noting that $[\alpha, \beta] = -[\beta, \alpha]$.

(i) Commutators of the form $[\theta_{1\lambda}, \theta_{j\sigma}](j \geq 1)$ with $\lambda, \sigma \in \{\pm 1, \pm 3\}$:

- $[\theta_{1\lambda}, \theta_{1\sigma}](a) = 0$ since $\theta_{1*}(a) = a$ for all values of *;
- $[\theta_{1\lambda}, \theta_{2\sigma}](a) = a + 4(\lambda b) - (a + 4b) = 4(\lambda - 1)b = 0$ since $\lambda - 1$ is even;
- $[\theta_{1\lambda}, \theta_{3\sigma}](a) = 0$ since $\theta_{1\lambda}(a) = a = \theta_{3\sigma}(a)$;
- $[\theta_{1\lambda}, \theta_{4\sigma}](a) = a + 4\lambda b - (a + 4b) = 0$;

(ii) Commutators of the form $[\theta_{2\lambda}, \theta_{j\sigma}](j \geq 2)$ with $\lambda, \sigma \in \{\pm 1, \pm 3\}$:

- $[\theta_{2\lambda}, \theta_{2\sigma}](a) = (a + 4b) + 4(\lambda b) - \{a + 4b + 4(\sigma b)\} = 4(\lambda - \sigma)b = 0$;
- $[\theta_{2\lambda}, \theta_{3\sigma}](a) = a + 4b - \{a + 4(a + \sigma b)\} = 4(1 - \sigma)b = 0$;
- $[\theta_{2\lambda}, \theta_{4\sigma}](a) = (a + 4b) + 4(\lambda b) - \{a + 4b + 4(a + \sigma b)\} = 4(\lambda - \sigma)b = 0$;

(iii) Commutators of the form $[\theta_{3\lambda}, \theta_{j\sigma}](j \geq 3)$ with $\lambda, \sigma \in \{\pm 1, \pm 3\}$:

- $[\theta_{3\lambda}, \theta_{3\sigma}](a) = 0$ since $\theta_{3*}(a) = a$ for all values of *;
- $[\theta_{3\lambda}, \theta_{4\sigma}](a) = a + 4(a + \lambda b) - (a + 4b) = 4(\lambda - 1)b = 0$;

(iv) Commutators of the form $[\theta_{4\lambda}, \theta_{4\sigma}]$ with $\lambda, \sigma \in \{\pm 1, \pm 3\}$:

- $[\theta_{4\lambda}, \theta_{4\sigma}](a) = (a + 4b) + 4(a + \lambda b) - \{a + 4b + 4(a + \sigma b)\} = 4(\lambda - \sigma)b = 0$;

Thus we have that all commutators involving pairs of θ's map $a \mapsto 0$. Now consider the corresponding situation for the ϕ's.

(v) Commutators of the form $[\phi_{1\mu}, \phi_{j\nu}](j \geq 1)$ with $\mu, \nu \in \{0, \pm 1, 2\}$:

- $[\phi_{1\mu}, \phi_{1\nu}](a) = 4(2\mu b) - 4(2\nu b) = 0$;
- $[\phi_{1\mu}, \phi_{2\nu}](a) = 4(a + 2\mu b) = 0$;
- $[\phi_{1\mu}, \phi_{3\nu}](a) = 4(2\mu b) - 4(a + 2\nu b) = 0$;
- $[\phi_{1\mu}, \phi_{4\nu}](a) = 4(2\mu b) = 0$;

(vi) Commutators of the form $[\phi_{2\mu}, \phi_{j\nu}](j \geq 2)$ with $\mu, \nu \in \{0, \pm 1, 2\}$:

- $[\phi_{2\mu}, \phi_{2\nu}](a) = 0$ since $\phi_{2*}(a) = 0$ for all $* \in \{0, \pm 1, 2\}$;

- $[\phi_{2\mu}, \phi_{3\nu}](a) = 4(a + 2\mu b) = 0$;
- $[\phi_{2\mu}, \phi_{4\nu}](a) = 0 - 0 = 0$;

(vii) Commutators of the form $[\phi_{3\mu}, \phi_{j\nu}](j \geq 3)$ with $\mu, \nu \in \{0, \pm 1, 2\}$:

- $[\phi_{3\mu}, \phi_{3\nu}](a) = 4(a + 2\mu b) - 4(a + 2\nu b) = 0$;
- $[\phi_{3\mu}, \phi_{4\nu}](a) = -4(2\mu b) = 0$;

(viii) Commutators of the form $[\phi_{4\mu}, \phi_{4\nu}]$ with $\mu, \nu \in \{0, \pm 1, 2\}$:

- $[\phi_{4\mu}, \phi_{4\nu}](a) = 0$ since $\phi_{4*}(a) = 0$ for all $* \in \{0, \pm 1, 2\}$;

Thus we have that all commutators involving pairs of ϕ's map $a \mapsto 0$. Now consider the remaining "mixed" situations.

(ix) Commutators of the form $[\theta_{1\lambda}, \phi_{j\nu}](j \geq 1)$ with $\lambda \in \{\pm 1, \pm 3\}, \mu \in \{0, \pm 1, 2\}$

- $[\theta_{1\lambda}, \phi_{1\nu}](a) = 4(\lambda b) - 4b = 4(\lambda - 1)b = 0$;
- $[\theta_{1\lambda}, \phi_{2\nu}](a) = 0 - 0 = 0$;
- $[\theta_{1\lambda}, \phi_{3\nu}](a) = 4(\lambda b) - 4b = 4(\lambda - 1)b = 0$;
- $[\theta_{1\lambda}, \phi_{4\nu}](a) = 0 - 0 = 0$;

(x) Commutators of the form $[\theta_{2\lambda}, \phi_{j\nu}](j \geq 1)$ with $\lambda \in \{\pm 1, \pm 3\}, \mu \in \{0, \pm 1, 2\}$

- $[\theta_{2\lambda}, \phi_{1\nu}](a) = 4\lambda b - \{4b + 4(2\mu b)\} = 4(\lambda - 1)b = 0$;
- $[\theta_{2\lambda}, \phi_{2\nu}](a) = 0 - \{0 + 4(a + 2\mu b)\} = 0$;
- $[\theta_{2\lambda}, \phi_{3\nu}](a) = 4(\lambda b) - \{4b + 4(a + 2\mu b)\} = 4(\lambda - 1)b = 0$;
- $[\theta_{2\lambda}, \phi_{4\nu}](a) = 0 - \{0 + 4(2\mu b)\} = 0$;

(xi) Commutators of the form $[\theta_{3\lambda}, \phi_{j\nu}](j \geq 1)$ with $\lambda \in \{\pm 1, \pm 3\}, \mu \in \{0, \pm 1, 2\}$

- $[\theta_{3\lambda}, \phi_{1\nu}](a) = 4(a + 4b) - 4b = 4(\lambda - 1)b = 0$;
- $[\theta_{3\lambda}, \phi_{2\nu}](a) = 0 - 0$;
- $[\theta_{3\lambda}, \phi_{3\nu}](a) = 4(a + \lambda b) - 4b = 4(\lambda - 1)b = 0$;
- $[\theta_{3\lambda}, \phi_{4\nu}](a) = 0 - 0$;

(xii) Commutators of the form $[\theta_{4\lambda}, \phi_{j\nu}](j \geq 1)$ with $\lambda \in \{\pm 1, \pm 3\}, \mu \in \{0, \pm 1, 2\}$

- $[\theta_{4\lambda}, \phi_{1\nu}](a) = 4(a + \lambda b) - \{4b + 4(2\mu b)\} = 4(\lambda - 1)b = 0$;
- $[\theta_{4\lambda}, \phi_{2\nu}](a) = 0 - \{0 + 4(a + 2\mu b)\} = 0$;
- $[\theta_{4\lambda}, \phi_{3\nu}](a) = 4(a + \lambda b) - \{4b + 4(a + 2\mu b)\} = 4(\lambda - 1)b = 0$;
- $[\theta_{4\lambda}, \phi_{4\nu}](a) = 0 - \{0 + 4(2\mu b)\} = 0$.

Since the "mixed" commutators also map $a \mapsto 0$, we conclude that every commutator in Φ maps $a \mapsto 0$ so that the subgroup $\langle a \rangle$ is certainly invariant under the action of commutators from Φ.

Calculations for Example 3.94 Let $G = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle a_3 \rangle$, where the elements a_i have order 2^i. Define the following mappings $G \to G$:

- $e_{ii} : a_i \mapsto a_i,\ a_j \mapsto 0$ if $i \neq j$;
- for $i < j,\ \sigma_{ij} : a_i \mapsto 2^{j-i} a_j,\ a_k \mapsto 0$ if $k \neq i$;
- for $j < i,\ \tau_{ij} : a_i \mapsto a_j,\ a_k \mapsto 0$ if $k \neq i$.

Consider the subring Φ generated (as a ring) by $\{e_{11}, (e_{22} + e_{33}), \sigma_{12}, \sigma_{13}, \tau_{21}, \sigma_{23}, \tau_{31}, \tau_{32}\}$. It is easy to check that the ring generators are linearly independent of additive orders respectively

$2, 2^3, 2, 2, 2, 2^2, 2, 2^2$, so that additively they generate a subgroup of order 2^{12}. Hennecke [59] has shown that this subring acts fully transitively on G and has order 2^{13}, so that additively Φ is not the direct sum of the subgroups generated by the elements listed above. However, the product $\tau_{32}\sigma_{23} = 2e_{22}$ is an element of Φ and it follows easily that the enlarged set $S = \{2e_{22}, e_{11}, (e_{22} + e_{33}), \sigma_{12}, \sigma_{13}, \tau_{21}, \sigma_{23}, \tau_{31}, \tau_{32}\}$ of linearly independent elements generates the ring Φ additively since the subgroup generated by S has order 2^{13}. Hence to check the possible actions of commutators from Φ on G, it suffices to consider commutators involving the elements of S. Moreover, since a commutator $[\alpha, \beta] = -[\beta, \alpha]$, we can reduce the calculations by half.

We consider actions of commutators from S on the subgroup $\langle a_2 \rangle$.

(i) Commutators involving τ_{32}:

- $[\tau_{32}, \tau_{31}](a_2) = \tau_{32}(0) - \tau_{31}(0) = 0$;
- $[\tau_{32}, \sigma_{23}](a_2) = \tau_{32}(2a_3) - \sigma_{23}(0) = 2a_2 - 0 = 2a_2$;
- $[\tau_{32}, \tau_{21}](a_2) = \tau_{32}(a_1) - \tau_{21}(0) = 0$;
- $[\tau_{32}, \sigma_{13}](a_2) = \tau_{32}(0) - \sigma_{13}(0) = 0$;
- $[\tau_{32}, \sigma_{12}](a_2) = \tau_{32}(0) - \sigma_{12}(0) = 0$;

  $[\tau_{32}, e_{33}](a_2) = \tau_{32}(0) - e_{33}(0) = 0$;

  $[\tau_{32}, e_{22}](a_2) = \tau_{32}(a_2) - e_{22}(0) = 0$;
- Hence $[\tau_{32}, (e_{33} + e_{22})](a_2) = 0$;
- $[\tau_{32}, e_{11}](a_2) = \tau_{32}(0) - e_{11}(0) = 0$;
- $[\tau_{32}, 2e_{22}](a_2) = 2[\tau_{32}, e_{22}](a_2) = 0$.

(ii) Commutators involving τ_{31}:

- $[\tau_{31}, \sigma_{23}](a_2) = \tau_{31}(2a_3) - \sigma_{23}(0) = 2a_1 = 0$;
- $[\tau_{31}, \tau_{21}](a_2) = \tau_{31}(a_1) - \tau_{21}(0) = 0$;
- $[\tau_{31}, \sigma_{13}](a_2) = \tau_{31}(0) - \sigma_{13}(0) = 0$;
- $[\tau_{31}, \sigma_{12}](a_2) = \tau_{31}(0) - \sigma_{12}(0) = 0$;

  $[\tau_{31}, e_{33}](a_2) = \tau_{31}(0) - e_{33}(0) = 0$;

  $[\tau_{31}, e_{22}](a_2) = \tau_{31}(a_2)) - e_{22}(0) = 0$;
- Hence $[\tau_{31}, (e_{33} + e_{22})](a_2) = 0$;
- $[\tau_{31}, e_{11}](a_2) = \tau_{31}(0) - e_{11}(0) = 0$;
- $[\tau_{31}, 2e_{22})](a_2) = 2[\tau_{31}, e_{22})](a_2) = 0$.

(iii) Commutators involving σ_{23}:

- $[\sigma_{23}, \tau_{21}](a_2) = \sigma_{23}(a_1) - \tau_{21}(2a_3) = 0$;
- $[\sigma_{23}, \sigma_{13}](a_2) = \sigma_{23}(0) - \sigma_{12}(2a_3) = 0$;

  $[\sigma_{23}, e_{33}](a_2) = \sigma_{23}(0) - e_{33}(2a_3) = -2a_{33}$;

  $[\sigma_{23}, e_{22}](a_2) = \sigma_{23}(a_2) - e_{22}(2a_3) = 2a_3$;
- Hence $[\sigma_{23}, (e_{33} + e_{22})] = -2a_3 + 2a_3 = 0$;
- $[\sigma_{23}, e_{11}](a_2) = \sigma_{23}(0) - e_{11}(2a_3) = 0$;
- $[\sigma_{23}, 2e_{22}](a_2) = 2[\sigma_{23}, \varepsilon_{22}](a_2) = 0$;

(iv) Commutators involving τ_{21}:

- $[\tau_{21}, \sigma_{13}](a_2) = \tau_{21}(0) - \sigma_{13}(a_1) = 4a_3$;
- $[\tau_{21}, \sigma_{12}](a_2) = \tau_{21}(0) - \sigma_{12}(a_1) = -2a_2 = 2a_2$;

$[\tau_{21}, e_{33}](a_2) = \tau_{21}(0) - e_{33}(a_1) = 0;$
$[\tau_{21}, e_{22}](a_2) = \tau_{21}(a_2) - e_{22}(a_1) = -a_1 = a_1;$

- Hence $[\tau_{21}, (e_{33} + e_{22}](a_2) = -a_1 = a_1;$
- $[\tau_{21}, e_{11}](a_2) = \tau_{21}(0) - e_{11}(a_1) = -a_1 = a_1;$
- $[\tau_{21}, 2e_{22}](a_2) = 2[\tau_{21}, e_{22}](a_2) = 2a_1 = 0.$

(v) Commutators involving σ_{13}:

- $[\sigma_{13}, \sigma_{12}](a_2) = \sigma_{13}(0) - \sigma_{12}(0) = 0;$
  $[\sigma_{13}, e_{33}](a_2) = \sigma_{13}(0) - e_{33}(0) = 0;$
  $[\sigma_{13}, e_{22}](a_2) = \sigma_{13}(a_2) - e_{22}(0) = 0;$
- Hence $[\sigma_{13}, (e_{33} + e_{22})](a_2) = 0;$
- $[\sigma_{13}, e_{11}](a_2) = \sigma_{13}(0) - e_{11}(0) = 0;$
- $[\sigma_{13}, 2e_{22}](a_2) = 2[\sigma_{13}, e_{22}] = 0.$

(vi) Commutators involving σ_{12}:

$[\sigma_{12}, e_{33}](a_2) = \sigma_{12}(0) - e_{33}(0) = 0;$
$[\sigma_{12}, e_{22}](a_2) = \sigma_{12}(a_2) - e_{22}(0) = 0;$

- Hence $[\sigma_{12}, (e_{33} + e_{22})](a_2) = 0;$
- $[\sigma_{12}, e_{11}](a_2) = \sigma_{12}(0) - e_{11}(0) = 0;$
- $[\sigma_{12}, 2e_{22}](a_2) = 2[\sigma_{12}, e_{22}](a_2) = 0.$

(vii) Commutators involving $(e_{22} + e_{33})$:

$[e_{33}, e_{11}](a_2) = e_{33}(0) - e_{11}(0) = 0;$
$[e_{22}, e_{11}](a_2) = e_{22}(0) - e_{11}(a_2) = 0;$

- Hence $[(e_{22} + e_{33}), e_{11}](a_2) = 0.$

(viii) Commutators involving e_{11}:

- $[e_{11}, 2e_{22}](a_2) = 2[e_{11}, e_{22}](a_2) = 0.$

It follows from the calculations above that the images of a_2 under the elements of Φ belong to the set $\{0, a_1, 2a_2, 4a_3\}$. Thus the subgroup $\langle 2a_2 \rangle$ is mapped to 0 by Φ; in particular $\langle 2a_2 \rangle$ is invariant under commutators from Φ, as required.

3.5. **On commutator fully transitive Abelian groups.** Throughout the present subsection, let all groups be additive Abelian groups and let all unexplained notions and notations follow those from [44, 47] and [71].

To simplify the notation, and to avoid any risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism **ring** of a group G, and $\mathrm{End}(G) = \mathrm{E}(G)^+$ for the endomorphism **group** of a group G. Likewise, the endomorphism ψ is called *commutator* if it can be represented as $\psi = [\alpha, \beta] = \alpha\beta - \beta\alpha$ for some endomorphisms α, β of G. Commutators of endomorphisms rings of groups and certain other questions connected with them were studied in the papers from [16] to [22].

Moreover, we shall denote by $\mathrm{Comm}(G)$ the **subring** of $\mathrm{E}(G)$ containing the same identity and generated by the commutator endomorphisms. In view of the equality $[\alpha,\beta] = -[\beta,\alpha]$, an element $\phi \in \mathrm{Comm}(G)$ will have the form $\phi = \sum_{\text{finite}} c_{i_1}c_{i_2}\dots c_{i_k}$, where every c_{i_j} is a commutator in $\mathrm{E}(G)$ for $i_j \in \mathbb{N}$ and $1 \le j \le k \in \mathbb{N}$.

Analogically, we let $\mathrm{comm}(G)$ denote the **subgroup** of $\mathrm{End}(G)$ generated by the commutator endomorphisms; so $\varphi \in \mathrm{comm}(G)$ has the form $\varphi = \sum_{i=1}^{n} c_i$ for some finite n, where each c_i is a commutator in $\mathrm{End}(G)$. Since 1 can be represented as a finite sum of finite products of commutators, it is immediately seen that the same holds for $c_i = 1 \cdot c_i = c_i \cdot 1$ and thus $\mathrm{comm}(G) \subseteq \mathrm{Comm}(G)$.

As usual, mimicking [78, Section 27], $H_G(g)$ denotes the *height matrix* of the element g of a group G. In case that the group G is a p-group, instead of $H_G(g)$, it can be considered the *Ulm indicator* $U_G(g)$ of the element g, while if the group G is torsion-free it can be considered the *characteristic* $\chi_G(g)$. Also, $o(g)$ will denote the *order* of the element g, i.e., the least $n \in \mathbb{N}$ with $ng = 0$ or ∞ if such an n does not exist. We also define the relation $\preceq$ as follows: for $m, n \in \mathbb{N} \cup \{\infty\}$ we suppose that $m \preceq n \Leftrightarrow$ either $n \mid m$ or $m = \infty$.

Let R be an associative unital ring, let G be a group, and let $\phi\colon R \to \mathrm{E}(G)$ be a ring homomorphism. We shall define the action of R on G by the equality $r(g) = \phi(r)(g)$. Analogously as above, we denote by $\mathrm{Comm}(R)$ and $\mathrm{comm}(R)$ the subring of R and the subgroup of R^+, respectively, generated by all commutators of R. So, we come to the following notion:

Main Definition. A group G is said to be *R-commutator fully transitive* if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists $\varphi \in \mathrm{Comm}(R)$ with $\varphi(x) = y$. If φ is chosen from $\mathrm{comm}(R)$, then the group is called *R-strongly commutator fully transitive.*

In what follows we will consider several times the examined group as a module on its endomorphism ring. In particular, when $R = \mathrm{E}(G)$ and $R^+ = \mathrm{End}(G)$, one can obtain the following two concepts:

Definition 1. A group G is said to be *commutator fully transitive* (briefly written as a *cft-group*) if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists $\phi \in \mathrm{Comm}(G)$ with $\phi(x) = y$.

Definition 2. A group G is said to be *strongly commutator fully transitive* (briefly written as a *scft-group*) if, given $0 \neq x, y \in G$ with $H_G(x) \leqslant H_G(y)$ and $o(x) \preceq o(y)$, there exists $\varphi \in \mathrm{comm}(G)$ with $\varphi(x) = y$.

Note that if the group is reduced, then the condition $o(x) \preceq o(y)$ in both Definitions 1 and 2 can be eliminated in conjunction with [54, Proposition 2.23].

However, the later usage of that condition is basically motivated by the existence of divisible direct factors. It is also clear that any scft-group is a cft-group.

Notice that in [D12] were studied the so-termed *projectively fully transitive* p-groups,i.e., the p-groups G having the property that, for any $x, y \in G$ with $U_G(x) \leqslant U_G(y)$, there exists $\varphi \in Proj(G)$ such that $\varphi(x) = y$, where $Proj(G)$ is the subring of $\mathrm{E}(G)$ generated by the idempotents of $\mathrm{E}(G)$. There were also explored *strongly projectively fully transitive p-groups* defined in a similar way replacing $Proj(G)$ by $\Pi(G)$, which is the subgroup of $\mathrm{End}(G)$ generated by all the idempotents additively. We shall often cite and use in what follows some results of [D12].

Once again, throughout the text, the word group will denote an *additively* written *Abelian* group. In this context, our terminology not explicitly explained herein is standard and follows the excellent monographs of Fuchs [44, 47] and the book of Kaplansky [71], where all mappings are written on the left. A good source in this subject is [15] too. Likewise, if A, B are groups and $H \subseteq A$, then let $\mathrm{Hom}(A,B)H = \sum\limits_{f \in \mathrm{Hom}(A,B)} f(H)$. Standardly, for this subsection $\mathbb{Z}_n$ denotes the cyclic group of order n, whereas the ring of integers modulo n is denoted by $\mathbb{Z}_{(n)}$.

Our work is motivated mainly by [D12] and [D13]. Here we wish to consider the situation when the projection endomorphisms are replaced by commutator endomorphisms and thus to find the similarity and the discrepancy in both of them. We just emphasize that there is no absolute analogy in both cases.

It is clear that if $\mathrm{Comm}(G) = \mathrm{E}(G)$ (resp., $\mathrm{comm}(G) = \mathrm{End}(G)$), then the fully transitive group G is a cft-group (resp., a scft-group), so we consider firstly this situation. We shall say that a group G is a *commutator-generated* group (or a CG-group for short) if $\mathrm{Comm}(G) = \mathrm{E}(G)$; reciprocally, we say that G is a *commutator-sum* group (or a CS-group for short) if $\mathrm{comm}(G) = \mathrm{End}(G)$. It is self-evident that a CS-group is a CG-group because $\mathrm{End}(G) \subseteq \mathrm{E}(G)$. Likewise, it is apparent that a group with commutative endomorphism ring is neither a CG-group nor a CS-group; for a more concrete information concerning groups with commutative endomorphism ring, we refer the interested reader to both [108] and [106] – compare also with results from Chapter III.

However, the next construction demonstrates that there exist CG-groups which are not CS-groups.

Example 3.99. There is a CG-group that is not a CS-group.

Proof. Suppose $R = \{\frac{m}{2^n} \mid m, n \in \mathbb{Z}\}$ is a ring of the rational fractions, which denominators consist of degrees of the number 2; thus $S = R \oplus Ri \oplus Rj \oplus Rk$

with $i^2 = j^2 = k^2 = -1$ is the ring of all quaternion of the ring R. For any $r \in R$ we have $[\frac{r}{2}i, j] = rk$, so $Rk \subseteq \mathrm{comm}(S)$. Similarly, $Ri \subseteq \mathrm{comm}(S)$ and $Rj \subseteq \mathrm{comm}(S)$. Thereby $\mathrm{comm}(S) = Ri \oplus Rj \oplus Rk$. Since $k^2 = [\frac{1}{2}i, j]^2 = -1$, one sees that $R \subseteq \mathrm{Comm}(S)$, i.e., $S = \mathrm{Comm}(S)$. Furthermore, according to Corner's realization theorem [78, Theorem 29.2] there exists a countable torsion-free group G with $\mathrm{E}(G) \cong S$. Hence $\mathrm{Comm}(G) = \mathrm{E}(G)$ and, consequently, G is a CG-group. However, it is routinely checked that $1 \notin \mathrm{comm}(S)$, whence $\mathrm{comm}(G) \neq \mathrm{End}(G)$, and therefore G is not a CS-group, as asserted. □

The following fact is rather elementary but is crucial for our further applicable purposes.

Remark 1. Notice also that if $G = A \oplus B$, then any endomorphism $\delta \in \mathrm{E}(G)$ such that $\delta \restriction A = f \in \mathrm{Hom}(A, B)$ and $\delta \restriction B = 0_B$ can be represented like this:

$$\delta = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}.$$

The last observation suggests the following obvious technicality which will be used in the sequel.

Lemma 3.100. *If $G = A_1 \oplus \ldots \oplus A_n$, where all A_i are CG-groups (resp., CS-groups), then G is a CG-group (resp., a CS-group).*

The next statement is also of further applicable interest.

Lemma 3.101. *([90]) If $G = A^{(\kappa)}$, where the cardinal κ is infinite, then comm$(G) = \mathrm{End}(G)$.*

We sketch here an idea of its proof only for the sake of completeness and for the reader's convenience: In fact, one may apply Theorem 13 of [90], where it was proved that if N is a right R-module over a ring R, I is an infinite set and $M = N^{(I)}$, then the equality $End_R(M) = [End_R(M), End_R(M)]$ holds, where $[S, S]$ is the additive subgroup in S, generated by commutators of all elements of the ring S.

As a useful consequence, we derive:

Corollary 3.102. *(1) If $G = A \oplus B$, where A is a fully invariant subgroup of G, then G is a CG-group (resp., a CS-group) if and only if both A and B are CG-groups (resp., CS-groups). In particular, if $G = D \oplus R$, where D is divisible and R is reduced, then G is a CG-group (resp., a CS-group) if and only if both D and R are CG-groups (resp., CS-groups).*

(2) If $G = \bigoplus A_i$, where each A_i is a fully invariant subgroup of G, then G is a CG-group (resp., a CS-group) if and only if every A_i is a CG-group (resp., a CS-group).

Proof. (1) Since A is a fully invariant subgroup in G, any $\varphi \in \mathrm{E}(G)$ can be represented as $\varphi = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$, where $\alpha \in \mathrm{E}(A)$, $\beta \in \mathrm{E}(B)$ and $\gamma \in \mathrm{Hom}(B, A)$. But $\alpha \in \mathrm{Comm}(A)$ and $\beta \in \mathrm{Comm}(B)$, so Remark 1 before Lemma 3.100 works to conclude that $\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \in \mathrm{Comm}(G)$. Thus $\varphi \in \mathrm{Comm}(G)$ and hence we obtain both claims, as desired.

(2) It is elementary. $\square$

It is worthwhile noticing that point (2) of Corollary 3.102 reduces the study of torsion CG-groups and CS-groups to the primary case.

Proposition 3.103. *(1) Let $G = C \oplus B$, where $C \neq 0$ is a free group and B is a CG-group having a direct summand isomorphic to C. Then G is a CG-group.*

(2) If A is a free group, then $A^{(\kappa)}$ is a CG-group for any cardinal $\kappa \geqslant 2$.

Proof. (1) Since B has a direct summand isomorphic to C, then for each $\alpha \in \mathrm{E}(C)$ there exist $\zeta \in \mathrm{Hom}(C, B)$ and $\xi \in \mathrm{Hom}(B, C)$ such that $\alpha = \xi\zeta$. But

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \zeta & 0 \end{pmatrix}.$$

Thus it is easily seen that $\mathrm{E}(G) = \mathrm{Comm}(G)$, as required.

Point (2) follows in the same manner. $\square$

It is well known that every divisible group D has the following representation $D = D_0 \oplus (\bigoplus_{p \in \Pi} D_p)$, where D_0 is its torsion-free part such that $D_0 = 0$ or $D_0 \cong \mathbb{Q}^{(m)}$ for some cardinal $m \geqslant 1$, while Π is a subset of the set of all prime numbers such that if $\Pi \neq \varnothing$ and $p \in \Pi$, then $D_p \cong \mathbb{Z}_{p^\infty}^{(k_p)}$ for some cardinal $k_p \geqslant 1$ where $m = \mathrm{rank}(D_0)$ and $k_p = \mathrm{rank}(D_p)$.

Combining Corollary 3.102 and Proposition 3.103, we immediately deduce:

Corollary 3.104. *A divisible group D is a CG-group if and only if either $D_0 = 0$ or $\mathrm{rank}(D_0) \geqslant 2$, and if $\Pi \neq \varnothing$ then $\mathrm{rank}(D_p) \geqslant 2$ for any $p \in \Pi$.*

Note that in [21] was investigated the so-termed E-commutant G' of a group G, that is, $G' = \langle [\alpha, \beta]G \,|\, \alpha, \beta \in \mathrm{E}(G) \rangle$. According to [21, Lemma 8] if $G = A \oplus B$, then $G' = \langle \mathrm{Hom}(A, B)A, \mathrm{Hom}(B, A)B, A', B' \rangle$. It is clear that if G is a CG-group or a CS-group, then $G = G'$, whereas the converse fails. Indeed, if $G = \mathbb{Q}\oplus(\mathbb{Z}\oplus\mathbb{Z})$, then $G = G'$, but by Corollary 3.102 the group G is not a CG-group (and hence it is not a CS-group as well).

Notice also that, if $G = A \oplus A$, where A is a group with commutative endomorphism ring, then G is not a CS-group. In fact, if $\varphi, \psi \in \mathrm{E}(G)$, $\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\psi = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$, then $[\varphi, \psi] = \begin{pmatrix} \beta\gamma_1 - \beta_1\gamma & * \\ * & \gamma\beta_1 - \gamma_1\beta \end{pmatrix}$. Since the ring $\mathrm{E}(A)$ is commutative, $\beta\gamma_1 - \beta_1\gamma = -(\gamma\beta_1 - \gamma_1\beta)$. It is now plainly seen that the matrices of this type do not generate additively all the ring $M(2, \mathrm{E}(A))$.

On the other hand, any bounded p-group A represents as

$$A = A_1 \oplus \ldots \oplus A_n, \tag{3.1}$$

where each subgroup A_i is isomorphic to a direct sum of some number of the group $\mathbb{Z}_{p^{n_i}}$ $(i = 1, \ldots, k)$ and $1 \leqslant n_1 < \ldots < n_k$.

So, we come to

Proposition 3.105. *The bounded p-group from (1) is a CG-group if and only if every its component A_i is a decomposable group, that is, none of its components A_i is a cyclic group.*

Proof. "**Necessity.**" Assume that some subgroup A_i is an indecomposable, i.e., it is a cyclic group of order p^{n_i} (note that its endomorphism ring is commutative). Therefore $A = A_i \oplus B$, where $B = B_1 \oplus B_2$, $B_1 = \bigoplus_{j=1}^{i-1} A_j$, $B_2 = \bigoplus_{j=i+1}^{k} A_j$ ($B_1 = 0$ or $B_2 = 0$ if, resp., $i = 1$ or $i = k$). If $\varphi, \psi \in \mathrm{E}(A)$ with $\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\psi = \begin{pmatrix} \varepsilon & \zeta \\ \eta & \theta \end{pmatrix}$, then in view of commutativity of the ring $\mathrm{E}(A_i)$ we obtain that

$$[\varphi, \psi] = \begin{pmatrix} \kappa & \lambda \\ \mu & \nu \end{pmatrix}, \tag{3.2}$$

where κ is a composition of homomorphisms from $\mathrm{Hom}(A_i, B)$ and $\mathrm{Hom}(B, A_i)$, respectively, $\lambda \in \mathrm{Hom}(B, A_i)$, $\mu \in \mathrm{Hom}(A_i, B)$, $\nu \in \mathrm{E}(B)$. It is easy to check that any finite product of commutators is of the form of (2). However, $\mathrm{Hom}(A_i, B_2)A_i \subseteq p^{n_{i+1}-n_i}B_2$, $\mathrm{Hom}(B_1, A_i)B_1 \subseteq p^{n_i - n_{i-1}}A_i$, where $n_{i+1} - n_i \geqslant 1$, $n_i - n_{i-1} \geqslant 1$. Hence $\mathrm{Im}\,\kappa \subseteq pA_i$, which assures that A is not a CG-group.

"**Sufficiency.**" It follows directly from Lemma 3.100 and Proposition 3.103. □

Now we will exhibit separable p-groups which are not CG-groups (and, consequently, are not CS-groups).

Example 3.106. Suppose R is a commutative ring whose additive group is the completion of a free p-adic module of at most countable rank. Then there exists an unbounded separable p-group G_R which is not a CG-group.

Proof. With Corner's realization theorem from [28] at hand (see also [27, 29] and [78, Theorem 28.11]), we conclude that there is an unbounded separable p-group G_R with $\mathrm{E}(G_R) = R \oplus E_S(G_R)$, where $E_S(G_R)$ is the ideal of small endomorphisms of G_R. Since $\mathrm{E}(G_R)/E_S(G_R) \cong R$ is a commutative ring, we deduce that G_R is not a CS-group, because if we assume that the ring $\mathrm{E}(G_R)$ is generated as a ring (resp., additively) by its commutators, then the same is true for $\mathrm{E}(G_R)/E_S(G_R)$ that is obviously wrong.

Such a ring R, for instance, can be taken to be $\widehat{\mathbb{Z}}_p \times \ldots \times \widehat{\mathbb{Z}}_p = \widehat{\mathbb{Z}}_p^{(n)}$, for a finite n, where $\widehat{\mathbb{Z}}_p$ is the ring of all p-adic integers. Notice that if $R = \widehat{\mathbb{Z}}_p$, then with the aid of [78, Proposition 28.12] the group G_R has to be an essentially indecomposable p-group. □

Proposition 3.107. *If A is a reduced separable p-group with a basic subgroup of $2 \leqslant$ rank $\leqslant 2^{\aleph_0}$, then for any infinite ordinal $\alpha < \omega^2$ there is a p-group G with $p^\alpha G = A$ such that G is not a CG-group.*

Proof. Again using Corner's realization theorem from [28], we construct a group G with $p^\alpha G = A$ and $\mathrm{E}(G)_A = \{\varphi \restriction A \mid \varphi \in \mathrm{E}(G)\} = \Phi$, where Φ is any complete separable p-adic subalgebra of $\mathrm{E}(A)$. If A is unbounded, then the choice $\Phi = \widehat{\mathbb{Z}}_p$ is possible, too. Since $\widehat{\mathbb{Z}}_p$ is commutative, the ring $\mathrm{E}(G)$ cannot be generated by its commutators since $\mathrm{E}(G)_A$ is a ring homomorphic image of $\mathrm{E}(G)$.

Let now A be bounded, and write $A = B \oplus C$, where $B = \mathbb{Z}_{p^{n_1}} \oplus \mathbb{Z}_{p^{n_2}}$ and $n_1 \leqslant n_2$. Let Φ be the algebra of matrices of the form $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$, where $r \in \mathbb{Z}_{(p^{n_1})}$ and $s \in \mathbb{Z}_{(p^{n_2})}$; because of finiteness Φ is complete and separable. Once again in view of commutativity of Φ, the ring $\mathrm{E}(G)$ cannot be generated by its commutators and consequently, in any case, G is not a CG-group, as expected. □

Proposition 3.108. *The next two statements are true:*

(i) If a p-group G is a CG (resp., a CS)-group, then so is also p^nG for any finite n;

(ii) If a p-group G is a CG (resp., a CS)-group and the quotient $G/p^{\alpha}G$ is totally projective for some ordinal α, then $p^{\alpha}G$ is a CG (resp., a CS)-group.

