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THE JNR PROPERTY AND THE BOREL STRUCTURE OF A BANACH SPACE

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Abstract. In this paper we study the property of having a countable cover by sets of small local diameter (SLD for short). We show that in the context of Banach spaces (JNR property) it implies that the Banach space is Čech-analytic. We also prove that to have the JNR property, to be σ-fragmentable and to have the same Borel sets for the weak and the norm topologies, they all are topological invariants of the weak topology. Finally, by means of "good" injections into $c_0(\Gamma)$, we give a great class of Banach spaces with the JNR property.

Introduction. Let us begin with two definitions from [13].

Definition 0.1. *Let (X,τ) be a topological space and let d be a metric on X. The space X is said to be σ-fragmented by the metric d if, foreach $\varepsilon > 0$,*

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it is possible to write

$$X = \bigcup_{i=1}^{\infty} X_i^{\varepsilon},$$

where each set X_i^{ε} *has the property that each non-empty subset of* X_i^{ε} *has a non-empty relatively open subset of* d*-diameter less than* ε*.*

When X *is a Banach space,* τ *is the weak topology and* d *is the* $\|\cdot\|$*, we shall say that* X *is* σ*-fragmentable.*

Definition 0.2. *Let* (X, τ) *be a topological space and let* d *be a metric on* X*. It is said that* X *has a countable cover by set of small local diameter (SLD) if for every* $\varepsilon > 0$ *there exists a decomposition*

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for each $n \in \mathbb{N}$ *every point of* X_n^{ε} *has a relatively* τ*-neighbourhood of* d*-diameter less than* ε*.*

When X *is a Banach space,* τ *is the weak topology and* d *is the* $\|\cdot\|$*, we shall say that* X *has the JNR property.*

The notion of σ-*fragmentable* topological spaces, was introduced and studied by Jayne, Namioka and Rogers, in a series of papers [13, 14, 15], arriving at the concept of spaces having a countable cover by sets of small local diameter, what we call *SLD* property, which has been studied in [13, 18, 19, 22, 23].

A norm on a Banach space is said to be a *Kadec* norm if the weak and the norm topologies agree on the unit sphere. In [3, 4], Edgar shows that in a Banach space that admits an equivalent Kadec norm the following hold:

1. $Borel(X, \|\cdot\|) = Borel(X, weak)$.

2. $X \in Borel(X^{**}, w^*)$.

In [8], Hansell introduced the concept of *descriptive* Banach spaces as those that the norm has a *network* which is σ-relatively discrete with respect to the weak topology. He shows that spaces with an equivalent Kadec norm are descriptive and that in descriptive Banach spaces 1) and 2) above hold, hence improving Edgar's results.

In [13], Jayne, Namioka and Rogers show that a Banach space with an equivalent Kadec norm has the JNR property.

In this paper we present as main results the following:

Theorem A. *Let X be a Banach space with the JNR property, then properties 1) and 2) above hold.*

Theorem B. *On a Banach space X, the following properties are topological invariants of the weak topology: to be σ-fragmentable, to have the JNR property, to have the same Borel sets for the weak and the norm topologies.*

For the invariance of the σ-fragmentability see [8, 20].

As a result of our Proposition 1.9, it is easy to show that for a Banach space having the JNR property and being descriptive turn out to be equivalent [19], p. 257. It must be said that all known σ-fragmentable Banach spaces have the JNR property [2, 8].

For recent results on the relationship between the JNR property and renormability of the space see [10, 18, 19, 23].

1. The JNR property. Let X be a topological space. As usual, we shall denote by $\mathcal{F}$ the family of closed subsets of X and $\mathcal{G}$ the open sets, and by $\mathcal{F}_\sigma$ countable unions of sets from $\mathcal{F}$. Our first result shows that for two metrics, σ-fragmentability and SLD are equivalent.

Proposition 1.1. *Let (X,d) be a metric space and ϱ be another metric defined on X. The following conditions are equivalent:*

i) (X,d) is σ-fragmented by ϱ;

ii) (X,d) has ϱ-SLD.

When the sets in i) can be taken to be differences of d-closed sets (or more generally d-$\mathcal{F}_\sigma$-sets), then the sets in ii) can be taken to be d-$\mathcal{F}_\sigma$-sets.

Proof. ii)$\Rightarrow$ i) Is clear by definition.

i)$\Rightarrow$ ii) Given $\varepsilon > 0$, let $\{C_i^\varepsilon\}_{i\in\mathbb{N}}$ be a cover of X given by the σ-fragmentability of the space.

Fix $i \in \mathbb{N}$. Because of the σ-fragmentability, there exists a family of d-open sets, $\{U_\alpha^i : 0 \le \alpha < \mu_i\}$, covering C_i^ε such that for $0 \le \alpha < \mu_i$ we have

$$C_i^\varepsilon \cap U_\alpha^i \setminus \bigcup_{0\le\beta<\alpha} U_\beta^i \neq \emptyset, \text{ and } \varrho\text{-diam}(C_i^\varepsilon \cap U_\alpha^i \setminus \bigcup_{0\le\beta<\alpha} U_\beta^i) < \varepsilon.$$

For each $n, i \in \mathbb{N}$ and $\alpha \in [0,\mu_i)$, let $F_{\alpha,i}^n = \{x \in X : d(x, X\setminus U_\alpha^i) \ge \frac{1}{n}\}$ and $H_{\alpha,i}^n = (C_i^\varepsilon \cap F_{\alpha,i}^n \setminus \bigcup_{0\le\beta<\alpha} U_\beta^i)$. It is clear that ϱ-diam$(H_{\alpha,i}^n) < \varepsilon$. Now for $\alpha \neq \beta$ the sets $H_{\alpha,i}^n$ and $H_{\beta,i}^n$, when non-empty, are separated by a d-distance at least $\frac{1}{n}$. So for each $n \in \mathbb{N}$ the family $\{H_{\alpha,i}^n : 0 \le \alpha < \mu_i\}$ is discrete in (X,d).

Set $H_i^n = \cup\{H_{\alpha,i}^n : 0 \leq \alpha < \mu_i\}$. We have

$$\bigcup_{n=1}^{\infty} H_i^n = \bigcup_{n=1}^{\infty} \bigcup_{0\leq\alpha<\mu_i} H_{\alpha,i}^n = \bigcup_{n=1}^{\infty} \bigcup_{0\leq\alpha<\mu_i} (C_i^\varepsilon \cap F_{\alpha,i}^n \setminus \bigcup_{0\leq\beta<\alpha} U_\beta^i) =$$

$$= \bigcup_{0\leq\alpha<\mu_i} (C_i^\varepsilon \cap U_\alpha^i \setminus \bigcup_{0\leq\beta<\alpha} U_\beta^i) = C_i^\varepsilon,$$

and therefore

$$X = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} H_i^n.$$

Let us see that for each $n, i \in \mathbb{N}$ the set H_i^n has local ϱ-diameter less than ε. Take $x \in H_i^n$. So for some $\alpha \in [0, \mu_i)$ $x \in H_{\alpha,i}^n$.

Since the family $\{H_{\alpha,i}^n : 0 \leq \alpha < \mu_i\}$ is discrete in (X, d) there must be a d-open neighbourhood V of x such that $V \cap H_{\beta,i}^n = \emptyset$ for $\beta \neq \alpha$. So

$$\varrho\text{-diam}(V \cap H_i^n) = \varrho\text{-diam}(V \cap H_{\alpha,i}^n) < \varepsilon.$$

Now if the C_i are differences of d-closed sets, and since the $F_{\alpha,i}^n$ are d-closed, we have that the $H_{\alpha,i}^n$ are differences of d-closed sets. So each H_i^n is the discrete union of sets which are differences of d-closed sets. Since d-open sets are F_σ-sets, it follows that the H_i^n's are also F_σ-sets. $\square$

Our first example of a Banach space with the JNR property can be used to prove the JNR property in several cases. If (X_γ, τ_γ) is a family of topological spaces we denote by τ_{pro} the product topology of these spaces.

Proposition 1.2. *Let $(X_\gamma, \|\cdot\|_\gamma)$ be a family of normed spaces (not necessarily complete) and τ_γ topologies defined on them. If for each $\gamma \in \Gamma$ the space (X_γ, τ_γ) is $\|\cdot\|_\gamma$-SLD, then $(\Sigma_{c_0}\{X_\gamma : \gamma \in \Gamma\}, \tau_{pro})$ is $\|\cdot\|_\infty$-SLD.*

Proof. Set $X = \Sigma_{c_0}\{X_\gamma : \gamma \in \Gamma\}$.

Given $\varepsilon > 0$, let $(A_\gamma^{n,p})_{n=1}^\infty$ be countable covers of X_γ by sets of local $\|\cdot\|_\gamma$-diameter less than $\frac{1}{p} < \varepsilon$.

For each $m, k \in \mathbb{N}$ with $\frac{1}{m} < \frac{\varepsilon}{4}$, let us define

$$X_k^m = \{x \in X : |\gamma \in \Gamma : \|x_\gamma\|_\gamma > \frac{1}{m}| = k\}.$$

Given $x \in X_k^m$ define $A_{m,k}(x) = \{\gamma \in \Gamma; \|x_\gamma\|_\gamma > \frac{1}{m}\}$. Now for all $p \in \mathbb{N}$ with $\frac{1}{p} < \varepsilon$, set

$$X_k^{m,p} = \{x \in X_k^m; \|x_{\gamma_i}\|_{\gamma_i} - \frac{1}{p} > \frac{1}{m},\ \text{for } \gamma_i \in A_{m,k}(x)\}.$$

$$A^{m,k,p}_{n_1,\dots,n_k} = \{x \in X^{m,p}_k; x_{\gamma_i} \in A^{n_i,p}_{\gamma_i}, \text{ for all } \gamma_i \in A_{m,k}(x)\}.$$

By the definition of X, it is easy to see that

$$X = \bigcup_{m,p;\frac{1}{m},\frac{1}{p}<\frac{\varepsilon}{2}} \bigcup_{k=1}^{\infty} \bigcup_{(n_1,\dots,n_k)\in\mathbb{N}^{[\mathbb{N}]}} A^{m,k,p}_{n_1,\dots,n_k}$$

Let us see now that it is a cover by sets of local $\|\cdot\|_\infty$-diameter less than ε.

So, take any $m, k, p, n_1, \dots, n_k$, with $\frac{1}{m}, \frac{1}{p} < \varepsilon$, and $x \in A^{m,k,p}_{n_1,\dots,n_k}$. By definition, there exist $\gamma_1, \dots, \gamma_k \in A_{m,k}(x)$ and τ_{γ_i}-open neighbourhoods of x_{γ_i}, say $U_{\gamma_i} \subset X_{\gamma_i}$ with $\|\cdot\|_{\gamma_i}$-diam$(U_{\gamma_i} \cap A^{n,p}_{\gamma_i}) < \frac{1}{p}$.

If we denote by P_γ the canonical projection onto the space X_γ, the set $V = P^{-1}_{\gamma_1}(U_{\gamma_1}) \cap \dots P^{-1}_{\gamma_k}(U_{\gamma_k})$ is τ_{pro}-open and $\|\cdot\|_\infty$-diameter$(V \cap A^{m,k,p}_{n_1,\dots,n_k}) < \varepsilon$.

To show the last claim take $x, y \in V \cap A^{m,k,p}_{n_1,\dots,n_k}$. For $\gamma \in A_{m,k}(x)$ we have

$$\|y_\gamma\|_\gamma > |\|x_\gamma\|_\gamma - \|x_\gamma - y_\gamma\|_\gamma| > \frac{1}{m}$$

which implies that $A_{m,k}(y) = A_{m,k}(x)$. So for $\gamma \in A_{m,k}(x)$ we have

$$\|x_\gamma - y_\gamma\|_\gamma < \frac{1}{p} < \frac{\varepsilon}{2}$$

and if $\gamma \notin A_{m,k}(x)$ we obtain

$$\|x_\gamma - y_\gamma\|_\gamma \leq \|x_\gamma\|_\gamma + \|y_\gamma\|_\gamma \leq \frac{1}{m} + \frac{1}{m} \leq \frac{\varepsilon}{2}$$

as we wanted. □

In [13], the authors showed that if a Banach space has an equivalent Kadec norm, then it has the JNR property by sets that are differences of weakly closed sets. As a corollary of our next result we can see that it always happens whenever the space has the JNR property. Also, Theorem 1.3 will be needed when proving the third assertion in Theorem B.

Theorem 1.3. *Let (X, τ) be a topological space and ϱ be a lower semi-continuous metric on X. If (X, τ) has the ϱ-SLD property, then (X, τ) has the ϱ-SLD property by differences of τ-closed sets. Moreover, if the ϱ-topology is finer than τ, then the sets can be taken to be ϱ-closed.*

Proof. Given $\varepsilon > 0$, there exists a sequence of sets X^ε_n covering the space, having local ϱ-diameter less than ε. Let us define the following sets

$$Y^\varepsilon_n = \{x \in \overline{X^\varepsilon_n}^\tau : \text{ exists } U \in \tau, x \in U \text{ with } \varrho - diam(\overline{X^\varepsilon_n}^\tau \cap U) \leq \varepsilon\}$$

We claim that the sequence $(Y_n^\varepsilon)_{n=1}^\infty$ is a countable cover of X by differences of τ-closed sets of local diameter less than ε.