Proof. (i) Follows from the fact that the mapping $\phi\colon \mathrm{E}(G) \to \mathrm{E}(p^n G)$, defined by $\phi(f) = f \restriction p^n G$, is a ring epimorphism in accordance with [44, Proposition 113.3].

(ii) It can be verified similarly by referring to [61]. $\square$

To give an example of a splitting group which is not a CG (resp., a CS)-group, say $G = A \oplus T$, where A is torsion-free group and T is a torsion group, it is enough to choose either A or T to be not a CG (resp., a CS)-group.

As usual, the letters $G_t = t(G)$ stand for the *torsion part* of any group G.

Proposition 3.109. *The following two statements hold:*

(1) If R is a countable commutative ring, the additive group R^+ of which is reduced torsion-free, then there exists a countable reduced mixed group G_R such that the factor-group $G_R/t(G_R)$ is divisible and G_R is not a CG-group. Moreover, if R^+ has rank n, then the torsion-free rank of G_R is equal to $2n$.

(2) For any infinite cardinal m there are 2^m reduced mixed non-isomorphic groups G such that $G/t(G)$ is divisible and G is not a CG-group.

Proof. (1) According to [78, Corollary 30.5], there is a group G such that $\mathrm{End}(G) = R \oplus \mathrm{E}_t(G)$, where $\mathrm{E}_t(G)$ is an ideal of $\mathrm{E}(G)$. If $Comm(G) = \mathrm{E}(G)$, then the factor-ring $\mathrm{E}(G)/\mathrm{E}_t(G)$ also possesses this property that contradicts its commutativity.

(2) Referring to [78, Corollary 30.6], we can take $R = \mathbb{Z}$. $\square$

It is well known that any completely decomposable torsion-free group G can uniquely be decomposed up to isomorphism as $G = \bigoplus_{s \in \Omega} G_s$, where G_s are homogeneous completely decomposable groups called *homogeneous components* of G, and Ω is some set of types.

Proposition 3.110. *The nest two assertions are valid:*

(1) The completely decomposable torsion-free group G is a CG-group if and only if each of its homogeneous component has rank $\geqslant 2$.

(2) The vector torsion-free group $G = \prod_{s \in \Omega} G_s$, where G_s is a direct product of groups of rank 1 and type s (notice once again that Ω is some set of types), is a CG-group if and only if $rank(G_s) \geqslant 2$ for each $s \in \Omega$.

Proof. (1) **Necessity.**" The subgroup $G(s) = \bigoplus_{\tau \geqslant s} G_\tau = G_s \oplus (\bigoplus_{\tau > s} G_\tau)$ is a fully invariant direct summand of G. So, by Corollary 3.102, both $G(s)$ and G_s are CG-groups. Consequently, $rank(G_s) \geqslant 2$.

"**Sufficiency.**" Let $G = G_1 \oplus G_2$, where G_1 is a direct sum of G_s that have either infinite rank or even finite one, and G_2 is a direct sum of G_s that has odd rank. According to Lemma 3.100, it is enough to show that G_1 and G_2 are CG-groups. The group G_1 is a direct sum of two mutually isomorphic direct summands and thus Proposition 3.103 allows us to conclude that G_1 is a CG-group. The group G_2 can be presented in the form $G_2 = A_1 \oplus A_2 \oplus A_3 \oplus B$, where $A_1 \cong A_2 \cong A_3$ and each homogeneous component of B (if $B \neq 0$) has even rank, so G_2 is also a CG-group.

(2) It can be verified similarly. □

We now come to our basic results concerning commutator fully transitive groups. So, the next lines are devoted to the exploration of the two new classes of groups named (strongly) commutator fully transitive groups.

We begin with a trivial but useful assertion.

Lemma 3.111. *The following two claims are fulfilled:*

(1) Let $G = A \oplus B$ be a cft-group and let A be a direct summand such that either $\mathrm{Hom}(A, B) = 0$ *or* $\mathrm{Hom}(B, A) = 0$. *Then A is also a cft-group.*

(2) If $G = \bigoplus_{i\in I} A_i$ is a reduced torsion-free group and either $\mathrm{Hom}(A_i, A_j) = 0$ *or* $\mathrm{Hom}(A_j, A_i) = 0$ *for any $i, j \in I$ with $i \neq j$, then G is a cft-group if and only if $pA_i \neq A_i$ implies $pA_j = A_j$ for each prime p and all $i, j \in I$ with $i \neq j$.*

Proof. Point (1) is pretty obvious. Since each cft-group is fully transitive, the necessity of (2) follows from the corresponding result for fully transitive groups (see, for example, [54, Theorem 3.20]). As for the sufficiency, we elementarily observe that these groups are of necessity fully transitive and hence we refer to Lemma 3.115 below. □

We are now able to prove the following:

Proposition 3.112. *A divisible group D is a cft-group if and only if $D_0 = 0$ or $rank(D_0) \geqslant 2$, and if $\Pi \neq \varnothing$, then $rank(D_p) \geqslant 2$ for any $p \in \Pi$.*

Proof. Necessity follows directly from Lemma 3.111, whereas to treat the sufficiency we employ the fact that any divisible group is a fully transitive group and since by Corollary 3.104 such a group is a CG-group, it follows immediately that it must be a cft-group. □

It is worthwhile noticing that neither $\mathbb{Q}$ nor $\mathbb{Z}_{p^\infty}$ are cft-groups. In fact, in [108] and [106] was shown that these two divisible groups have commutative endomorphism rings, as the first one is the only divisible torsion-free group having this property (for any set Π of prime numbers, the group $\bigoplus_{p\in\Pi} \mathbb{Z}_{p^\infty}$ also has commutative endomorphism ring). Since these two groups are fully transitive, we thus obtain two examples of fully transitive groups that are not cft-groups.

As it is well-known, any separable p-group is a fully transitive group. But each fully transitive group with commutative endomorphism ring is obviously not a cft-group; for instance, owing to [108], $\mathbb{Z}_{p^n}$ is a separable (and even a p^n-bounded) p-group that is not cft.

So it is interesting to find a concrete example of a reduced inseparable fully transitive group which is not a cft-group. This is subsumed by the following two constructions:

Example 3.113. There exist two types of non-separable fully transitive p-groups which are not cft-groups.

Proof. (i) Using once again Corner's realization theorem from [28], we construct a p-group G such that $p^\omega G = \mathbb{Z}_{p^n}$ and $\mathrm{E}(G) \restriction p^\omega G = \mathbb{Z}_{(p^n)}$. Since $\mathrm{E}(G)$ acts fully transitively on $p^\omega G$, the group G is fully transitive. However, $\mathrm{E}(G) \restriction p^\omega G$ is commutative and thereby $p^\omega G$ is fully invariant in G. Therefore, $Comm(G) \restriction p^\omega G = 0$, i.e., G is not a cft-group, as expected.

(ii) Let $H = \mathbb{Z}_p \oplus \mathbb{Z}_p = \langle a \rangle \oplus \langle b \rangle$ and $\phi \in \mathrm{E}(H)$ such that $\phi(a) = b$, $\phi(b) = a + b$; Φ is a subring in $\mathrm{E}(H)$ generated by I, ϕ, where I is the identity on H and p is a prime of the form $p = 5n + 2$. If G is a group such that $p^\omega G = H$ and $\mathrm{E}(G) \restriction H = \Phi$, then it was shown in Proposition 3.5 (ii) from [D12] that G is a fully transitive group. Arguing as in (i), we detect that G is not a cft-group, as promised. ☐

The next statement illustrates that cft-groups are not closed under the formation of direct summands.

Corollary 3.114. *A direct summand of a cft-group need not necessarily be a cft-group.*

Proof. By virtue of Proposition 3.112, the sums $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ and $\mathbb{Q} \oplus \mathbb{Q}$ are cft-groups, but as we commented above neither $\mathbb{Z}_{p^\infty}$ nor $\mathbb{Q}$ are cft-groups.

On the other hand, concerning the reduced inseparable case, let G be one of the exhibited non cft-groups groups in Example 3.113. Then $G \oplus G$ is by Theorem 3.118 below a cft-group, as needed. ☐

It is clear that the direct sum $\bigoplus_{i \in I} A_i$ of cft-groups A_i $(i \in I)$ with an infinite index set I is a cft-group if and only if for each finite subset $J \subseteq I$ there is such a finite $S \subseteq I$ that $J \subseteq S$ and $\bigoplus_{i \in S} A_i$ is a cft-group.

Lemma 3.115. *Let $A = \bigoplus_{i \in I} A_i$ be a fully transitive group, where every A_i is a cft-group for $i \in I$. Then A is a cft-group.*

Proof. Assume that $H_A(a) \leqslant H_A(b)$ for some $0 \neq a, b \in A$. It is necessary to show that there exists $\alpha \in Comm(A)$ with property that $\alpha(a) = b$. Since a and b can be written as a finite sum of elements of some A_i, it is possible to assume that I is finite and, in particular, that $|\,I\,| = 2$; whence we write $A = A_1 \oplus A_2$. But by assumption $\alpha(a) = b$ for some $\alpha \in \mathrm{E}(A)$. Given $\pi_i \colon A \to A_i$ are projections for $i = 1, 2$, we have that $a = a_1 + a_2$, $b = b_1 + b_2$, where $a_i, b_i \in A_i$ such that $(\pi_1 + \pi_2)\alpha(a) = \pi_1\alpha(a_1) + \pi_2\alpha(a_1) + \pi_1\alpha(a_2) + \pi_2\alpha(a_2) = b_1 + b_2$, $\pi_1\alpha(a_1) + \pi_1\alpha(a_2) = b_1$ and $\pi_2\alpha(a_1) + \pi_2\alpha(a_2) = b_2$. However, $\pi_i\alpha\pi_i \in \mathrm{E}(A_i) = Comm(A_i) \subseteq Comm(A)$, and $\pi_1\alpha\pi_2, \pi_2\alpha\pi_1 \in Comm(A)$ by Remark 1 stated before Lemma 3.100, as required. □

As two helpful consequences, we yield:

Proposition 3.116. *If $G = D \oplus R$ is a group, where D is a divisible subgroup and R is a reduced subgroup, then G is a cft-group if and only if D and R are cft-groups.*

Proof. Necessity follows from Lemma 3.111. As for the sufficiency, any divisible group is fully transitive and by hypothesis R is also fully transitive. So G is fully transitive, and it remains only to apply Lemma 3.115. □

Corollary 3.117. *Let G be either a p-group or a homogeneous torsion-free group. If G is a cft-group, then so is $G^{(\kappa)}$ for any cardinal κ.*

Proof. Every commutator fully transitive group is obviously fully transitive and hence we apply either [43] or [78] to get that $G^{(\kappa)}$ is fully transitive. Henceforth, Lemma 3.115 successfully applies to infer that $G^{(\kappa)}$ is, in fact, a cft-group, as required. □

The last assertion can be somewhat refined like this:

Theorem 3.118. *Let $\kappa > 1$ and let G be either a p-group or a torsion-free homogeneous group. Then the following condition are equivalent:*

(a) G is fully transitive;
(b) $G^{(\kappa)}$ is fully transitive;
(c) $G^{(\kappa)}$ is cft.

Proof. Foremost, assume that G is a p-group. The equivalence between (a) and (b) was proved in [43]. The implication $(c) \Rightarrow (b)$ is obvious. Now, to show that $(b) \Rightarrow (c)$ holds, we employ Proposition 3.103 to get that $G^{(\kappa)}$ is a CG-group, so that $G^{(\kappa)}$ as a fully transitive CG-group must be a cft-group.

Next, assume that G is a torsion-free group. The same method as in the primary case also works, as the equivalence of (a) and (b) was noted in [78, Section 25, Exercise 12]. $\square$

With Proposition 3.103 (2) at hand we can deduce the following statement.

Proposition 3.119. *Let $\kappa > 1$. Then the group $G^{(\kappa)}$ is cft if and only if $G^{(\kappa)}$ is fully transitive.*

Now, we need the following preliminary technical claim.

Lemma 3.120. *Let G be a separable p-group and let $B = \bigoplus_{i=1}^{\infty} B_i$ be its basic subgroup, where $B_i \cong \bigoplus_{m_i} \mathbb{Z}_{p^{n_i}}$ and $n_1 < n_2 < \cdots$. Then G is a cft-group if and only if $m_i > 1$ for every i such that $B_i \neq 0$.*

Proof. "**Necessity.**" It can be proved in the same manner as Proposition 3.105.

"**Sufficiency.**" Assuming that $U_G(a) \leqslant U_G(b)$, we can embed a and b in a finite direct summand A of G, say $G = A \oplus B$, because G is separable (see [44]). Adding to A, if it is necessary, a cyclic direct summand from B, we derive in view of Proposition 3.103 that A is a CG-group. Since A is also fully transitive, it follows that $\alpha(a) = b$ for some $\alpha \in Comm(A) \subseteq Comm(G)$, as desired. $\square$

Remember that the n-th *invariant* $f_n(A)$ *of Ulm-Kaplansky* of a p-group A is the cardinal number $f_n(A) = rank((p^n A)[p]/(p^{n+1}A)[p])$. Under this point of view, Lemma 3.120 confirms that a separable p-group G is a cft-group if and only if $f_n(B) \neq 0$ implies that $f_n(B) > 1$ for each n, where B is its basic subgroup (see [44, Section 37, Exercise 9]). Since by [44, Section 34, Exercise 2] we know that $f_n(G) = f_n(B)$, as valuable consequences we have:

Corollary 3.121. *The next two points are valid:*

(1) Suppose G is a separable p-group. Then G is cft if, and only if, for each natural n, $f_n(G) \neq 0$ implies that $f_n(G) > 1$.

(2) A separable p-group is cft if and only if its basic subgroup is cft.

We shall say that $\mathrm{E}(G)$ acts *commutator fully transitively* on the first Ulm subgroup $p^\omega G$ of a p-group G (resp., a torsion-free group) if, given $x, y \in p^\omega G$ with $U_G(x) \leqslant U_G(y)$ (resp., $\chi_G(x) \leqslant \chi_G(y)$), there is $\phi \in \mathrm{Comm}(G)$ with $\phi(x) = y$.

The following is a key technical instrument for our further applications.

Lemma 3.122. *A p-group G is a cft-group if and only if $G/p^\omega G$ is a cft-group and $\mathrm{E}(G)$ acts commutator fully transitively on $p^\omega G$.*

Proof. Since G is cft, it readily follows that the ring $\mathrm{E}(G)$ should act commutator fully transitively on $p^\omega G$. Next, according to Corollary 3.121, a separable p-group is a cft-group if and only if its basic subgroup is a cft-group. Since any basic subgroup of $G/p^\omega G$ is isomorphic to a basic subgroup of G (see, e.g., [44]), the "necessity" is proved.

In order to prove "sufficiency", note that we shall use the idea for the proof from Lemma 2.1 in [28] and Lemma 3.11 in [D12]; in fact, it is necessary only to make some small changes in the argumentation. To that aim, consider $x, y \in G$ with $U_G(x) \leqslant U_G(y)$. Let r, s be the least non-negative integers such that $p^r x, p^s y \in p^\omega G$; if $r = 0$, then $x, y \in p^\omega G$ and we are done, so let $r > 0$. We may choose an integer $m > \max\{ht_G(p^{r-1}x), ht_G(p^{s-1}y)\}$; if $s - 1 < 0$ we omit the final term $ht_G(p^{s-1}y)$.

Furthermore, if $p^r x = p^{r+m}x_0$, then $x = x_1 + p^m x_0$, where $p^r x_1 = 0$. Note that $o(x_1) = p^r$ since $p^t x = p^{t+m}x_0$ for $t < r$ is a contradiction to the choice of m and $ht_G(p^{r-1}x_1) = ht_G(p^{r-1}x)$. Thus $\langle x_1\rangle \cap p^m G = 0$. Let now A be a $p^m G$-high subgroup with $x_1 \in A$, and hence A as being a bounded pure subgroup of G is its direct summand, say $G = A \oplus H$ for some complement $H \supseteq p^m G$. Let $\pi\colon G \to H$ be the corresponding projection to this decomposition. Since A is isomorphic to a direct summand of $G/p^\omega G$, then by what we have shown in the proof of Lemma 3.120, A is a cft-group. Note that $s \leqslant r$, $p^r x, p^r y \in p^\omega G \subseteq H$ and $U_G(p^r x) \leqslant U_G(p^r y)$, so $\phi_0(p^r x) = p^r y$ for some $\phi_0 \in Comm(G)$; moreover we consider that $\phi_0 \restriction A = 0$, i.e., $\phi_0 = \phi_0\pi$. If $y_0 = \phi_0(x_0)$, then $p^r y = \phi_0(p^r x) = p^{r+m}y_0$ and $y = y_1 + p^m y_0$ for certain y_1 with the property $p^r y_1 = 0$.

Let $y_1 = a_1 + h_1$, where $a_1 \in A$, $h_1 \in H$. Then $U_G(y_1) = U_G(a_1) \cap U_G(h_1) \geqslant U_G(x_1)$ and so $U_G(x_1) \leqslant U_G(a_1), U_G(h_1)$. Thus $\theta(x_1) = a_1$ for some $\theta \in Comm(A)$, so that $\theta \in Comm(G)$ with $\theta \restriction H = 0$.

Since A is a bounded summand of G, we can certainly find an endomorphism ϕ' of G with $\phi'(x_1) = h_1$. Set $\psi = \pi\phi'(1 - \pi)$ and observe that $\psi(x_1) = h_1$. Since $\psi(H) = 0$ and $\psi(A) \subseteq H$, by the remark before Lemma 3.143 we have that $\psi \in Comm(G)$. Set $\phi_1 = \theta + \psi$ and note that $\phi_1(x_1) = a_1 + h_1 = y_1$, $\phi_1 = \phi_1(1 - \pi)$.

Finally, we set $\phi = \phi_0 + \phi_1$, so that $\phi \in Comm(G)$. Now $\phi(x) = \phi_0(x) + \phi_1(x)$ and, because $x = x_1 + p^m x_0$, we obtain $\phi_0(x) = \phi_0\pi(x) = \phi_0(p^m x_0) = p^m\phi_0(x_0) = p^m y_0$, $\phi_1(x) = \phi_1(1 - \pi)(x) = \phi_1(x_1) = y_1$. Thus $\phi(x) = y_1 + p^m y_0 = y$. □

As two immediate consequences, we derive the following three statements:

Corollary 3.123. *(1) Let the p-groups A and B are cft-groups. If B is separable, then $A \oplus B$ is a cft-group.*

(2) Let A be a cft p-group and B its basic subgroup. Then $A \oplus B$ is a cft-group.
(3) If A is a separable p-group, then $A^{(\kappa)}$ is a cft-group for any cardinal $\kappa > 1$.

Proof. (1) It is enough to check that the separable p-group $(A/p^{\omega}A) \oplus B$ satisfies condition (1) of Corollary 3.121, but this follows immediately from Lemma 3.122 because $A/p^{\omega}A$ and B are cft-groups. Points (2) and (3) follow from (1) and Lemma 3.122. □

Corollary 3.124. *Let G be a p-group so that $p^{\omega}G \cong \mathbb{Z}_{p^{\kappa}}$, where $1 \leqslant \kappa \leqslant \infty$. Then G is not a cft-group.*

Using Corollaries 3.121 and 3.124 it is not difficult to construct a totally projective p-group which is not a cft-group satisfying a specific condition of its Ulm-Kaplansky invariants. In fact, construct a group G as in Example 3.113 (ii) such that the factor-group $G/p^{\omega}G$ is a direct sum of cyclic groups which is cft. Thus we get a fully transitive p-group G, which is necessarily totally projective and which is not cft, with the property that if the σ-th invariant of Ulm-Kaplansky $f_{\sigma}(G) \neq 0$ then $f_{\sigma}(G) > 1$, where $1 \leqslant \sigma \leqslant \omega$.

Proposition 3.125. *If G is a cft p-group, then $p^{\beta}G$ is a cft-group for all ordinals β.*

Proof. Follows directly from the fact that the inequality $U_{p^{\beta}G}(x) \leqslant U_{p^{\beta}G}(y)$ holds precisely when $U_G(x) \leqslant U_G(y)$ for any $x, y \in p^{\beta}G$, and that $p^{\beta}G$ is a fully invariant subgroup of G. □

Proposition 3.126. *Suppose that G is a p-group, B is its basic subgroup and n is a natural number. If both p^nG and B are cft-groups, then G is a cft-group.*

Proof. Set $H = p^nG$. In view of Lemma 3.122, it suffices to show that $\mathrm{E}(G)$ acts commutator fully transitively on $p^{\omega}G = p^{\omega}H$. If $x, y \in p^{\omega}G$, then $U_G(x) \leqslant U_G(y)$ uniquely when $U_H(x) \leqslant U_H(y)$, and so $\alpha(x) = y$ for some $\alpha \in Comm(H)$. According to [44, Proposition 113.3], every endomorphism of H is induced by some endomorphism of G, which assures that each element of $Comm(H)$ is induced by some element of $Comm(G)$. □

Recall that the p-groups G_1 and G_2 form a *fully transitive pair*, if, for every non-zero $x \in G_i$, $y \in G_j$ $(i, j \in \{1, 2\})$ with $U_{G_i}(x) \leqslant U_{G_j}(y)$, there exists $\alpha \in \mathrm{Hom}(G_i, G_j)$ such that $\alpha(x) = y$. In [43] it was proved that if $\{G_i\}_{i \in I}$ is a family of p-groups such that for each $i, j \in I$ the pair $\{G_i, G_j\}_{i,j \in I}$ is fully transitive, then $\bigoplus_{i \in I} G_i$ is a fully transitive group. Notice that, in [54], in order to describe direct sums of fully transitive groups, it was incorporated the notion *systems of groups with condition of monotonicity* for height matrix. Likewise, in [17] some sufficient conditions were specified under which any system of torsion-free groups satisfied the condition of monotonicity.

Note also that it can be proved as in [51, Lemma 2.2] that the p-group G is a cft-group if and only if, for all $0 \neq x, y \in G$ with $py = 0$ and $U_G(x) \leqslant U_G(y)$, there is $\alpha \in \mathrm{E}(G)$ such that $\alpha(x) = y$ (the statement holds by induction on the order of y thus: supposing $o(y) = p^{n+1}$ and $U_G(x) \leqslant U_G(y)$, if $\varphi(px) = py$ and $\psi(x) = y - \varphi(x)$ then $(\varphi + \psi)x = y$; it is similarly seen that if G is a p-group, then, for all $a \in A$, $b \in G$ with $H_A(a) \leqslant H_G(b)$, there exists $f \in \mathrm{Hom}(A, G)$ with the property that $f(a) = b$ if and only if such a homomorphism f exists for all $a \in A$, $b \in G[p]$ with $H_A(a) \leqslant H_G(b)$).

The following somewhat strengthens Theorem 1.1 from [51].

Proposition 3.127. *If G_i $(i \in I)$ is a cft p-group, then the torsion group $H = t(\prod_{i\in I} G_i)$ is a cft p-group if and only if for each $i, j \in I$ the pair (G_i, G_j) is fully transitive.*

Proof. "**Necessity.**" It is obvious.

"**Sufficiency.**" Suppose that $U_H(x) \leqslant U_H(y)$ for $x = (\ldots, x_i, \ldots), y = (\ldots, y_i, \ldots) \in H$ and $py = 0$ (see Remark 1 stated in the previous pages). Since $ht_H(x) = \inf\{ht_{G_i}(x_i), i \in I\}$, there exists $i \in I$ such that $ht_H(x) = ht_{G_i}(x_i)$, so we consider that $i = 1$. Since $py = 0$, $U_{G_1}(x_1) \leqslant U_H(y) \leqslant U_{G_i}(y_i)$ for all i and so there are $\alpha_i \colon G_1 \to G_i$ such that $\alpha_i(x_1) = y_i$, $i \in I$.

The matrix $\varphi = \begin{pmatrix} 0 & 0 & \ldots \\ \alpha_2 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots \\ \alpha_2 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}\begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$ $- \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}\begin{pmatrix} 0 & 0 & \ldots \\ \alpha_2 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \in comm(\prod_{i\in I} G_i)$. Therefore, one deduces that $\alpha_1 \in Comm(G_1) \subseteq Comm(\prod_{i\in I} G_i)$, $(\alpha_1 + \varphi)x = y$ and the restriction $\alpha_1 + \varphi$ to H is an endomorphism of H. $\square$

Since any two separable or totally projective p-groups form a fully transitive pair, one can document the following:

Corollary 3.128. *If G_i $(i \in I)$ is either a cft separable or a totally projective p-group, then the torsion group $H = t(\prod_{i\in I} G_i)$ is cft.*

Proposition 3.129. *Let $G = A \oplus T$, where A is a torsion-free reduced group and T is a torsion reduced group. Then G is a cft-group if and only if both A and T are cft-groups.*

Proof. "**Necessity.**" It follows from Lemma 3.111.

"**Sufficiency.**" Assume that $H(x) \leqslant H(y)$ for some $x = a+b$, $y = c+d$, where $a, c \in A$ and $b, d \in T$. Then $H(x) \leqslant H(c)$, $H(x) \leqslant H(d)$. It is enough to show that there exist such $\alpha, \beta \in \mathrm{Comm}(G)$ that $\alpha(x) = c$, $\beta(x) = d$.

To that aim, let $H(a+b) \leqslant H(c)$. Notice that $H(a) \leqslant H(c)$. In fact, if $h_p(b) = \infty$, then $H_p(a+b) = H_p(a)$; but if $h_p(b) < \infty$ then $h_p(p^k b) = \infty$ for some natural k, whence $h_p(p^k(a+b)) = h_p(p^k a) \leqslant h_p(p^k c)$ gives $h_p(a) \leqslant h_p(c)$ and $H_p(a) \leqslant H_p(c)$. So, $\alpha(a) = c$ for some $\alpha \in \mathrm{Comm}(A) \subseteq \mathrm{Comm}(G)$.

Suppose now $H(a+b) \leqslant H(d)$. Then $d = d_1 + \ldots + d_k$, where $d_i \in t_{p_i}(T)$ and $H(d) \leqslant H(d_i)$ for each $(i = 1, \ldots, k)$, so we can consider that $d \in t_p(T)$ for some p. According to the comments before Proposition 3.127, we may assume that $d \in T[p]$.

If $H(b) \leqslant H(d)$, then the condition on T forces that such a β can be found.

Assume now that $H(a) \leqslant H(d)$. If $h_p(d) = \infty$, then $H(b) \leqslant H(d)$, so let $h_p(d) < \infty$. Set $h_p(a) = n$. Since T is reduced, then in $t_p(T)$ there exists a cyclic direct summand $\langle z \rangle$ such that $o(z) > p^n$. Thus $H(a) \leqslant H(p^n z) \leqslant H(d)$. Consequently, there exists a homomorphism $f \colon \langle a \rangle_* \to \langle z \rangle \subseteq T$ defined by $f(a) = p^n z$, where $\langle a \rangle_*$ is the pure subgroup in A containing a. Since $\langle z \rangle$ as a bounded group is algebraically compact, the homomorphism f extends to a homomorphism $\varphi \in \mathrm{Hom}(A, T)$. But $\gamma(p^n z) = d$ for some $\gamma \in \mathrm{E}(T)$, so that $\gamma\varphi(a) = d$ and, according to Remark 1 above listed before Lemma 3.100, we obtain that $\gamma\varphi \in comm(G)$.

Finally note that since $h_q(d) = \infty$ for any prime $q \neq p$ and $H_p(d) = (h_p(d), \infty, \ldots)$, then the inequalities $H(a) \nleqslant H(d)$, $H(b) \nleqslant H(d)$ are impossible. □

We conclude the work on commutator full transitivity with the following observation: Imitating [21], a subgroup C of a group G is said to be *commutator invariant* if $f(C) \subseteq C$ for every $f \in \mathrm{E}(G)$ which is of the form $f = [\phi, \psi]$, where $\phi, \psi \in \mathrm{E}(G)$. Moreover, following [D13], a p-group G is said to be *commutator socle-regular* if, for each commutator invariant subgroup C of G, there exists an ordinal α (depending on C) such that the equality $C[p] = (p^\alpha G)[p]$ holds.

What we now offer is the following property of cft-groups:

Proposition 3.130. *Every cft p-group is commutator socle-regular.*

Proof. Suppose that C is an arbitrary commutator invariant subgroup of G and $\alpha = \min\{ht_G(z) \mid z \in C[p]\}$, whence $C[p] \leqslant (p^\alpha G)[p]$. Next, choose $x \in C[p]$ with $ht_G(x) = \alpha$, so that $U_G(x) = (\alpha, \infty, \ldots)$. Letting now $y \in (p^\alpha G)[p]$ be an arbitrary element, we deduce that $U_G(y) = (\beta, \infty, \ldots)$, where $\beta \geqslant \alpha$. Since G is a cft-group, there is $\phi \in \mathrm{Comm}(G)$ such that $\phi(x) = y$. But, because ϕ is a linear combination of products of commutators and C is commutator invariant in G, we have that $y = \phi(x) \in C[p]$. Since y was arbitrary, we infer that $(p^{\alpha}G)[p] \leqslant C[p]$ and hence we obtain the desired equality. □

We now approach to strongly commutator fully transitive groups. Many of the results proved so far in this subsection can be proved for scft-groups as well. In fact, this can be said for Lemma 3.111, Proposition 3.112, Lemma 3.115, Proposition 3.116, Corollary 3.117 and Proposition 3.125 – see the corresponding statements below, formulated for scft-groups.

The following lemma shows that there exists a scft-group which is not a CS-group (compare with remarks after Corollary 3.104 as well).

Lemma 3.131. *The following two points hold:*

(1) If $G = (\mathbb{Z}_{p^n})^{(\kappa)}$, where $\kappa > 1$, then G is a scft-group.

(2) If G is a homogeneous torsion-free separable group and $rank(G) > 1$, then G is a scft-group.

Proof. (1) Let $U_G(a) \leqslant U_G(b)$ for $0 \neq a, b \in G$ and write $a = a_1 + \ldots + a_n$, $b = b_1 + \ldots + b_m$, where $a_i \in A_{j_i}$, $b_s \in A_{j_s}$ and $A_{j_i}, A_{j_s} \cong \mathbb{Z}_{p^n}$. If $ht(a_{i_0}) = min\{ht(a_1), \ldots, ht(a_n)\}$, then $U_G(a_{i_0}) = U_G(a) \leqslant U_G(b_s)$ for each $s = 1, \ldots, m$. If $i_0 \neq s$ for some s, then according to the statement before Lemma 3.100, $\phi(a_0) = b_s$ for some $\phi \in comm(G)$. But if $s_0 = i_0$ for some $1 \leqslant s_0 \leqslant m$, then since the additional direct summand B contains a direct summand isomorphic to A_{i_0}, as in Proposition 3.103 there exist $\zeta \in \mathrm{Hom}(A_{i_0}, B)$ and $\xi \in \mathrm{Hom}(B, A_{i_0})$ such that $b_{s_0} = \alpha(a_{s_0}) = \xi\zeta(a_{s_0})$. If now $\varphi = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \zeta & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \zeta & 0 \end{pmatrix}\begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}$, then $\varphi(a_{i_0}) = b_{i_0}$, as required.

(2) If $\chi(a) \leqslant \chi(b)$, then a can be embedded in a direct summand of rank 1. So the proof is similar to that in (1). □

If G is a separable torsion-free group, then according to [6, Corollary 7.12] it is fully transitive if and only if for all its direct summands A and B of rank 1, if type $t(A) \neq t(B)$, then the condition $pA \neq A$ implies $pB = B$ for each prime number p. According to [78, Section 19, Exercise 7] any fully transitive separable group G can be presented as $G = \bigoplus_{i \in I} G_i$, where all G_i are homogeneous separable groups and the condition $pG_i \neq G_i$ implies $pG_j = G_j$ for each $i, j \in I$ with $i \neq j$.

So, we come now to the following:

Corollary 3.132. *The next five statements are valid:*

(1) Let G be a separable p-group and let $B = \bigoplus_{i=1}^{\infty} B_i$ be its basic subgroup, where $B_i \cong \bigoplus_{m_i} \mathbb{Z}_{p^{n_i}}$ and $n_1 < n_2 < \cdots$. Then G is a scft-group if and only if $m_i > 1$ for every i such that $B_i \neq 0$.

(2) Let A be a scft p-group and B its basic subgroup. Then $A \oplus B$ is a scft-group.

(3) If G is a separable p-group or a torsion-free fully transitive separable group (in in particular, G is a homogeneous separable group), then $G^{(\kappa)}$ is a scft-group for any cardinal $\kappa > 1$.

(4) A separable torsion-free group G is scft if and only if $G = \bigoplus_{i \in I} G_i$, where all G_i are decomposable homogeneous separable groups and the condition $pG_i \neq G_i$ implies $pG_j = G_j$ for each $i, j \in I$ with $i \neq j$.

(5) A vector non-zero torsion-free group G is scft if and only if $G = \prod_{i \in I} G_i$, where all G_i is a direct product of groups of rank 1 one and same type, $rank(G_i) > 1$ and the condition $pG_i \neq G_i$ implies $pG_j = G_j$ for each $i, j \in I$ with $i \neq j$.

Proof. Points (1) and (2) have similar proof to that of Lemma 3.120. Point (3) follows from (1) and Lemma 3.131. To show the validity of clause (4), it is necessary to use certain well-known facts about fully transitive torsion-free groups (see, for instance, [17, Theorem 11]). Finally, point (5) can be proved similarly to (4). □

Proposition 3.133. *Let A be a homogeneous torsion-free group and κ is a infinite cardinal. Then $A^{(\kappa)}$ is a scft-group if and only if A is fully transitive.*

Proof. It follows from Lemma 3.101 and from the fact that the group $A^{(\kappa)}$ is fully transitive if and only if the group A is fully transitive. □

Recall that if p is a prime number, then the p-rank $rank_p(A)$ of the group A is identified as the rank of its factor-group A/pA. In conjunction with [44], any reduced algebraically compact torsion-free group $G \neq 0$ can be represented as $G = \prod_{p \in \Pi} G_p$, where $G_p \neq 0$ is a p-adic algebraically compact group and Π is a certain set of prime numbers.

Corollary 3.134. *The reduced algebraically compact torsion-free group $G = \prod_{p \in \Pi} G_p$ is scft if and only if $rank_p(G_p) > 1$ for each $p \in \Pi$.*

Proposition 3.135. *A divisible group D is a scft-group if and only if $D_0 = 0$ or $rank(D_0) \geqslant 2$, and if $\Pi \neq \varnothing$, then $rank(D_p) \geqslant 2$ for any $p \in \Pi$.*

Proof. "**Necessity.**" It follows from Lemma 3.111.

"**Sufficiency.**" As in Lemma 3.131, D_0 and D_p are scft-groups, so that $\bigoplus_p D_p$ is a scft-group. Since any divisible group is fully transitive by Lemma 3.115, we obtain that $D = D_0 \oplus (\bigoplus_p D_p)$ is a scft-group. □

By a simple combination of methods of proofs in Corollary 3.132 and Lemma 3.122, it can be proved the following.

Lemma 3.136. *A p-group G is a scft-group if and only if $G/p^\omega G$ is a scft-group and $\mathrm{E}(G)$ acts strongly commutator fully transitively on $p^\omega G$.*

Corollary 3.137. *If A is a bounded scft p-group, then there is a scft p-group G with $p^\omega G = A$.*

Proof. As in Proposition 3.107, we construct with the aid of Corollary 3.124 a group G with the properties that $p^\omega G = A$ and $\{\varphi \restriction A \mid \varphi \in \mathrm{E}(G)\} = \mathrm{E}(A)$ such that $G/p^\omega G$ is a scft-group. □

It follows from Lemma 3.120 and Corollary 3.132 (1) that a separable p-group is cft if and only if it is scft. Under certain additional circumstances on the endomorphism ring of the group, this can be slightly extended to the following:

Proposition 3.138. *Let G be a p-group and let $\mathrm{E}(G) \restriction p^\omega G = \mathrm{E}(p^\omega G)$. Then the following two points hold:*

(1) G is cft if and only if $G/p^\omega G$ and $p^\omega G$ are cft.

(2) G is scft if and only if $G/p^\omega G$ and $p^\omega G$ are scft.

Proof. (1) Combining both Lemma 3.122 and Proposition 3.125, the necessity follows at once.

As for the sufficiency, one sees that $\mathrm{E}(p^\omega G)$ and thus $\mathrm{E}(G)$ both act commutator fully transitively on $p^\omega G$. So, again Lemma 3.122 works to get that G is cft, as formulated.

(2) It follows by the same token with the aid of Lemma 3.136 accomplished with a similar statement for Ulm subgroups of scft-groups as that Proposition 3.125. □

As a consequence, we yield:

Corollary 3.139. *Suppose G is a p-group of length $\leq \omega \cdot 2$ such that $\mathrm{E}(G) \restriction p^\omega G = \mathrm{E}(p^\omega G)$. Then G is cft if and only if G is scft.*

Proof. Since $p^\omega G$ is separable, we just apply Proposition 3.138 and the comments on separable groups stated before it. □

It seems at the current stage that it is extremely difficult to construct if possible a cft-group that is not scft. It is worthwhile noticing that the same problem is unresolved yet for projectively fully transitive and strongly projectively fully transitive p-groups, respectively (cf. [D12]).

Nevertheless, we succeed to show the following:

Example 3.140. There exists a ring S such that there is a S-commutator fully transitive group which is not S-strongly commutator fully transitive.

Proof. Let p be a prime number and set $T = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0, (n, p) = 1\}$; it is obvious that T is a subring of the ring $\mathbb{Q}$ consisting of all rational numbers. Putting $S = T \oplus Ti \oplus Tj \oplus Tk$ with $i^2 = j^2 = k^2 = -1$ as the ring of quaternions of the ring T, it is not too hard to verify that the group S^+ is a homogeneous fully decomposable group of rank 4. Therefore, S^+ is a strongly commutator fully transitive by Lemma 3.131. Further, as in Example 3.99, one may infer that $Comm(S) = S$. It is well known that $S \cong \mathrm{E}_S(S_S)$. Since any endomorphism of the module S_S acts as left multiplicand on elements of the ring S and since for each nonzero element of S there is an integer multiple invertible, all nonzero endomorphisms of S_S are monomorphisms. In addition, the group S^+ is torsion-free.

Let now $0 \neq a, b \in S^+$ and $\chi(a) \leqslant \chi(b)$. Then $ua = p^n \cdot 1$ and $vb = p^m \cdot 1$ for some invertible elements $u, v \in S$, where $n \leqslant m$. Thus $b = (p^{m-n}v^{-1}u)a$, i.e., $b \in Comm(\mathrm{E}_S(S^+))a$. But $a \notin comm(\mathrm{E}_S(S^+))a$ for any $0 \neq a \in S^+$, because only $1 \in S$ sends a to a. So, we obtain the construction of a group which S-commutator fully transitive but not S-strongly commutator fully transitive, where $S \cong \mathrm{E}_S(S_S)$ and $i \colon \mathrm{E}_S(S_S) \to \mathrm{E}(S^+)$ is the identical embedding. □

In contrast to fully transitive groups, for projectively fully transitive groups not any direct summand is again projectively fully transitive (see Corollary 3.9 in [D12]). The same appears for scft-groups, so the direct summand of a scft-group is also not a scft-group; for a proof we use ideas from Propositions 4.10 and 4.11 in [D12] that are exactly the results from Subsection 3.3 on the preceding pages.

Proposition 3.141. *If $p^\omega G$ is an elementary group for a p-group G, then G is fully transitive if and only if $G \oplus G$ is a scft-group.*

Proof. The sufficiency is immediate since direct summands of fully transitive groups are fully transitive.

Suppose now that G is fully transitive. Set $H = G \oplus G$ and consider the elements $(a, b), (c, d) \in p^\omega H$. Assume first that $a, b \neq 0$. Since all non-zero elements of $p^\omega G$ have the same Ulm sequence $(\omega, \infty, \ldots)$, there are endomorphisms $\gamma, \delta \in \mathrm{E}(G)$ with the property $\gamma(b) = c$ and $\delta(a) = d$. The matrix $\Delta = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}$ maps (a, b) to (c, d). According to Remark 1 before Lemma 3.100, one sees that $\Delta \in comm(H)$. Let now $a \neq 0$, $b = 0$ and $\alpha(a) = c$, $\delta(a) = d$, where $\alpha \in \mathrm{E}(G)$. Then the matrix $\Lambda = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix}$ maps

$(a, 0)$ to (c, d). Here as in Lemma 3.131 we observe that $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in comm(H)$ and $\begin{pmatrix} 0 & 0 \\ \delta & 0 \end{pmatrix} \in comm(H)$. □

Proposition 3.142. *There is a non scft p-group G with elementary first Ulm subgroup such that $G \oplus G$ is a scft-group.*

Proof. It is enough to take any group from Example 3.113 (thus $n = 1$ if consider point (i)) and then we refer to Proposition 3.141. □

We close the work with some left-open questions of interest.

Problem 1. Find conditions on a (torsion-free) fully transitive group A under which A is a cft (resp., a scft)-group.

Problem 2. Construct, if possible, a cft-group which is not a scft-group.

Problem 3. Find conditions on a totally projective p-group G such that it is a cft (resp., a scft)-group.

Problem 4. Find conditions on a CG-group (resp., a CS-group) such that it is a cft (resp., a scft)-group.

Problem 5. To what extent there exist indecomposable torsion-free CG-groups which are not CS-groups?

Problem 6. Let A_i $(i \in I)$ be a system of reduced groups, $\mathbf{K}$ is an ideal of the Boolean algebra of all subsets of I. Find a suitable necessary and/or sufficient condition when the $\mathbf{K}$-direct sum $\bigoplus_{\mathbf{K}} A_i$ (in particular, the direct product $\prod_{i \in I} A_i$) is a cft-group (resp., a scft-group).

Remark 2. In the proof of Proposition 3.3 of [D12] on lines 5,6 the word "**idempotent**" should be written and read as "**product of idempotent**".

3.6. **On abelian groups having all proper fully invariant subgroups isomorphic.** Throughout the present subsection, let all groups into consideration be *additively* written and *abelian*. Our notations and terminology from group theory are mainly standard and follow those from [44, 47] and [71]. For instance, if p is a prime integer and G is an arbitrary group, $p^nG = \{p^ng \mid g \in G\}$ denotes the *p^n-th power subgroup* of G consisting of all elements of p-height greater than or equal to $n \in \mathbb{N}$, $G[p^n] = \{g \in G \mid p^ng = 0, n \in \mathbb{N}\}$ denotes the *p^n-socle* of G, and $G_p = \cup_{n<\omega} G[p^n]$ denotes the *p-component* of the *torsion part* $tG = \oplus_p G_p$ of G.

On the other hand, if G is a torsion-free group and $a \in G$, then let $\chi_G(a)$ denote the *characteristic* and let $\tau_G(a)$ denote the *type* of a, respectively. Specifically, the class of equivalence in the set of all characteristics is just called *type* and we write τ. If $\chi_G(a) \in \tau$, then we write $\tau_G(a) = \tau$, and so $\tau(G) = \{\tau_G(a) \mid 0 \neq a \in G\}$ is the set of types of all non-zero elements of G. The set $G(\tau) = \{g \in G \mid \tau(g) \geqslant \tau\}$ forms a pure fully invariant subgroup of the torsion-free group G. Recall that a torsion-free group G is called *homogeneous* if all its non-zero elements have the same type.

Concerning ring theory, suppose that all rings which we consider are *associative* with *identity* element. For any ring R, the letter R^+ will denote its *additive group*. To simplify the notation and to avoid a risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism ring of G and $\mathrm{End}(G) = \mathrm{E}(G)^+$ for the endomorphism group of G.

As usual, a subgroup F of a group G is called *fully invariant* if $\phi(F) \subseteq F$ for any $\phi \in \mathrm{E}(G)$. In addition, if ϕ is an invertible endomorphism (= an automorphism), then F is called *a characteristic* subgroup, while if ϕ is an idempotent endomorphism (= a projection), then F is called *a projection invariant* subgroup.

Classical examples of important fully invariant subgroups of an arbitrary group G are the defined above subgroups p^nG and $G[p^n]$ for any natural n as well as tG and the maximal divisible subgroup dG of G; actually dG is a fully invariant direct summand of G (see, for instance, [44]).

We shall say that a group G has only *trivial fully invariant subgroups* if $\{0\}$ and G are the only ones. Same appears for characteristic and projection invariant subgroups, respectively.

The following notions are our major tools.

Definition 1. A non-zero group G is said to be an *IFI-group* if either it has only trivial fully invariant subgroups, or all its non-trivial fully invariant subgroups are isomorphic otherwise.

Definition 2. A non-zero group G is said to be an *IC-group* if either it has only trivial characteristic subgroups, or all its non-trivial characteristic subgroups are isomorphic otherwise.

Definition 3. A non-zero group G is said to be an *IPI-group* if either it has only trivial projection invariant subgroups, or all its non-trivial projection invariant subgroups are isomorphic otherwise.

Note that Definition 3 implies Definition 1 and Definition 2 implies Definition 1. In other words, any IPI-group is an IFI-group and any IC-group is an IFI-group; in fact every fully invariant subgroup is both characteristic and projection invariant.

Definition 4. A non-zero group G is called a *strongly IFI-group* if either it has only trivial fully invariant subgroups, or all its non-zero fully invariant subgroups are isomorphic otherwise.

Definition 5. A non-zero group G is called a *strongly IC-group* if either it has only trivial characteristic subgroups, or all its non-zero characteristic subgroups are isomorphic otherwise.

Definition 6. A non-zero group G is called a *strongly IPI-group* if either it has only trivial projection invariant subgroups, or all its non-zero projection invariant subgroups are isomorphic otherwise.

Notice that Definition 6 implies Definition 4 and Definition 5 implies Definition 4.

On the other hand, it is obvious that Definition 4 implies Definition 1, whereas the converse fails as the next example shows: In fact, construct the group $G \cong \mathbb{Z}(p) \oplus \oplus_{\aleph_o}\mathbb{Z}(p^2)$. Since it is fairly clear that $G \neq pG$, $G \neq G[p]$ and $G = G[p^2]$, we deduce that $pG \cong \oplus_{\aleph_0}\mathbb{Z}(p) \cong G[p]$ that are the only proper fully invariant subgroups of G. However, $G \ncong G[p]$, as required. Thus there exists a p-primary IFI-group which is not a strongly IFI-group, as asserted.

However, in the torsion-free case, Definitions 1 and 4 are tantamount (see Proposition 3.144 below).

Moreover, each subgroup of an indecomposable group is projection invariant, so that an indecomposable group is an IPI-group if and only if it is either a cyclic group of order p for some prime p, or is isomorphic to the additive group of integers $\mathbb{Z}$.

It is worthwhile noticing in the current context that in [55] and [56] were studied p-groups which are isomorphic to their fixed proper fully invariant subgroup (see also cf. [54]) as well as in [5] were examined the so-called *IP-groups* that are isomorphic to their fixed pure subgroup.

Our purpose here is to explore some crucial properties of the defined above new classes of groups. The chief results are stated and proved in the next section.

As usual, $\oplus_m G = G^{(m)}$ will denote the *external* direct sum of m copies of the group G, where m is some ordinal (finite or infinite). The following statement asserts that in a special case the three classes from Definitions 1, 2 and 3 do coincide.

Theorem 3.143. *Let G be a p-group and let $m \geqslant 2$ be an ordinal. Then $G^{(m)}$ is an IFI-group if, and only if, G is an IC-group if and only if G is an IPI-group.*

Proof. The statement follows directly by results from [35] and [36], where it is shown that in this case characteristic and projection invariant subgroups are fully invariant. $\square$

Remark 1. In [18] and [20] were considered some other special properties of projection invariant subgroups and, in addition, when they are fully invariant (see [88] too). The results established there can also be applied successfully to the proof of Theorem 3.143.

Proposition 3.144. *Let G be a torsion-free group. Then G is an IFI-group if, and only if, G is a strongly IFI-group.*

Proof. One direction being elementary, we assume now that G is a torsion-free IFI-group containing all non-trivial fully invariant subgroups isomorphic. So, for all primes p, we have that $G[p] = \{0\}$, and consequently $G \cong G/\{0\} = G/G[p] \cong pG \neq \{0\}$. If $G = pG$ for any prime p, it follows from [44] that G is a torsion-free divisible group, whence it does not contain proper fully invariant subgroups, and so we are finished.

Suppose now that $G \neq pG$ for some prime p. Since $pG \neq \{0\}$ is fully invariant in G for every p, it follows by definition that each other non-trivial fully invariant subgroup of G must be isomorphic to pG, and hence to G. So, all non-zero fully invariant subgroups of G (including the full group G) must be mutually isomorphic, i.e., G is a strongly IFI-group, as claimed. $\square$

For torsion (strongly) IFI-groups we can obtain a complete description; however the torsion-free case is rather more complicated. We first need a series of technical claims.

The next technicality is quite easy, but we provide a proof only for the sake of completeness and for the readers' convenience.

Lemma 3.145. *(a) A fully invariant subgroup of an IFI-group is an IFI-group.*

(b) A fully invariant subgroup of a strongly IFI-group is a strongly IFI-group.

Proof. (a) Let G be an IFI-group with a fully invariant subgroup F. If either $F = \{0\}$ or $F = G$, we are done. Suppose now that K and L are two different proper fully invariant subgroups of F. Since they are obviously proper fully invariant in G, we deduce that $K \cong L$, as required.

(b) The same idea as that in the preceding point successfully works to get the claim. $\square$

Before proceed by proving our main characterization theorem, we need one more useful observation:

Proposition 3.146. *A non-zero IFI-group is either divisible or reduced.*

Proof. If $dG = \{0\}$ or $dG = G$, we are finished. If now $G \neq dG \neq \{0\}$, we have that $G[p] = \{0\}$ or that $G[p] = G$ for any p, because otherwise $dG \cong G[p]$ assures that $dG = \{0\}$, a contradiction. In the latter case, $dG = \{0\}$, again a contradiction. So, let $G[p] = \{0\}$ for all p. But $G \neq pG$ for some p; if not G should be divisible – contrary to our assumption. Therefore, $G \cong G/\{0\} = G/G[p] \cong pG$ and thus $dG \cong pG \cong G$, which is false since G is not divisible. $\square$

In accordance to the last statement, since divisible groups are well-classified (cf. [44]), we will henceforth consider only *reduced* groups.

Theorem 3.147. *The following two points hold:*

(i) A non-zero group G is an IFI-group if, and only if, one of the following holds:

- *For some prime p either $pG = \{0\}$, or $p^2G = \{0\}$ with $r(G) = r(pG)$.*
- *G is a homogeneous torsion-free IFI-group of an idempotent type.*

(ii) A non-zero torsion group G is a strongly IFI-group if, and only if, it is an elementary p-group for some prime p.

Proof. (i) Suppose first that G is torsion, that is, $G = tG$. If $G = G[p]$, the assertion follows. So, assume now that $G \neq G[p]$. We next claim that $G = G[p^2]$ or, equivalently, $p^2G = \{0\}$. If $G \neq G[p^2]$, then $G[p^2] \cong G[p]$ which is untrue, so that the claim is sustained. But moreover $G[p]$ and $(pG)[p] = pG$ are both non-trivial fully invariant in G, whence they should be isomorphic. In addition, appealing to [44], one may derive that $r(G) = r(G[p]) = r((pG)[p]) = r(pG)$, as stated.

Reciprocally, if G is an elementary p-group, it contains only trivial fully invariant subgroups and thus we are done. So, let G be p^2-bounded. It is well known in this case that the only proper fully invariant subgroups of G are $G[p]$ and $pG = (pG)[p]$. Now, the rank condition allows us to infer that they are isomorphic, as required. This completes the proof of the torsion case.

Assume now that G is torsion-free, i.e., $G \neq tG = \{0\}$. Since $G(\tau) \cong G$, we can write $G = G(\tau)$. But $G(\chi)$ is also fully invariant in G for a characteristic $\chi \in \tau$ and, because $G \cong G(\chi)$, we conclude that the type τ must be an idempotent, that is, $\tau^2 = \tau$, as claimed. This completes the torsion-free case.

Finally, we will show that an IFI-group cannot be mixed. In fact, applying Lemma 3.145, tG is an IFI-group. By what we have shown above, tG has to be a p^2-bounded p-group for some prime p. This means that G splits, that is, $G = tG \oplus R$ where R is torsion-free (see, for instance, [44]). Since both $tG \neq \{0\}$ and $p^2R = p^2G \neq \{0\}$ are obviously non-trivially fully invariant in G (this is because $G \neq tG$ and $p^2R = G = R \oplus tG$ ensures that $tG = \{0\}$ which is against our assumption, they must be mutually isomorphic. But this is manifestly wrong, because p^2R remains torsion-free while tG is torsion, which gives the desired contradiction. This completes the proof of the mixed case.

(ii) If G possesses only two trivial fully invariant subgroups, we are done. Suppose now that $G_p \neq \{0\}$ for some prime p. Since both $G \neq \{0\}$ and $G[p] \neq \{0\}$ are fully invariant in G, they should be isomorphic, so that G must be an elementary p-group, as asserted.

Conversely, it is apparent that each elementary p-group G, where p is a prime, is a strongly IFI-group because it has only two fully invariant subgroups, namely $\{0\}$ and G. $\square$

In conjunction with the last statement we will hereafter be interested only in torsion-free groups.

It is self-evident that any rank one torsion-free group of an idempotent type is an IFI-group; these groups are realized as subgroups of the additive group of rational numbers $\mathbb{Q}$. Thus a question related to torsion-free IFI-groups, which immediately arises, is the following: Is it true that all homogeneous torsion-free groups of an idempotent type are IFI-groups? Unfortunately, this problem has a negative resolution; especially there is a homogeneous torsion-free group of an idempotent type with arbitrary rank greater than 1 which is not an IFI-group. In fact, the following concrete example is true:

Example 3.148. There exists a homogeneous torsion-free group of an idempotent type with arbitrary large rank > 1 that is not an IFI-group.

Proof. Letting $\mathbb{Q}_p$ be the ring of all rational numbers with denominator q such that $(q, p) = 1$, we employ [44, Paragraph 110, Exercize 7] or [26] to find that there is a reduced indecomposable torsion-free group G of rank 2 with endomorphism ring $\mathrm{E}(G) = \mathbb{Q}_p$ – compare also with Chapter III. Therefore, G is a $\mathbb{Q}_p$-module, whence G is a homogeneous torsion-free group of an idempotent type with the property that $\mathrm{E}(G)a \cong \mathbb{Q}_p^+$ for any $0 \neq a \in G$, but $\mathrm{E}(G)a \not\cong G$. However, one may see that G is not an IFI-group.

Even more, the following generalized construction holds: Suppose that $L = \mathbb{Q}_p^{(n)} = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ (n-times), where n is a natural. So, using the construction of G demonstrated above, there exists a reduced indecomposable torsion-free group G_n of rank $2n$ with endomorphism ring $\mathrm{E}(G) = L$, and hence $G_1 = G$. Consequently, $\mathrm{rank}(\mathrm{E}(G_n)a) \leqslant n$ for any $0 \neq a \in G_n$ and thus $\mathrm{E}(G_n)a \not\cong G_n$. So, for each n, we have constructed a homogeneous group of an idempotent type and rank $2n$ which is not an IFI-group. Set $A = G^{(\kappa)} \oplus \mathbb{Q}_p^+$, where κ is an arbitrary cardinal. It is not too hard to see that the group A is endocyclic, that is, $A = \mathrm{E}(A)a$ for some $a \in A$, although G is obviously not endocyclic. Furthermore, $\mathrm{Hom}(G, \mathbb{Q}_p^+) = 0$ because otherwise G will have a direct summand isomorphic to $\mathbb{Q}_p^+$ which will contradict the fact that G is indecomposable. Thus, $\mathrm{Hom}(G^{(\kappa)}, \mathbb{Q}_p^+) = 0$, i.e., $G^{(\kappa)}$ is a fully invariant subgroup in A. But it is clear that $G^{(\kappa)} \not\cong A$, as wanted. Therefore, if $\kappa = l$ is a natural, for each l we have constructed a homogeneous group of an idempotent type and rank $2l + 1$ which is not an IFI-group. For an infinite ordinal κ such a group has exactly rank κ.

Note also that if $2 \leqslant \kappa \leqslant 2^{\aleph_0}$ and B is a pure subgroup of rank κ of the group $\mathbb{J}_p$ of p-adic integers, then $B^{(m)} \oplus \mathbb{Q}_p^+$ is also not an IFI-group for any cardinal m. □

It follows directly from the proof of Theorem 3.147 that the following is true:

Proposition 3.149. *Suppose G is a divisible group. Then G is an IFI-group if, and only if, it is a torsion-free group.*

As an explicit example to this fact, it is worthwhile noticing that $\mathbb{Q}$ is a torsion-free divisible group of rank 1, whence it is an IFI-group.

Since the divisible part is always a fully invariant subgroup of the whole group, then the (torsion-free) IFI-group is either divisible or reduced. That is why, we may hereafter assume that all groups are *reduced*.

Observe also that Theorem 3.147 gives a chance to describe some partial classes of IFI-groups. So, the following holds:

Corollary 3.150. *(1) A coperiodical group is an IFI-group if, and only if, it is either an elementary p-group, or is a torsion-free p-adic algebraically compact group, for some single prime p.*

(2) A vector torsion-free group is an IFI-group if, and only if, it is a direct product of groups of rank 1 with the same idempotent type.

Proof. (1) Applying Theorem 3.147, such a group should be either an elementary p-group or a torsion-free group. In the latter case, in accordance with [44, Corollary 54.5], a torsion-free coperiodical group is algebraically compact (for more details the interested reader can see cf. [44] too). Since each its non-zero p-adic component is fully invariant, we are done.

(2) Owing to [44, Lemma 96.4] such a group should be homogeneous and separable, whence it is an IFI-group. □

As already illustrated in the proof of point (2) of Corollary 3.150, since any non-zero fully invariant subgroup of a group G is of the form nG, where $n \in \mathbb{N}$, it easily follows that every separable homogeneous torsion-free group of idempotent type is an IFI-group.

Furthermore, recall that a torsion-free group A is called *fully transitive* if, for each two elements $0 \neq a, b \in A$ with $\chi_A(a) \leqslant \chi_A(b)$, there exists $f \in \mathrm{E}(A)$ such that $f(a) = b$. This class of groups is quite large and, for instance, it contains algebraically compact torsion-free groups and homogeneous separable groups (see, for example, [78]). Using this definition, the last claim about separable homogeneous torsion-free groups stated above can be somewhat extended like this:

Proposition 3.151. *Every homogeneous fully transitive torsion-free group of an idempotent type is an IFI-group.*

Proof. In Paragraph 25, Exercise 11 of [78] was proved that every fully invariant subgroup of a torsion-free group G has the form nG for some integer $n \geqslant 0$ if and only if G is a homogeneous torsion-free fully transitive group of an idempotent type. And since nG is isomorphic to G, all non-trivial fully invariant subgroups are mutually isomorphic, so that the assertion follows. □

On the other hand, if an almost completely decomposable group (for the definition we refer the reader to [83]) is an IFI-group, then by virtue of Theorem 3.147 it is homogeneous of an idempotent type. Likewise, excepting the case where it is isomorphic to its regulator, an almost completely decomposable IFI-group A should be a completely decomposable homogeneous group with the property that each its fully invariant subgroup has the form nA.

As a consequence to Proposition 3.151 we obtain the following (see also Problem 1 below).

Corollary 3.152. *A direct summand of a fully transitive torsion-free IFI-group is again a fully transitive IFI-group.*

Proof. In view of Theorem 3.147, the group G should be homogeneous of an idempotent type. Moreover, it follows from [78] that any direct summand of a fully transitive torsion-free group is again a torsion-free fully transitive group. But it must also be homogeneous of an idempotent type, so that Proposition 3.151 is applicable to get the claim. □

In [54] a group G is called *an H-group* if any its fully invariant subgroup F has the form $F = \{a \in G \mid H(a) \geqslant M\}$, where $\mathrm{H}(a)$ is the *height matrix* of the element a and M is some $\omega \times \omega$-matrix with ordinal numbers and symbol ∞ for entries. Likewise, it is shown there that every H-group is a fully transitive group and that a p-group is a H-group if and only if it is fully transitive. However, there are fully transitive torsion-free groups that are not H-groups. Nevertheless, torsion-free homogeneous fully transitive groups are necessarily H-groups.

The next assertions shed some light about the relationships between IFI-groups and H-groups (compare also with Theorem 3.161 below).

Proposition 3.153. *Suppose that G is a torsion-free H-group. Then G is an IFI-group if, and only if, G is a homogeneous group of an idempotent type.*

Proof. The necessity follows directly from Theorem 3.147. Since as observed above H-groups are fully transitive, the sufficiency follows directly from Proposition 3.151. □

Mimicking [105], a ring R with identity is said to be an *E-ring* if $\mathrm{Hom}_{\mathbb{Z}}(R,R) = \mathrm{Hom}_R(R,R)$, where $\mathbb{Z}$ is the ring of integers. Note that every E-ring is necessarily commutative. The additive groups of E-rings are just called *E-groups.* Notice also that the group A is an E-group if and only if $A \cong \mathrm{End}(A)$ and the ring $\mathrm{E}(A)$ is commutative. Furthermore, if R is a commutative ring, then the right R-module A is said to be an *E-module* if $\mathrm{Hom}_{\mathbb{Z}}(R,A) = \mathrm{Hom}_R(R,A)$.

We also recall that the commutative ring R with identity is called *a principal ideal ring* if each its ideal is principal, that is, it is of the form xR for some element $x \in R$.

Theorem 3.154. *Suppose $A \neq 0$ is a torsion-free group whose non-zero endomorphisms are monomorphisms. Then A is an IFI-group if, and only if, A is an E-group and $E(A)$ is a principal ideal ring.*

Proof. "**Necessity.**" Set $R = \mathrm{E}(A)$. For each $0 \neq a \in A$ the map of R^+ onto Ra, defined by $R^+\beta \mapsto \beta a$, gives the group isomorphism $R^+ \cong Ra$. Thus $A \cong Ra \cong R^+$. Let $f\colon R^+ \to A$ be an isomorphism. Now, the map $\psi\colon ra \mapsto f(ra) - r(f(a))$ defines for each fixed $0 \neq a \in A$ a group homomorphism $Ra \to A$ with non-zero kernel. Therefore, $\psi = 0$ which forces an isomorphism $Ra \cong A$. So, $f(ra) = r(f(a))$ and, hence, f is an R-modular isomorphism. But $R^+ \cong A$ implies the equality $\mathrm{Hom}_{\mathbb{Z}}(R,R) = \mathrm{Hom}_R(R,R)$, that is, the ring R is an E-ring. Every ideal I of R as a submodule of an E-module R_R is an E-module as well. Consequently, the isomorphism $I^+ \cong Ra$ is an R-modular isomorphism and so the ideal I is principal, i.e., R is a principal ideal ring.

"**Sufficiency.**" Since $A \cong \mathrm{End}(A)$, then we can determine on A the structure of the ring $\mathrm{E}(A)$, so that all non-zero fully invariant subgroups of A can be considered as the ideals of the ring $\mathrm{E}(A)$. According to the condition on additive groups, such ideals are obviously isomorphic to $\mathrm{End}(A)$, as required. □

The next implication is simple but useful.

Lemma 3.155. *If A is a torsion-free IFI-group, then all its non-zero central endomorphisms are monomorphisms.*

Proof. If α is a central endomorphism and $\ker\alpha \neq 0$, then $\ker\alpha \cong A$ and therefore there exists a monomorphism $f \in \mathrm{E}(A)$ such that $\alpha f = 0$. But $\alpha f = f\alpha$, whence $\alpha = 0$ as needed. □

It follows from [16, Lemma 1.3] that in any quasi-homogeneous torsion-free fully transitive group all non-zero central endomorphisms are monomorphisms. Besides, in [22] were found some necessary and sufficient conditions for groups to be torsion-free fully transitive, provided that their endomorphism ring is commutative.

A group is said to be *irreducible* if it does not have proper pure fully invariant subgroups. So, elementary p-groups can be considered as irreducible. If now A is a torsion-free IFI-group of finite rank, then any its pure fully invariant subgroups coincides with the full group A; in particular, the group A is irreducible.

We are now concentrated on the IFI-groups without torsion elements having finite rank. The following assertion sheds some light on the endomorphism ring structure of such groups.

Proposition 3.156. *If A is a torsion-free IFI-group of finite rank, then the following conditions are equivalent:*

(1) all non-zero endomorphisms of A are monomorphisms;

(2) A is a strongly indecomposable group;

(3) $E(A)$ is a commutative ring.

Proof. The implication (1) $\Rightarrow$ (2) is obvious, while the implication (1) $\Rightarrow$ (3) follows from Theorem 4.60. As for the validity of the implication (3) $\Rightarrow$ (1), it was noted above. Now, we will show that (2) $\Rightarrow$ (1) is true. In fact, in a strongly indecomposable torsion-free IFI-group of finite rank any pure fully invariant subgroup coincides with the whole group. Consequently, according to Corollary 5.14 from [78], all its non-zero endomorphisms are monomorphisms which guarantees the wanted implication. □

A combination of Theorem 3.154 and Proposition 3.156 gives the following:

Corollary 3.157. *Suppose $A \neq 0$ is a strongly indecomposable torsion-free IFI-group of finite rank. Then A is an E-group and* $\mathrm{E}(A)$ *is a principal ideal ring.*

Homogeneous fully transitive torsion-free groups A of an idempotent type are *endocyclic groups*, that is, $A = \mathrm{E}(A)a$ for a certain element $a \in A$; in conjunction with Proposition 3.151 they are also IFI-groups. All fully invariant subgroups of such a group A are submodules of the R-module ${}_RA$, where $R = \mathrm{E}(A)$. If in the determination of the torsion-free IFI-group we require an R-module isomorphism, then under the validity of the isomorphism $A \cong Ra$, where $0 \neq a \in A$, the group A is endocyclic. Moreover, a more general class form the so-called *endofinite groups* that are groups considered as finitely generated modules over their endomorphism rings.

So, we proceed by proving the following statement.

Theorem 3.158. *Suppose A is an irreducible endofinite torsion-free group, the center C of $E(A)$ is a principal ideal domain and the module ${}_CA$ has rank $\leqslant \aleph_0$. Then A is an IFI-group. Besides, if the group A is decomposable, then it is both an IC-group and an IP-group.*

Proof. According to [58] (see also [78, Corollary 8.6]) one sees that A is a free C-module. If now H is a fully invariant subgroup of A, then H is a submodule of the module ${}_CA$. Since C is a principal ideal domain, then H is also a free C-module (same rank as ${}_CA$ under the truthfulness of the fully invariance of the subgroup H). Consequently, the module ${}_CH$ is isomorphic to ${}_CA$, and hence we have the group isomorphism $H \cong A$, as desired.

The second part is immediate. □

We shall say that R is *a ring with property* $(*)$ if R^+ is a torsion-free group and the factor-ring R/pR is a domain for any prime number p such that $pR \neq R$. With [78, Lemma 44.6] at hand it will follow that in such a ring R the equality $\chi(ab) = \chi(a) + \chi(b)$ holds for any $a, b \in R$.

The following technicality is pivotal.

Lemma 3.159. *Let R be an E-ring with property $(*)$. Then the following conditions are equivalent:*

(1) R^+ is irreducible;

(2) any element of R is an integer multiplied by invertible;

(3) R^+ is a homogeneous fully transitive group.

Proof. "(1) $\Rightarrow$ (2)". Since R^+ is irreducible then it is homogeneous, and since $\chi(1)$ is the least characteristic then the type of R^+ is an idempotent. Supposing that $I = xR$ is a main ideal, we write $x = nx_0$ where $\chi(x_0) = \chi(1)$ and $J = x_0R$. If $y = x_0z \in J$, where $z \in R$, then $\chi(y) = \chi(x_0) + \chi(z) = \chi(z)$ because $\chi(x_0)$ is a characteristic consisting only of 0 and ∞. So, if $p^kt = y$, then $z \in p^kR$ and $y \in p^kJ$. Equivalently, J^+ is a pure fully invariant subgroup in R^+ because R is an E-ring. Consequently, $J = R$ ensures that the element x_0 is invertible, as required.

"(2) $\Rightarrow$ (3)". Since any element of R is an integer multiplied by invertible, the group R^+ is homogeneous of an idempotent type. Let $0 \neq a, b \in R^+$ and $\chi(a) \leqslant \chi(b)$. Assuming $na_0 = a$, where a_0 is invertible, we obtain that $nb_0 = b$. Therefore, $b_0a_0^{-1}a = b$, as wanted.

"(3) $\Rightarrow$ (1)". It is obvious. □

It is worthwhile noticing that, since the multiplication of elements of a ring by its invertible elements is an automorphism, all conditions of Lemma 3.159 are also equivalent to the fact that R^+ is a homogeneous transitive group. Besides, note that the ring R from Lemma 3.159 is a principal ideal domain.

Proposition 3.160. *Any countable irreducible and endofinite torsion-free group, for which the center of its endomorphism ring is a principal ideal domain with property $(*)$, is both a fully transitive and transitive group.*

Proof. Let A be such a group and let C be the center of $\mathrm{E}(A)$. In accordance with [78, Theorem 8.7], C is an E-ring. With [78, Corollary 8.6] at hand, A is a free C-module. However, as a direct summand of A, the group C^+ is irreducible. Thus the proof goes on by virtue of Lemma 3.159 and [78, Corollary 40.5]. □

The next statement also describes certain cases of IFI-groups (compare with Proposition 3.156 above).

Theorem 3.161. *For a torsion-free group G of finite rank, for which the center C of $E(G)$ is a ring satisfying property $(*)$, the following four conditions are equivalent:*

(1) G is an IFI-group;

(2) G is an irreducible endofinite group and C is a principal ideal E-ring;

(3) $G \cong (C^+)^{(n)}$, where n is some natural number and C^+ is a strongly indecomposable E-group of finite rank;

(4) G is a homogeneous fully transitive group of idempotent type.

Proof. "(1) $\Rightarrow$ (2)" and "(1) $\Rightarrow$ (3)". We have noted before the statement of Proposition 3.156 that G is irreducible as well as we have proved in Lemma 3.155 that all non-zero endomorphisms in C are monomorphisms. Since $G \cong \mathrm{E}(G)a$ for any fixed $0 \neq a \in G$, the subgroup $\mathrm{E}(G)a$ has finite index in G because of the finite rank of G. So G is an endofinite group. Invoking [78, Corollary 8.8] and Corollary 3.157, C is a principal ideal E-ring, and G is quasi-isomorphic to $(C^+)^{(n)}$ for some n. As being a quasi-summand of G, the group C^+ is irreducible, so referring to Lemma 3.159 and [78, Corollary 8.6] we obtain that $G \cong (C^+)^{(n)}$ that substantiates the proof of these two implications.