To show that they cover X, take $x \in X$. There exists $n \in \mathbb{N}$ and $U \in \tau$, open neighbourhood of x such that $x \in X_n^\varepsilon \cap U$ having this set diameter less than ε. Since ϱ is τ-lower semi-continuous we have $\varrho - diam(\overline{X_n^\varepsilon \cap U}^\tau) \leq \varepsilon$ and since U is τ-open, $\overline{X_n^\varepsilon}^\tau \cap U \subset \overline{X_n^\varepsilon \cap U}^\tau$. Therefore $\varrho - diam(\overline{X_n^\varepsilon}^\tau \cap U) \leq \varepsilon$ which implies $x \in Y_n^\varepsilon$. So $X = \cup_n Y_n^\varepsilon$.

If $x \in Y_n^\varepsilon$ there exists U such that

$$\varrho - diam(Y_n^\varepsilon \cap U) \leq \varrho - diam(\overline{X_n^\varepsilon \cap U}^\tau) \leq \varepsilon.$$

Now for each $x \in Y_n^\varepsilon$ call U_x the open set given by the definition of Y_n^ε. It is easy to see that

$$Y_n^\varepsilon = \overline{X_n^\varepsilon}^\tau \cap \bigcup_{x \in Y_n^\varepsilon} U_x$$

which shows that Y_n^ε is the difference of τ-closed sets.

If $\tau \preceq \varrho$, since $Y_n^\varepsilon = F_n^\varepsilon \cap G_n^\varepsilon$ with F_n^ε τ-closed (hence ϱ-closed) and G_n^ε τ-open (hence ϱ-open), and since in a metric space open sets are $\mathcal{F}_\sigma$ sets, $Y_n^\varepsilon = F_n^\varepsilon \cap (\cup_m F_{m,n}^\varepsilon) = \cup_m (F_n^\varepsilon \cap F_{m,n}^\varepsilon)$.

So $X = \cup_{m,n}(F_n^\varepsilon \cap F_{m,n}^\varepsilon)$ being these sets ϱ-closed and of local diameter less than ε. $\square$

In the next proof we will need some definitions on families of sets. So, recall that a family of subsets $\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma}$, of a topological space: is called *discrete* if every point of the space, has an open neighbourhood that meets, at most, one element of the family; *isolated* if it is discrete in its union; *σ-discretely (isolatedly) decomposable* if each $A_\gamma = \cup_n \{A_\gamma^n\}$ so that each $\{A_\gamma^n\}_{\gamma\in\Gamma}$ is discrete (isolated). In a metric space (X, d) a family $\mathcal{A} = \{A_\gamma\}_{\gamma\in\Gamma}$ is called ε-discrete (or metrically discrete with separating distance ε) if $d(x, y) \geq \varepsilon$ whenever $x \in A_\gamma$, $y \in A_\beta$, with $\gamma \neq \beta$.

As usual, we denote by $\mathcal{F} \cap \mathcal{G}$ the sets which are the intersection of a closed and an open set.

Theorem 1.4. *Let X be a Banach space with the JNR property. Then X is a $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$ in $(X^{**}, weak^*)$ and therefore $X \in Borel(X^{**}, w^*)$. Moreover, any $\|\cdot\|$-closed subset of X is of the same type.*

P r o o f. For each $p \in \mathbb{N}$, let $\{U_{\alpha,n}^p : \alpha \in \Gamma_{n,p}\}$ be refinement of a cover of X by balls of diameter less than $\frac{1}{p}$ such that for all $n \in \mathbb{N}$ the families $\{U_{\alpha,n}^p : \alpha \in \Gamma_{n,p}\}$ are $\|\cdot\|$-discrete. We may suppose that each $\{U_{\alpha,n}^p : \alpha \in \Gamma_{n,p}\}$ is a metrically discrete family with separating distances $\delta_{n,p} > 0$.

For $n,p\in\mathbb{N}$, let $\{C^p_{n,m}:m\in\mathbb{N}\}$ be a countable cover of X with the local diameter of each $C^p_{n,m}<\delta_{n,p}$. Then for each $n,m,p\in\mathbb{N}$ the family $\{U^p_{\alpha,n}\cap C^p_{n,m}\}_{\alpha\in\Gamma_{n,p}}$ is weakly isolated, in fact it is weakly discrete in $C^p_{n,m}$. So if we look at X as a subset of (X^{**},w^*), the family $\{U^p_{\alpha,n}\cap C^p_{n,m}\}_{\alpha\in\Gamma_{n,p}}$ is w^*-isolated. Thus, for each $x\in U^p_{\alpha,n}\cap C^p_{n,m}$ there exists a w^*-open neighbourhood of x in X^{**}, say $U^{n,m,p}_{x,\alpha}$, such that

$$U^{n,m,p}_{x,\alpha}\cap(U^p_{\alpha,n}\cap C^p_{n,m})\neq\emptyset,\ \text{and}$$

$$U^{n,m,p}_{x,\alpha}\cap(U^p_{\beta,n}\cap C^p_{n,m})=\emptyset,\ \text{for}\ \beta\neq\alpha.$$

Set

$$G^{m,p}_{\alpha,n}=\bigcup_{x\in U^p_{\alpha,n}\cap C^p_{n,m}}U^{n,m,p}_{x,\alpha}.$$

$G^{m,p}_{\alpha,n}$ is clearly a w^*-open set with $G^{m,p}_{\alpha,n}\supset(U^p_{\alpha,n}\cap C^p_{n,m})$ and for $\beta\neq\alpha$, $G^{m,p}_{\alpha,n}\cap(U^p_{\beta,n}\cap C^p_{n,m})=\emptyset$. Set

$$M_{\alpha,n,m,p}=\overline{(U^p_{\alpha,n}\cap C^p_{n,m})}^{\,w^*}\cap G^{m,p}_{\alpha,n}.$$

We show that

$$X=\bigcap_p\bigcup_m\bigcup_n\{\bigcup_{\alpha\in\Gamma_{n,p}}M_{\alpha,n,m,p}\}.$$

Notice that, since $\operatorname{diam}(U^p_{\alpha,n}\cap C^p_{n,m})\leq\frac{1}{p}$, we have

$$\operatorname{diam}(\overline{U^p_{\alpha,n}\cap C^p_{n,m}}^{\,w^*})\leq\frac{1}{p}.$$

Let

$$x^{**}\in X^{**}\cap\bigcap_p\bigcup_m\bigcup_n(\bigcup_{\alpha\in\Gamma_{n,p}}M_{\alpha,n,m,p}).$$

For each $p\in\mathbb{N}$ there exist $n,m\in\mathbb{N}$ and $\alpha\in\Gamma_{n,p}$ such that

$$x^{**}\in\overline{(U^p_{\alpha,n}\cap C^p_{n,m})}^{\,w^*}\cap G^{p,m}_{\alpha,n},$$

so there exists $x_p\in U^p_{\alpha,n}\cap C^p_{n,m}$ such that $\|x^{**}-x_p\|\leq\frac{1}{p}$. Thus we have

$$\|\cdot\|\text{-}\lim_{p\to\infty}x_p=x^{**}$$

and therefore $x^{**}\in X$.