"(2) $\Rightarrow$ (3)". Follows from [78, Corollary 11.5].

"(3) $\Rightarrow$ (4)". By virtue of [78, Corollary 8.10] the group $\mathrm{End}(C^+)$ is an irreducible group, and hence in view of Lemma 3.159 the group $C^+ \cong \mathrm{End}(C^+)$ is a fully transitive group, so G as being isomorphic to the direct sum of copies of a fixed fully transitive group is again fully transitive, as required.

"(4) $\Rightarrow$ (1)". Follows directly from Proposition 3.151. $\square$

Remark 2. Note that Theorem 3.147 guarantees the validity only of a part of implication (1) $\Rightarrow$ (4), namely that G is a homogeneous group of an idempotent type.

Now we will consider the question of when an arbitrary direct sum of IFI-groups is again an IFI-group. Before doing this, it is worthy of noticing that any IFI-group G has no a non-trivial fully invariant direct summand (i.e., a fully invariant direct summand $\neq 0, G$). To that goal, Theorem 3.147 settles this when G is a torsion group. Letting now G be torsion-free, we write in a way of contradiction that $G = A \oplus B$, where $A \neq 0$ is fully invariant in G. Then $A \cong G$, so one can infer that $A = A_1 \oplus B_1$, where $A_1 \cong A$ and $B_1 \cong B$. But this allows us to conclude that $\mathrm{Hom}(A, B) \neq 0$, and so the desired claim follows.

Proposition 3.162. *Suppose A_i $(i \in I)$ is a system of non-zero torsion-free IFI-groups. Then $G = \bigoplus_{i\in I} A_i$ is an IFI-group if, and only if, at most one of the following two conditions is valid:*

(1) for any pair $i, j \in I$ and for each $0 \neq a \in A_i$ there exists $\varphi \in Hom(A_i, A_j)$ with the property $\varphi(a) \neq 0$;

(2) $\bigoplus_{j\in J_K} A_j \cong G$ *for each* $K \subseteq I$ $(K \neq \varnothing, I)$, *where* $J_K = \{j \in I \mid \bigcap_{f\in \mathrm{Hom}(A_j,A_k),\, k\in K} \ker f \neq 0\}$.

Proof. "**Necessity.**" Assume that $J_K \neq \varnothing$ for some $\varnothing \neq K \subsetneqq I$. Thus $G = B \oplus C$, where $B = \bigoplus_{j\in J_K} A_j$ and $C = \bigoplus_{i\in I\setminus J_K} A_i$. Set

$$H_j = \bigcap_{f\in \mathrm{Hom}(A_j,A_k),\, k\in K} \ker f,$$

where $j \in J_K$. It is clear that $H = \bigoplus_{j\in J_K} H_j$ is a fully invariant subgroup in B. But if $i \in I \setminus J_K$, then for any $0 \neq a \in A_i$ there exist $k \in K$ and $f \in \mathrm{Hom}(A_i, A_k)$ with the property $f(a) \neq 0$. So H is a fully invariant subgroup of G and H_j is a fully invariant subgroup of A_j for each $j \in J_K$, respectively. Therefore, $H \cong G$ and hence $B \cong H \cong G$, as required.

"**Sufficiency.**" If H is a fully invariant subgroup of G, then it is well known that $H = \bigoplus_{i\in I}(H \cap A_i)$, where every $H \cap A_i$ is a fully invariant subgroup of A_i. If now condition (1) holds, then $H \cap A_i \neq 0$ for any $i \in I$. However, $H \cap A_i \cong A_i$ which assures that $H \cong G$, as needed.

If we set $K = \{k \in I \mid H \cap A_k = 0\} \neq \varnothing$ and $J = \{j \in I \mid H \cap A_j \neq 0\}$, then one sees that $J \cup K = I$ and $J \cap K = \varnothing$, so that

$$J = J_K = \{s \in I \mid \bigcap_{f\in \mathrm{Hom}(A_s,A_k),\, k\in K} \ker f \neq 0\}.$$

Next, in the presence of condition (2), we conclude that $H \cap A_j \cong A_j$ for each $j \in J$ and consequently $H \cong G$, as required. □

We notice the obvious fact that condition (1) in Proposition 3.162 is not equivalent to $\mathrm{Hom}(A_i, A_j) \neq 0$ for any $i, j \in I$; in fact, it is weaker than that inequality because in (1) the homomorphism φ depends on the choice of the element a.

The next assertion however shows that under some additional circumstances on the family $\{A_i\}_{i\in I}$, the last statement can be somewhat reversed.

Proposition 3.163. *Let* A_i $(i \in I)$ *be a system of non-zero irreducible IFI-groups. Then* $G = \bigoplus_{i\in I} A_i$ *is an IFI-group if, and only if,* $Hom(A_i, A_j) \neq 0$ *for any* $i, j \in I$.

Proof. "**Necessity.**" Assume that $\varphi(a) = 0$ for some $0 \neq a \in A_i$ and for each $\varphi \in \mathrm{Hom}(A_i, A_j)$. If we set $B = \bigoplus_{k \in I \setminus \{j\}} A_k$ and $C = A_j$, then $a \in H = \bigcap_{f \in \mathrm{Hom}(B,C)} \ker f$, where it is readily checked that H is a pure fully invariant subgroup of G. So, it follows that $H = \bigoplus_{i \in I}(H \cap A_i)$, where $H \cap A_i$ are fully invariant pure subgroups of A_i and thus $H \cap A_i = A_i$ if $H \cap A_i \neq 0$. Consequently, H is a non-zero fully invariant direct summand of G that contradicts the remark listed before Proposition 3.162. In particular, $\mathrm{Hom}(A_i, A_j) \neq 0$ for any $i, j \in I$, as desired.

"**Sufficiency.**" By hypothesis, it follows that either all A_i are elementary p-groups for a fixed prime natural p and hence G is an elementary p-group, or all A_i are torsion-free groups. In second case these A_i are irreducible groups. So, for any $0 \neq a, b \in A_i$, we find $f \in \mathrm{E}(A_i)$ with the property that $f(a) = kb$ for some natural number k. Thus, if H is a fully invariant subgroup of G, then $H \cap A_i \neq 0$ for every $i \in I$. As in the proof of Proposition 3.162 we deduce that $H \cong G$ whence G is an IFI-group, as claimed. □

As an immediate consequence to Proposition 3.162, we also derive:

Corollary 3.164. *If G is an IFI-group, then $G^{(m)}$ is also an IFI-group for any ordinal m.*

It was proved in [54, Corollary 3.24] that if G is a homogeneous fully transitive torsion-free group and $\mathbf{K}$ is an arbitrary ideal of the Boolean algebra of all subsets of a certain set of indices I, then the $\mathbf{K}$-direct sum $\bigoplus_{\mathbf{K}} G$ remains a fully transitive group. If, additionally, the type of G is an idempotent, then $\bigoplus_{\mathbf{K}} G$ will also be homogeneous as a pure subgroup of the homogeneous group G^I (see, for instance, [44, Lemma 96.4]).

We thus deduce the following statement:

Proposition 3.165. *If G is a fully transitive torsion-free IFI-group, then any $\mathbf{K}$-direct sum $\bigoplus_{\mathbf{K}} G$ is an IFI-group.*

Recall that a torsion-free group is called *strongly irreducible* if any its non-zero fully invariant subgroup has bounded index. Utilizing [44, Proposition 92.1], we directly obtain the following:

Proposition 3.166. *Any strongly irreducible group G, satisfying the condition $|G/pG| \leqslant p$ for each prime p, is an IFI-group.*

We close the work with some questions of certain interest and importance.

Problem 1. Is a direct summand of an IFI-group again an IFI-group?

Problem 2. Do there exist IFI-groups that are not fully transitive (in particular, that are not H-groups)?

Problem 3. Do there exist non irreducible and non endocyclic torsion-free IFI-groups?

Problem 4. Does there exist a strongly irreducible endocyclic group which is not an IFI-group?

Problem 5. If possible, construct an IFI-group that is neither an IC-group nor an IPI-group.

We come now to our other key section.

4. Generalizations of simply presented Abelian p-groups

We shall distinguish here three subsections as follows:

4.1. **An application of set theory to $(\omega+n)$-totally $p^{\omega+n}$-projective abelian p-groups.** By the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration. Our group theoretic terminology and notation will generally follow that found in [44, 47]. In particular, $p^\omega G$ denotes the first Ulm subgroup of a group G consisting of all elements of infinite height, and $p^{\omega+n}G = p^n(p^\omega G)$. The cyclic group of order p^k will be denoted by $\mathbb{Z}_{p^k}$ and the infinite cocyclic group will be denoted by $\mathbb{Z}_{p^\infty}$. We will say a group G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. A group G is a *dsc-group* if it is isomorphic to a direct sum of countable groups. In particular, we are not assuming that our dsc-groups are necessarily reduced; in fact, they are a direct sum of a divisible group and a reduced group where the second summand is a dsc-group in the sense of [44]. Following [65] and [67], a group G is said to be a Σ-*group* if one (and hence every) *high* subgroup of G is Σ-cyclic (where a subgroup X of G is high if it is maximal with respect to the property $X \cap p^\omega G = \{0\}$).

It was asked in [65] and [67] whether or not subgroups of Σ-groups are again Σ-groups. In general, a subgroup of a Σ-group is not necessarily a Σ-group (see Example 2 of [86]). We will say G is a *totally* Σ-*group* if every subgroup of G is also a Σ-group. Our first objective is to give several different characterizations of this class (Theorem 4.6). For example, G is a totally Σ-group iff it is the direct sum of a countable group and a Σ-cyclic group. Alternatively, we will say that G is ω-*totally* Σ-*cyclic* if every separable subgroup S of G is Σ-cyclic. It is elementary that G is a totally Σ-group iff it is ω-totally Σ-cyclic (Proposition 4.1).

The class of ω-totally Σ-cyclic groups can be described in other ways. For example, it coincides with the class of *ω-totally pure-complete groups*, i.e., those groups all of whose separable subgroups are pure-complete (where a group X is pure-complete if for every subgroup $S \subseteq X[p]$ there is a pure subgroup $P \subseteq X$ such that $P[p] = S$). It also coincides with the class of *$\omega + n$-totally dsc-groups*, i.e., those groups all of whose $p^{\omega+n}$-bounded subgroups are dsc-groups.

Expanding slightly on the example of Megibben in [86], if H is any group (e.g., a torsion-complete group), then there is a group G such that $p^{\omega}G = H$ and $G/p^{\omega}G$ is Σ-cyclic. Since for any high subgroup Z of G there is an embedding $Z \to G/p^{\omega}G$, Z must be Σ-cyclic, so that G will be a Σ-group containing H. On the other hand, if H is not countable, then G will not be a totally Σ-group. We sharpen this observation by showing that any separable group S can be embedded as a subgroup in a group G of length $\omega + 1$ which is a Σ-group (but not a totally Σ-group - Proposition 4.9).

More generally, if $\mathbf{C}$ is a class of groups and α is an ordinal, we will say that G is *α-totally* $\mathbf{C}$ if every p^{α}-bounded subgroup of G is a member of $\mathbf{C}$. Again, it is elementary that G is α-totally $\mathbf{C}$ iff every subgroup of G has the property that all of its *p^{α}-high* subgroups are in $\mathbf{C}$ (where a subgroup X of a group Y is p^{α}-high iff it is maximal with respect to the property that $X \cap p^{\alpha}Y = \{0\}$). In fact, we will mainly be concerned with the case where $n < \omega$, $\alpha = \omega + n$ and $\mathbf{C}$ is the class of *$p^{\omega+n}$-projective groups*; recall that G is $p^{\omega+n}$-projective if $p^{\omega+n}\mathrm{Ext}(G, X) = 0$ for all X, or equivalently, if there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic (see, e.g., [97]). So, a group is p^{ω}-projective iff it is Σ-cyclic. It follows easily that the class of $p^{\omega+n}$-projectives is closed under arbitrary subgroups. In addition, if G_1 and G_2 are $p^{\omega+n}$-projectives, then G_1 and G_2 are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric (i.e., there is an isomorphism that preserves the height functions on the two groups; see [46]). So, if $\mathbf{C}$ is the class of $p^{\omega+n}$-projective groups and $\alpha = \omega + n$, we have that a group G is $\omega + n$-totally $p^{\omega+n}$-projective iff every $p^{\omega+n}$-bounded subgroup X of G is $p^{\omega+n}$-projective. And since a group is p^{ω}-projective iff it is Σ-cyclic, a group is ω-totally p^{ω}-projective iff it is ω-totally Σ-cyclic.

Note that if $p^{\omega+n}G = \{0\}$, then G is $\omega + n$-totally $p^{\omega+n}$-projective iff it is $p^{\omega+n}$-projective. It is also straightforward to verify that the class of $\omega + n$-totally $p^{\omega+n}$-projectives contains the class of ω-totally Σ-cyclic groups (Corollary 4.8). We will say an $\emptyset + n$-totally $p^{\omega+n}$-projective group G is *proper* if it does not belong to either of these two classes; i.e., iff it is not $p^{\omega+n}$-projective and not ω-totally Σ-cyclic. In particular, there are no proper ω-totally p^{ω}-projectives. For $0 < n < \omega$

we study the question of whether there are, in fact, any proper $\varnothing+n$-totally $p^{\omega+n}$-projective groups. In fact, we show that this question is equivalent to a natural construction expressible using *valuated vector spaces* (see, for example, [101] and [45]).

If V is a group, then a *valuation* on V is a function $v : V \to \mathbf{O}_\infty$ (where $\mathbf{O}_\infty$ is the class of all ordinals plus the symbol ∞), such that for all $x, y \in V$, $v(x \pm y) \geq \min\{v(x), v(y)\}$ and $v(px) > v(x)$. It follows that for every $\alpha \in \mathbf{O}_\infty$, $V(\alpha) = \{x \in V : v(x) \geq \alpha\}$ is a subgroup of V. If V and W are valuated groups, then a homomorphism $\phi : V \to W$ will be said to be *valuated* if $v(x) \leq v(\phi(x))$ for all $x \in V$, and an *isometry* if it is bijective and preserves all values. Note that if G is any group and H is a subgroup of G, then the height function on G restricts to a valuation on H. The category of valuated groups clearly has direct sums.

Naturally, a valuated group V is a valuated vector space if $pV = \{0\}$. In particular, the socle of a group will always be a valuated vector space. The valuated vector space V will be said to be *separable* if $V(\omega) = \{x \in V : v(x) \geq \omega\} = \{0\}$ and *free* if it is isometric to the valuated direct sum of valuated vector spaces of rank one. If W is a subspace of V, then the *corank* of W is the dimension of V/W. A subspace E of V will be called *cofree* if there is a valuated decomposition $V = E \oplus F$, where F is free [in other words, V is algebraically the internal direct sum of E and F, and $v(x + y) = \min\{v(x), v(y)\}$ for all $x \in E$ and $y \in F$].

If κ is an infinite cardinal, then a valuated vector space V will be said to be *κ-coseparable* if it is separable and every subspace W of corank strictly less than κ contains a subspace $E \subseteq W$ that is cofree in V. We will really only be concerned with the cases where $\kappa = \aleph_0$ or $\aleph_1$. A κ-coseparable valuated vector space will be said to be *proper* if it is not free. In [38] the existence of a proper $\aleph_1$-coseparable valuated vector space was shown to be equivalent to a question involving the structure of abelian groups, and to be independent of ZFC. We conclude this subsection by showing that for $0 < n < \omega$, the existence of a proper $\aleph_0$-coseparable valuated vector space is equivalent to the existence of a proper $\omega+n$-totally $p^{\omega+n}$-projective group, and we prove that both of these propositions are independent of ZFC (Theorem 4.24).

We start with the comprehensive study of $\omega+n$-totally $p^{\omega+n}$-projective groups. First, we begin with the following elementary assertion:

Proposition 4.1. *If G is a group, α is an ordinal and $\mathbf{C}$ is a class of groups, then G is α-totally $\mathbf{C}$ iff every subgroup $T \subseteq G$ has the property that every p^α-high subgroup of T is a member of $\mathbf{C}$.*

Proof. Suppose G is α-totally $\mathbf{C}$ and T is an arbitrary subgroup of G. If S is a p^α-high subgroup of T, then S is p^α-bounded, so by hypothesis, S is in $\mathbf{C}$. So, one direction has been established.

Conversely, suppose every p^α-high subgroup of a subgroup of G is in $\mathbf{C}$. If S is any p^α-bounded subgroup of G, then S is a p^α-high subgroup of itself, so it must be in $\mathbf{C}$, so that G is α-totally $\mathbf{C}$. □

The following is a special case of a general result on extending homomorphisms on nice subgroups.

Lemma 4.2. *Suppose G and H are groups, H has infinite cardinality κ, P is a subgroup of H such that H/P is Σ-cyclic and there is an injective homomorphism $\phi : P \to G$ such that (1) for every $x \in P$, $ht_G(\phi(x)) \geq ht_H(x)$; and (2) for every $m < \omega$, $(p^m G)[p]/(p^m G \cap \phi(P))[p]$ has cardinality at least κ. Then ϕ extends to an injective homomorphism $\Phi : H \to G$.*

Proof. Suppose, for a moment, that the quotient H/P possesses cardinality κ – in fact, by adding a Σ-cyclic summand if needed, there is clearly no loss of generality in assuming that H/P has cardinality κ. So, suppose $H/P \cong \bigoplus_{i<\kappa} \langle x_i + P\rangle$, and for $\alpha < \kappa$ let $H_\alpha = P + \langle x_i : i < \alpha\rangle$; thus $H = \cup_{\alpha<\kappa} H_\alpha$. We inductively extend ϕ to an injection $\phi_\alpha : H_\alpha \to G$, so that $\beta < \alpha$ implies that ϕ_α agrees with ϕ_β on H_β. Assume we have constructed ϕ_β for all $\beta < \alpha$. If α is a limit, then we clearly need just take unions. On the other hand, suppose α is isolated and $x_{\alpha-1} + P$ has order p^m in H/P. It follows that $p^m x_{\alpha-1} \in P$, and so $\phi(p^m x_{\alpha-1})$ is defined. By condition (1), we have $ht_G(\phi(p^m x_{\alpha-1})) \geq m$; let $u \in G$ satisfy $p^m u = \phi(p^m x_{\alpha-1})$. Clearly, $[\langle u\rangle + \phi_{\alpha-1}(H_{\alpha-1})]/\phi(P)$ has rank $|\alpha| < \kappa$. By condition (2), there is an element

$$w \in (p^{m-1}G)[p] - (\langle u\rangle + \phi_{\alpha-1}(H_{\alpha-1})) \,. \qquad (*)$$

Choose $z \in G$ such that $p^{m-1}z = w$. We let ϕ_α agree with $\phi_{\alpha-1}$ on $H_{\alpha-1}$ and $\phi_\alpha(x_{\alpha-1}) = u + z$. To show ϕ_α is an injection, suppose $y \in H_\alpha$ and $\phi_\alpha(y) = 0$. Let $y = a + kx_{\alpha-1}$, where $a \in H_{\alpha-1}$ and k is an integer. We first claim that $p^m|k$: If this failed, then for some integer ℓ we would have $\ell k \equiv p^{m-1}$ modulo the order of z. Therefore,

$$0 = \phi_\alpha(\ell y) = \phi_\alpha(\ell a + \ell k x_{\alpha-1}) = \phi_{\alpha-1}(\ell a) + \ell k u + \ell k z = \phi_{\alpha-1}(\ell a) + \ell k u + w.$$

This implies that $w = -\ell k u - \phi_{\alpha-1}(\ell a)$, which contradicts $(*)$. We can, therefore, conclude that $p^m|k$, so that $kx_{\alpha-1} \in P$, and hence, $y \in H_{\alpha-1}$. Since $\phi_{\alpha-1}$ is injective, we have $y = 0$, as required.

Letting $\Phi = \cup_{\alpha<\kappa}\phi_\alpha$ completes the proof. □

Recall that G is ω-totally $p^{\omega+n}$-projective means that every separable subgroup of G is $p^{\omega+n}$-projective.

Proposition 4.3. *If $n < \omega$ and G is ω-totally $p^{\omega+n}$-projective, then $p^{\omega+n}G$ is countable.*

Proof. Suppose on the contrary that $p^{\omega+n}G$ is uncountable. Let H be a separable group of cardinality $\aleph_1$ which is $p^{\omega+n+1}$-projective, but not $p^{\omega+n}$-projective. [To construct such a group, let A be a separable group of cardinality $\aleph_1$ with a countable basic subgroup, so that A is not p^{α}-projective for any ordinal α. If C is a Σ-group of rank and final rank $\aleph_1$, then by Theorem 8 of [33], $A \oplus C$ has a subgroup H of the required form.]

Let P be a subgroup of H such that $p^{\omega+n+1}P = \{0\}$ and H/P is Σ-cyclic. Since $p^{\omega+n}G$ is uncountable, there is a subgroup P' of $p^{\omega}G$ which is isomorphic to P such that $(p^{\omega}G)[p]/P'[p]$ is uncountable. By virtue of Lemma 4.2, the isomorphism of P and P' extends to an embedding $H \to G$. Since H is separable and not $p^{\omega+n}$-projective, we can conclude that G is not ω-totally $p^{\omega+n}$-projective, contrary to our assumption. $\square$

Since an $\omega+n$-totally $p^{\omega+n}$-projective group is ω-totally $p^{\omega+n}$-projective, we have the following:

Corollary 4.4. *If $n < \omega$ and G is $\omega+n$-totally $p^{\omega+n}$-projective, then $p^{\omega+n}G$ is countable.*

Corollary 4.5. *If $n < \omega$ and G is ω-totally $p^{\omega+n}$-projective, then G is a dsc-group iff $G/p^{\omega}G$ is Σ-cyclic.*

Proof. By using Lemma 78.1 of [44], if G is a dsc-group, then $G/p^{\omega}G$ is Σ-cyclic. Conversely, Proposition 4.3 ensures that $p^{\omega+n}G$ is countable, so that $p^{n}(p^{\omega}G)$ is a dsc-group, and hence, so is $p^{\omega}G$. If, in addition, $G/p^{\omega}G$ is Σ-cyclic, then G must be a dsc-group. $\square$

The following characterizes the class of groups that are ω-totally Σ-cyclic (= ω-totally p^{ω}-projective).

Theorem 4.6. *If G is a group, then the following are equivalent:*

(a) G is a totally Σ-group;
(b) G is ω-totally Σ-cyclic;
(c) G is a Σ-group and $p^{\omega}G$ is countable;
(d) $G/p^{\omega}G$ is Σ-cyclic and $p^{\omega}G$ is countable;
(e) $G \cong C \oplus M$, where C is countable and M is Σ-cyclic;

(f) G is ω-totally pure-complete;
(g) For all $n < \omega$, G is an $\omega + n$-totally dsc-group;
(h) For some $n < \omega$, G is an $\omega + n$-totally dsc-group.

Proof. By Proposition 4.1, (a) and (b) are equivalent, and we begin by verifying that these imply (c); so suppose that G is a totally Σ-group. Clearly, if G is a totally Σ-group, then it is a Σ-group. By Proposition 4.3, the subgroup $p^\omega G$ is countable.

Next assume the validity of (c), so that G is a Σ-group with countable $p^\omega G$, and we show that $G/p^\omega G$ is Σ-cyclic, as required in (d). Suppose Z is a high subgroup of G, so that Z is Σ-cyclic. Since $p^\omega G$ embeds as an essential subgroup of G/Z, it follows that G/Z is countable. Since there is a surjection $G/Z \to G/[Z+p^\omega G]$, it follows that latter group is also countable. However, since there exists a short exact sequence

$$0 \to Z \to G/p^\omega G \to G/[Z + p^\omega G] \to 0,$$

it follows that $G/p^\omega G$ is Σ-cyclic (see, for example, Corollary 3.1 of [37]).

The equivalence of (d) and (e) is another elementary exercise in the theory of totally projective groups (again, see Chapter XII of [44]). So, suppose G satisfies (d) and (e), and we verify that (b) holds as well. If S is any separable subgroup of G, then $S/(S \cap p^\omega G)$ embeds in $G/p^\omega G$, and since $G/p^\omega G$ is Σ-cyclic, it follows that $S/(S \cap p^\omega G)$ is Σ-cyclic. Since $S \cap p^\omega G$ is countable, it follows that S is Σ-cyclic, as required (see, for instance, Theorem 4.2 of [37]).

To establish the equivalence of (b) and (f), note that if G is ω-totally Σ-cyclic and X is a separable subgroup of G, then X must be Σ-cyclic. Since any Σ-cyclic group is pure-complete, it follows that G is ω-totally pure-complete. Conversely, suppose that G is not ω-totally Σ-cyclic, so it has a separable subgroup S which is not Σ-cyclic. By virtue of the "core class property" from [8], one may infer that S contains a subgroup X which is $p^{\omega+1}$-projective but not Σ-cyclic. But by Theorem 2 of [66], a pure-complete $p^{\omega+1}$-projective group must be Σ-cyclic, so that S is not pure-complete. It follows, in turn, that G is not ω-totally pure-complete, proving the result.

Finally, turning to the equivalence of (g) and (h) with the other conditions, suppose first that G is ω-totally Σ-cyclic. It follows that every subgroup of G must also be ω-totally Σ-cyclic, and hence a dsc-group. In particular, for every $n < \omega$, every $p^{\omega+n}$-bounded subgroup of G is a dsc-group, i.e., G is an $\omega+n$-totally dsc-group and thereby (g) now follows successfully.

Clearly (g) implies (h), so assume (h) holds for some positive integer n. It follows that every separable subgroup of G is a separable dsc-group, i.e., every separable subgroup of G is Σ-cyclic. This shows that (h) implies (b), thus completing the proof. $\square$

Remark 4.7. In [86, Theorem 7], Megibben noted that, in our terminology, a group which is isomorphic to a direct sum of a countable group and a Σ-cyclic group is a totally Σ-group; Theorem 4.6 gives the converse of this observation.

Corollary 4.8. *If $n < \omega$ and G is ω-totally Σ-cyclic, then G is $\omega + n$-totally $p^{\omega+n}$-projective.*

Proof. If G is ω-totally Σ-cyclic, then it is an $\omega + n$-totally dsc-group, and since a $p^{\omega+n}$-bounded dsc-group is $p^{\omega+n}$-projective, G must be $\omega + n$-totally $p^{\omega+n}$-projective. $\square$

We now observe that every separable group can be embedded in a Σ-group of minimal length.

Proposition 4.9. *If S is any separable group, then there is a Σ-group G of length $\omega + 1$ containing S as a subgroup.*

Proof. Suppose T is any dsc-group of length $\omega + 1$ for which there is an isomorphism $\phi : p^\omega T \to S[p]$. Let

$$X = \{(x, \phi(x)) : x \in p^\omega T\} \subseteq T \oplus S, \text{ and } G = [T \oplus S]/X.$$

Since $T \cong ([T \oplus \{0\}] + X)/X \subseteq G$ and $S \cong ([\{0\} \oplus S] + X)/X \subseteq G$, we may identify S and T with subgroups of G so that $G = S+T$ and $p^\omega T = S[p] = T \cap S$. Since $p^\omega T \subseteq p^\omega G$ and

$$\begin{aligned} G/p^\omega T &\cong [T + S]/[T \cap S] \\ &\cong (T/[T \cap S]) \oplus (S/[T \cap S]) \\ &= (T/p^\omega T) \oplus (S/S[p]) \\ &\cong (T/p^\omega T) \oplus pS \end{aligned}$$

is separable, it follows that $p^\omega G = p^\omega T$, so that G has length $\omega + 1$.

If Z is a high subgroup of G, then $Z \cap S[p] = Z \cap p^\omega T = \{0\}$, so that $Z \cap S = \{0\}$, as well. Since $Z \cap S = \{0\}$ is the kernel of the composite homomorphism

$$Z \hookrightarrow G = T + S \to (T + S)/S \cong T/(S \cap T) = T/p^\omega T,$$

it follows that this is an embedding. However, since $T/p^\omega T$ is Σ-cyclic, we have that Z is also Σ-cyclic, so that G is a Σ-group, as expected. $\square$

Note that in Proposition 4.9, if S is not Σ-cyclic, then G is a Σ-group which is not a totally Σ-group.

The following property of valuated vector spaces is well-known: If $\phi : V \to F$ is a valuated vector space homomorphism and F is separable and free, then the kernel of ϕ is cofree in V. [See, for example, Lemma 1 of [66]. If W is this kernel, then the separability of F implies that W is nice in V, that is, every coset has an element of maximal value, and the quotient valuated vector space V/W is separable. Since F is the union of bounded subspaces B_k, for $k < \omega$, it follows that V/W will be the union of bounded subspaces $\phi^{-1}(B_k)/W$, again for $k < \omega$. This means that V/W is also free, so that V is isometric to the valuated direct sum $W \oplus (V/W)$.]

We now introduce two useful functors. If G is a group, we let $K(G) = (G/p^\omega G)[p]$ and $K_0(G) = \{(G[p] + p^\omega G)/p^\omega G\} \subseteq K(G)$. Note that $K_0(G)$ is dense in $K(G)$ in the induced p-adic topology. [If $x + p^\omega G \in K(G)$ and $m < \omega$, then $px \in p^\omega G$, so there is a $y \in G$ such that $p^{m+1}y = px$. It follows that $x + p^\omega G = (x - p^m y + p^\omega G) + (p^m y + G)$ so that $K(G) = K_0(G) + K(G)(m)$.] Another way to interpret this notion is to check that the map $x + p^\omega G \mapsto px + p^{\omega+1}G$ is a well-defined surjective homomorphism $K(G) \to p^\omega G/p^{\omega+1}G$, and that $K_0(G)$ is the kernel of this map; thus $K(G)/K_0(G) \cong p^\omega G/p^{\omega+1}G$.

Lemma 4.10. *Suppose G is a group such that $G/p^\omega G$ is $p^{\omega+1}$-projective. Then the following are equivalent:*

(a) There is a group decomposition $G = H \oplus M$ where H is separable and $M/p^\omega M$ is Σ-cyclic;

(b) $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective;

(c) $K_0(G) \subseteq K(G)$ contains a cofree subspace of $K(G)$.

Proof. We first show (a) implies (c). If $G \cong H \oplus M$ is as described, then clearly $H[p]$ maps to a subspace of $K_0(G)$, and $K(G)$ is isometric to $H[p] \oplus (M/p^\omega M)[p]$, where the latter summand is free. This proves (c).

Suppose now that (c) holds, and we will prove (b) does, as well. Let $K(G)$ be the valuated direct sum $E \oplus F$, where $E \subseteq K_0(G)$ and F is free. Since $G_1 = G/p^\omega G$ is $p^{\omega+1}$-projective, there is a subgroup $P \subseteq K(G)$ such that G_1/P is Σ-cyclic. If $Q = P \cap E$, then Q is the kernel of the valuated homomorphism $P \subseteq K(G) \to F$, so that it follows that there is a valuated decomposition $P = Q \oplus F'$, where F' is free. Let C be a Σ-cyclic group such that there is an isometry $\phi : F' \to C[p]$. Letting $\phi(Q) = 0$ then gives a valuated homomorphism $P \to C[p]$, and since G_1/P is Σ-cyclic, this extends to a homomorphism $\phi : G_1 \to C$ such that $P \cap ker(\phi) = Q$. It therefore follows that the map $G_1 \to (G_1/P) \oplus C$ given by $g \mapsto (g+P, \phi(g))$ has Q as its kernel, so that G_1/Q is also Σ-cyclic. Replacing P by Q, then we may assume that $P \subseteq E \subseteq K_0(G)$. This implies that there is a subgroup $P_0 \subseteq G[p]$ such that $P_0 \cap p^\omega G = \{0\}$ and $P = [P_0 \oplus p^\omega G]/p^\omega G$. We then let

$$P_1 = ([P_0 \oplus p^{\omega+1} G]/p^{\omega+1}G) \oplus (p^\omega G/p^{\omega+1}G) \subseteq G/p^{\omega+1}G.$$

It follows that $pP_1 = \{0\}$ and that $(G/p^{\omega+1}G)/P_1 \cong G_1/P$ is Σ-cyclic, so $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective, as required.

Finally, we assume that (b) holds and prove (a). Note that there is a decomposition:

$$G/p^{\omega+1}G = H \oplus Y,$$

where H is separable and $p^{\omega+1}$-projective, and Y is a dsc-group (see, e.g., [48]). We define $L, M \subseteq G$ by the conditions $p^{\omega+1}G = L \cap M$, $L/p^{\omega+1}G = H$ and $M/p^{\omega+1}G = Y$. Note that $G/M \cong H$ is separable, so that $p^\omega G \subseteq M$. This implies that, for every $x \in p^{\omega+1}G$, there is a $y \in p^\omega G \subseteq M$, such that $py = x$. We now prove by induction on m that $p^\omega G \subseteq p^m M$, which we have just observed holds for $m = 0$. Suppose next that it holds for m and $z \in p^\omega G$. Considering $G/p^{\omega+1}G \cong (L/p^{\omega+1}G) \oplus (M/p^{\omega+1}G)$, there is a $w \in M$ such that $x_1 = p^{m+1}w - z \in p^{\omega+1}G$. This means that $x_1 = py_1$ for some $y_1 \in p^\omega G \subseteq p^m M$. Therefore, $y_1 = p^m u$ for some $u \in M$, so that $z = p^{m+1}w - x_1 = p^{m+1}w - py_1 = p^{m+1}(w-u) \in p^{m+1}M$, as required. We can conclude that $p^\omega G \subseteq p^\omega M \subseteq p^\omega G$, so that $p^\omega G = p^\omega M$, and hence $p^{\omega+1}M = p^{\omega+1}G$.

We, therefore, have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & p^{\omega+1}M & \to & L & \to & H & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & M & \to & G & \to & H & \to & 0 \end{array}$$

By Proposition 56.1 of [44], it follows that the bottom row is $p^{\omega+1}$-pure, and since H is $p^{\omega+1}$-projective, we have $G \cong H \oplus M$. Finally, $M/p^\omega M \cong Y/p^\omega Y$ is Σ-cyclic. □

As a consequence, we have the following:

Corollary 4.11. *Suppose that G is a group and $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.*

(i) If $p^\omega G$ is countable, then G is the direct sum of a separable $p^{\omega+1}$-projective group and a countable group.

(ii) If $p^{\omega+1}G$ is countable, then G is the direct sum of a $p^{\omega+1}$-projective group and a countable group.

Proof. Since $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective, it easily follows that $G/p^{\omega}G$ is $p^{\omega+1}$-projective. Applying Lemma 4.10(a), one may write that $G = H \oplus M$, where H is a separable $p^{\omega+1}$-projective group and M is a group with the property that $M/p^{\omega}M$ is Σ-cyclic. Regarding (i), since $p^{\omega}M$ is countable, it follows that M can be written as a direct sum of a Σ-cyclic group and a countable group. So, (i) is sustained.

As for (ii), it is easy to see that $M/p^{\omega+1}M$ is a dsc-group and $p^{\omega+1}M$ is countable. Therefore, M is itself a dsc-group which, because of the countability of $p^{\omega+1}M$, can be decomposed as a direct sum of a dsc-group of length $\omega + 1$, which is certainly $p^{\omega+1}$-projective, and a countable group. □

The following example illustrates that neither statement in Corollary 4.11 holds for $n \geq 2$.

Example 4.12. There is a group G such that $G/p^{\omega+2}G$ is $p^{\omega+2}$-projective and $p^{\omega}G$ is countable which is not the direct sum of a $p^{\omega+2}$-projective group and a countable group.

Proof. Suppose A is an unbounded separable $p^{\omega+2}$-projective group with the property that every summand of A which is Σ-cyclic must be bounded (an example of which was constructed by Cutler and Missel in [34]). Since any unbounded $p^{\omega+1}$-projective group has unbounded Σ-cyclic summands, it follows that A is not $p^{\omega+1}$-projective. Let $P \subseteq A[p^2]$ be a subgroup such that A/P is Σ-cyclic.

We claim that $(p^m A)[p]$ is not contained in P for any $m < \omega$: Assume this fails for some m. If we let $P_0 = (p^m A \cap P)/(p^m A)[p]$, it follows that $pP_0 = \{0\}$. In addition, $(p^m A/(p^m A)[p])/P_0 \cong p^m A/(p^m A \cap P)$ embeds in A/P, so, in particular, it is Σ-cyclic. This implies that $p^{m+1}A \cong p^m A/(p^m A)[p]$ is $p^{\omega+1}$-projective; which in turn would imply that A is $p^{\omega+1}$-projective, which is not the case.

This last claim implies that we can construct a dense subsocle $D \subseteq A[p]$ containing $P[p]$ such that $A[p]/D$ has rank 1. Let L be a subgroup of A containing P that is maximal with respect to $L[p] = D$. It follows that L is pure and dense in A and there is an isomorphism $\varphi : A/L \cong \mathbb{Z}_{p^\infty}$. Let

$$G = \{(a, z) : a \in A, z \in \mathbb{Z}_{p^\infty} \text{ and } \varphi(a) = p^3 z\}.$$

It readily follows that $p^{\omega}G = \{0\} \oplus \mathbb{Z}_{p^\infty}[p^3]$, which we denote by J, and that $G/J \cong A$. In addition, let $P' = P \oplus \{0\} \subseteq G$.

Note that $J \cap P' = \{0\}$, and so $P' \oplus J$ can also be viewed as a p^3-bounded subgroup of G containing $p^{\omega+2}G = p^2 J$. Since

$$(G/p^{\omega+2}G)/[(P' \oplus J)/p^{\omega+2}G] \cong G/[P' \oplus J] \cong A/P$$

is Σ-cyclic and $p^2[(P' \oplus J)/p^{\omega+2}G] = \{0\}$, $G/p^{\omega+2}G$ is $p^{\omega+2}$-projective.

On the other hand, if $G = C \oplus G'$, where C is countable and G' is $p^{\omega+2}$-projective, then $p^{\omega+2}G' = \{0\}$, so that $p^{\omega+2}C = p^{\omega+2}G \neq \{0\}$. In particular, $p^m C \neq \{0\}$ for all $m < \omega$. However,

$$A \cong G/J = G/p^{\omega}G \cong (C/p^{\omega}C) \oplus (G'/p^{\omega}G'),$$

where $C/p^{\omega}C$ is an unbounded Σ-cyclic, which contradicts the fact that A has no unbounded Σ-cyclic summands. $\square$

We can now extend Theorem 4.6(b) $\Leftrightarrow$ (e) for $n = 1$ in the following way:

Proposition 4.13. *An $\omega + 1$-totally $p^{\omega+1}$-projective group G is a direct sum of a $p^{\omega+1}$-projective group and a countable group iff $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.*

Proof. If G is the direct sum of a $p^{\omega+1}$-projective group and a countable group, say $H \oplus C$, then it plainly follows that $G/p^{\omega+1}G \cong H \oplus (C/p^{\omega+1}C)$ is $p^{\omega+1}$-projective, as well.

Conversely, let G be an $\omega+1$-totally $p^{\omega+1}$-projective group such that $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective. Employing Proposition 4.3 and Corollary 4.11(ii), we deduce the desired decomposition of G. $\square$

Though the next lines contain a discussion of the structure of proper $\omega + n$-totally $p^{\omega+n}$-projective groups, we pause for a moment for a few general observations on κ-coseparable valuated vector spaces. It can be easily verified that the class of κ-coseparable valuated vector spaces is closed under valuated direct sums and summands, and that it contains all the separable free valuated vector spaces. In particular, if the separable valuated vector space V is the valuated direct sum $W \oplus F$, where F is free, then V is κ-coseparable iff W is κ-coseparable. In addition, a separable valuated vector space V is $\aleph_0$-coseparable iff every subspace $W \subseteq V$ of corank one contains a cofree subspace (this follows easily since the intersection of a finite collection of cofree subspaces is also cofree).

The following result is our main tool in analyzing proper $\omega + n$-totally $p^{\omega+n}$-projective groups. Since non-free separable valuated vector spaces are usually not $\aleph_0$-coseparable, it puts a serious limitation on the structure of proper $\omega+n$-totally $p^{\omega+n}$-projectives, showing that they are relatively rare phenomena.

Theorem 4.14. *Suppose $n < \omega$ and G is a proper $\omega + n$-totally $p^{\omega+n}$-projective group. If V is a separable valuated vector space for which there is an injective valuated homomorphism $V \to G[p]$, then V is $\aleph_0$-coseparable.*

Proof. We may clearly assume V is unbounded and our valuated injection $V \to G[p]$ is an inclusion such that for all $x \in V$, $v(x) \leq ht_G(x)$. If necessary, we may replace G by $p^m G$ and V by $V(m)$, so that there is no loss of generality in assuming that the rank and final rank of G is some cardinal κ, and that the rank of V is at most κ. Since $p^{\omega+n}G \neq \{0\}$, we can find some non-zero $x \in (p^{\omega+n}G)[p]$. Observe first that if $x \in V$, then $\langle x\rangle$ is a valuated summand of V, and if $V = \langle x\rangle \oplus V_0$, then V is $\aleph_0$-coseparable iff V_0 is. Replacing V by V_0, we may therefore assume that $x \notin V$. Find $y \in p^\omega G$ such that $p^n y = x$, so that $\langle y\rangle \cap V = \{0\}$.

Let Y be a high subgroup of G, so that Y is $p^{\omega+n}$-projective and there is a (ht_G-)valuated decomposition $G[p] = Y[p] \oplus (p^\omega G)[p]$. It follows from Corollary 26 of [66] that $Y[p]$ is isometric to $Q \oplus F$, where F is a free valuated vector space of final rank κ. Consider the valuated composition $V \to G[p] \to Y[p] = Q \oplus F \to F$ whose kernel is $V_1 = V \cap (Q + (p^\omega G)[p])$. We can conclude that V is isometric to $V_1 \oplus F'$ where F' is free; therefore, V is $\aleph_0$-coseparable iff V_1 is. Replacing V by V_1, we may assume $V \subseteq Q + (p^\omega G)[p]$, so that $F \cap V = \{0\}$. This means that if $m < \omega$, that $(p^m G)[p]/(p^m G \cap V)[p]$ has cardinality κ, since it contains a copy of $F(m)$.

Let D be a subspace of V of corank one; we need to exhibit a subspace of D which is cofree in V. If D is not dense in V, then D will be a valuated summand of V, so it will be cofree. We may therefore assume that D is dense in V. Suppose $z \in V - D$, and let $P = D \oplus \langle z + y\rangle \subseteq G$. Note that there is a surjective homomorphism $\rho : P \to V$ which is the identity on D and maps $z + y$ to z; the kernel of this homomorphism is clearly $\langle py\rangle \subseteq P$. We define a valuation v_P on P as follows: Suppose $u \in P$; if $u = 0$, then let $v_P(u) = \infty$; otherwise, if $\rho(u) \neq 0$, then let $v_P(u) = v(\rho(x))$; finally, if $\rho(u) = 0$, then $u = p^k q(py)$, where $(p, q) = 1$ and $k < n$, and we let $v_P(u) = \omega + k$. It is straightforward to check that v_P is a valuation, and if $v_P(u)$ is infinite and $\beta < v_P(u)$, then there is a $w \in P$ such that $pw = u$ and $\beta \leq v_P(w)$. By a variation on a construction in [101], there is a group H of rank at most κ containing P as a subgroup such that

(1) the height function on H agrees with v_P on P;

(2) H/P is Σ-cyclic of rank at most κ.

[Let H be generated by P and a set of elements x_u, for $u \in P - P(\omega)$, subject to the relations $p^{v_P(u)}x_u = u$.]

It follows that $p^\omega H = P(\omega) = \langle py\rangle$, so G is $p^{\omega+n}$-bounded. It is easy to verify that for all $u \in P$, $v_P(u) \leq ht_G(u)$, so by Lemma 4.2, the inclusion $P \subseteq G$ extends to an embedding $H \to G$.

Since G is $\omega + n$-totally $p^{\omega+n}$-projective, we can conclude that H is $p^{\omega+n}$-projective. Therefore, there is a subgroup $R \subseteq H[p^n]$ such that H/R is Σ-cyclic. Note that $P(\omega) \subseteq R \cap P \subseteq P[p^n] = D \oplus \langle py\rangle$, so that if $E = \rho(R \cap P)$, then $E \subseteq D$. In addition, E is the kernel of the valuated composition: $V \cong P/P(\omega) \to$

$H/P(\omega) \to H/R$. Since $(H/R)[p]$ is free, it follows that $E \subseteq D$ is cofree in V, as required. $\square$

A separable valuated vector space V is *efi* (for *essentially finitely indecomposable*) iff it does not have a valuated summand which is an unbounded free valuated vector space. In particular, an unbounded efi valuated vector space cannot be $\aleph_0$-coseparable. Therefore, we have the following direct consequence of Theorem 4.14.

Corollary 4.15. *Suppose G is a proper $\omega+n$-totally $p^{\omega+n}$-projective and V is an unbounded valuated vector space that is efi. Then there does not exist a valuated injection $V \to G[p]$.*

We have seen by Theorem 4.6 that if G is ω-totally Σ-cyclic, then $p^\omega G$ is countable. More generally, by Corollary 4.4, if $n > 0$ and G is $\omega + n$-totally $p^{\omega+n}$-projective, then $p^{\omega+n}G$ must be countable, but $p^\omega G$ does not have to be countable: for example, if G is $p^{\omega+n}$-projective (such as a $p^{\omega+n}$-bounded dsc-group), the group $p^\omega G$ can be made as large as we please.

We now investigate the question of the countability of $p^\omega G$ for *proper* $\omega + n$-totally $p^{\omega+n}$-projective groups by focussing on the cardinal measure of the first Ulm subgroup.

Let σ be the smallest cardinal such that there is a separable valuated vector space of cardinality σ which is *not* $\aleph_0$-coseparable. Since any countable separable valuated vector space is free, and hence $\aleph_0$-coseparable, we can conclude that $\sigma \geq \aleph_1$.

Corollary 4.16. *If G is a proper $\omega+n$-totally $p^{\omega+n}$-projective, then $r(p^\omega G) < \sigma$.*

Proof. Let V be a separable valuated vector space of cardinality σ which is not $\aleph_0$-coseparable. If $r(p^\omega G) \geq \sigma$, then there is an injective group homomorphism $V \to (p^\omega G)[p] \subseteq G[p]$, which certainly does not decrease values, contradicting Theorem 4.14. $\square$

It is clear that the class of ω-totally Σ-cyclic groups is closed under (countable) direct sums. On the other hand, this property does not generalize to integers $0 < n < \omega$.

Corollary 4.17. *If $0 < n < \omega$, then the class of $\omega + n$-totally $p^{\omega+n}$-projective groups is not closed under (finite) direct sums.*

Proof. Let A be a $p^{\omega+1}$-bounded dsc-group such that $r(p^\omega A) \geq \sigma$. Then A is $p^{\omega+1}$-projective, and hence $p^{\omega+n}$-projective, and hence $\omega + n$-totally $p^{\omega+n}$-projective.

Next, let M be a countable reduced group such that $p^{\omega+n}M \neq 0$. Then M is ω-totally Σ-cyclic, and hence $\omega+n$-totally $p^{\omega+n}$-projective.

Note that if $G = A \oplus M$ were $\omega+n$-totally $p^{\omega+n}$-projective, then since G is not $p^{\omega+n}$-projective and $p^{\omega}G$ is not countable, it would have to be proper. Since $r(p^{\omega}G) \geq \sigma$, however, this would contradict Corollary 4.16. □

Let us notice that Corollary 4.16 implies that we would like to know whether $\sigma = \aleph_1$. To investigate this question, we extend our brief detour into the theory of valuated vector spaces. If λ is a cardinal number, let D_λ be a valuated vector space of dimension λ such that $v(x) = \emptyset$ for all non-zero $x \in D_\lambda$. Let $\phi_\lambda : F_\lambda \to D_\lambda$ be a surjective homomorphism, where F_λ is a free separable valuated vector space of cardinality $\lambda \cdot \aleph_0$ such that if M_λ is the kernel of ϕ_λ, then $\omega = \max\{v(x+y) : y \in M_\lambda\}$ for every $x \in F_\lambda - M_\lambda$ (i.e., M_λ is a dense subspace of F_λ of corank λ). If V and W are valuated vector spaces, let $\mathrm{Hom}_v(V, W)$ denote the collection of all valuated homomorphisms $f : V \to W$.

Lemma 4.18. *Suppose κ is an infinite cardinal and V is a separable valuated vector space. Then V is κ-coseparable iff for every cardinal $\lambda < \kappa$,*

$$\mathrm{Hom}_v(V, F_\lambda) \to \mathrm{Hom}_v(V, D_\lambda)$$

is surjective, i.e., for every homomorphism $f : V \to D_\lambda$ (which is automatically valuated) there is a valuated homomorphism $g : V \to F_\lambda$ such that $f = \phi_\lambda \circ g$. If $\kappa = \aleph_0$, then this need only be true for $\lambda = 1$.

Proof. We will concentrate on the case where $\kappa = \aleph_0$ and $\lambda = 1$, which is the only one we will use in the rest of the work. (The general case follows in an almost identical way.) Suppose V is $\aleph_0$-coseparable and $f : V \to D_1$ is a homomorphism. If W is the kernel of f, then it follows that V/W has rank at most one. Since V is $\aleph_0$-coseparable, it follows that $V = E \oplus F$, where $E \subseteq W$ and F is free. Since F is free, there is a valuated homomorphism $g : F \to F_1$ such that $f|_F = \phi_1 \circ g$. If we then define $g(E) = 0$, then it follows that $f = \phi_1 \circ g$.

Conversely, suppose V satisfies this homological condition and W is a subspace of V of corank one. Then there is a valuated composite homomorphism $f : V \to V/W \to D_1$ with kernel W. If $g : V \to F_1$ is the valuated homomorphism satisfying $f = \phi_1 \circ g$, then letting E be the kernel of g, it follows that $E \subseteq W$. Since F_1 is separable and free, it follows that E is cofree in V, as required. □

The following gives a great deal of information about the size of σ.

Proposition 4.19. *The following relations hold:*

(a) $\sigma \leq c = 2^{\aleph_0}$;

(b) If $2^{\aleph_0} < 2^{\aleph_1}$, then $\sigma = \aleph_1$.

Proof. Regarding (a), let B be a countable separable unbounded free valuated vector space. If $V = \overline{B}$ is the p-adic completion of B, then V has cardinality c. Since $\overline{B}$ is efi, it follows that it is not $\aleph_0$-coseparable, so that $\sigma \leq c$.

Turning to (b), again let B be a countable separable unbounded free valuated vector space, but this time, let V be a subspace of $\overline{B}$ containing B of cardinality $\aleph_1$. We claim that V is not $\aleph_0$-coseparable. To that end, consider the valuated sequence

$$0 \to M_1 \to F_1 \to D_1 \to 0$$

from Lemma 4.18. Note that any valuated homomorphism $g : V \to F_1$ is determined by its restriction to B. It follows that $\mathrm{Hom}_v(V, F_1)$ has cardinality at most $2^{\aleph_0}$. On the other hand, since any homomorphism $f : V \to D_1$ is valuated, the cardinality of $\mathrm{Hom}_v(V, D_1)$ is $2^{\aleph_1}$. It follows that $\mathrm{Hom}_v(V, F_1) \to \mathrm{Hom}_v(V, D_1)$ is not surjective, so that V is not $\aleph_0$-coseparable. This implies that $\sigma = \aleph_1$, as required. □

Combining the Corollary 4.16 and Proposition 4.19, we derive:

Corollary 4.20. *If $2^{\aleph_0} < 2^{\aleph_1}$ (e.g., in any set-theoretic environment in which CH is valid) and G is a proper $\omega+n$-totally $p^{\omega+n}$-projective group, then $p^{\omega}G$ must be countable.*

A simple combination of Theorem 4.6 and Corollary 4.20 leads us to the following supplement to Corollary 4.5:

Corollary 4.21. *If $2^{\aleph_0} < 2^{\aleph_1}$ and G is a proper $\omega + n$-totally $p^{\omega+n}$-projective group, then G is not a dsc-group.*

We will have use for the following technical observation:

Lemma 4.22. *Suppose κ is an infinite cardinal, V is a κ-coseparable valuated vector space, W is a separable valuated vector space and $\phi : W \to V$ is a valuated vector space homomorphism with finite kernel J. Then W is also κ-coseparable.*

Proof. Since J is finite, there is a valuated decomposition $W = J \oplus W'$. It follows that W is κ-coseparable iff W' is κ-coseparable, so, without loss of generality, we may assume $J = \{0\}$, $W = W'$ and ϕ is injective (note that ϕ may increase values computed in W and V). Considering Lemma 4.18, if $\lambda < \kappa$ is a cardinal and $f_W : W \to D_\lambda$ is a homomorphism, then there is a homomorphism $f_V : V \to D_\lambda$ such that $f_W = f_V \circ \phi$. Since V is κ-coseparable, there is a valuated homomorphism $g_V : V \to F_\lambda$ such $f_V = \phi_\lambda \circ g_V$. If $g_W = g_V \circ \phi$, it follows that $f_W = f_V \circ \phi = \phi_\lambda \circ g_V \circ \phi = \phi_\lambda \circ g_W$, so that W is κ-coseparable, as required. □

A group G will be said to be *special* if it is isomorphic to a direct sum $H \oplus M$, where:

(a) H is a separable $p^{\omega+1}$-projective and $H[p]$ is a $\aleph_0$-coseparable valuated vector space;

(b) M is a dsc-group and $p^\omega M$ is finite.

Clearly, a special group is reduced, and, in fact, $p^{\omega+n}G = \{0\}$ for some $n < \omega$. Since M can be decomposed as a direct sum of a Σ-cyclic group and a countable group, we may assume M is countable.

Theorem 4.23. *The following hold:*

(a) A group G is special iff $p^\omega G$ is finite, $G/p^\omega G$ is $p^{\omega+1}$-projective and $K(G)$ is $\aleph_0$-coseparable.

(b) The class of special groups is closed under arbitrary subgroups.

(c) Any special group is $\omega + n$-totally $p^{\omega+n}$-projective for all $0 < n < \omega$.

Proof. Regarding (a), if $G \cong H \oplus M$ is special, then clearly $p^\omega G \cong p^\omega M$ is finite, and $G/p^\omega G \cong H \oplus (M/p^\omega M)$ is $p^{\omega+1}$-projective. Note that $K(G)$ is isometric to the valuated sum $H[p] \oplus K(M)$, and since the first summand is $\aleph_0$-coseparable and the second summand is separable and free, it follows that $K(G)$ is also $\aleph_0$-coseparable.

Suppose now that G satisfies the conditions listed in the last half of (a). Since $K(G)$ is $\aleph_0$-coseparable and $K(G)/K_0(G) \cong p^\omega G/p^{\omega+1}G$ is finite, it follows from Lemma 4.10 that $G \cong H \oplus M$, where H is a separable $p^{\omega+1}$-projective and M is such that $p^\omega M$ is finite and $M/p^\omega M$ is Σ-cyclic, thus a dsc-group, so that (a) follows.

Turning to (b), suppose G is special and A is some subgroup of G. Since $p^\omega A \subseteq p^\omega G$ and the latter is finite, it follow that $p^\omega A$ is finite, as well. Next note that there is an induced homomorphism $\phi : A/p^\omega A \to G/p^\omega G$ which restricts to a homomorphism $K(A) \to K(G)$. The kernel of ϕ is $[A \cap p^\omega G]/p^\omega A$ which is finite (so that it embeds in a finite summand of $A/p^\omega A$), and it follows easily that $A/p^\omega A$ is $p^{\omega+1}$-projective. Finally, since $K(G)$ is $\aleph_0$-coseparable and $K(A) \to K(G)$ has finite kernel, it follows from Lemma 4.22 that $K(A)$ is $\aleph_0$-coseparable. This proves that A is special and concludes the proof of (b).

Finally, to show (c), if $0 < n < \omega$, G is special, and A is a $p^{\omega+n}$-bounded subgroup of G, then in view of (b) we have that A is also special. If follows that $A \cong H' \oplus M'$, where H' is $p^{\omega+1}$-projective, and M' is a countable group with $p^{\omega+n}M' = \{0\}$. Since H' and M' are $p^{\omega+n}$-projective, we can conclude that A is $p^{\omega+n}$-projective and hence G is $\omega + n$-totally $p^{\omega+n}$-projective. $\square$

We come now to our main theorem on proper $\omega + n$-totally $p^{\omega+n}$-projectives.

Theorem 4.24. *The equivalence of the following three statements is a theorem in ZFC:*

(a) There is a proper $\omega + n$-totally $p^{\omega+n}$-projective group for some $0 < n < \omega$.

(b) There is a proper $\aleph_0$-coseparable valuated vector space.

(c) There is a separable $p^{\omega+1}$-projective group A which is not Σ-cyclic such that whenever G is a group with $p^\omega G \cong \mathbb{Z}_p$ and $G/p^\omega G \cong A$, then G must also be $p^{\omega+1}$-projective.

On the other hand, all three are undecidable in ZFC; in particular, they all hold in a model of MA+$\neg$CH, whereas they all fail in a model of V=L.

Proof. We begin by showing that (b) implies (a); to that end, suppose V is a non-free $\aleph_0$-coseparable valuated vector space. Then there is a group H containing $V \subseteq H[p]$ such that the valuation on V agrees with the height function on H, and for which H/V is Σ-cyclic. Note that such an H will be separable and $p^{\omega+1}$-projective. As we have observed several times in the past, $H[p]$ is isometric to $V \oplus F$, where F is a free valuated vector space. It therefore follows that $H[p]$ is also a proper $\aleph_0$-coseparable valuated vector space. If M is any countable group such that $p^\omega M$ is finite and $p^{\omega+n}M \neq \{0\}$, then $G = H \oplus M$ will be special, and hence $\omega + n$-totally $p^{\omega+n}$-projective, by Theorem 4.23(c). Since M is not $p^{\omega+n}$-bounded and H is not Σ-cyclic, G is necessarily proper, thus proving (a).

We next verify that (a) implies (b), so suppose G is a proper $\omega + n$-totally $p^{\omega+n}$-projective group. Suppose first that $p^\omega G$ is uncountable. In this case, Corollary 4.16 implies that $\sigma > \aleph_1$. However, if we let V be any separable valuated vector space of rank $\aleph_1$ with a countable basic subspace, then V is clearly not free, but since $r(V) < \sigma$, V must be a proper $\aleph_0$-coseparable valuated vector space, proving (b) in this case.

On the other hand, assume that $p^\omega G$ is countable. Let H be a high subgroup of G. Note that if H is Σ-cyclic, then G must be a Σ-group. However, since $p^\omega G$ is countable, Theorem 4.6(c) $\Rightarrow$ (b) would imply that G is ω-totally Σ-cyclic, contrary to assumption. It follows that $H[p]$ is not free. Since there is obviously a valuated injection $H[p] \to G[p]$, it follows from Theorem 4.14 that $H[p]$ is $\aleph_0$-coseparable, which establishes (b).

Assume now that (b) holds, and we will prove (c). Let V be a proper $\aleph_0$-coseparable valuated vector space. It follows that there is a separable $p^{\omega+1}$-projective group A containing V as a subgroup where the height function on A coincides with the valuation endowed on V, and such that A/V is Σ-cyclic. If G is any group with $p^\omega G \cong \mathbb{Z}_p$ and $G/p^\omega G \cong A$, it follows from Theorem 4.23(a) that G is special. Therefore, $G \cong H \oplus M$, where H is a separable $p^{\omega+1}$-projective group and M is countable. Since G is $p^{\omega+1}$-bounded, so is M, so that G is necessarily $p^{\omega+1}$-projective.

Conversely, suppose that (c) holds, and we establish (b). Let $V = A[p]$; since A is not Σ-cyclic, V is not free. Suppose D is a subspace of V of corank one. If there is an $m < \omega$ such that $V(m) \subseteq D$, then D is already cofree, so assume D is dense in V. If L is a pure subgroup of A with $L[p] = D$, then there is a surjective homomorphism $\phi : A \to \mathbb{Z}_{p^\infty}$ with kernel L. Let

$$G = \{(a, z) : a \in A, z \in \mathbb{Z}_{p^\infty} \text{ and } \phi(a) = pz\}.$$

It follows that $G/p^\omega G \cong A$, $p^\omega G$ is cyclic of order p and $D \cong K_0(G)$. Since G must be $p^{\omega+1}$-projective, by Lemma 4.10(c), $D = K_0(G)$ contains a cofree subspace of $K(G)$, so that $V = A[p] = K(G)$ must be a proper $\aleph_0$-coseparable valuated vector space.

We next show that all of them are valid in a model of MA+$\neg$CH. In fact, in this set-theoretic context, by Theorem 3.4(a) and 3.3 of [38], there is a proper $\aleph_1$-coseparable valuated vector space. Since an $\aleph_1$-coseparable valuated vector space is also $\aleph_0$-coseparable, we have that (b) holds.

Finally, arguing as in [38], we show that (c) does not hold in a model of V=L. Suppose, therefore, that A satisfies (c). Note that if G is some group such that $p^\omega G \cong \mathbb{Z}_p$ and $G/p^\omega G \cong A$, then G is $p^{\omega+1}$-projective, so that $G \cong C \oplus S$, where C is a dsc-group and S is separable. If $H_{\omega+1}$ is the generalized Prüfer group of length $\omega+1$, there is clearly a homomorphism $G \to C \to H_{\omega+1}$, which is non-zero on $p^\omega G \cong p^\omega C$. In the presence of V=L, by Theorem 2.2 of [89], the group A must be Σ-cyclic, contrary to assumption, as pursued. □

The last proof actually shows the following:

Corollary 4.25. *If there is a proper $\omega + n$-totally $p^{\omega+n}$-projective for some $0 < n < \omega$, then there is a proper $\omega + n$-totally $p^{\omega+n}$-projective for all $0 < n < \omega$.*

Corollary 4.26. *In V=L, if $n < m < \omega$ and G is $\omega + n$-totally $p^{\omega+n}$-projective, then it is $\omega + m$-totally $p^{\omega+m}$-projective.*

Proof. Since by Theorem 4.24 the group G cannot be proper, it must either be $p^{\omega+n}$-projective or ω-totally Σ-cyclic. In either case it will be $\omega+m$-totally $p^{\omega+m}$-projective. □

There are still unanswered questions that pertain to the structure of proper $\omega + n$-totally $p^{\omega+n}$-groups, at least in those set-theoretic environments in which they exist. For example, we have the following:

Problem 1: In ZFC, does $\sigma = \aleph_1$?

Problem 2: In ZFC, if G is a proper $\omega + n$-totally $p^{\omega+n}$-projective, does it follow that $p^{\omega}G$ is necessarily countable?

By Corollary 4.16, an affirmative answer to Problem 1 implies an affirmative answer to Problem 2.

Problem 3: In ZFC, if $n < m < \omega$ and G is $\omega + n$-totally $p^{\omega+n}$-projective, must it also be $\omega + m$-totally $p^{\omega+m}$-projective?

Problem 4: If $n < \omega$, describe the class of ω-totally $p^{\omega+n}$-projectives (which contains the class of $\omega + n$-totally $p^{\omega+n}$-projectives).

4.2. **On n-simply presented abelian p-groups.** Throughout, by the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration of the subsection. Our terminology and notation will be based upon [44] and [47]. For example, if α is an ordinal, then a group G will be said to be *p^{α}-projective* if $p^{\alpha}\mathrm{Ext}(G, X) = \{0\}$ for all groups X. We will denote the height of an element $x \in G$ by $|x|_G$. We will say G is *Σ-cyclic* if it is isomorphic to a direct sum of cyclic groups.

The *totally projective* groups have a central position in the study of abelian p-groups (see Chapter XII of [44] or Chapter VI of [53]). One reason for their importance is the number of different ways they can be characterized; recall that a group G is totally projective if any one of the following equivalent conditions is satisfied:

(1) G is *simply presented*;
(2) G is *balanced projective*, i.e., $\mathrm{Bext}(G, X) = \{0\}$ for all groups X;
(3) $G/p^{\alpha}G$ is p^{α}-projective for every ordinal α;
(4) G has a *nice system*;
(5) G has a *nice composition series*.

It is worth pointing out that, unlike the treatment in [44], we do not require a simply presented group to be reduced.

In a somewhat different direction, if n is a non-negative integer (that will be fixed for the remainder of this section), then the group G is $p^{\omega+n}$-projective iff there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic (see, e.g., [97]). So, a group is p^{ω}-projective iff it is Σ-cyclic. It follows easily that the class of $p^{\omega+n}$-projectives is closed under arbitrary subgroups. In addition, if G_1 and G_2 are $p^{\omega+n}$-projectives, then G_1 and G_2 are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric (i.e., there exists an isomorphism that preserves the height functions on the two subgroups as computed in the whole groups; see [46]).

A number of papers have been written over the years that combine elements of these two important components of the study of abelian p-groups (see, for example, [48], [49] and [74]). In this and a subsequent section, we will consider several other interesting ways to combine them.

Generalizing (1), a group G will be said to be *n-simply presented* if there is a subgroup $P \subseteq G[p^n]$ such that G/P is simply presented. Such a subgroup will be called *n-simply representing*. It follows, therefore, that the class of n-simply presented groups includes both the simply presented groups and the $p^{\omega+n}$-projective groups.

In terms of homological algebra, we say a short exact sequence $0 \to X \to Y \to G \to 0$ is *n-balanced exact* if it represents an element of $p^n\mathrm{Bext}(G, X)$. Generalizing (2), we say G is *n-balanced projective* if every such n-balanced exact sequence splits. We show that G is n-balanced projective iff it is a summand of a group that is n-simply presented, and that there are enough n-balanced projectives (Theorem 4.27). We also show that a separable group G is n-simply presented iff it is n-balanced projective iff it is $p^{\omega+n}$-projective (Proposition 4.28).

If G is $p^{\omega+n}$-projective and P is a subgroup of $G[p^n]$ such that G/P is Σ-cyclic, then P will, in fact, be *nice* in G (i.e., every coset $x+P$ will contain an element of maximal height). This leads to a further generalization of (1): We say the group G is *strongly n-simply presented* if it has an n-simply representing subgroup which is nice.

Continuing in the language of homological algebra, we say a short exact sequence $0 \to X \to Y \xrightarrow{\phi} G \to 0$ is *strongly n-balanced exact* if it is balanced and there is a height-preserving homomorphism $\nu : G[p^n] \to Y[p^n]$ such that $\phi \circ \nu$ is the identity on $G[p^n]$ (note that if $n \geq 1$, then the latter condition already implies that the sequence is balanced - see, for instance, [44, Proposition 80.2]). In other words, we are requiring that the induced exact sequence, $0 \to X[p^n] \to Y[p^n] \to G[p^n] \to 0$, is split in the category of valuated groups. We can, therefore, consider the class of *strongly n-balanced projectives.*

In parallel with the above, we next show that a group G is strongly n-balanced projective iff it is a summand of a group that is strongly n-simply presented, and that there are enough strongly n-balanced projectives (Theorem 4.30). We also show that a $p^{\omega+n}$-bounded group G is strongly n-simply presented iff it is strongly n-balanced projective iff it is $p^{\omega+n}$-projective (Proposition 4.31).

One of the most useful and important results in the study of totally projective groups is a theorem of Nunke from [98] which states that if λ is an ordinal, then a group G is totally projective iff $p^\lambda G$ and $G/p^\lambda G$ are both totally projective (see, for example, [53, Theorem 74]). The same property was independently proved by Crawley-Hales for simply presented groups (see [30] and [31]). It is not hard to see that if G is (strongly) n-simply presented or (strongly) n-balanced projective, then $p^\lambda G$ and $G/p^\lambda G$ must share the corresponding property (Theorem 4.36(a) and Proposition 4.37(a)). The converse is rather more complicated. We show that if $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are strongly n-simply presented or strongly n-balanced projective, then so is G (Theorem 4.36(b) and Proposition 4.37(b)). On the other hand, for ordinals not of the form $\lambda + n$ (e.g., limit ordinals), we show that this can fail for strongly n-simply presented groups (Example 4.39).

The next part of the subsection is devoted to showing that for an arbitrary ordinal λ, if $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented or n-balanced projective, then the same can be said of G (Theorem 4.43 and Corollary 4.45). This surprisingly very difficult proof requires a detailed examination of the behavior of bounded subgroups P of G for which G/P is simply presented.

These properties allow us to conclude that for any group G of length strictly less than ω^2, that G is (strongly) n-simply presented iff it is (strongly) n-balanced projective (Corollaries 4.38 and 4.46). In other words, the (strongly) n-simply presented groups of length less than ω^2 are closed under taking direct summands. Later, we will establish some further statements of this sort.

A group G is $p^{\omega+n}$-projective iff there is a Σ-cyclic group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$ (see, e.g., [46]). The proof of this property depends solely on the fact that T is Σ-cyclic iff p^nT is Σ-cyclic. Similarly, we say G is *n-co-simply presented* if there is a simply presented group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$. Since T is also simply presented iff p^nT is simply presented, the same proof shows that G is n-simply presented iff it is n-co-simply presented.

We begin by describing the summands of the n-simply presented groups.

Theorem 4.27. *The group G is n-balanced projective iff it is a summand of a group that is n-simply presented. There are enough n-balanced projectives.*

Proof. Suppose first that G is n-simply presented, and hence n-cosimply presented, i.e. there is a simply presented (and hence balanced projective) group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$. For any group A we have an exact sequence

$$\to \mathrm{Hom}(T, A) \to \mathrm{Hom}(Q, A) \xrightarrow{\phi} \mathrm{Ext}(G, A) \xrightarrow{\mu} \mathrm{Ext}(T, A).$$

It follows that $\mu(\mathrm{Bext}(G, A)) \subseteq \mathrm{Bext}(T, A) = \{0\}$, so that $\mathrm{Bext}(G, A) \subseteq \phi(\mathrm{Hom}(Q, A))$. Since $p^n\mathrm{Hom}(Q, A) = \{0\}$, we can conclude that $p^n\mathrm{Bext}(G, A) =$

$\{0\}$, so that G is n-balanced projective. Therefore, any direct summand of a group that is n-simply presented is also n-balanced projective.

We now show the converse, and at the same time we show that there are enough n-balanced projectives. Let $0 \to X \to Y \to G \to 0$ be a balanced projective resolution of G (i.e., it is balanced and Y is simply presented). Consider the pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G[p^n] & = & G[p^n] & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & X & \longrightarrow & Z & \longrightarrow & G & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle p^n} & & \\
0 & \to & X & \longrightarrow & Y & \longrightarrow & G & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G/p^nG & = & G/p^nG & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Obviously, the upper short exact row is n-balanced exact. We claim that Z is n-simply presented: Note that Y is simply presented and $p^n(Y/\gamma(Z)) = \{0\}$. It follows from general properties of simply presented groups, therefore, that $\gamma(Z)$ is simply presented (or see Lemma 4.34(a) below). Since the middle column determines an isomorphism $Z/G[p^n] \cong \gamma(Z)$, we can infer that $G[p^n]$ is an n-simply representing subgroup of Z, i.e., Z is n-simply presented, as claimed.