Now since the family $\{M_{\alpha,n,m,p} : \alpha \in \Gamma_{n,p}\}$ is isolated and it consists of $\mathcal{F} \cap \mathcal{G}$ sets, $\cup\{M_{\alpha,n,m,p} : \alpha \in \Gamma_{n,p}\}$ is also a $\mathcal{F} \cap \mathcal{G}$ set ([8, Lemma 3.3]).

If F is a norm closed subset of X, consider the families

$$\{F \cap U^p_{\alpha,n} \cap C^p_{n,m}\}_{\alpha \in \Gamma_{n,p}}$$

and follow the proof. In this case the vectors x_p's belong to F and, since F is closed, the limit also belongs to F. □

Proof of Theorem A. 2) follows from Theorem 1.4. Let us prove 1).

To do so, let A be a norm-closed subset of X. By Theorem 1.4, $A \in Borel(X^{**}, weak^*)$, i.e. there exists $B \subset X^{**}$, $B \in Borel(X^{**}, weak^*)$ and $A = B \cap X$. But the weak topology on X coincides with the restriction to X of the weak* topology on X^{**} and therefore A is a weak-Borel subset of X. □

In fact, 1) in Theorem A follows also from the following result, from where we will be able to specify the class of any norm open set. We would like to mention that Proposition 1.5 will be used also in the proof of Theorem B.

Proposition 1.5. *Let (X, τ) be a topological space with the ϱ-SLD property by differences of τ-closed sets, for some metric ϱ. Then every ϱ-open set is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set in (X, τ), that is, a countable union of differences of τ-closed sets.*

Proof. For every $n \in \mathbb{N}$, let $(X_{n,m})_{m\in\mathbb{N}}$ be a countable cover of X by sets of local ϱ-diameter less than $\frac{1}{n}$, and with each $X_{n,m}$ being the difference of two τ-closed sets. Reordering the indexes, call them $(X_r)_{r\in\mathbb{N}}$. Now for each $n \in \mathbb{N}$, define $\Gamma(n) = \{r \in \mathbb{N}; X_r$ has local ϱ-diameter less than $\frac{1}{n}\}$. It is easy to see that for each $n \in \mathbb{N}$, the set $\cup\{X_r; r \in \Gamma(n)\}$ covers X.

Now, let G be a norm open set. Set $G_n = \{x \in X; B_\varrho(x; \frac{1}{n}) \subset G\}$.

Fix $n \in \mathbb{N}$. For each $r \in \Gamma(n)$, and every $x \in G_n \cap X_r$, since the set X_r has local ϱ-diameter less than $\frac{1}{n}$, there exists a τ relatively open subset $U(x)$ of X_r containing x, such that, ϱ-diam$(U(x)) < \frac{1}{n}$. Thus $x \in U(x) \subset G$. Hence, for $r \in \Gamma(n)$, the set

$$U_{n,r} = \cup\{U(x) : x \in G_n \cap X_r\}$$

is a relatively open subset of X_r. So $U_{n,r}$ is a $(\mathcal{F} \cap \mathcal{G})$ set in τ. It contains $G_n \cap X_r$ and is contained in G. Now the set $\cup\{U_{n,r}; n \in \mathbb{N}, r \in \Gamma(n)\}$ coincides with G and it is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set in τ. □

Corollary 1.6. *Let X be a Banach space with the JNR property. Then any $\|\cdot\|$ open subset of X is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set in the weak topolgy.*

In particular, $Borel(X, \|\cdot\|) = Borel(X, weak)$.

Problem 1.7. *Let X be a Banach space such that any norm open set is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set for the weak topology. Does it have the JNR property?*

The proof of Proposition 1.5 can be adapted to give the following:

Proposition 1.8. *Let (X, τ) be a topological space such that any open set is an $\mathcal{F}_\sigma$-set. Let ϱ be a metric on X and suppose that X has the SLD property by sets of additive class α, then each ϱ-open subset of X is of additive class α.*

One way to characterize the SLD property is through discrete families.

Proposition 1.9. *Let (X, τ) be a topological space and d a metric on it. Then (X, τ) has d-SLD if, and only if, any d-discrete family of subsets of X is σ-isolatedly decomposable for τ.*

Proof. Let us assume that (X, τ) has d-SLD. For each $n \in \mathbb{N}$, consider $X = \cup\{C_i^n; i \in \mathbb{N}\}$, where the sets C_i^n have local d-diameter less than $\frac{1}{n}$.

Let $\mathcal{A} = \{A_s\}_{s\in S}$ be a discrete family in (X, d). Then there exist $\frac{1}{m}$-discrete families $\{A_s^m\}_{s\in S}$ such that $A_s = \cup\{A_s^m; m \in \mathbb{N}\}$, for $s \in S$.

Write $\{A_s^{i,m}\} = A_s^m \cap C_i^m$, for $m, i \in \mathbb{N}$ and $s \in S$ and fix $i, m \in \mathbb{N}$. Take $x \in C_i^m$. There exists a τ-open neighbourhood of x, say U, such that $\operatorname{diam}(U \cap C_i^m) < \frac{1}{m}$. Therefore U meets at most one element of the family $\{A_s^{i,m}\}_{s\in S}$. So $\{A_s^{i,m}\}_{s\in S}$ is discrete in (C_i^m, τ) and we have

$$A_s = \bigcup_{m=1}^{\infty} A_s^m = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{\infty} A_m^s \cap C_i^m = \bigcup_{i,m} A_s^{i,m}.$$

Hence $\mathcal{A}$ is σ-isolatedly decomposable.

Let us show the converse. Given $\varepsilon > 0$, let $\{U_\alpha^n : \alpha \in \Gamma\}$ be a σ-discrete open refinement of an open cover of X by balls of radius less than $\frac{\varepsilon}{2}$. By hypothesis, for each $n \in \mathbb{N}$, $U_\alpha^n = \cup\{B_\alpha^{n,m} : m \in \mathbb{N}\}$, where for each $n, m \in \mathbb{N}$ the family $\{B_\alpha^{n,m}\}_{\alpha\in\Gamma}$ is τ-isolated.

Write $F_m^n = \cup\{B_\alpha^{n,m} : \alpha \in \Gamma\}$; obviously, $X = \cup\{F_m^n : n, m \in \mathbb{N}\}$.

We now show that for each $n, m \in \mathbb{N}$, the set F_m^n has local diameter less than ε.

Take $x \in F_m^n$. Then $x \in B_{\alpha_0}^{n,m}$ and therefore there exists a τ-open neighbourhood U of x such that $U \cap B_\alpha^{n,m} = \emptyset$ for $\alpha \neq \alpha_0$. So $\operatorname{diam}(U \cap F_m^n) = \operatorname{diam}(U \cap B_{\alpha_0}^{n,m}) \leq \operatorname{diam}(B_{\alpha_0}^{n,m}) < \varepsilon$. $\square$

This characterization leads us to the following concept introduced by Hansell [8].

Definition 1.10. *Let X be a Banach space. We shall say that X is descriptive if the norm topology admits a network which is σ-isolated for the weak topology.*

The concept of *network* was introduced by Arkangel'skii. By a *network* for a space X we mean any collection $\mathcal{M}$ of subsets of X (not necessarily open) such that each open set is a union of sets from $\mathcal{M}$.

Remark 1.11. It is not difficult to show after Proposition 1.9 that on a Banach space, the JNR property and being descriptive are equivalent notions [19]. Hansell showed in [8], that a Banach space is descriptive if, and only if, the weak topology has a σ-isolated network. Thus showing that being descriptive depends only on the weak topology of the space.

2. SLD maps. That characterization of the SLD property in terms of discrete decompositions of families leads us to the study the following class of maps.

Definition 2.1. *Let (X, ϱ) and (Y, d) be metric spaces. We shall say that a one-to-one map $T : X \to Y$ is a (ϱ, d)-SLD map if it maps ϱ-discrete families of subsets of X into d-σ-discretely decomposable families of subsets of Y.*

Under the conditions of Definition 2.1, if we denote by T_d the topology (metric) on X generated by the sets $T^{-1}(U)$ with $U \subset Y$ d-open, we can prove:

Proposition 2.2. *Let $T : (X, \varrho) \to (Y, d)$ be a one-to-one map between metric spaces. T is SLD if, and only if (X, T_d) has ϱ-SLD.*

P r o o f. The proof follows the same line as in Proposition 1.9 and the fact that in any metric space σ-isolatedly decomposable families are σ-discretely decomposable (Hansell [6], Remark 1.3).

Theorem 2.3. *Let X, Y be two Banach spaces and $f : X \longrightarrow Y$ be a continuous linear injection. Define $\varphi = f^{-1} : f(X) \longrightarrow X$. Then the following conditions are equivalent:*

i) φ is of Borel class α and f is SLD;

ii) $(X, f_{\|\cdot\|_Y})$ has the $\|\cdot\|_X$-SLD by sets of additive class α (for the topology $f_{\|\cdot\|_Y} = \{f^{-1}(U); U \text{ is } \|\cdot\|_Y-\text{open in } Y\}$).

P r o o f. ii)$\Rightarrow$ i) By Proposition 2.2 we have that f is SLD.

Now let G be a norm open subset of X. Since the topology $f_{\|\cdot\|_Y}$ in X verifies that any open set is an $\mathcal{F}_\sigma$ set, we apply Proposition 1.8 and obtain that G is of additive class α in $(X, f_{\|\cdot\|_Y})$. Hence the set $\varphi^{-1}(G)$ is of additive class α in $(f(X), \|\cdot\|_Y)$ and therefore φ is of Borel class α.

i)$\Rightarrow$ ii) Let us assume $\alpha > 0$ (for $\alpha = 0$ is obvious). Given $\varepsilon > 0$, let $\{U_\gamma^n : \gamma \in A\}$ be an open refinement of an open cover of X by balls of radius less than $\frac{\varepsilon}{2}$, such that for all $n \in \mathbb{N}$, the family $\{U_\gamma^n : \gamma \in A\}$ is norm discrete. Consider $B_\gamma^n = f(U_\gamma^n)$. Then for each $n \in \mathbb{N}$ the family $\{B_\gamma^n\}_{\gamma\in A}$ is σ-discretely decomposable, so for every $n, i \in \mathbb{N}$ there exist discrete families $\{B_\gamma^{i,n}\}_{\gamma\in A}$ such that

$$B_\gamma^n = \bigcup_{i=1}^{\infty} B_\gamma^{i,n}.$$

Note that since the sets U_γ^n are open in X and φ is of Borel class α, the sets $B_\gamma^n = \varphi^{-1}(U_\gamma^n)$ are of additive class α.

Now fix $i, n \in \mathbb{N}$. The family $\{\overline{B_\gamma^{i,n}} \cap B_\gamma^n\}_{\gamma\in A}$ is discrete in $f(X)$ and its sets are of additive class α.

Set

$$\Xi_{i,n} = \bigcup_{\gamma\in A} (\overline{B_\gamma^{i,n}} \cap B_\gamma^n).$$

Then $\Xi_{i,n}$ is a discrete union of sets of additive class α and therefore is itself of additive class α. (For a proof of this fact see [17], page 358, Theorem 1). It is obvious that

$$f(X) = \bigcup_{i,n} \Xi_{i,n}.$$

Define $C_{i,n} = f^{-1}(\Xi_{i,n})$. The sets $C_{i,n}$'s are of additive class α and they form a countable cover of $(X, f_{\|\cdot\|_Y})$.

Now take $x \in C_{i,n}$ and $f(x) = y \in \Xi_{i,n}$. So $y \in \overline{B_{\gamma_0}^{i,n}} \cap B_{\gamma_0}^n$ for only one $\gamma_0 \in A$. Thus there exists an open neighbourhood U of y in Y such that $U \cap (\overline{B_\gamma^{i,n}} \cap B_\gamma^n) = \emptyset$ for $\gamma \neq \gamma_0$.

Write $V = f^{-1}(U)$. V is a $f_{\|\cdot\|_Y}$-open neighbourhood of x and

$$\operatorname{diam}(V \cap C_{i,n}) = \operatorname{diam}(f^{-1}(U \cap \Xi_{i,n})) = \operatorname{diam}(f^{-1}(U \cap \overline{B_{\gamma_0}^{i,n}} \cap B_{\gamma_0}^n)) \leq$$

$$\leq \operatorname{diam}(f^{-1}(B_\gamma^n)) = \operatorname{diam}(U_\gamma^n) < \varepsilon.$$

Therefore $C_{i,n}$ has local diameter less than ε. $\square$

Let us see that SLD maps are what we need to transfer the SLD property.

Theorem 2.4. *Let (X,τ_1) and (Y,τ_2) be topological spaces and let ϱ_1,ϱ_2 be metrics defined on X and Y, respectively, with $\tau_2 \preceq \varrho_2$. Suppose that there exists a one-to-one map $T : X \longrightarrow Y$ and that (Y,τ_2) has the ϱ_2-SLD property. Then the following conditions are equivalent:*

i) T is (ϱ_1,ϱ_2)-SLD;

ii) (X, T_{τ_2}) has the ϱ_1-SLD property;

iii) (X, T_{τ_2}) is σ-fragmented by ϱ_1.

When (Y,τ_2) is σ-fragmented by ϱ_2, i)⇔iii) holds.