By the first part of the proof, we can deduce that Z is n-balanced projective; and since $0 \to X \to Z \to G \to 0$ is n-balanced exact, there are enough n-balanced projectives.

Finally, if G is n-balanced projective, then there is a splitting $Z \cong G \oplus X$, so that G is a summand of a group which is n-simply presented, as required. $\square$

Proposition 4.28. *If G is a separable (i.e., p^ω-bounded) group, then the following are equivalent:*

(a) G is n-simply presented;

(b) G is n-balanced projective;

(c) G is $p^{\omega+n}$-projective.

Proof. We begin by showing that (a) and (c) are equivalent. Observe first that if G is $p^{\omega+n}$-projective, then there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic, and hence totally projective. It follows immediately that G must be n-simply presented (this argument does not use the separability of G). Conversely, suppose P is an n-simply representing subgroup of G. If $\overline{P}$ is the closure of P in the p-adic topology of G, then $\overline{P} \subseteq G[p^n]$ and $\overline{P}/P = p^\omega(G/P)$. Therefore, $G/\overline{P} \cong (G/P)/p^\omega(G/P)$ will also be Σ-cyclic, so that G is $p^{\omega+n}$-projective.

Next, observe that since the collection of $p^{\omega+n}$-projective groups is closed under direct summands, by Theorem 4.27, the equivalence of (a) and (b) follows from the equivalence of (a) and (c). □

We will extensively employ concepts related to *valuated groups* and *valuated vector spaces* which can be found, for example, in [101] and [45], and which we briefly review. Let $\mathcal{O}$ be the class of ordinals and $\mathcal{O}_\infty = \mathcal{O} \cup \{\infty\}$, where we agree that $\alpha < \infty$ for all $\alpha \in \mathcal{O}_\infty$. A *valuation* on a group V is a function which assigns to every $x \in V$ an element $|x|_V \in \mathcal{O}_\infty$ such that for every $x, y \in V$, $|x \pm y|_V \geq \min\{|x|_V, |y|_V\}$ and $|px|_V > |x|_V$. As a result, for all $\alpha \in \mathcal{O}_\infty$, $V(\alpha) = \{x \in V : |x|_V \geq \alpha\}$ is a subgroup of V with $pV(\alpha) \subseteq V(\alpha + 1)$.

A homomorphism between two valuated groups will be said to be *valuated* if it does not decrease values and an *isometry* if it is bijective and preserves values. If V_i, $i \in I$, is a collection of valuated groups, then the usual direct sum $V = \bigoplus_{i\in I} V_i$ has a natural valuation, where $V(\alpha) = \bigoplus_{i\in I} V_i(\alpha)$ for every $\alpha \in \mathcal{O}_\infty$.

If W is any subgroup of V, then restricting $|\ |_V$ to W turns W into a valuated group with $W(\alpha) = W \cap V(\alpha)$ for all $\alpha \in \mathcal{O}_\infty$. A *valuated vector space* W is a p-bounded valuated group, so each $W(\alpha)$ will be a subspace of W; further, we say W is *free* if it is isometric to a valuated direct sum of cyclic groups (of order p). If V is a valuated group, then its socle $V[p]$ is a valuated vector space. Clearly, any group is a valuated group, using the height function as its valuation.

If V is a valuated group, then in [101] a functorial group $\mathcal{G}(V)$ was defined such that

(a) V is a nice subgroup of $\mathcal{G}(V)$;
(b) the valuation on V agrees with the height valuation on $\mathcal{G}(V)$;
(c) $\mathcal{G}(V)/V$ is simply presented;
(d) $V(\alpha) = \{0\}$ iff $p^\alpha\mathcal{G}(V) = \{0\}$.

It follows that if V is p^n-bounded, then $\mathcal{G}(V)$ is strongly n-simply presented.

We extend this construction in the following way: If G is a group and $n \geq 1$, then let $H(G) = \mathcal{G}(G[p^n])$.

Lemma 4.29. *Suppose G is a group and $n \geq 1$.*

(a) The identity map $G[p^n] \to G[p^n]$ extends to a homomorphism $\pi : H(G) \to G$;

(b) If $K(G)$ is the kernel of π, then $0 \to K(G) \to H(G) \to G \to 0$ is strongly n-balanced exact.

Proof. (a) This follows from the fact that $G[p^n]$ is nice in $H(G)$, $H(G)/G[p^n]$ is simply presented and the identity map clearly does not decrease heights (see, for example, Corollary 81.4 of [44]).

(b) Observe first that if $x \in G[p]$, then $|x|_G = |x|_{H(G)} \leq |x|_{\pi(H(G))} \leq |x|_G$, so that $|x|_{\pi(H(G))} = |x|_G$. It follows that $\pi(H(G))$ is an isotype subgroup of G containing $G[p]$, so that $\pi(H(G)) = G$ and π is surjective.

Next, the identity map $G[p^n] \to G[p^n]$ induces a valuated splitting

$$H(G)[p^n] \cong K(G)[p^n] \oplus G[p^n],$$

and (b) follows. $\square$

We have the following analogue of Theorem 4.27.

Theorem 4.30. *The group G is strongly n-balanced projective iff it is a summand of a group that is strongly n-simply presented. There are enough strongly n-balanced projectives.*

Proof. Note that if $n = 0$, the result is well-known, so assume $n \geq 1$. If G is strongly n-simply presented, then it has a nice n-simply representing subgroup N. Suppose $E : 0 \to X \to Y \to G \to 0$ is strongly n-balanced exact and ϕ is the right homomorphism; by definition, then, there is a valuated splitting $\nu : G[p^n] \to Y[p^n]$. Again referring to Corollary 81.4 of [44], the restriction of ν to $N \to Y[p^n]$ extends to a homomorphism $h : G \to Y$ such that $\phi \circ h$ is the identity on N. It follows that $1_G - \phi \circ h$ is zero on N, so it induces a homomorphism $G/N \to G$. However, since G/N is simply presented and E is balanced, there is a homomorphism $G/N \to Y$ such that if h' is the composition $G \to G/N \to Y$, then $1_G - \pi \circ h = \pi \circ h'$. Since $1_G = \pi \circ (h + h')$, it follows that E splits, so that G is strongly n-balanced projective.

Therefore, any summand of a strongly n-simply presented group is strongly n-balanced projective, and by Lemma 4.29(b), there are enough strongly n-balanced projectives.

Conversely, if G is strongly n-balanced projectives, then it must be a summand of $H(G)$, which is strongly n-simply presented. $\square$

The following is an analogue of Proposition 4.28:

Proposition 4.31. *If G is a $p^{\omega+n}$-bounded group, then the following are equivalent:*

(a) G is strongly n-simply presented;

(b) G is strongly n-balanced projective;

(c) G is $p^{\omega+n}$-projective.

Proof. Note that if $n = 0$, all these statements simply say that G is Σ-cyclic, so assume $n \geq 1$. As in Proposition 4.28, the result will follow once we show the equivalence of (a) and (c). Note if (c) holds, then there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic. If follows that this P will necessarily be nice in G, so that G is strongly n-simply presented.

Conversely, suppose P is a nice n-simply presenting subgroup of G and $A = G/P$. Note $p^\omega G \subseteq G[p^n]$, so that $P' = p^\omega G + P$ is p^n-bounded. In addition,

$$G/P' = G/[p^\omega G + P] \cong (G/P)/[p^\omega G + P]/P = A/p^\omega A$$

is Σ-cyclic, and it follows that G is $p^{\omega+n}$-projective, as required. □

We say G is *strongly n-co-simply presented* if there is a simply presented group T and a nice subgroup $Q \subseteq T[p^n]$ such that $G \cong T/Q$. Though a group is n-simply presented iff it is n-cosimply presented, we make the following observation:

Example 4.32. There is a group G which is strongly 1-co-simply presented, which is not strongly 1-balanced projective (and so not strongly 1-simply presented).

Proof. Let M be some separable non-Σ-group with basic subgroup B, and let $0 \to Y \to X \to M \to 0$ be a pure-projective resolution of M, where we assume $Y \subseteq X$ and $X/Y = M$. Let $P = Y[p]$, so that $Z = X/P$ is a separable $p^{\omega+1}$-projective group which is not Σ-cyclic. Next, let D be the subgroup of X containing Y such that $D/Y = B$. There is a splitting, $D \cong Y \oplus B$, hence $E = D/P \cong (Y/P) \oplus B \cong pY \oplus B$ will also be Σ-cyclic. Note that D is pure and dense in X, so that E is pure and dense in Z.

Let C be a group such that $p^\omega C$ is a direct sum of cyclic groups of order p for which $C/p^\omega C$ can be identified with X such that $D = [D' + p^\omega C]/p^\omega C$ for some high subgroup D' of C. [For example, if $K = X/D$, we may let $C = \{(x, k) \in X \oplus K : x + D = pk\}$.] If $P' \subseteq D'$ satisfies $P = [P' + p^\omega C]/p^\omega C$, then we let $G = C/P'$.

The following can now be checked:

(a) C is a dsc-group of length $\omega + 1$ (since $C/p^\omega C = X$ is Σ-cyclic).

(b) P' is nice in C (since $P' \cap p^\omega C = \{0\}$ and $[P' + p^\omega C]/p^\omega C = P$ is nice in $X = C/p^\omega C$).

(c) G is strongly 1-co-simply presented (by (a) and (b)).

(d) $E' = D'/P'$ is a high subgroup of G (since D' is a maximal subgroup of C containing P' intersecting $p^\omega C$ trivially, we have E' is a maximal subgroup of G intersecting $p^\omega G = [p^\omega C + P']/P'$ trivially).

(e) $G[p]$ is a free valuated vector space ($E' = D'/P' \cong D/P = E$ is Σ-cyclic and there is an isometry $G[p] = E'[p] \oplus p^\omega G$ where both terms are, in fact, free as valuated vector spaces).

(f) G is not strongly 1-balanced projective: If G were strongly 1-balanced projective, then by Proposition 4.31 it would be $p^{\omega+1}$-projective. In this case, (e) would imply that G is totally projective (since $G[p]$ will be isometric to the socle of a totally projective group and $p^{\omega+1}$-projective groups with isometric socles are, in fact, isomorphic). However, this contradicts the fact that

$$\begin{aligned} G/p^\omega G &= [C/P']/[(p^\omega C + P')/P'] \\ &\cong C/(p^\omega C + P') \\ &\cong [C/p^\omega C]/[(p^\omega C + P')/p^\omega C] = X/P = Z \end{aligned}$$

is not Σ-cyclic. $\square$

Since the group in this example is strongly 1-co-simply presented, it is also 1-co-simply presented and hence 1-simply presented and 1-balanced projective, i.e., the classes of 1-simply presented and 1-balanced projective groups properly contain the classes of strongly 1-simply presented and strongly 1-balanced projective groups, respectively. In other words, though "0-simply presented" = "strongly 0-simply presented" = "0-balanced projective" = "strongly 0-balanced projective" = "simply presented," for $n \geq 1$ the containments "strongly n-simply presented" $\subset$ "n-simply presented" and "strongly n-balanced projective" $\subset$ "n-balanced projective" are proper. This also implies that for $n \geq 1$, there are strongly n-balanced short exact sequences that are not n-balanced short exact.

We now begin with Nunke-esque results. We first collect a number of routine observations in the following:

Lemma 4.33. *Suppose λ is an ordinal, G is a group with a subgroup $P \subseteq G[p^n]$, $A = G/P$ and X is the subgroup of G containing P such that $X/P = p^{\lambda+n}A$. Then*

(a) $p^{\lambda+n}G + P \subseteq X \subseteq p^\lambda G + P$;

(b) there is a short exact sequence

$$0 \to p^{\lambda+n}G/[p^{\lambda+n}G \cap P] \to p^{\lambda+n}A \to B_1 \to 0$$

where B_1 is bounded;

(c) there is a short exact sequence

$$0 \to B_2 \to A/p^{\lambda+n}A \to G/[p^\lambda G + P] \to 0$$

where $B_2 \subseteq p^\lambda(A/p^{\lambda+n}A)$ *is bounded.*

Proof. (a) Clearly $[p^{\lambda+n}G + P]/P \subseteq p^{\lambda+n}A$, so that $p^{\lambda+n}G + P \subseteq X$. On the other hand, there is a short exact sequence

$$0 \to [p^\lambda G + P]/p^\lambda G \to G/p^\lambda G \to G/[p^\lambda G + P] \to 0.$$

Since $p^\lambda(G/p^\lambda G) = \{0\}$ and $p^n([p^\lambda G + P]/p^\lambda G) = \{0\}$, we have $p^{\lambda+n}(G/[p^\lambda G + P]) = \{0\}$.

Now, there is another short exact sequence

$$0 \to [p^\lambda G + P]/P \to A \to G/[p^\lambda G + P] \to 0.$$

It therefore follows that $p^{\lambda+n}A \subseteq [p^\lambda G+P]/P$, giving $X \subseteq p^\lambda G+P$, as required.

(b) There is a short exact sequence

$$0 \to [p^{\lambda+n}G + P]/P \to X/P \to X/[p^{\lambda+n}G + P] \to 0.$$

Clearly, the first two terms agree with those in (b), and we let B_1 be the third term. By (a), $p^n X \subseteq p^n(p^\lambda G + P) \subseteq p^{\lambda+n}G + P$, showing that B_1 is bounded.

(c) There is a short exact sequence

$$0 \to [p^\lambda G + P]/X \to G/X \to G/[p^\lambda G + P] \to 0.$$

There is also an isomorphism $G/X \cong (G/P)/(X/P) = A/p^{\lambda+n}A$. Finally,

$$B_2 = [p^\lambda G + P]/X = [p^\lambda G + X]/X \subseteq p^\lambda(G/X) \cong p^\lambda(A/p^{\lambda+n}A)$$

is p^n-bounded. $\square$

The following well-known technicality is even more straightforward.

Lemma 4.34. *Let* λ *be an ordinal and* Z *be a group.*

(a) Suppose Y *is a subgroup of* Z *such that* Z/Y *is bounded. Then* Z *is simply presented iff* Y *is simply presented.*

(b) Suppose Y *is a bounded subgroup of* $p^\lambda Z$*. Then* $p^\lambda Z$ *is bounded and* Z *is simply presented iff* $p^\lambda(Z/Y)$ *is bounded and* Z/Y *is simply presented.*

Proof. (a) Suppose Z is simply presented. Then $p^\omega Y = p^\omega Z$ is simply presented, and $Y/p^\omega Y$ embeds in $Z/p^\omega Z$, which is Σ-cyclic. Therefore, $Y/p^\omega Y$ is Σ-cyclic, and Y is simply presented.

Conversely, if Y is simply presented, then for some k, $p^k Z \subseteq Y$, so that by the first part of the proof, $p^k Z$ is simply presented, implying that Z is simply presented, as well.

(b) Note that $p^\lambda Z$ is bounded iff $p^\lambda(Z/Y) = p^\lambda Z/Y$ is bounded. In this case, it follows that Z is simply presented iff $Z/p^\lambda Z$ is simply presented iff $(Z/Y)/p^\lambda(Z/Y)$ is simply presented iff Z/Y is simply presented. $\square$

We put these together in the following:

Lemma 4.35. *Suppose λ is an ordinal, G is a group with a subgroup P such that $p^nP = \{0\}$ and $A = G/P$. Then*

(a) If $m < \omega$, then P is an n-simply representing subgroup of G iff $p^mG \cap P$ is an n-simply representing subgroup of p^mG;

(b) A is simply presented iff both $p^\lambda G/[p^\lambda G \cap P]$ and $G/[p^\lambda G + P]$ are simply presented.

Proof. In can be checked that (a) is a consequence of Lemma 4.34(a). As to (b), by Lemma 4.33(b) and Lemma 4.34(a), it follows that $p^{\lambda+n}A$ is simply presented iff $p^{\lambda+n}G/[p^{\lambda+n}G \cap P]$ is simply presented, and by (a) this is equivalent to $p^\lambda G/[p^\lambda G \cap P]$ being simply presented. Again, by Lemma 4.33(c) and Lemma 4.34(b), $A/p^{\lambda+n}A$ is simply presented iff $G/[p^\lambda G+P]$ is simply presented. Since A is simply presented iff $p^{\lambda+n}A$ and $A/p^{\lambda+n}A$ are simply presented, the result follows. $\square$

Theorem 4.36. *Suppose λ is an ordinal and G is a group.*

(a) If G is (strongly) n-simply presented, then both $p^\lambda G$ and $G/p^\lambda G$ are (strongly) n-simply presented.

(b) If both $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are (strongly) n-simply presented, then G is (strongly) n-simply presented.

Proof. Suppose first that G is n-simply presented. Let P be an n-simply representing subgroup of G. By Lemma 4.35(b),

$$(G/p^\lambda G)/([p^\lambda G + P]/p^\lambda G) \cong G/[p^\lambda G + P]$$

and $p^\lambda G/[p^\lambda G \cap P]$ are simply presented. In addition, since

$$p^n[p^\lambda G \cap P] = \{0\} = p^n\left([p^\lambda G + P]/p^\lambda G\right),$$

we can conclude that $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented.

Observe that if G were actually *strongly* n-simply presented, then we could assume P is nice in G, and it would follow that $P \cap p^\lambda G$ is nice in $p^\lambda G$ and $[p^\lambda G + P]/p^\lambda G$ is nice in $G/p^\lambda G$, so that these two groups are, in fact, strongly n-simply presented, as well.

Turning to (b), suppose that P_1 is a subgroup of G containing $p^{\lambda+n}G$ for which $P_1/p^{\lambda+n}G$ is an n-simply representing subgroup of $G/p^{\lambda+n}G$. Let Y be a maximal p^n-bounded summand of $p^\lambda G$, so that there is a decomposition $p^\lambda G = X \oplus Y$. Let H be a $p^{\lambda+n}$-high subgroup of G containing Y (i.e., H is maximal with respect to intersecting $p^{\lambda+n}G$ trivially).

We next claim that $(G/p^{\lambda+n}G)[p^n] = (X \oplus H[p^n])/p^{\lambda+n}G$: Note that $X[p] = (p^{\lambda+n}G)[p]$, so that $X \cap H = \{0\}$; this means that $X \oplus H[p^n]$ really is an internal direct sum. Since $p^n X \subseteq p^{\lambda+n}G$ and $p^n H[p^n] = \{0\}$, inclusion in one direction is clear. So assume $z \in G$ and $p^n z \in p^{\lambda+n}G$; we need to show that $z \in X \oplus H[p^n]$. If $x \in X$ is chosen so that $p^n x = p^n z$, then replacing z by $z - x$, we may assume $p^n z = 0$. Next, since $G[p] = (p^{\lambda+n}G)[p] \oplus H[p]$, H is pure in G and $(p^{\lambda+n}G) = p^n X$, it follows that $G[p^n] = X[p^n] \oplus H[p^n]$. Therefore, $z = x' + h$, where $x' \in X[p^n] \subseteq X$ and $h \in H[p^n]$, as required.

It follows from the last paragraph that $P_1 \subseteq X \oplus H[p^n]$. Let

$$P_2 = (X + P_1) \cap H[p^n] \subseteq G[p^n].$$

Clearly, $P_2 \subseteq H$ implies that $P_2 \cap p^{\lambda+n}G = \{0\}$. In addition, $P_1 \subseteq X \oplus H[p^n]$ also implies that

$$X + P_1 = X + [(X + P_1) \cap H[p^n]] = X + P_2.$$

We can therefore conclude that $p^\lambda G + P_1 = p^\lambda G + P_2$.

Next, let P_3 be an n-simply representing subgroup of $(p^{\lambda+n}G)[p^n]$. We then let $P = P_2 + P_3$, so that $P \subseteq G[p^n]$. We clearly have $p^{\lambda+n}G \cap P = P_3$, so that $p^{\lambda+n}G/[p^{\lambda+n}G \cap P]$ is simply presented. By Lemma 4.35(a), we also can conclude that $p^\lambda G/[p^\lambda G \cap P]$ is simply presented. In addition, $p^\lambda G + P = p^\lambda G + P_2 = p^\lambda G + P_1$.

Note that $G/P_1 \cong (G/p^{\lambda+n}G)/(P_1/p^{\lambda+n}G)$ is simply presented; and since $p^n(P_1/p^{\lambda+n}G) = \{0\}$, it follows that $p^\lambda(G/P_1)$ is bounded (by p^{2n}). Therefore, $[p^\lambda G + P_1]/P_1$ is a bounded subgroup of $p^\lambda(G/P_1)$. Applying Lemma 4.34(b) to G/P_1, we can deduce that

$$G/[p^\lambda G + P] = G/[p^\lambda G + P_1] \cong (G/P_1)/([p^\lambda G + P_1]/P_1)$$

is simply presented. Therefore, by Lemma 4.35(b), we have that G/P is simply presented, as desired.

Suppose $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are actually *strongly* n-simply presented. In this case, we can choose $P_3 = p^{\lambda+n}G \cap P$ to be nice in $p^{\lambda+n}G$ and $P_1/p^{\lambda+n}G$ to be nice in $G/p^{\lambda+n}G$. If, as above, $P = P_2 + P_3$, then $P \cap p^{\lambda+n}G = P_3$ being nice in $p^{\lambda+n}G$ and $p^\lambda G/p^{\lambda+n}G$ being bounded readily imply that $P \cap p^\lambda G$ is nice in $p^\lambda G$. In addition,

$$\begin{aligned} [p^\lambda G + P]/p^\lambda G &= [p^\lambda G + P_1]/p^\lambda G \\ &\cong [(p^\lambda G/p^{\lambda+n}G) + (P_1/p^{\lambda+n}G)]/(p^\lambda G/p^{\lambda+n}G) \end{aligned}$$

is nice in $G/p^\lambda G \cong (G/p^{\lambda+n}G)/p^\lambda(G/p^{\lambda+n}G)$. Together, these assure that P is nice in G, hence G is strongly n-simply presented. $\square$

We can easily extend the last result to summands.

Proposition 4.37. *Suppose C is a group and λ is an ordinal.*

(a) If C is (strongly) n-balanced projective, then both $p^\lambda C$ and $C/p^\lambda C$ are (strongly) n-balanced projective.

(b) If $p^{\lambda+n}C$ and $C/p^{\lambda+n}C$ are (strongly) n-balanced projective, then C is (strongly) n-balanced projective.

Proof. For (a), note that if C is (strongly) n-balanced projective, then C is a summand of a (strongly) n-simply presented group G. It follows that $p^\lambda C$ and $C/p^\lambda C$ are summands of $p^\lambda G$ and $G/p^\lambda G$, respectively, and since the latter two groups are (strongly) n-simply presented, it follows that $p^\lambda C$ and $C/p^\lambda C$ are (strongly) n-balanced projectives.

Turning to (b), suppose $p^{\lambda+n}C$ and $C/p^{\lambda+n}C$ are (strongly) n-balanced projective. Observe first that there are groups Z and Y such that $p^{\lambda+n}C \oplus Z$ and $(C/p^{\lambda+n}C) \oplus Y$ are (strongly) n-simply presented. It follows that $p^{\lambda+n}Y \cong p^{\lambda+n}((C/p^{\lambda+n}C) \oplus Y)$ is (strongly) n-simply presented. Construct a group X such that $p^{\lambda+n}X \cong Z$ and $X/p^{\lambda+n}X$ is simply presented (see, for instance, [53]).

The proof will be complete if we can show $C \oplus X \oplus Y$ is (strongly) n-simply presented. Note that $p^{\lambda+n}(C \oplus X \oplus Y) \cong (p^{\lambda+n}C \oplus Z) \oplus p^{\lambda+n}Y$ is (strongly) n-simply presented. In addition, since $X/p^{\lambda+n}X$ is simply presented,

$$\begin{aligned}&(C \oplus X \oplus Y)/p^{\lambda+n}(C \oplus X \oplus Y) \cong \\ &\quad ((C/p^{\lambda+n}C) \oplus Y)/p^{\lambda+n}((C/p^{\lambda+n}C) \oplus Y) \oplus (X/p^{\lambda+n}X)\end{aligned}$$

is (strongly) n-simply presented. Therefore, by Theorem 4.36, $C \oplus X \oplus Y$ is (strongly) n-simply presented, as required. $\square$

Corollary 4.38. *Suppose G is a group of length strictly less than ω^2. Then the following conditions are equivalent:*

(a) G is strongly n-simply presented;

(b) G is strongly n-balanced projective;

(c) For every non-negative integer m, the factor group $p^{\omega\cdot m+n}G/p^{\omega\cdot(m+1)+n}G$ is $p^{\omega+n}$-projective.

Proof. First of all, note that there are only a finite number of non-zero factor groups $p^{\omega\cdot m+n}G/p^{\omega\cdot(m+1)+n}G$, and it follows from Theorem 4.36 that G is strongly n-simply presented (respectively, strongly n-balanced projective) iff each of these factor groups share that property. Finally, by Proposition 4.31, these two conditions are equivalent on each of these factors, and further, they are equivalent to condition (c). $\square$

We now illustrate that the full Nunke-like property does *not* hold for *strongly* n-simply presented groups.

Example 4.39. There is a group G for which $p^\omega G$ and $G/p^\omega G$ are strongly 1-simply presented, but G itself is not strongly 1-simply presented (or even strongly 1-balanced projective).

Proof. Consider the group G of Example 4.32. In discussing this example, it was noted that $G' = G/p^\omega G$ is $p^{\omega+1}$-projective, and hence strongly 1-simply presented. Since G is $p^{\omega+1}$-bounded, we also can conclude that $p^\omega G$ is Σ-cyclic, and hence strongly 1-simply presented. We know, however, that G is not strongly 1-balanced projective. □

In fact, in the last example, we really could be more general. If G is any group of length $\omega+1$ which is not $p^{\omega+1}$-projective, but for which $G/p^\omega G$ is $p^{\omega+1}$-projective, then G will satisfy our requirements.

The purpose in what follows is to verify that, as opposed to the case of strongly n-simply presented groups, the full Nunke-like property holds for the larger class of n-simply presented groups. Before doing so, however, we first take a fairly extended detour into the realm of valuated vector spaces.

A valuated vector space is said to be *subfree* if it embeds as an isometric subspace of a free valuated vector space. The following property is well-known, but we however will record it here only for the sake of completeness.

Lemma 4.40. *If H is a totally projective group, then $H[p]$ is subfree.*

Proof. We verify this by induction on the length of H, which we denote by γ. If γ is a limit, then H is isomorphic to a direct sum $\bigoplus_{\alpha<\gamma} H_\alpha$, where $p^\alpha H_\alpha = \{0\}$. By induction, each $(H_\alpha)[p]$ is subfree, hence the same holds for H. Assuming $\gamma = \beta + 1$ is isolated, if H' is a p^β-high subgroup of H, then there is an isometry $H[p] = H'[p] \oplus p^\beta H$. Clearly $p^\beta H$ is free. In addition, $H'[p]$ embeds isometrically in $(H/p^\beta H)[p]$. Since $H/p^\beta H$ is totally projective of length β, by induction, its socle is subfree. Hence $H'[p]$ is also subfree, so that $H[p]$ is subfree, establishing the result. □

By a *graded vector space*, we will mean a collection of vector spaces indexed by the ordinals, $U = [U_\alpha]_{\alpha<\infty}$, such that there is an ordinal λ with $U_\alpha = \{0\}$ for all $\alpha \geq \lambda$; the smallest such ordinal λ we call the *length* of U. The definition of a graded homomorphism or isomorphism follows naturally and the resulting category of graded vector spaces clearly has direct sums. We say $x \in U$ if there is an α such that $x \in U_\alpha$ and if $x \neq 0$ we write $|x|_U = \alpha$. We say U is *admissible* if its *Ulm function* $f_U(\alpha) = r(U_\alpha)$ is admissible in the usual sense. Let $R(U) = \sum_{\alpha<\infty} r(U_\alpha)$, and if β is an ordinal, let $R_\beta(U) = \sum_{\beta\leq\alpha<\beta+\omega} r(U_\alpha)$.

Our motivating example is where V is a valuated vector space (e.g., the socle of some group) and $U(V)$ is the graded vector space $[U_\alpha(V)]_{\alpha<\infty} = [V(\alpha)/V(\alpha+1)]_{\alpha<\infty}$. We let $R(V) = R(U(V))$ and $R_\beta(V) = R_\beta(U(V))$. If $\mathcal{L}$ is a subset of a valuated vector space V, then for each ordinal α we let $\mathcal{L}_\alpha = \{x \in \mathcal{L} : |x|_V = \alpha\}$ and we let $\mathrm{span}(\mathcal{L})$ be the graded vector space $[\mathrm{span}(\mathcal{L}_\alpha)]_{\alpha<\infty}$ (where we are identifying each element of $\mathcal{L}_\alpha$ with its image in $U_\alpha(V)$). We say $\mathcal{L}$ is *linearly independent* if $\mathcal{L}_\alpha$ is linearly independent in $U_\alpha(V)$ for every α, and a *basis* if, in addition, $U(V) = \mathrm{span}(\mathcal{L})$. If $\mathcal{L}$ is linearly independent, let $R(\mathcal{L}) = |\mathcal{L}| = R(\mathrm{span}(\mathcal{L}))$, and if β is an ordinal, let $R_\beta(\mathcal{L}) = |\{x \in \mathcal{L} : \beta \leq |x|_V < \beta+\omega\}| = R_\beta(\mathrm{span}(\mathcal{L}))$. We say $\mathcal{L}$ is *admissible* if the function $f_{\mathcal{L}}(\alpha) = |\mathcal{L}_\alpha|$ is an admissible function.

Lemma 4.41. *If V is a subfree valuated vector space and $V(\infty) = \{0\}$, then $R(V) = r(V)$.*

Proof. Suppose V is a valuated subspace of the free valuated vector space W and λ is the length of V. We induct on λ, so assume the result holds for all subfree valuated vector spaces of smaller length. Replacing W by $W/W(\lambda)$, we may assume $W(\lambda) = \{0\}$.

Fix a decomposition $W = \oplus_{\alpha<\lambda} B_\alpha$, where $|x|_W = \alpha$ for all $0 \neq x \in B_\alpha$. For $\gamma \leq \lambda$, let $W_\gamma = \oplus_{\alpha<\gamma} B_\alpha \subseteq W$ and $V_\gamma = V \cap W_\gamma$.

Case 1 - λ is a limit ordinal:

Using the induction hypothesis, it can be seen that

$$r(V) = \sup\{r(V_\gamma) : \gamma < \lambda\} = \sup\{R(V_\gamma) : \gamma < \lambda\} = R(V).$$

Case 2 - $\lambda = \gamma + 1$ is an isolated ordinal:

Again, there is a valuated decomposition $V \cong V(\gamma) \oplus (V/V(\gamma))$, where the first term is already free and the second term is sub-free of smaller length. Applying the induction hypothesis to the second term gives the result. □

We now verify an important technical observation.

Lemma 4.42. *Suppose U is a graded vector space whose length is a limit ordinal λ, $\kappa \geq |\lambda|$ is a cardinal and $R_\beta(U) = \kappa$ for all $\beta < \lambda$ (so that $R(U) = \kappa$ as well, and U is admissible). Let I be a set of cardinality κ, and for each $i \in I$, let W_i be a graded subspace of U with $R(W_i) = \kappa$. Then U is an (internal) direct sum, $\bigoplus_{i\in I} V_i$, where each V_i is admissible of length $\lambda_i < \lambda$, and $V_i \cap W_i \neq \{0\}$.*

Proof. Identify I with κ. Suppose we have defined s_i and t_i for all $i < \ell < \kappa$ satisfying

(a1) $s_i \in W_i$ for all $i < \ell$;

(b1) if $\mathcal{L}_\ell = \{s_i : i < \ell\}$ and $\mathcal{M}_\ell = \{t_i : i < \ell\}$, then $\mathcal{L}_\ell \cup \mathcal{M}_\ell$ is linearly independent;

(c1) for each $i < \ell$, $|s_i|_U \leq |t_i|_U < |s_i|_U + \omega$.

To define s_ℓ, note that $R(W_\ell) = \kappa$ and $|\mathcal{L}_\ell \cup \mathcal{M}_\ell| < \kappa$, so we can find an $s_\ell \in W_\ell$ which is not in span$(\mathcal{L}_\ell \cup \mathcal{M}_\ell)$. Since $R_{|s_\ell|_U}(U) = \kappa$, we can therefore find a t_ℓ so that (b1) and (c1) are valid for $\ell' = \ell + 1$. Therefore, by induction, we can define these elements so that (a1), (b1) and (c1) hold for all $i < \ell = \kappa$, and we let $\mathcal{L} = \{s_i : i < \kappa\}$ and $\mathcal{M} = \{t_i : i < \kappa\}$.

Note that if $\beta < \lambda$ is an ordinal, then (c1) implies that $R_\beta(\mathcal{L}) \leq R_\beta(\mathcal{M})$. Expand $\mathcal{M}$ to a set $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{L} = \emptyset$ and $\mathcal{L} \cup \mathcal{P}$ is a basis for U. Observe that (c1) implies that for all $\beta < \lambda$, $R_\beta(\mathcal{L}) \leq R_\beta(\mathcal{P})$; and $R_\beta(U) = \kappa$ implies that $R_\beta(\mathcal{P}) = \kappa$. This means that we can decompose $\mathcal{P}$ into the disjoint union of admissible subsets $\mathcal{P}_i$, for $i < \kappa$, of length λ_i, where $|s_i|_U \leq \lambda_i < \lambda$ (just construct them such that $R_\beta(\mathcal{P}_i) = \kappa$ for all $\beta < \lambda_i$). Letting $W_i = \text{span}(\mathcal{P}_i \cup \{s_i\})$ proves the result. □

This brings us to the objective of this section.

Theorem 4.43. *Suppose G is a group and λ is any ordinal. Then G is n-simply presented iff $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented.*

Proof. By Theorem 4.36(a), if G is n-simply presented, then $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented. On the other hand, if $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented, then clearly $p^{\lambda+n}G = p^n(p^\lambda G)$ is n-simply presented. If we let $G' = G/p^{\lambda+n}G$, it follows from Theorem 4.36(b) that G is n-simply presented iff G' has that property. If $\lambda = \mu + m$, where μ is a limit ordinal and $m < \omega$, then

$$G'/p^\mu G' = (G/p^{\lambda+n}G)/p^\mu(G/p^{\lambda+n}G) \cong (G/p^\lambda G)/p^\mu(G/p^\lambda G)$$

is n-simply presented. In addition, $p^\mu G' = p^\mu G/p^{\mu+m+n}G$ is bounded. Our result, therefore, can be reduced to the following special case. Because of its importance, we formulate it separately. □

Theorem 4.44. *If G is a group and λ is a limit ordinal such that $p^\lambda G$ is bounded and $G/p^\lambda G$ is n-simply presented, then G is n-simply presented.*

Proof. We begin with some simplifying assumptions. Consider a subgroup Q of G containing $p^\lambda G$ such that $Q/p^\lambda G$ is an n-simply representing subgroup of $G/p^\lambda G$.

As in the proof of Proposition 4.28, it is easily checked that if $\overline{Q}$ is the closure of Q in the p^λ-topology of G (which uses the subgroups $p^\alpha G$ for $\alpha < \lambda$ as a neighborhood basis of 0), then $\overline{Q}/p^\lambda G$ is also an n-simply representing subgroup of $G/p^\lambda G$. We may therefore assume that Q is closed, so that $p^\lambda(G/Q) = \{0\}$.

Next, observe that if $\alpha < \lambda$ is an ordinal, then by Theorem 4.36,

$$G/p^{\alpha+n}G \cong (G/p^\lambda G)/p^{\alpha+n}(G/p^\lambda G)$$

is n-simply presented, and so G is n-simply presented iff $p^{\alpha+n}G$ is n-simply presented. This means that we may replace G by $p^{\alpha+n}G$, if necessary. For example, the λ-*final rank* of G is the infimum of the set $\{r(p^\alpha G) : \alpha < \lambda\}$, and we may clearly assume that G has rank and λ-final rank equal to some infinite cardinal κ. Note also that if $\kappa = \aleph_0$, then G is countable, and hence trivially n-simply presented. We may therefore assume that κ is uncountable.

In fact, we can refine these conditions.

Assumption: If $\pi_Q : G \to G/Q$ is the natural homomorphism, then for some positive integer k we have

$$r(\pi_Q(G[p^{k-1}])) < \kappa \text{ and } r(\pi_Q((p^\alpha G)[p^k])) = \kappa$$

for all $\alpha < \lambda$.

To verify that we can make this Assumption, for any ordinal α and integer $k \geq 1$, let $\rho(\alpha, k) = r(\pi_Q((p^\alpha G)[p^k]))$. Since $p^\lambda G$ and $Q/p^\lambda G$ are bounded, there is an $m < \omega$ such that $p^m Q = \{0\}$. For $k > m$, the fact that the rank and λ-final rank of G both equal κ implies that $\rho(\alpha, k) = \kappa$ for all $\alpha < \lambda$. For each $k \leq m$, we can find an $\alpha_k < \lambda$ such that $\rho(\alpha, k)$ is constant for all $\alpha_k < \alpha < \lambda$. If $\beta = \max\{\alpha_k : k \leq m\}$, we then replace G by $p^{\beta+n}G$ and Q by $p^{\beta+n}G \cap Q$ and we can let k be the smallest integer such that $\rho(\beta + n, k) = \kappa$.

For the rest of the proof, Q and k will be defined as in the Assumption. The next definition is the key concept in verifying the full Nunke property for n-simply presented groups. A subgroup P of G containing $p^\lambda G$ is an (n, λ, κ)-*subgroup* if $P/p^\lambda G$ is an n-simply representing subgroup of $G/p^\lambda G$, and if $\pi_P : G \to G/P$ is the usual homomorphism, then there is a decomposition

$$H_P = G/P = \bigoplus_{i \in I} Y_i$$

where $|I| = \kappa$, such that

(a2) Y_i has length strictly less than λ;

(b2) If $K \subseteq I$ with $|K| < \kappa$, and $\alpha < \lambda$, then there is an $x \in (p^\alpha G)[p] - P$ such that $\pi_P(x) = y + z$, where $0 \neq y \in \bigoplus_{i \in I-K} Y_i$, $z \in \bigoplus_{i \in K} Y_i$ and $|y|_{H_P} \leq |z|_{H_P}$.

Intuitively, an (n, λ, κ) subgroup is one where, for all $\alpha < \lambda$, $H_P[p]$ has "enough" elements of the form $x_P = \pi_P(x)$, where $x \in (p^\alpha G)[p]$; we are essentially demanding that they are "spread widely" among the summands of some decomposition. The next statement is a refinement of a construction that appeared in [66].

Claim A: G has an (n, λ, κ)-subgroup.

Clearly, $|\lambda| \leq \kappa$, so if I is a set of cardinality κ there is a function $\phi : I \to \lambda$ such that for all $\alpha < \lambda$, the set of $i \in I$ such that $\phi(i) = \alpha$ also has cardinality κ. Denote $\phi(i)$ by α_i. Consider the graded vector space $U = U(H_Q[p])$, where for each $i \in I$ we let

$$W_i = U(H_Q[p] \cap \pi_Q((p^{\alpha_i} G)[p^k])) \subseteq U.$$

By Lemma 4.40, U, and hence W_i, is subfree. So, by Lemma 4.41 and our Assumption, we know that

$$R(W_i) = r(H_Q[p] \cap \pi_Q((p^{\alpha_i} G)[p^k])) = \kappa.$$

Consequently, in view of Lemma 4.42, we can conclude that there is a decomposition $U = \bigoplus_{i \in I} V_i$ where each V_i is an admissible graded space of length $\lambda_i < \lambda$, and an element $s_i \in (p^{\alpha_i} G)[p^k]$ such that $s_{Q,i} = \pi_Q(s_i)$ represents a non-zero element of V_i.

Let T_i be a simply presented group such that $U(T_i)$ is isomorphic to V_i. If $T = \bigoplus_{i \in I} T_i$, then the usual approach to the classification of totally projective groups (see, for example, Lemma 77.1 of [44]) implies these isomorphisms are induced by a group isomorphism $T \to H_Q$ which we interpret as an equality. All of this work has the following consequence:

(a3) If $i \in I$ and we consider $s_i \in (p^{\alpha_i} G)[p^k] - Q$, then $s_{Q,i} \in H_Q[p]$ is a non-zero element whose i^{th} coordinate has minimal value in our decomposition $H_Q = \bigoplus_{i \in I} T_i$.

If $J \subseteq I$, we let $\Sigma_J = \bigoplus_{i \in J} T_i$. By our Assumption, $\pi_Q(G[p^{k-1}]))$ has rank strictly less than κ, so there is a subset $J \subseteq I$ such that $|J| < \kappa$ and $\pi_Q(G[p^{k-1}])) \subseteq \Sigma_J$. We let R be the subgroup of G containing Q such that $R/Q = (\Sigma_J)[p^{k-1}]$; clearly $G[p^{k-1}] \subseteq R$. The proof of Claim A will be completed by the next observation.

Claim B: $P = p^{k-1} R$ is an (n, λ, κ)-subgroup of G.

We break this into a sequence of statements.

Subclaim B1: $p^nP = p^{n+k-1}R \subseteq p^\lambda G$.

Note that $p^{k-1}R \subseteq Q$, so that $p^{n+k-1}R \subseteq p^nQ \subseteq p^\lambda G$, as required.

Subclaim B2: $p^{k-1}G/P \cong (\bigoplus_{i\in I-J} T_i) \oplus (\bigoplus_{j\in J} p^{k-1}T_j)$.

We have a sequence of isomorphisms,

$$\begin{aligned} p^{k-1}G/p^{k-1}R &\cong (G/G[p^{k-1}])/(R/G[p^{k-1}]) \cong G/R \cong (G/Q)/(R/Q) \\ &\cong (\Sigma_I)/(\Sigma_J)[p^{k-1}] \cong (\Sigma_{I-J}) \oplus (p^{k-1}\Sigma_J), \end{aligned}$$

which clearly gives the Subclaim.

Note that $p^{k-1}(G/P) = p^{k-1}G/P$, so that G/P is simply presented. It follows that for all $i \in I$, we can construct a group Y_i such that

(a4) there is an isomorphism $\bigoplus_{i\in I} Y_i \cong G/P$ such that
(b4) it restricts to an isomorphism of $p^{k-1}Y_j$ and $p^{k-1}T_j$ whenever $j \in J$;
(c4) it restricts to an isomorphism of $p^{k-1}Y_i$ and T_i whenever $i \in I - J$.

Again, interpret this isomorphism as an equality. Since (a2) apparently holds, the following completes the proof of Claims A and B.

Subclaim B3: (b2) holds.

Given $K \subseteq I$ with $|K| < \kappa$ and $\alpha < \kappa$, find an $\ell \in I - (J \cup K)$ such that $\alpha_\ell = \alpha$. Let $x = p^{k-1}s_\ell \in p^\alpha G[p]$. Note that in the isomorphism of Subclaim B2 we have

$$\begin{array}{ccccccc} G/Q & \to & G/R & \cong & p^{k-1}G/P & \subseteq & G/P \\ | & & | & & | & & | \\ \bigoplus_{i\in I} T_i & \to & (\bigoplus_{i\in I-J} T_i) \oplus (\bigoplus_{j\in J} p^{k-1}T_j) & \cong & \bigoplus_{i\in I} p^{k-1}Y_i & \subseteq & \bigoplus_{i\in I} Y_i \end{array}$$

It follows from (a3) that the ℓ^{th} coordinate of $s_{Q,\ell} = s_\ell + Q$ has the minimum height in $H_Q = \bigoplus_{i\in I} T_i$. This shows that the ℓ^{th} coordinate of $x_P = x + P$ has minimum height in $H_P = \bigoplus_{i\in I} Y_i$. Therefore, (b2) must hold for this x.

All of the above work was intended to establish the following, from which we can conclude that Theorem 4.44 holds by inducting on n.

Claim C: There is a subgroup $S \subseteq G[p]$ such that if $G' = G/S$, then $G'/p^\lambda G'$ is $n-1$-simply presented.

By Claim A, there is an (n, λ, κ)-subgroup P of G, and we continue to use the notation given there; so, for example, if $x \in G$, we let $x_P = x + P \in H_P$. In addition, if $J \subseteq I$, we now let $\Sigma_J = \bigoplus_{i\in J} Y_i \subseteq H_P$. Let $P_1 = \{w \in P : pw \in p^\lambda G\}$

and $(w_\gamma, \alpha_\gamma)$ for $\gamma < \kappa$ be an enumeration of $P_1 \times \lambda$ (where we just repeat terms if $|P_1 \times \lambda| < \kappa$). We inductively define elements $x_\gamma \in (p^{\alpha_\gamma} G)[p] - P$ and $u_\gamma \in p^{\alpha_\gamma} G$ with the following properties:

(a5) If $K_\gamma \subseteq I$ is the union of the supports of $x_{P,\delta} = x_\delta + P$ and $u_{P,\delta} = u_\delta + P$ for all $\delta < \gamma$, then $x_{P,\gamma} = y_\gamma + z_\gamma$, where $0 \neq y_\gamma \in \Sigma_{I-K_\gamma}$ and $z_\gamma \in \Sigma_{K_\gamma}$, and $|y_\gamma|_{H_P} \leq |z_\gamma|_{H_P}$;
(b5) $pu_\gamma = pw_\gamma \in p^\lambda G$;
(c5) $|u_\gamma|_G > |x_{P,\gamma}|_{H_P}$;
(d5) $\operatorname{supp}(u_{P,\gamma}) \cap \operatorname{supp}(x_{P,\gamma}) = \emptyset$.

The existence of a x_γ that satisfies (a5) follows from (b2). Having chosen x_γ, let $\beta < \lambda$ be chosen large enough so that $p^\beta Y_i = \{0\}$ for any $i \in \operatorname{supp}(x_{P,\gamma})$. If we then choose $u_\gamma \in G$ satisfying (b5) such that $|u_\gamma|_G > \beta$, then it is easy to check that (c5) and (d5) will hold, as well.

For $\gamma < \kappa$, let $r_\gamma = x_\gamma - u_\gamma + w_\gamma$. Note that $pr_\gamma = px_\gamma - p(u_\gamma - w_\gamma) = 0$, so that if $S = \langle r_\gamma : \gamma < \kappa \rangle$, then $S \subseteq G[p]$.

Claim D: $[P_1 + S]/S \subseteq p^\lambda(G/S) = p^\lambda G'$.

Let $w \in P_1$, so that $w + S \in G/S = G'$. For any $\alpha < \lambda$, let $\gamma < \kappa$ be chosen such that $w = w_\gamma$, $\alpha = \alpha_\gamma$. Then

$$w + S = u_\gamma - x_\gamma + S \in [p^\alpha G + S]/S \subseteq p^\alpha(G/S) = p^\alpha G'.$$

Since this holds for all $\alpha < \lambda$, we can conclude that $w + S \in p^\lambda G'$.

Observe that Claim D implies that $p^{n-1}[P + S]/S \subseteq [P_1 + S]/S \subseteq p^\lambda G'$. Therefore, Claim C, and hence the entire result, will follow once we establish our next statement.

Claim E: $G'/([P + S]/S) \cong G/[P + S]$ is simply presented.

Note first that $G/[P + S] \cong (G/P)/([P + S]/P)$. For each $\gamma \leq \kappa$, let $K_\gamma \subseteq I$ again denote the union of the supports of $x_{P,\delta}$ and $u_{P,\delta}$ for all $\delta < \gamma$, so that $[P + S]/P \subseteq \Sigma_{K_\kappa}$. Next, define

$$S_\gamma = \langle r_{P,\delta} : \delta < \gamma \rangle = \langle x_{P,\delta} - u_{P,\delta} : \delta < \gamma < \lambda \rangle \subseteq \Sigma_{K_\gamma} \subseteq \Sigma_I = H_P = G/P.$$

If $H_\gamma = \Sigma_{K_\gamma}/S_\gamma$, then $G/[P + S] \cong H_\kappa \oplus \Sigma_{I-K_\kappa}$. Since H_κ is the direct limit of the H_γ, for $\gamma < \kappa$, Claim E, and hence the entire result, will once again follow from our next statement.

Claim F: For every $\gamma < \kappa$, there is a split short exact sequence

$$0 \to H_\gamma \to H_{\gamma+1} \to L_\gamma \to 0$$

where

$$L_\gamma = \Sigma_{(K_{\gamma+1}-K_\gamma)}/\langle y_\gamma\rangle$$

is a p^λ-bounded simply presented group.

To verify this, note that in moving from γ to $\gamma+1$, by (a5) and (d5), we have added two types of summands Y_i; those corresponding to elements $i \in \mathrm{supp}(u_{P,\gamma}) - K_\gamma$, and those corresponding to elements $i \in \mathrm{supp}(y_\gamma)$. Also, in going from S_γ to $S_{\gamma+1}$ we included exactly one more relator,

$$r_{P,\gamma} = x_{P,\gamma} - u_{P,\gamma} = y_\gamma + z_\gamma - u_{P,\gamma}.$$

Note that including $r_{P,\gamma}$ has the effect of identifying $y_\gamma \in \Sigma_{\mathrm{supp}(y_\gamma)}$ with

$$u_{P,\gamma} - z_\gamma \in \Sigma_{K_\gamma \cup \mathrm{supp}(u_{P,\gamma})}.$$

In more detail, observe that $|y_\gamma|_{H_P} = |x_{P,\gamma}|_{H_P} \leq |z_\gamma - u_{P,\gamma}|_{H_P}$. Since the subgroup $\langle y_\gamma\rangle$ of $\Sigma_{\mathrm{supp}(y_\gamma)}$ has order p, it is nice and $\Sigma_{\mathrm{supp}(y_\gamma)}/\langle y_\gamma\rangle$ is simply presented. Therefore, the assignment $y_\gamma \mapsto z_\gamma - u_\gamma$ extends to a homomorphism $\phi : \Sigma_{\mathrm{supp}(y_\gamma)} \to \Sigma_{K_\gamma \cup \mathrm{supp}(u_{P,\gamma})}$. It follows that

$$(a, b) \mapsto (a + \phi(b), b)$$

is an automorphism of

$$\Sigma_{K_{\gamma+1}} \cong \Sigma_{K_\gamma \cup \mathrm{supp}(u_{P,\gamma})} \oplus \Sigma_{\mathrm{supp}(y_\gamma)}$$

(where $\phi^{-1}(a, b) = (a - \phi(b), b)$), which takes $S_\gamma \oplus \langle y_\gamma\rangle$ to $S_{\gamma+1}$. Therefore,

$$\Sigma_{K_{\gamma+1}}/S_{\gamma+1} \cong (\Sigma_{K_\gamma}/S_\gamma) \oplus (\Sigma_{(K_{\gamma+1}-K_\gamma)}/\langle y_\gamma\rangle).$$

This proves Claim F, and hence the entire result. □

Our next statement follows as in the proof of Proposition 4.37.

Corollary 4.45. *If C is a group and λ is an ordinal, then C is n-balanced projective iff both $p^\lambda C$ and $C/p^\lambda C$ are n-balanced projective.*

And as in the proof of Corollary 4.38 we have

Corollary 4.46. *Suppose G is a group of length strictly less than ω^2. Then the following conditions are equivalent:*

(a) G is n-simply presented;

(b) G is n-balanced projective;

(c) For every non-negative integer m, the Ulm factor $p^{\omega\cdot m}G/p^{\omega\cdot(m+1)}G$ is $p^{\omega+n}$-projective.

We begin now with the following quite natural question:

Problem 1. Is a group G (strongly) n-simply presented iff it is (strongly) n-balanced projective?

It seems plausible that this is true, at least for groups of countable length. It is also plausible that it holds for one class of groups, but not for the other. It is worthwhile restating that, according to Theorems 4.27 and 4.30, Problem 1 is tantamount to asking whether the (strongly) n-simply presented groups are closed under summands.

The following generalizes Corollaries 4.38 and 4.46.

Proposition 4.47. *Suppose G is a group such that $p^\lambda G$ is (strongly) n-simply presented for some $\lambda < \omega^2$. Then G is (strongly) n-balanced projective iff it is (strongly) n-simply presented.*

Proof. One direction being an immediate consequence of Theorems 4.27 and 4.30, we consider the converse. If $p^\lambda G$ is (strongly) n-simply presented, then by Theorem 4.36(a), $p^{\lambda+n}G$ is (strongly) n-simply presented. If, in addition, G is (strongly) n-balanced projective, then by Proposition 4.37(a) we can conclude $G/p^{\lambda+n}G$ is (strongly) n-balanced projective. Since $\lambda+n < \omega^2$, by Corollary 4.38 and Corollary 4.46, $G/p^{\lambda+n}G$ is (strongly) n-simply presented. An appeal to Theorem 4.36(b) completes the argument. □

A homomorphism $f : G \to A$ is said to be ω_1-*bijective* if its kernel and cokernel are countable. This condition has proven useful in a number of contexts (see, for example, [3] and [37]). The following applies this idea to our investigation.

Proposition 4.48. *Suppose $f : G \to A$ is an ω_1-bijective homomorphism.*

(a) If G is n-simply presented, then A is n-simply presented.

(b) If G is n-balanced projective, then A is n-balanced projective.

Proof. (a) Suppose K is the kernel of f, P is an n-simply representing subgroup of G and $Q = f(P) \subseteq A[p^n]$. If $f' : G/P \to A/Q$ is the induced homomorphism, then the kernel of f' is $[P+K]/P$, which is countable. In addition, the cokernels of f and f' are isomorphic, and hence they are both countable. Therefore, f' is also an ω_1-bijection. It follows from Theorem 2.4 of [37] that A/Q is simply presented, so that Q is an n-simply representing subgroup of A.

(b) If X is a group such that $G \oplus X$ is n-simply presented, then the induced homomorphism $f \oplus 1_X$ shows that $A \oplus X$ is also n-simply presented, so that A is n-balanced projective. □

In Example 2.3 of [37], a group G which is *not* (0-)simply presented was constructed with a countable (and, in fact, pure) subgroup K such that G/K is (0-)simply presented. It follows that the converses of both parts of Proposition 4.48 fail. On the other hand, we do have the following partial result in this direction.

Proposition 4.49. *Suppose G is a group, $p^\omega G$ is countable, and K is a countable and nice subgroup of G. If G/K is n-simply presented, then G is n-simply presented.*

Proof. The niceness of K in G implies that there is a short exact sequence

$$0 \to K/[p^\omega G \cap K] \to G/p^\omega G \to (G/K)/p^\omega(G/K) \to 0.$$

Since G/K is n-simply presented, so is $(G/K)/p^\omega(G/K)$; and since it is separable, we can infer from Proposition 4.28 that it is $p^{\omega+n}$-projective. Since the left-hand group is countable, by Theorem 4.2 of [37], we can conclude that $G/p^\omega G$ is $p^{\omega+n}$-projective, i.e., n-simply presented. Since $p^\omega G$ is clearly n-simply presented, by Theorem 4.43, so is G. □

Corollary 4.50. *Suppose K is a countable subgroup of G. If G/K is separable and n-simply presented, then G is n-simply presented.*

Proof. Since K is nice in G and $p^\omega G \subseteq K$ is countable, the conclusion follows directly from Proposition 4.49. □

The analogue of the last corollary fails for strongly n-simply presented groups. As noted previously, there are many $p^{\omega+n}$-bounded groups G for which $K = p^\omega G$ is countable, $G/p^\omega G$ is $p^{\omega+n}$-projective (and so strongly n-simply presented), such that G is *not* $p^{\omega+n}$-projective (and hence not strongly n-simply presented).

In parallel with the above, a homomorphism $f : G \to A$ is *ω-bijective* if its kernel and cokernel are finite. It is easy to check that in this case, G is simply presented *iff* A is simply presented. The proof of Proposition 4.48(a) then shows that G is n-simply presented *iff* A is n-simply presented. Since finite subgroups are always nice, that argument also shows that if G is strongly n-simply presented, then the same is true of A. On the other hand, in the examples mentioned in the last paragraph, $p^\omega G$ can easily be chosen to be finite, showing that the converse of this statement fails.

Proposition 4.51. *If $A = B \oplus C$ is n-simply presented, where $p^\lambda C$ is countable for some $\lambda < \omega^2$, then B is n-simply presented.*

Proof. Since $p^\lambda A$ is n-simply presented, $p^\lambda C$ is countable and $p^\gamma B \cong p^\lambda A/p^\lambda C$, it follows from Proposition 4.48(a) that $p^\lambda B$ is n-simply presented. Since B is clearly n-balanced projective, the result follows from Proposition 4.47. $\square$

A similar argument gives our last observation.

Proposition 4.52. *If $A = B \oplus C$ is strongly n-simply presented, where $p^\lambda C = \{0\}$ for some $\lambda < \omega^2$, then B is strongly n-simply presented.*

We close the present work with the following special case of Problem 1, which is parallel to Proposition 4.51:

Problem 2. If $A = B \oplus C$ is strongly n-simply presented and C is countable, does it follow that B is also strongly n-simply presented?

4.3. **On ω_1-n-simply presented abelian p-groups.** Throughout the present subsection, let all groups into examination be p-torsion abelian written additively as is the custom when discussing such groups. Also, let $n \geq 0$ be a non-negative integer. Most of the used notions and notations are standard and can be seen in the classical sources [44, 47] and [53]. For the more specific terminology the interested reader can read [37, 38] and [D11] (actually, representing the statements from the previous subsection). For instance, we will abbreviate G as a *dsc-group* if it is a *direct sum of countable groups*. Besides, imitating [D11] (compare with the preceding subsection, too) a group G is called *n-simply presented* if there is $P \leq G[p^n]$ such that G/P is simply presented. When P is nice in G, such groups are said to be *strongly n-simply presented* or *nicely n-simply presented*. The last is a common generalization of the well-known concept of $p^{\omega+n}$-projectivity due to Nunke where G is *$p^{\omega+n}$-projective* whenever there exists a p^n-bounded subgroup $P \leq G$ such that G/P is *Σ-cyclic* (= a direct sum of cyclics). Later on, Keef enlarged in [75] that notion to the so-called *ω_1-$p^{\omega+n}$-projective* groups that are groups G for which there exist countable (nice) subgroups C such that G/C are $p^{\omega+n}$-projective.

This article is an extension of n-simply presented groups in the spirit of (the previous generalizations of) ω_1-$p^{\omega+n}$-projective groups. It is organized as follows: In the first part, i.e. here, we put the main definitions. In the second one, we prove some useful preliminary assertions and state some background material, and in the third one we state with proofs the major results in the subject. Next, in the final stage, we prove a series of statements concerning the important full Nunke-esque property, and we close in the remaining part with some unsettled challenging questions.

Definition 1. The group G is called *ω_1-n-simply presented* if there is a countable subgroup K of G such that G/K is n-simply presented. In addition, if K is finite, G is said to be *ω-n-simply presented*.

When $n = 0$, and as a result G/K is simply presented, we will just say that G is *ω_1-simply presented*. But if K is nice in G, G is just simply presented (see [37] or [38]). Likewise, ω-n-simply presented groups are precisely n-simply presented.

When K is a priory chosen nice in G, one may state:

Definition 2. The group G is called *nicely ω_1-n-simply presented* if there is a countable nice subgroup N of G such that G/N is n-simply presented.

When $n = 0$, and hence G/N is simply presented, we observe with the aid of [37, 38] that G must be simply presented, too.

Definition 3. The group G is said to be *strongly ω_1-n-simply presented* if there exists a countable subgroup C of G such that G/C is strongly n-simply presented. In addition, if C is finite, we will say that G is *strongly ω-n-simply presented*.

In case that C is taken a priory nice in G, one can state:

Definition 4. The group G is said to be *strongly nice ω_1-n-simply presented* if there exists a countable nice subgroup M of G such that G/M is strongly n-simply presented.

Apparently, because $p^{\omega+n}$-projective groups are strongly n-simply presented, the ω_1-$p^{\omega+n}$-projectives, defined as in [75], are themselves strongly nice ω_1-n-simply presented. Moreover, strongly ω-n-simply presented groups are strongly nice ω_1-n-simply presented, because finite subgroups are always nice. As indicated in [D11], strongly ω-n-simply presented groups need not be strongly n-simply presented.

Also, it is clear that Definition 4 yields Definition 2 and Definition 3 implies Definition 1. Likewise, some enlargements of this kind for the n-totally projective groups from [76] can be given as well.

On the other vein, Hill and Megibben gave in [63] the definition of *a c.c. group* as a group G such that $p^\omega(G/C)$ is countable whenever $C \leq G$ is a countable subgroup. For our applicable purposes we shall now enlarge this concept to the so-called *α-countably groups* where α is an arbitrary ordinal. This is necessary because the approach used in [D11] does not work here because $p^{\alpha+n}(G/C)$ is not always contained in $(p^\alpha G + C)/C$ if $p^n C \neq \{0\}$.

Definition 5. We will say that the group G is *α-countably* if for any its countable subgroup C the factor-group $p^\alpha(G/C)/(p^\alpha G + C)/C$ is always countable.

Note also that if either C is a nice subgroup or $\alpha \in \mathbb{N}$, the factor-group $p^\alpha(G/C)/(p^\alpha G + C)/C$ equals to zero, and so Definition 5 is satisfied in both situations.

When $\alpha = \omega$, the posed condition is equivalent to the countability of the quotient $[\cap_{i<\omega}(p^i G+C)]/[\cap_{i<\omega} p^i G]$ which in turn is tantamount to the countability of the quotient $[\cap_{i<\omega}(p^i G + C)]/(\cap_{i<\omega} p^i G + C)$. Apparently, c.c. groups are always ω-countably. To treat the converse relationship, one sees that if $p^\omega G$ is countable, then every ω-countably group is a c.c. group, and thus these two notions do coincide. In particular, weakly ω_1-separable groups (which are of necessity separable), are ω-countably as well as ω-countably separable groups are weakly ω_1-separable.

Definition 6. We will say that the group G is *α-boundary* if for any its countable subgroup C the factor-group $p^\alpha(G/C)/[(p^\alpha G + C)/C]$ is always bounded.

In particular, there is a natural number m such that the inclusion $p^{\alpha+m}(G/C) \subseteq (p^\alpha G + C)/C$ holds.

Note also that if either C is a nice subgroup or $\alpha \in \mathbb{N}$ then the quotient $p^\alpha(G/C)/[(p^\alpha G+C)/C]$ equals to zero, as well as if $p^m C = \{0\}$ then the inclusion $p^{\alpha+m}(G/C) \subseteq (p^\alpha G + C)/C$ holds appealing to Lemma 3.1 of [D11], and thus in all cases Definition 6 is fulfilled.

We will begin with our preliminary and background material. So, the following two technicalities possess a central role for our further applications.

Lemma 4.53. *Suppose that α is an ordinal, and that G and F are groups where F is finite. Then the following formula is fulfilled:*

$$p^\alpha(G + F) = p^\alpha G + [F \cap p^\alpha(G + F)] \subseteq p^\alpha G + F.$$

Proof. We will use a transfinite induction on α. First, if $\alpha - 1$ exists, we have

$$p^\alpha(G + F) = p(p^{\alpha-1}(G + F)) = p(p^{\alpha-1}G + [F \cap p^{\alpha-1}(G + F)]) =$$

$p(p^{\alpha-1}G)+p(F\cap p^{\alpha-1}(G+F)) \subseteq p^\alpha G+[F\cap p(p^{\alpha-1}(G+F))] = p^\alpha G+[F\cap p^\alpha(G+F)]$.

Since the reverse inclusion "$\supseteq$" is obvious, we obtain the desired equality.

If now $\alpha - 1$ does not exist, we have that $p^\alpha(G + F) = \cap_{\beta<\alpha}(p^\beta(G + F)) \subseteq \cap_{\beta<\alpha}(p^\beta G + F) = \cap_{\beta<\alpha}p^\beta G + F = p^\alpha G + F$. In fact, the second sign "=" follows like this: Given $x \in \cap_{\beta<\alpha}(p^\beta G+F)$, we write that $x = g_{\beta_1} + f_1 = \cdots = g_{\beta_s} + f_s = \cdots$ where $f_1, \cdots, f_s \in F$ are the all elements of F; $g_{\beta_1} \in p^{\beta_1}G, \cdots, g_{\beta_s} \in p^{\beta_s}G$ with $\beta_1 < \cdots < \beta_s < \cdots$.

Since F is finite, while the number of equalities is infinite due to the infinite cardinality of α, we infer that $g_{\beta_s} \in p^\beta G$ for any ordinal $\beta < \alpha$ which means that $g_{\beta_s} \in \cap_{\beta<\alpha}p^\beta G = p^\alpha G$. Thus $x \in \cap_{\beta<\alpha}p^\beta G + F = p^\alpha G + F$, as claimed. Furthermore, $p^\alpha(G+F) \subseteq (p^\alpha G+F)\cap p^\alpha(G+F) = p^\alpha G+[F\cap p^\alpha(G+F)]$ which is obviously equivalent to an equality. □

Lemma 4.54. *Let N be a nice subgroup of a group G. Then*

(i) $N + R$ is nice in G for every finite subgroup $R \leq G$;

(ii) N is nice in $G + F$ for each finite group F.

Proof. (i) For any limit ordinal γ, we deduce that $\cap_{\delta<\gamma}(N + R + p^\delta G) \subseteq R + \cap_{\delta<\gamma}(N + p^\delta G) = R + N + p^\gamma G$, as required. Indeed, the relation "$\subseteq$" follows like this: Given $x \in \cap_{\delta<\gamma}(N + R + p^\delta G)$, we write $x = a_1 + r_1 + g_1 = \cdots = a_s + r_s + g_s = \cdots = a_k + r_1 + g_k = \cdots$, where $a_1, \cdots, a_k \in N$; $r_1, \cdots, r_k \in R$; $g_1 \in p^{\delta_1}G, \cdots, g_k \in p^{\delta_k}G$ with $\delta_1 < \cdots < \delta_k$. So $a_1 + g_1 = \cdots = a_k + g_k = \cdots \in \cap_{\delta<\gamma}(N + p^\delta G)$ and hence $x \in R + \cap_{\delta<\gamma}(N + p^\delta G)$, as requested.

(ii) Since N is nice in G, we may write $\cap_{\delta<\gamma}[N + p^\delta G] = N + p^\gamma G$ for every limit ordinal γ. Furthermore, with Lemma 4.53 at hand, we subsequently deduce that

$$\cap_{\delta<\gamma}[N + p^\delta(G + F)] = \cap_{\delta<\gamma}[N + p^\delta G + (F \cap p^\delta(G + F))] \subseteq$$

$$\cap_{\delta<\gamma}(N+p^\delta G)+[F\cap p^\gamma(G+F)] = N+p^\gamma G+[F\cap p^\gamma(G+F)] = N+p^\gamma(G+F).$$

In fact, the inclusion "$\subseteq$" follows thus: Given $x \in \cap_{\delta<\gamma}[N+p^\delta G+(F\cap p^\delta(G+F))]$, we write $x = a_1+g_1+f_1 = \cdots = a_s+g_s+f_s = \cdots = a_k+g_k+f_1 = \cdots$, where $a_1, \cdots, a_k \in N$; $g_1 \in p^{\delta_1}G, \cdots, g_k \in p^{\delta_k}G$; $f_1 \in F \cap p^{\delta_1}(G + F), \cdots, f_k \in F \cap p^{\delta_k}(G+F)$ with $\delta_1 < \cdots < \delta_k$. Hence $a_1+g_1 = \cdots = a_k+g_k = \cdots \in \cap_{\delta<\gamma}(N+p^\delta G)$ and because the number of the f_i's ($1 \leq i \leq k$) is finite whereas the number of equalities is not, we can deduce that $f_1 \in \cap_{\delta<\gamma}(F \cap p^{\delta}(G+F)) = F \cap p^{\gamma}(G+F)$, as needed. □

It was pointed out in [D11] that G is (strongly) n-simply presented if and only if $p^i G$ is (strongly) n-simply presented whenever $i \in \mathbb{N}$ - see also Lemma 1.3 from [76]. Before continue, we pause for the following mere observations.

Moreover, it was proved in Proposition 5.2 of [D11] that if $f : G \to A$ is an ω_1-bijection, that is a (bijective) homomorphism whose kernels and co-kernels are both countable, then G being n-simply presented implies the same for A. In particular, if G is n-simply presented, then G/H is n-simply presented whenever H is a countable subgroup. However, the converse does not hold; nevertheless it could be true provided in addition that H is nice in G - compare with Corollary 4.60 listed below (for more details about that type of results the reader can see [38] too). When $n = 0$, i.e., in the case of simply presented groups, this is fulfilled (see [37] and [38]); however for any $n \geq 1$ we have doubts about its validity; thus will exist a nicely ω_1-n-simply presented group (with uncountable first Ulm subgroup) which is not n-simply presented – compare with Theorem 4.59 stated in the sequel.

We proceed in this way with the following particular case of the aforementioned result from [D11]; nevertheless we give a more conceptual and easy proof needed for applicable purposes.

Lemma 4.55. *If T is n-simply presented and G/T is countable, then G is n-simply presented.*

Proof. Write $G = T + K$ where K is countable. With [D11] at hand, there exists $P \leq T[p^n]$ such that T/P is simply presented. Furthermore, $G/P = (T/P) + (K+P)/P$ where $(K+P)/P \cong K/(K \cap P)$ is countable. Thus Theorem 2.4 of [37] can be successfully employed to show that G/P is simply presented, as required. □

Likewise, it was only pointed out in [D11] without a proof that if $\varphi : G \to A$ is an ω-bijective homomorphism, that is, a homomorphism whose kernel and co-kernel are finite, then G is n-simply presented if and only if A is n-simply presented – we shall give a suitable confirmation to this fact below. Besides, if G is strongly n-simply presented, then A is strongly n-simply presented, whereas the converse is not true as some concrete examples from [D11] show; compare also with the comments given below.

However, the defined above new group classes inherit ω-bijections as we will prove in the sequel (for another treatment see also cf. [37]). First, one more preliminary claim is needed (see also Lemma 1.3 of [76] for a more general treatment considering bounded factors).

Lemma 4.56. *If S is a subgroup of a group G such that G/S is finite, then G is (strongly) n-simply presented if and only if S is (strongly) n-simply presented.*

Proof. Write $G = S + F$ where $F \leq G$ is finite.

Suppose first that S is strongly n-simply presented. With [D11] in hand, there is $Z \leq S[p^n]$ which is nice in S such that S/Z is simply presented. We therefore have that $G/Z = [S/Z] + [(F + Z)/Z]$ where $(F + Z)/Z \cong F/(F \cap Z)$ is finite. By what we have remarked above, G/Z should be simply presented. But Z is nice in G utilizing Lemma 4.54 (ii), as required.

Reciprocally, let G be strongly n-simply presented. Since $p^tG = p^tS$ for some $t \in \mathbb{N}$ and in [D11] was established that any group A is strongly n-simply presented if and only if so is p^tA, one may derive that S is strongly n-simply presented. Actually, this idea also provides a new verification of sufficiency considered above.

The same method works for n-simply presented groups as well. $\square$

Continuing this approach, we can state that if $G = S + B$, where $B \leq G$ is bounded, then G is (strongly) n-simply presented if and only if S is (strongly) n-simply presented. In addition, it seems that the same procedure does not to work for G/S being bounded.