Proof. i)⇒ ii) Given $\varepsilon > 0$, let $\mathcal{A} = \{\mathcal{A}^n\} = \{A^n_\gamma; \gamma \in \Gamma(n)\}$ be a σ-discrete open refinement of an open cover by balls of radius less than $\frac{\varepsilon}{2}$. Since T is a SLD map, for each $n \in \mathbb{N}$, the family $T(\mathcal{A}^n)$ is ϱ_2-σ-discretely decomposable, i.e., for each $B^n_\gamma = T(A^n_\gamma)$, $B^n_\gamma = \cup_m B^{n,m}_\gamma$, and for fixed $n,m \in \mathbb{N}$, $\{B^{n,m}_\gamma\}_{\gamma\in\Gamma(n)}$ is ϱ_2-discrete. Now, since (Y,τ_2) has ϱ_2-SLD, by Proposition 1.9, each $\{B^{n,m}_\gamma\}_{\gamma\in\Gamma(n)}$ is σ-isolatedly decomposable (for τ_2), hence $B^{n,m}_\gamma = \cup_p\{B^{n,m,p}_\gamma\}$, with $\{B^{n,m,p}_\gamma\}_{\gamma\in\Gamma(n)}$ τ_2-isolated.

Now for each $n,m,p \in \mathbb{N}$, set $A^{n,m,p}_\gamma = T^{-1}(B^{n,m,p}_\gamma)$. It is easy to see that the family $\{A^{n,m,p}_\gamma\}_{\gamma\in\Gamma(n)}$ is T_{τ_2}-isolated.

Define $X^{n,m,p} = \cup\{A^{n,m,p}_\gamma; \gamma \in \Gamma(n)\}$. Since it is a isolated union and each $A^{n,m,p}_\gamma$ is contained in an open ball of radius less than $\frac{\varepsilon}{2}$, it is clear that each $X^{n,m,p}$ has local ϱ_1-diameter less than ε. It is also easy to see by construction that $X = \cup\{X^{n,m,p} : n,m,p \in \mathbb{N}\}$. Since this can be done for every $\varepsilon > 0$, we conclude with (X, T_{τ_2}) has ϱ_1-SLD.

ii)⇒ iii) It is obvious.

iii)⇒ i) Since $\tau_2 \preceq \varrho_2$, the hypothesis implies that (X, T_{ϱ_2}) is ϱ_1-σ-fragmented, and that, by Proposition 1.1, is equivalent to T being (ϱ_1,ϱ_2)-SLD.

In the case (Y,τ_2) is ϱ_2-σ-fragmented, iii)⇒i) follows the same line as above.

Now let us show that i)⇒ iii). Given $\varepsilon > 0$, let $\{U^n_\alpha; \alpha \in \Gamma, n \in \mathbb{N}\}$ be an open refinement of an open cover of X by balls of ϱ_1-radius less than $\frac{\varepsilon}{2}$ such that for all $n \in \mathbb{N}$ the family $\{U^n_\alpha; \alpha \in \Gamma\}$ is ϱ_1-discrete.

For each $n \in \mathbb{N}$, $T(U^n_\alpha) = \cup\{B^{n,m}_\alpha; n,m \in \mathbb{N}\}$, with $\{B^{n,m}_\alpha\}_{\alpha\in\Gamma}$ being ϱ_2-discrete. Then there exist $\frac{1}{k}$-discrete families $\{B^{n,m,k}_\alpha\}_{\alpha\in\Gamma}$ with $B^{n,m}_\alpha = \cup\{B^{n,m,k}_\alpha; k \in \mathbb{N}\}$.

Now for each $k \in \mathbb{N}$ there is a cover of $X = \cup\{C_r^k; r \in \mathbb{N}\}$ given by the σ-fragmentability such that each set C_r^k is fragmented down to $\frac{1}{k}$.

Set $D_{\alpha,r}^{n,m,k} = C_r^k \cap B_\alpha^{n,m,k}$ and take $A_{\alpha,r}^{n,m,k} = T^{-1}(D_{\alpha,r}^{n,m,k}) \subset U_\alpha^n$. If we define $C_r^{n,m,k} = \cup\{A_{\alpha,r}^{n,m,k}; \alpha \in \Gamma\}$ then $X = \cup\{C_r^{n,m,k}; n,m,r,k \in \mathbb{N}\}$. We are going to show that if $n,m,r,k \in \mathbb{N}$ are fixed, any nonempty subset of $C_r^{n,m,k}$ has a nonempty relatively T_{τ_2}-open subset of ϱ_1-diameter less than ε.

So take any $\emptyset \neq A \subset C_r^{n,m,k}$. We have

$$\emptyset \neq T(A) \subset \bigcup_{\alpha\in\Gamma}\{D_{\alpha,r}^{n,m,k}\} \subset C_r^k.$$

By the σ-fragmentability there exists a τ_2-open set U, such that ϱ_2-diam$(T(A) \cap U) \leq \frac{1}{k}$. So $T(A) \cap U \subset B_\alpha^{n,m,k}$ for only one $\alpha \in \Gamma$. Hence, $T(A) \cap U \subset D_{\alpha,r}^{n,m,k}$ for only one $\alpha \in \Gamma$. Therefore, $A \cap T^{-1}(U) \subset U_\alpha^n$ and that makes ϱ_1-diam$(A \cap T^{-1}(U)) \leq \varepsilon$. $\square$

When T in Theorem 2.4, is $\tau_1 - \tau_2$ continuous, i) implies (X, τ_1) has ϱ_1-SLD (or σ-fragmented), so as a corollary we can obtain the following result:

Proposition 2.5. *Let X and Y be Banach spaces. Let $T : X \to Y$ be a weakly continuous SLD map. Then:*

i) If Y has the JNR property, then so has X.

ii) If Y is σ-fragmentable, then so is X.

The following result is from [18] and it is very useful when trying to construct SLD maps.

Proposition 2.6. *Let $(X, \|\cdot\|_X)$ be a Banach space and (Y, ϱ) a metric space. Let $T : X \longrightarrow Y$ be a one-to-one map. If, for every $x \in X$ there exists a separable subspace Z_x of X with $x \in \overline{\text{span}\,\{Z_{x_n}; n \in \mathbb{N}\}}^{\|\cdot\|_X}$, whenever (x_n) is a bounded sequence in X with $T(x_n)$ converging to $T(x)$ in ϱ (in particular when (x_n) converges weakly to x, or pointwise when X=C(K)), then T is a $(\|\cdot\|_X, \varrho)$-SLD map.*

The part i) in the following result is due to Kenderov and Moors [16].

Corollary 2.7. *Let X be a Banach space.*

i) X is σ-fragmentable if, and only if, there exists a metric ϱ finer than the weak topology such that $(X, weak)$ is σ-fragmented by ϱ.

ii) X has the JNR property if, and only if, there exists a metric ϱ finer than the weak topology such that $(X, weak)$ has ϱ-SLD.

Proof. We only have to prove the if part. Let us consider the identity map $Id : (X, \|\cdot\|_X) \to (X, \varrho)$. If x_n converges to x in the ϱ metric, x_n must converge to x in the weak topology. Proposition 2.6 says, Id is an SLD map, therefore Theorem 2.4 gives the conclusion. $\square$

These maps have been used by several authors: Spahn in [24] constructed a SLD map from a WCG Banach space into a $c_0(\Gamma)$, Hansell [7] called them co-σ-discrete maps, and Moltó, Orihuela and Troyanski [18] gave as a definition the equivalence in Proposition 2.2.

Proof of Theorem B. Let $\phi : X \to Y$ be a weak-homeomorphism.

Let (x_n) be a bounded sequence in X and $x \in X$ such that $\phi(x_n) \to \phi(x)$ in $\|\cdot\|_Y$, hence $\phi(x_n) \to \phi(x)$ weakly. Thus, $x_n \to x$ weakly and Proposition 2.6 applies. So our map ϕ and its inverse are SLD. Moreover, since there is weak to weak continuity, by Proposition 2.5, i) and ii) hold. To prove iii) assume that $Borel(Y, weak) = Borel(Y, \|\cdot\|_Y)$.

Denote by $B_{(X,\|\cdot\|_X)}$ the closed unit ball of $(X, \|\cdot\|_X)$. $B_{(X,\|\cdot\|_X)}$ is a w-closed subset of X and, since ϕ is a homeomorphism, $\phi(B_{(X,\|\cdot\|_X)})$ is a w-closed subset of Y. Thus the norm $\|\cdot\|_X$ is lower semi-continuous on $(X, \phi_{\|\cdot\|_Y})$. Since ϕ is SLD, by Theorem 1.3, $(X, \phi_{\|\cdot\|_Y})$ has the $\|\cdot\|_X$-SLD property by differences of $\phi_{\|\cdot\|_Y}$-closed sets. So if G is a $\|\cdot\|_X$-open subset of X, then, by Proposition 1.5, $G = \cup\{C_i; i \in \mathbb{N}\}$, where C_i is the difference of two $\phi_{\|\cdot\|_Y}$-closed sets for every $i \in \mathbb{N}$.

Set $B_i = \phi(C_i)$. Then the B_i's are differences of $\|\cdot\|_Y$-closed sets and therefore they are w-Borel sets. Thus $\phi^{-1}(B_i) = C_i$ are w-Borel in X. We conclude that G is a countable union of *weak*-Borel sets and therefore is itself a *weak*-Borel subset of X. $\square$

The σ-fragmentability counterpart of our next result is due to Kenderov and Moors, [16]. For the JNR property it answers a question of Haydon in [11]. See [21].

Corollary 2.8. *Let $(K_n)_{n=1}^{\infty}$ be a sequence of closed subsets of a compact Hausdorff space K such that $K = \cup K_n$. Then: if for each $n \in \mathbb{N}$ the space $(C(K_n), ptwise)$, (resp. $(C(K_n), weak)$) has the JNR property, then so has the space $(C(K), ptwise)$, (resp. $(C(K), weak)$).*

P r o o f. Define the map

$$T : C(K) \longrightarrow \Sigma_{c_0}(C(K_n), \|\cdot\|_\infty)$$

by the formula

$$T(f) = (\frac{1}{n} f|_{K_n})_{n=1}^{\infty}.$$

Since T is clearly pointwise to pointwise continuous, we only have to show that T SLD.

So take a bounded sequence $(f_m)_{m=1}^{\infty} \in C(K)$, and $f \in C(K)$ and suppose that $(T(f_m))$ converges to $T(f)$ in the norm of Σ_{c_0}. We need to show that f_m converges pointwise to f.

So take $x \in K$. There must be $n_0 \in \mathbb{N}$ such that $x \in K_{n_0}$. Given $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for all $k \geq m$ we have

$$\|T(f_k) - T(f)\|_\infty \leq \frac{\varepsilon}{n_0}, \text{ i.e., } \|\frac{1}{n} f_m|_{K_n} - \frac{1}{n} f|_{K_n}\|_\infty \leq \frac{\varepsilon}{n_0} \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$|\frac{1}{n_0} f_k(x) - \frac{1}{n_0} f(x)| \leq \frac{\varepsilon}{n_0} \text{ for all } k \geq m.$$

Thus f_m converges to f in the pointwise topology and by Proposition 2.6 we conclude that T is SLD. □

Let us present now some examples of constructions of SLD maps. The following proposition is almost clear by the definitions.

Proposition 2.9. *Let X, Y and Z be metric spaces and let f, g be SLD maps, $f : X \to Y$, $g : Y \to Z$. Then the map $h = g \circ f$ is SLD.*

Let us give the definition of a projectional resolution of the identity and some basic properties we shall need in our following result. They can be found in [1].

Definition 2.10. *Let X be a Banach Space. We denote by μ the smallest ordinal such that its cardinality $|\mu| = dens(X)$. A projectional resolution of identity, PRI for short, on X is a collection $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ of projections from X into X that satisfy, for every α with $\omega_0 \leq \alpha \leq \mu$, the following conditions:*

i) $\|P_\alpha\| = 1$;

ii) $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\alpha$ *if* $\omega_0 \leq \alpha \leq \beta \leq \mu$;

iii) $dens(P_\alpha(X)) \leq |\alpha|$;

iv) $\bigcup\{P_{\beta+1}(X) : \omega_0 \leq \beta < \alpha\}$ *is norm dense in* $P_\alpha(X)$;

v) $P_\mu = Id_X$.

Lemma 2.11. *Let* X *be a Banach space and* $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ *be a PRI on* X*. We put* $P_{\alpha+1} - P_\alpha = T_\alpha$*, for* $\omega_0 \leq \alpha < \mu$*. Then the following results hold.*

i) For every $x \in X$*, if* α *is a limit ordinal,* $\omega_0 \leq \alpha \leq \mu$*, we have* $P_\alpha(x) = \|\cdot\| - \lim_{\beta<\alpha} P_\beta(x)$.

ii) For every $x \in X$*,* $\{\|T_\alpha(x)\| : \alpha \in [\omega_0, \mu)\}$ *belongs to* $c_0([\omega_0, \mu))$.

iii) For every $x \in X$ *and* $\alpha \in [\omega_0, \mu), P_\alpha(x)$ *belongs to norm closed linear span of* $\{T_\beta(x) : \beta < \alpha\} \cup \{P_{\omega_0}(x)\}$.

We shall need the following:

Lemma 2.12. *Let* $(X, \|\cdot\|)$ *be a normed space and* $(T_\gamma)_{\gamma\in\Gamma}$ *be a family of bounded linear maps,* $T_\gamma : X \longrightarrow X$*, satisfying:*

i) $\forall x \in X$, $(\|T_\gamma(x)\|)_{\gamma\in\Gamma} \in c_0(\Gamma)$.

ii) $\forall x \in X$, $x \in \overline{span\{T_\gamma(x); \gamma \in \Gamma\}}^{\|\cdot\|}$.