As a final comment, we note that Lemma 1.9 of [75] asserts that the following two conditions are equivalent for any class $\mathbb{K}$ of abelian groups:

(*) Whenever S is a subgroup of a group G with G/S countable, then $G \in \mathbb{K}$ if, and only if, $S \in \mathbb{K}$.

(**) Whenever C is a countable subgroup of G, then $G \in \mathbb{K}$ if, and only if, $G/C \in \mathbb{K}$.

In addition they are equivalent for $\mathbb{K}$ to be closed under taking ω_1-bijections.

However, there is no equivalence if "countable" is replaced by "finite". Indeed, suppose $\mathbb{K}$ coincides with the class of strongly n-simply presented groups. It was proved in Lemma 4.56 that if G/S is finite, then G is strongly n-simply presented if and only if S is, so that condition (*) is satisfied. Nevertheless, the same cannot be said of (**); namely if G is strongly n-simply presented, then so is G/F for any finite subgroup F, but G/F being strongly n-simply presented does not imply the same for G - in fact, as noticed in [D11], taking $p^\omega G \cong \mathbb{Z}(p)$ and $G/p^\omega G$ to be $p^{\omega+n}$-projective, and hence strongly n-simply presented, it follows from Example 2.3 in [75] that G is ω-$p^{\omega+n}$-projective but not $p^{\omega+n}$-projective.

So, it will follow that the class of such groups G is closed under the formation of ω-bijections exactly when point (*) is fulfilled for finite subgroups. It should be better if points (*) and (**) are tantamount with the word "finite" in the claims; in other words they must hold together. It is noteworthy that (**) always implies (*); in fact if $G = S + F$ for some finite subgroup $F \leq G$, then $G/F = (S + F)/F \cong S/(S \cap F)$ where $S \cap F$ is finite, thus sustaining our affirmation.

So, we are now able to give the promised above proof of the following statement which is no longer true for *strongly* n-simply presented groups as indicated above – compare with Lemma 4.56 too.

Lemma 4.57. *A group G is n-simply presented if, and only if, G/F is n-simply presented for some finite subgroup F of G.*

Proof. As aforementioned, the "and only if" direction was proved in Proposition 5.2 of [D11].

To treat the "if" one, write $G/F/A/F \cong G/A$ is simply presented for some $A \leq G$ such that $p^n A \subseteq F \subseteq A$. Since $p^n A$ is finite, it is a routine technical exercise to check that $A = L + A[p^n]$ for some finite $L \leq A$. Furthermore, $G/A \cong (G/A[p^n])/(A/A[p^n])$ being simply presented with finite $A/A[p^n] \cong L/L[p^n]$ implies with the help of [37] or [38] that $G/A[p^n]$ is simply presented, as required. $\square$

We start here with stating and proving some basic results.

This work is mainly inspired by Proposition 5.3 of [D11] and it is a significant generalization of the stated above concept in [75]. So, in this light, we begin this section with some different characterizations of ω_1-n-simply presented groups.

Theorem 4.58. *The following points are equivalent:*

(i) G is ω_1-n-simply presented;

(ii) $G/(C \oplus L)$ is simply presented where C is a countable subgroup of G and L is a p^n-bounded subgroup of G;

(iii) G/L is ω_1-simply presented for some $L \leq G[p^n]$.

Proof. "$(i) \iff (ii)$". Foremost, letting (i) be fulfilled, given G/K is n-simply presented for some countable subgroup $K \leq G$. Thus there is A/K with $A \leq G$ and $p^n A \subseteq K$ such that G/A is simply presented. But it is well known that $A = C \oplus L$, and hence (ii) holds.

Conversely, assume that (ii) is true. Thus $G/(C \oplus L) \cong [G/C]/[(C \oplus L)/C]$ is simply presented, where $(C \oplus L)/C \cong L$ is p^n-bounded. Therefore G/C is n-simply presented, as required.

"$(ii) \iff (iii)$". First, assuming that (ii) is valid, we see that $G/(C \oplus L) \cong [G/L]/[(C \oplus L)/L]$ is simply presented where $(C \oplus L)/L \cong C$ is countable. So, G/L is ω_1-simply presented.

Reciprocally, let (iii) be true, so given G/L is ω_1-simply presented for some p^n-bounded subgroup L. Hence there is a countable subgroup B/L with $B \leq G$ such that $(G/L)/(B/L) \cong G/B$ is simply presented. Besides, $B = L+K$ for some countable $K \leq B$. Since $p^nL = \{0\}$, we write $L = L_1 \oplus L_2$ where L_2 is countable and $L \cap K \subseteq L_2$. Observe that $B = L_1 + (K + L_2)$ where L_1 is p^n-bounded and $K + L_2$ is countable. Moreover, $L_1 \cap (K + L_2) = \{0\}$; indeed take $a = b + c$ where $a \in L_1$, $b \in K$ and $c \in L_2$. Furthermore, $a - c \in L \cap K \subseteq L_2$, whence $a \in L_1 \cap L_2 = \{0\}$ and so $a = 0$. Finally, $B = L_1 \oplus (K + L_2)$ and thus $G/(C \oplus M)$ is simply presented for the countable $C = K + L_2$ and the p^n-bounded $M = L_1$, as stated. $\square$

Remark 1. Unfortunately there is no an absolute analogy with the corresponding result from [75]. In fact, the reciprocal conditions $G \cong S/(C \oplus L)$ where S is simply presented, C is countable, L is p^n-bounded and $G \cong T/K$ where T is n-simply presented and K is countable, are obviously equivalent to the fact that G is n-simply presented by utilizing [D11]. Similarly $G \cong V/Z$ for some ω_1-simply presented group V and its p^n-bounded subgroup Z is tantamount again to the fact that G is n-simply presented.

The next main result shows how nicely ω_1-n-simply presented groups differ from the n-simply presented ones. Specifically, the next statement somewhat extends both Corollary 4.7 and Proposition 5.3 of [D11]. As we will see below, it actually shows that p^λ-bounded n-simply presented groups are closed under $\aleph_0$-nice elongations whenever $\lambda < \omega^2$.

Theorem 4.59. *Nicely ω_1-n-simply presented groups of length $< \omega^2$ are n-simply presented.*

Proof. Suppose that G/K is n-simply presented for some countable nice subgroup $K \leq G$. We will transfinite induction on length$(G) = \lambda < \omega^2$.

First, assume that $\lambda = \omega$. Hence Proposition 4.61 listed below applies to derive that G is $p^{\omega+n}$-projective and hence n-simply presented.

Let us assume that the claim is true for groups of length $\leq \omega \cdot t$ for some $t \in \mathbb{N}$ and we shall prove it for groups G of length $\leq \omega \cdot (t+1) = \omega \cdot t + \omega$. Thus

$$(G/K)/p^{\omega\cdot t}(G/K) \cong G/(p^{\omega\cdot t}G + K) \cong [G/p^{\omega\cdot t}G]/[(p^{\omega\cdot t}G + K)/p^{\omega\cdot t}G]$$

is n-simply presented with the aid of Theorem 3.4 (a) in [D11]. Since $(p^{\omega\cdot t}G + K)/p^{\omega\cdot t}G \cong K/(K \cap p^{\omega\cdot t}G)$ is countable and nice in $G/p^{\omega\cdot t}G$, the induction hypothesis yields that $G/p^{\omega\cdot t}G$ is n-simply presented. In addition, $p^{\omega\cdot t}(G/K) \cong p^{\omega\cdot t}G/(p^{\omega\cdot t}G \cap K)$ is also n-simply presented owing again to Theorem 3.4 (a) of [D11]. However, $p^{\omega}(p^{\omega\cdot t}G) = p^{\omega\cdot t+\omega}G = \{0\}$ and $p^{\omega\cdot t}G\cap K$ is countable and nice in $p^{\omega\cdot t}G$. Consequently, $p^{\omega\cdot t}G/(p^{\omega\cdot t}G\cap K)$ is separable $p^{\omega+n}$-projective, whence $p^{\omega\cdot t}G$ is separable $p^{\omega+n}$-projective in accordance with step one given above. Finally, Theorem 4.4 from [D11] ensures that G is n-simply presented, as asserted.

Let now we consider the general case when length$(G) = \omega \cdot t + l$ with $t, l \in \mathbb{N} \cup \{0\}$. As above

$$G/K/p^{\omega\cdot t}(G/K) \cong [G/p^{\omega\cdot t}G]/[(p^{\omega\cdot t}G + K)/p^{\omega\cdot t}G]$$

is n-simply presented of length at most $\omega\cdot t$, where $(p^{\omega\cdot t}G+K)/p^{\omega\cdot t}G \cong K/(p^{\omega\cdot t}G\cap K)$ is its countable nice subgroup. By what we have shown above $G/p^{\omega\cdot t}G$ is n-simply presented of length not exceeding $\omega\cdot t$. Besides, $p^{\omega\cdot t}G$ is obviously bounded by p^l, so that we appeal to Theorem 4.5 in [D11] to infer after all that G is n-simply presented, as stated. □

For groups of length beyond or equal to ω^2 the validity of the last theorem remains left-open. We however conjecture that there exists a group G of length ω^2 which is nicely ω_1-n-simply presented but not n-simply presented; in fact, $p^{\omega}G$ should be uncountable.

As an immediate consequence, we yield:

Corollary 4.60. *Suppose K is a countable nice subgroup of a group G such that length$(G) < \omega^2$. Then G is n-simply presented if and only if G/K is n-simply presented.*

Proof. The necessity follows directly from Proposition 5.2 (a) in [D11], as the sufficiency follows directly from Theorem 4.59. □

Again, as remarked in [D11] and proved in Lemma 4.57, G is n-simply presented if and only if G/F is n-simply presented whenever $F \leq G$ is finite. Likewise, as indicated in [D11] and commented above, the same type affirmation does not hold for strongly n-simply presented groups and thus for strongly nice ω_1-n-simply presented groups. In fact, there are too many groups G of length $\omega + n$ such that

$p^\omega G$ is countable (and even finite) and $G/p^\omega G$ is $p^{\omega+n}$-projective but G is not $p^{\omega+n}$-projective - see [75] - such a group G is actually ω_1-$p^{\omega+n}$-projective (or even ω-$p^{\omega+n}$-projective).

We continue with some other structural affirmations.

Proposition 4.61. *If G is nicely ω_1-n-simply presented, then $G/p^\omega G$ is $p^{\omega+n}$-projective. In particular, separable nicely ω_1-n-simply presented groups are $p^{\omega+n}$-projective.*

Proof. According to Proposition 2.2 of [D11], $G/N/p^\omega(G/N) \cong G/(p^\omega G + N)$ is $p^{\omega+n}$-projective, where N is a countable nice subgroup of G. But $G/(p^\omega G+N) \cong [G/p^\omega G]/[(p^\omega G + N)/p^\omega G]$, where it is obvious that $(p^\omega G + N)/p^\omega G \cong N/(N \cap p^\omega G)$ is countable. Henceforth, we apply Theorem 4.2 from [37] to get the first claim. The second part is its trivial consequence. □

Corollary 4.62. *Suppose G is a group for which $p^\omega G$ is countable. Then G is nicely ω_1-n-simply presented if, and only if, it is ω_1-$p^{\omega+n}$-projective.*

Proof. In view of Proposition 4.61, $G/p^\omega G$ is $p^{\omega+n}$-projective. Hence, in virtue of [74], G is ω_1-$p^{\omega+n}$-projective, as expected.

The reverse implication is obvious. □

Remark 2. Another proof might be like this: Utilizing Proposition 5.3 from [D11], G must be n-simply presented. Therefore, [75] applies to conclude that G has to be ω_1-$p^{\omega+n}$-projective, as wanted.

Corollary 4.63. *Suppose that G is a group whose $p^{\omega+n}G$ is countable. Then G is strongly nice ω_1-n-simply presented if, and only if, it is ω_1-$p^{\omega+n}$-projective.*

Proof. Let G be strongly nice ω_1-n-simply presented. Owing to ([D11], Proposition 2.5), we have that the factor-group $G/M/p^{\omega+n}(G/M) \cong G/(p^{\omega+n}G + M) \cong [G/p^{\omega+n}G]/[(p^{\omega+n}G + M)/p^{\omega+n}G]$ is $p^{\omega+n}$-projective. However, $(p^{\omega+n}G + M)/p^{\omega+n}G \cong M/(M \cap p^{\omega+n}G)$ is obviously countable whence by the utilization of [75] we obtain that $G/p^{\omega+n}G$ is ω_1-$p^{\omega+n}$-projective. Thus again [75] is applicable to conclude that G is ω_1-$p^{\omega+n}$-projective, too, as asserted.

The converse part is immediate as mentioned in the introductory section. □

Remark 3. The above statements generalize the facts that every n-simply presented group is ω_1-$p^{\omega+n}$-projective provided its first Ulm subgroup is countable, and that any strongly n-simply presented group G is ω_1-$p^{\omega+n}$-projective provided that $p^{\omega+n}G$ is countable.

Moreover, in this aspect, is it true that ω_1-n-simply presented groups G with countable $p^\omega G$, as well as strongly ω_1-n-simply presented groups G with countable $p^{\omega+n}G$, are ω_1-$p^{\omega+n}$-projective?

This can be partially settled like this:

Proposition 4.64. *If G is an ω_1-n-simply presented and ω-countably group, then $G/p^\omega G$ is $p^{\omega+n}$-projective.*

Proof. Let G/K be n-simply presented where K is a countable subgroup of G. Therefore, we apply [D11] to show that $G/K/p^\omega(G/K) \cong G/\cap_{i<\omega}(p^iG+K) \cong [G/p^\omega G]/[\cap_{i<\omega}(p^iG+K)/p^\omega G]$ is $p^{\omega+n}$-projective. Since $\cap_{i<\omega}(p^iG+K)/p^\omega G$ is countable, [37] applies to get that $G/p^\omega G$ remains $p^{\omega+n}$-projective, as desired. □

As two immediate consequences, we deduce:

Corollary 4.65. *Suppose G is a c.c. group. Then G is ω_1-n-simply presented if, and only if, G is ω_1-$p^{\omega+n}$-projective.*

Proof. The sufficiency being elementary, we deal with the necessity. Since c.c. groups are obviously countably separable with countable first Ulm subgroup, Proposition 4.64 allows us to conclude with the help of [75] that G is ω_1-$p^{\omega+n}$-projective, as stated. □

So, we directly obtain:

Corollary 4.66. *Suppose G is a weakly ω_1-separable group. Then G is ω_1-n-simply presented if and only if G is $p^{\omega+n}$-projective.*

Furthermore, we come to the following:

Proposition 4.67. *Suppose that A is a group with a countable subgroup L. Then A is ω_1-n-simply presented if, and only if, A/L is ω_1-n-simply presented.*

Proof. First, let us assume that A be ω_1-n-simply presented, hence A/K is n-simply presented for some countable $K \leq A$. But

$$[A/L]/[(L+K)/L] \cong A/(L+K) \cong [A/K]/[(L+K)/K],$$

where the last factor-group $[A/K]/[(L+K)/K]$ is n-simply presented by Proposition 5.2 (a) of [D11] since $(L+K)/K$ is countable. Therefore, $[A/L]/[(L+K)/L]$ is n-simply presented with countable $(L+K)/L \cong K/(K\cap L)$, as wanted.

Reciprocally, let us now A/L be ω_1-n-simply presented, and so let C/L be a countable subgroup of A/L for some $C \leq A$ such that $(A/L)/(C/L) \cong A/C$ is n-simply presented. Observing that C is of necessity countable, we deduce via Definition 1 that A is ω_1-n-simply presented, as formulated. □

As an easy consequence, we deduce:

Corollary 4.68. *Suppose A is a group such that $p^\alpha A$ is countable for some ordinal α. Then A is ω_1-n-simply presented if, and only if, $A/p^\alpha A$ is ω_1-n-simply presented.*

We now arrive at the following:

Proposition 4.69. *Let A be a group with a subgroup G such that A/G is countable. Then A is ω_1-n-simply presented if, and only if, G is ω_1-n-simply presented.*

Proof. Write $A = G + C$ where C is countable and assume that G is ω_1-n-simply presented. Now Definition 1 insures that there is a countable subgroup K such that G/K is n-simply presented. Consequently, $A/K = (G/K) + (C + K)/K$. Employing Lemma 4.55, A/K is n-simply presented since $(C+K)/K$ is obviously countable. This gives that A is ω_1-n-simply presented, as desired.

Conversely, let us assume that A is ω_1-n-simply presented. Now, Proposition 4.67 guarantees that $(G + C)/C \cong G/(G \cap C)$ is ω_1-n-simply presented. But $G \cap C$ is countable and again Proposition 4.67 will work to get that G is ω_1-n-simply presented, as wanted. □

Remark 4. Actually Propositions 4.67 and 4.69 are equivalent and can be deduced one of other. The above arguments also give a simpler verification to ([75], Lemma 1.9).

We are now ready to prove the following central result:

Theorem 4.70. *The class of ω_1-n-simply presented groups is closed under the formation of ω_1-bijections, and is the smallest class containing n-simply presented groups with this property.*

In other words, if $f : G \to A$ is an ω_1-bijective homomorphism and G is an ω_1-n-simply presented group, then A is an ω_1-n-simply presented group, and ω_1-n-simply presented groups form the minimal class of groups possessing that property.

Proof. The first part follows by ([75], Lemma 1.9) plus Propositions 4.67 and 4.69.

The fact that it is the minimal class possessing that property follows using Proposition 1.10 of [75] and Theorem 4.58. □

Proposition 4.71. *Suppose A is a group with a finite subgroup F. Then A is*

(a) nicely ω_1-n-simply presented;

(b) strongly ω_1-n-simply presented;

(c) strongly nice ω_1-n-simply presented,

if, and only if, A/F is.

Proof. (a) Assume first that A is nicely ω_1-n-simply presented, i.e., there is a countable nice subgroup N such that A/N is n-simply presented. Observing as above that

$$[A/F]/[(F+N)/F] \cong A/(F+N) \cong [A/N]/[(F+N)/N],$$

and that $[A/N]/[(F+N)/N]$ is n-simply presented, it follows that A/F is nicely ω_1-n-simply presented because $(F+N)/F$ is countable and nice in A/F in accordance with Lemma 4.54 (i).

Conversely, given that A/F is nicely ω_1-n-simply presented, so there exists a countable nice subgroup C/F of A/F with $C \leq A$ such that $(A/F)/(C/F) \cong A/C$ is n-simply presented. Since F is nice in A, one can see that C is countable and nice in A (see [44, 47]), whence Definition 2 gives the claim.

(b) In view of the aforementioned results from [D11] concerning strongly n-simply presented groups and with Lemma 4.54 (i) in hand, the assertion follows using the tricks in the previous point (a).

(c) Follows using the arguments in the preceding two points (a) and (b). □

Proposition 4.72. *Let A be a group with a subgroup G such that A/G is finite. Then A belongs to (a), (b) or (c) of Proposition 4.71 if, and only if, G belongs to one of them.*

Proof. Repeating the same method as in Proposition 4.67 combined with Proposition 4.71, we complete the arguments. □

We are now ready to establish the following main result:

Theorem 4.73. *The classes of strongly ω_1-n-simply presented groups, nicely ω_1-n-simply presented groups and strongly nice ω_1-n-simply presented groups are closed under taking ω-bijections. Moreover, the class of strongly nice ω_1-n-simply presented groups is the smallest (minimal) class containing strongly n-simply presented groups possessing that property.*

Proof. The first part follows by our discussion in the second section (see again [75], Lemma 1.9) along with Propositions 4.71 and 4.72. For the second one, follows from Proposition 1.10 of [75] and some similar arguments to that of Theorem 4.58. □

Analyzing the proof of Proposition 4.71, we now detect that this proposition can be somewhat (considerably) extended to proper $\aleph_0$-nice elongations. Specifically, the following is fulfilled:

Proposition 4.74. *Let A be a group with a countable nice subgroup N. If A/N is either*

(a) nicely ω_1-n-simply presented;

(b) strongly ω_1-n-simply presented;

(c) strongly nice ω_1-n-simply presented,

then so is A.

We finish off here with the intersection between some well-known group classes. To that goal, the definition of an n-summable group can be seen in [38].

Theorem 4.75. *If G is both a strongly nice ω_1-n-simply presented group and an n-summable group, then G is a C_{ω_1}-group.*

Proof. By definition, there is a countable nice subgroup M such that G/M is strongly n-simply presented.

On another vein, Corollary 1.3 from [38] guarantees that G/M is also n-summable. We consequently apply Corollary 4.7 of [76] to get that G/M is a C_{ω_1}-group. Finally, Corollary 2.4 again in [38] gives the desired fact that G is a C_{ω_1}-group, too. □

As a valuable consequence, we derive:

Corollary 4.76. *Suppose that length$(G) < \omega_1$. Then G is strongly nice ω_1-n-simply presented and n-summable if, and only if, G is a dsc-group.*

Proof. The necessity follows immediately from Theorem 4.75 because C_{ω_1}-groups of countable length are dsc-groups.

The sufficiency is obvious. □

In order to approach to Nunke-like theorems, we will now prove some partial Nunke-esque results for the new group classes defined in the introductory section.

Proposition 4.77. *Suppose λ is an ordinal such that G is a λ-boundary group. If G is ω_1-n-simply presented, then so are $p^\lambda G$ and $G/p^\lambda G$.*

Proof. Let G/K be n-simply presented for some countable $K \leq G$. Consequently, Theorem 3.4 (a) in [D11] gives that $p^\lambda(G/K)$ is also n-simply presented. But by assumption $p^\lambda(G/K)/(p^\lambda G + K)/K$ is bounded, which means that $(p^\lambda G +$

$K)/K \cong p^\lambda G/(p^\lambda G \cap K)$ is n-simply presented as well. And since $p^\lambda G \cap K$ is countable, this substantiates that $p^\lambda G$ is ω_1-n-simply presented.

To prove the second statement, again Theorem 3.4 (a) of [D11] is applied to insure that $(G/K)/p^\lambda(G/K) \cong [G/K/(p^\lambda G+K)/K]/[p^\lambda(G/K)/(p^\lambda G+K)/K] = [G/K/(p^\lambda G + K)/K]/p^\lambda(G/K/(p^\lambda G + K)/K)$ is also n-simply presented. But $p^\lambda(G/K/(p^\lambda G + K)/K) = p^\lambda(G/K)/(p^\lambda G + K)/K$ is bounded, and therefore Theorem 4.5 of [D11] is in use to get that $G/K/(p^\lambda G+K)/K \cong G/(p^\lambda G+K) \cong [G/p^\lambda G]/(p^\lambda G + K)/p^\lambda G$ is n-simply presented. This means that $G/p^\lambda G$ is ω_1-n-simply presented, because $(p^\lambda G + K)/p^\lambda G \cong K/(K \cap p^\lambda G)$ is countable, as required. $\square$

Theorem 4.78. *Suppose that G is a λ-boundary group for which $G/p^\lambda G$ is n-simply presented for some ordinal λ. Then G is ω_1-n-simply presented if, and only, if $p^\lambda G$ is ω_1-n-simply presented.*

Proof. "$\Rightarrow$". It follows directly from Proposition 4.77.

"$\Leftarrow$". Let $p^\lambda G/T = p^\lambda(G/T)$ be n-simply presented for some countable subgroup T. Observe that

$$(G/T)/p^\lambda(G/T) = (G/T)/(p^\lambda G/T) \cong G/p^\lambda G$$

is n-simply presented. We therefore apply Theorem 4.4 from [D11] to get that G/T is n-simply presented, as desired. $\square$

Theorem 4.79. *Suppose G is a λ-boundary group for some ordinal λ such that $p^\lambda G$ is n-simply presented. Then G is ω_1-n-simply presented if, and only if, $G/p^\lambda G$ is ω_1-n-simply presented.*

Proof. **"Necessity".** It follows utilizing directly Proposition 4.77.

"Sufficiency". Given $(G/p^\lambda G)/(C/p^\lambda G) \cong G/C$ is n-simply presented for some countable $C/p^\lambda G$ with $C \leq G$. Write $C = p^\lambda G + K$ for some countable subgroup K. That is why $G/(p^\lambda G + K) \cong [G/K]/[(p^\lambda G + K)/K]$ is n-simply presented too. Now Theorem 3.4 (a) in [D11] can be used to derive that

$$[G/K/(p^\lambda G + K)/K]/p^\lambda(G/K/(p^\lambda G + K)/K) =$$

$$[G/K/(p^\lambda G + K)/K]/[p^\lambda(G/K)/(p^\lambda G + K)/K] \cong G/K/p^\lambda(G/K)$$

is n-simply presented. But on the other vein $(p^\lambda G + K)/K \cong p^\lambda G/(p^\lambda G \cap K)$, which leads with the help of Proposition 5.2 (a) in [D11] that they are n-simply presented because $p^\lambda G \cap K$ is countable. We therefore may employ Lemma 1.3

of [76] to infer that $p^\lambda(G/K)$ is n-simply presented. Next, the usage of Theorem 4.4 from [D11] guarantees that G/K is n-simply presented, as needed. □

Theorem 4.80. *Suppose G is a λ-countably group for some ordinal λ such that $p^\lambda G$ is n-simply presented. Then G is ω_1-n-simply presented if, and only if, $G/p^\lambda G$ is ω_1-n-simply presented.*

Proof. "$\Rightarrow$". Given G/K is an n-simply presented group for some countable $K \leq G$. Consequently, Theorem 3.4 (a) in [D11] forces that

$$G/K/p^\lambda(G/K) \cong [G/K/(p^\lambda G+K)]/p^\lambda(G/K)/(p^\lambda G+K)/K$$

$$= [G/K/(p^\lambda G+K)/K]/p^\lambda(G/K/(p^\lambda G+K)/K)$$

is also n-simply presented. Because of the countability of $p^\lambda(G/K)/(p^\lambda G + K)/K = p^\lambda(G/K/(p^\lambda G+K)/K)$, a simple appeal to Theorem 4.4 of [D11] leads to n-simply presentness of

$$[G/K]/[(p^\lambda G+K)/K] \cong G/(p^\lambda G+K) \cong [G/p^\lambda G]/[(p^\lambda G+K)/p^\lambda G].$$

And since $(p^\lambda G+K)/p^\lambda G \cong K/(K \cap p^\lambda G)$ is countable, we are done.

"$\Leftarrow$". Let $(G/p^\lambda G)/(C/p^\lambda G) \cong G/C$ be n-simply presented for some countable $C/p^\lambda G$ with $C \leq G$. Write $C = p^\lambda G + K$ for some countable subgroup K. So $G/(p^\lambda G + K) \cong [G/K]/[(p^\lambda G + K)/K]$ is n-simply presented; note that this gives with the aid of Theorem 3.4 (a) from [D11] that $p^\lambda(G/K/(p^\lambda G+K)/K) = p^\lambda(G/K)/(p^\lambda G+K)/K$ is n-simply presented - however we have by assumption that this quotient is countable. Moreover, again Theorem 3.4 (a) or Proposition 5.2 (a) in [D11] applies to conclude that

$$[G/K/(p^\lambda G+K)/K]/p^\lambda(G/K/(p^\lambda G+K)/K) \cong$$

$$[G/K/(p^\lambda G+K)/K]/p^\lambda(G/K)/(p^\lambda G+K)/K \cong (G/K)/p^\lambda(G/K)$$

is n-simply presented. But on the other hand $(p^\lambda G+K)/K \cong p^\lambda G/(p^\lambda G \cap K)$, which means by Proposition 5.2 (a) of [D11] that the second term, and hence the first one, are n-simply presented because $p^\lambda G \cap K$ is countable. We therefore may apply Lemma 4.55 to derive that $p^\lambda(G/K)$ is n-simply presented. Finally, utilizing Theorem 4.4 in [D11] to get after all that G/K is n-simply presented, as expected. □

Notice that we have not used in the necessity the condition that $p^\lambda G$ is n-simply presented, so that what immediately arises is whether or not this limitation can be dropped off in the formulation of the theorem.

Our next result generalizes in one way Corollary 4.68.

Proposition 4.81. *Suppose that G is a group such that $p^\lambda G$ is a dsc-group for some ordinal λ. If $G/p^\lambda G$ is ω_1-n-simply presented, then G is ω_1-n-simply presented.*

Proof. Let $(G/p^\lambda G)/(V/p^\lambda G) \cong G/V$ is n-simply presented for some countable factor $V/p^\lambda G$ with $V \leq G$. Thus using [37], V is a dsc-group as well. Assume that V' is a direct summand of V such that V/V' is countable. Hence $G/V \cong (G/V')/(V/V')$ being n-simply presented forces by Definition 1 that G/V' is ω_1-n-simply presented. Furthermore, suppose as before V" is a direct summand of V' such that V'/V" is countable. That is why, referring to Proposition 4.67, we conclude that G/V" remains ω_1-n-simply presented because $G/V' \cong (G/V")/(V'/V")$, etc. repeating the same procedure after a finite or infinite number of steps, we will obtain a countable subgroup $C \leq G$ such that G/C is ω_1-n-simply presented. A final employment of Proposition 4.67 assures the statement that G is ω_1-n-simply presented, after all. □

Proposition 4.82. *Let G be a group and α an ordinal. If G is nicely ω_1-n-simply presented, then $p^\alpha G$ and $G/p^\alpha G$ are nicely ω_1-n-simply presented.*

Proof. Let G/N be n-simply presented for some countable nice subgroup N of G. Hence, using [D11], $p^\alpha(G/N) = (p^\alpha G + N)/N \cong p^\alpha G/(p^\alpha G \cap N)$ is n-simply presented, where $p^\alpha G \cap N$ is countable and nice in $p^\alpha G$ (cf. [44]).

Moreover, $G/N/p^\alpha(G/N) \cong G/(p^\alpha G + N) \cong [G/p^\alpha G]/[(p^\alpha G + N)/p^\alpha G]$ is n-simply presented where $(p^\alpha G + N)/p^\alpha G \cong N/(p^\alpha G \cap N)$ is countable and $(p^\alpha G + N)/p^\alpha G$ is nice in $G/p^\alpha G$, because $N + p^\alpha G$ is so in G. □

An interesting consequence is the following one, extending Proposition 4.74 in some aspect.

Corollary 4.83. *Suppose that G is a group such that $p^\lambda G$ is a Σ-cyclic group for some ordinal λ. Then G is nicely ω_1-n-simply presented if, and only if, $G/p^\lambda G$ is nicely ω_1-n-simply presented.*

Proof. The "necessity" follows immediately from Proposition 4.82.

To treat the "sufficiency", let $(G/p^\lambda G)/(N/p^\lambda G) \cong G/N$ be n-simply presented for some countable nice subgroup $N/p^\lambda G$ with $N \leq G$. Thus $N = p^\lambda G + K$ is nice in G for some countable $K \leq N$. Since $p^\lambda G$ is a Σ-cyclic group, appealing to [37] we have that N is a direct sum of a countable group and a Σ-cyclic group; in fact writing $p^\lambda G = C_1 \oplus C_2$ where C_1 is Σ-cyclic, and C_2 is countable with $p^\lambda G \cap K \subseteq C_2$, one plainly follows that $N = C_1 \oplus (C_2 + K)$. Henceforth, $G/N \cong (G/C_1)/(N/C_1)$ being n-simply presented, where $N/C_1 \cong C_2 + K$ is countable and nice in G/C_1, yields by Definition 2 that G/C_1 is nicely ω_1-n-simply presented. Note that C_1 remains nice in G because it is nice in $p^\lambda G$. Referring to Proposition 4.71, G/C_1' remains nicely ω_1-n-simply presented, where $G/C_1 \cong (G/C_1')/(C_1/C_1')$ and C_1/C_1' is finite such that $C_1 \cong C_1' \oplus (C_1/C_1')$, etc. repeating the same procedure, after a finite or infinite number of steps, we will obtain a finite subgroup $F \leq G$ such that G/F is nicely ω_1-n-simply presented. A final application of Proposition 4.71 ensures the assertion that G is nicely ω_1-n-simply presented, after all. □

Proposition 4.84. *Suppose that $G/p^\lambda G$ is n-simply presented for some ordinal λ. Then G is nicely ω_1-n-simply presented if, and only if, $p^\lambda G$ is nicely ω_1-n-simply presented.*

Proof. The "and only if" part follows by a direct application of Proposition 4.82.

As for the "if" part, let $p^\lambda G/Y = p^\lambda(G/Y)$ be n-simply presented for some nice countable subgroup Y. Hence Y is also nice in G (see, e.g., [44]), and besides $G/p^\lambda G \cong (G/Y)/p^\lambda(G/Y)$ is n-simply presented by assumption. Now the application of Theorem 4.4 of [D11] leads us to G/Y is n-simply presented, as wanted. □

Remark 5. As a final notice, we emphasize that the same types results concerning Ulm subgroups and Ulm factors can be formulated and proved also for strongly (nice) ω_1-n-simply presented groups and strongly ω-n-simply presented groups, but we omit their representations in order to avoid the similarity of the considerations.

In closing, we shall state some left-open problems that still elude us (see also the listed problems at the end of the present dissertation which mainly treat analogous problematic).

Problem 1. Does it follow that if G is an ω_1-n-simply presented group with $p^\omega G = \{0\}$, then it is $p^{\omega+n}$-projective?

Problem 2. Let α be an ordinal. Does it follow that G is (nicely) ω_1-n-simply presented if and only if both $p^\alpha G$ and $G/p^\alpha G$ are (nicely) ω_1-n-simply presented?

Problem 3. Let α be an ordinal. Does it follow that G is strongly (nice) ω_1-n-simply presented if and only if $p^{\alpha+n}G$ and $G/p^{\alpha+n}G$ are both strongly (nice) ω_1-n-simply presented?

Problem 4. If α is an ordinal such that $p^{\alpha}G$ is countable, what is the structure of $p^{\alpha}(G/C)$ where $C \leq G$ is a countable subgroup? Does it follow that it is countable as well, or is simply presented, or something else?

Chapter V. Left-Open Problems

We shall here state some still unsettled intriguing questions as well as we shall restate for completeness of the exposition some already putted queries in the corresponding subsections quoted above.

Concerning ring theory (possibly non-commutative), recall that a ring R is said to be π-*regular* if, for each $a \in R$, there is a natural number n (depending on a) such that $a^n \in a^nRa^n$.

Problem 4.85. Does it follow that all weakly exchange (respectively, all exchange) rings whose units are sums of two idempotents are π-regular?

Problem 4.86. Suppose that R is a ring with $R = Id(R) + Id(R)$ such that $U(R) = 1 + Nil(R)$. Is it true that R is (weakly) exchange or even π-regular?

We close the queries on ring theory with

Problem 4.87. Let R be a ring and G a group. Is the group ring $R[G]$ a UU-ring iff R is a UU-ring and G is a 2-group? If *not*, find a necessary and sufficient condition for $R[G]$ to be UU only in terms of R, G and their sections.

Concerning Abelian group theory, we finish off our work with a few challenging problems of certain interest and importance, some of which are also relevantly stated for concreteness in the separate subsections alluded to above. So, we ask for the following:

Problem 4.88. Are reduced simply presented p-groups necessarily projectively fully transitive?

Problem 4.89. Suppose $n \in \mathbb{N}$. If the direct sum $G \oplus H$ is a strongly n-simply presented group for two groups G and H such that H is countable, does the complement G is also strongly n-simply presented?

We end all the work with our final query.

Problem 4.90. In the presence of **ZFC**, if G is a proper $(\omega + n)$-totally $p^{\omega+n}$-projective p-group for some $n \in \mathbb{N}$, does it follow that $p^\omega G$ is necessarily countable?

Chapter VI. References/Bibliography

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