Then there exists a SLD map $\Phi : X \longrightarrow \Sigma_{c_0}(T_\gamma(X), \|\cdot\|_\infty)$.

Proof. Define $\Phi : X \longrightarrow \Sigma_{c_0}(T_\gamma(X), \|\cdot\|_\infty)$ by $\Phi(x) = (T_\gamma(x))_{\gamma\in\Gamma}$. It is clear that Φ is a one-to-one linear map.

From i) we obtain that the set $\{\gamma \in \Gamma; T_\gamma(x) \neq 0\}$ is countable. So for every $x \in X$ we define a separable subspace of X, Z_x as

$$Z_x = \overline{\text{ span } \{T_\gamma(x); \gamma \in \Gamma\} }^{\|\cdot\|}.$$

Let us show that Φ is a SLD map. Let (x_n) be a bounded sequence in X such that $\Phi(x_n)$ converges to $\Phi(x)$ in Σ_{c_0} for some $x \in X$.

By ii), given $\varepsilon > 0$ there exists Γ_1 finite subset of Γ and real numbers α_i such that $\|x - \sum \alpha_i T_{\gamma_i}(x)\| < \frac{\varepsilon}{2}$ for $\gamma_i \in \Gamma_1$. Now since $\lim T_\gamma(x_n) = T_\gamma(x)$ for any $\gamma \in \Gamma$, it follows that $\lim_n \|x - \sum \alpha_i T_{\gamma_i}(x_n)\| \leq \frac{\varepsilon}{2}$. Thus there must be $n_0 \in \mathbb{N}$ such that $\|x - \sum \alpha_i T_{\gamma_i}(x_{n_0})\| < \varepsilon$, which means that $x \in \overline{\cup Z_{x_n}}$. We only have to apply Proposition 2.6. $\square$

Theorem 2.13. *Let* $\wp$ *be a class of Banach spaces such that the following holds:*

i) For every $X \in \wp$ there exists a PRI on X, $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$.

ii) $(P_{\alpha+1} - P_\alpha)(X) \in \wp$.

Then for every $X \in \wp$ there exists a set Γ and a bounded linear SLD map $T : X \longrightarrow c_0(\Gamma)$.

Proof. Let $X \in \wp$. We proceed by induction on $dens(X)$.

When X is separable, we have that (B_{X^*}, w^*) is metrizable and separable. So let $\{f_n : n \geq 1\}$ be a dense subset of (B_{X^*}, w^*) and define

$$T : X \longrightarrow c_0(\mathbb{N}) \text{ by } T(x) = \left(\frac{1}{n} f_n(x)\right)_{n=1}^{\infty}.$$

T is clearly a linear map, and it is one-to-one because $(f_n)_{n=1}^{\infty}$ is dense in B_{X^*}.

In order to show that T is SLD, we notice that discrete families in separable Banach spaces are countable. And it is clear that the image of any countable family is σ-discretely decomposable.

Let χ be an uncountable cardinal and $X \in \wp$ such that $dens(X) = \chi$. Suppose that the result is true for every $Y \in \wp$ with $dens(Y) < \chi$. Let μ be the smallest ordinal with cardinality $|\mu| = \chi$.

Let $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ be a *PRI* on X. For any $\alpha \in [\omega_0, \mu)$, we set $T_\alpha = P_{\alpha+1} - P_\alpha$ and $X_\alpha = T_\alpha(X)$. Then $X_\alpha \in \wp$ and $dens(X_\alpha) \leq |\alpha| < dens(X)$. Thus there exist sets Γ_α and bounded linear SLD maps $J_\alpha : X_\alpha \longrightarrow c_0(\Gamma_\alpha)$.

Since $P_{\omega_0}(X)$ is separable, there is also $J_0 : P_{\omega_0}(X) \longrightarrow c_0(\mathbb{N})$ sharing these properties. Set

$$\Gamma = \mathbb{N} \cup \bigcup_{\omega_0 \leq \alpha < \mu} \Gamma_\alpha \text{ disjoint union}$$

and define $T : X \longrightarrow \ell_\infty(\Gamma)$ by

$$T(x)(n) = J_0(P_{\omega_0}(x))(n) \text{ for } n \in \mathbb{N}, \text{ and}$$

$$T(x)(\gamma) = \frac{1}{2(\|J_\alpha\| + 1)} J_\alpha(T_\alpha(x))(\gamma) \text{ for } \gamma \in \Gamma_\alpha.$$

T is clearly a linear map and is continuous and indeed $T(X) \subset c_0(\Gamma)$. Also, by the properties of the PRI and the induction hypothesis T is an injection.

Let us define

$$\Phi : X \longrightarrow \Sigma_{c_0}(X_\alpha, \|\cdot\|_\infty) \text{ by } \Phi(x) = (T_\alpha(x))_{\alpha<\mu}$$

and

$$\psi : \Sigma_{c_0}(X_\alpha, \|\cdot\|_\infty) \longrightarrow c_0(\Gamma) \text{ by } \psi((x_\alpha)_{\alpha<\mu})(\gamma) = (J_\alpha(x_\alpha))(\gamma) \text{ for } \gamma \in \Gamma.$$

By Lemma 2.12, Φ is a SLD map. By the induction hypothesis and Theorem 2.4, the spaces $(X_\alpha, J_{\alpha,ptwise})$ have the SLD property and that implies, by Proposition 1.2, that $\Sigma_{c_0}(X_\alpha, \tau_{pro})$ has the SLD property as well.

On the other hand, it is easy to see that the topologies ψ_{ptwise} and τ_{pro} coincide on $\Sigma_{c_0}(X_\alpha)$, which means that the space $\Sigma_{c_0}(X_\alpha, \psi_{ptwise})$ has the SLD property and therefore, by Theorem 2.4 and the fact that $(c_0(\Gamma), ptwise)$ has $\|\cdot\|_\infty$-SLD (Proposition 1.2), ψ is SLD.

We only have to apply Proposition 2.9 to the maps Φ and ψ to obtain the desired result. $\square$

Theorem 2.13 can be applied to several classes of Banach spaces, for example WCD spaces, duals to Asplund spaces (we would obtain the SLD property for the weak topology, not for the weak*-topology in general) and $C(K)$ spaces with K being a Valdivia compact space. See either [1, 5] for the construction of PRI on these spaces. We can summarize as follows:

Corollary 2.14. *Let X be a Banach space of one of the following types: weakly countably determined, the dual of an Asplund space or a $C(K)$ space with K being a Valdivia compact space. Then there exists a bounded SLD linear map $T : X \longrightarrow c_0(\Gamma)$, pointwise to pointwise continuous when $X = C(K)$, and therefore $(X, weak)$, also $(C(K), pointwise)$, has the JNR property.*

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