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ON THE EXPONENTIAL BOUND OF THE CUTOFF RESOLVENT

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Abstract. A simpler proof of a result of Burq [1] is presented.

Let $\mathcal{O} \subset \mathbb{R}^n, n \geq 2$, be a bounded domain with C^∞ boundary Γ and connected complement $\Omega = \mathbb{R}^n \setminus \overline{\mathcal{O}}$. Consider in Ω the operator

$$\Delta_g := c(x)^2 \sum_{i,j=1}^{n} \partial_{x_i}(g_{ij}(x)\partial_{x_j}),$$

where $c(x), g_{ij}(x) \in C^\infty(\overline{\Omega})$, $c(x) \geq c_0 > 0$ and

$$\sum_{i,j=1}^{n} g_{ij}(x)\xi_i\xi_j \geq C|\xi|^2, \qquad \forall (x,\xi) \in T^*\Omega, \quad C > 0.$$

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We also suppose that $c(x)=1$, $g_{ij}(x)=\delta_{ij}$ for $|x|\geq \rho_0$ for some $\rho_0\gg 1$. Denote by G the selfadjoint realization of Δ_g in the Hilbert space $H=L^2(\Omega; c(x)^{-2}dx)$ with a domain of definition $D(G)=\{u\in H^2(\Omega), Bu|_\Gamma=0\}$, where either $B=Id$ (Dirichlet boundary conditions) or $B=\partial_\nu$ (Neumann boundary conditions). Consider the resolvent $R(\lambda):=(G+\lambda^2)^{-1}: H\to H$ defined for $\mathrm{Im}\,\lambda<0$, and introduce the cutoff resolvent $R_\chi(\lambda):=\chi R(\lambda)\chi$, where $\chi\in C_0^\infty(\mathbb{R}^n)$, $\chi(x)=1$ for $|x|\leq\rho_0+1$, $\chi(x)=0$ for $|x|\geq\rho_0+2$. It is well known that $R_\chi(\lambda)$ extends through the real axis as a meromorphic function the poles of which are called resonances. Using the Carleman estimates proved by Lebeau-Robbiano ([4] in the Dirichlet case and [5] in the Neumann one) Burq has proved the following result

Theorem ([1]). *There exist constants $C, C_1, C_2, \gamma>0$ so that $R_\chi(\lambda)$ extends holomorphically to the region*

$$\{\lambda\in\mathbb{C}: \mathrm{Im}\,\lambda\leq C_1 e^{-\gamma|\lambda|},\ |\mathrm{Re}\,\lambda|\geq C_2\}$$

and satisfies there the estimate

$$\|R_\chi(\lambda)\|_{\mathcal{L}(H)}\leq Ce^{\gamma|\lambda|}. \tag{1}$$

Furthermore, he applied this theorem to obtain uniform rate of the decay of the local energy. Denote by $u(t)$ the solution of the equation

$$\begin{cases} (\partial_t^2-\Delta_g)u(t)=0,\\ Bu|_\Gamma=0,\\ u(0)=f_1, \partial_t u(0)=f_2.\end{cases}$$

Given any compact $K\subset\overline{\Omega}$ and any $m>0$, set

$$p_m(t)=\sup\left\{\frac{\|\nabla_x u\|_{L^2(K)}+\|\partial_t u\|_{L^2(K)}}{\|\nabla_x f_1\|_{H^m(K)}+\|f_2\|_{H^m(K)}}, (0,0)\neq(f_1,f_2)\in[C^\infty(\overline{\Omega})]^2, \mathrm{supp}\, f_j\subset K\right\}.$$

Burq derived from (1) the following bounds

$$p_m(t)\leq C_m(\log t)^{-m}\qquad \text{for}\quad t\geq 2. \tag{2}$$

Note that another method allowing to derive (2) from (1) is presented in [6, Section 3].

The purpose of the present note is to give another proof of how the Carleman estimates of Lebeau-Robbiano imply (1). The first observation is that Theorem follows easily from the bound

$$\|R_\chi(\lambda)\|_{\mathcal{L}(H)} \leq \widetilde{C}e^{\gamma|\lambda|}, \quad \lambda \in \mathbb{R},\, |\lambda| \gg 1, \tag{3}$$

(e.g. see [2, Corollary 3.1]). In fact, it suffices to prove (3) for $\lambda \gg 1$ as the case $\lambda \ll -1$ can be treated similarly. So, in what follows λ will be real, $\lambda \gg 1$. Consider the Helmholtz equation

$$\begin{cases} (\Delta_g + \lambda^2)u &= v \quad \text{in} \quad \Omega, \\ Bu &= 0 \quad \text{on} \quad \Gamma, \\ u &- \quad \lambda - \text{outgoing}, \end{cases}$$

where $v \in C^\infty(\Omega)$, $\operatorname{supp} v \subset \Omega_{a_0} := \{x \in \Omega : |x| < a_0\}$, where $a_0 \gg 1$ is taken so that the support of the perturbation is contained in Ω_{a_0}. Clearly, (3) is equivalent to the estimate

$$\|u\|_{L^2(\Omega_{a_0})} \leq Ce^{\gamma\lambda}\|v\|_{L^2(\Omega)}. \tag{4}$$

Take $a > a_0$ to be fixed later on and denote $S = \{x \in \mathbb{R}^n : |x| = a\}$. Define the Neumann operator $N(\lambda) : H^1(S) \to L^2(S)$ by $N(\lambda)g := \lambda^{-1}\partial_{\nu'}w|_S$, where w solves the equation

$$\begin{cases} (\Delta + \lambda^2)w &= 0 \quad \text{in} \quad |x| > a, \\ w &= g \quad \text{on} \quad S, \\ w &- \quad \lambda - \text{outgoing}. \end{cases}$$

Here Δ denotes the free Laplacian and ν' denotes the outer unit normal to S. It is well known that for strictly convex S we have the bound

$$\|N(\lambda)\|_{\mathcal{L}(H^1(S), L^2(S))} \leq C \tag{5}$$

with a constant $C > 0$ independent of λ (e.g. see [3, Corollary 3.3]). Hereafter, given a domain K, $H^s(K)$ will denote the Sobolev space equipped with the semiclassical norm $\|f\|_{H^s(K)} := \|\Lambda_s f\|_{L^2(K)}$, where Λ_s is a $\lambda - \Psi DO$ on K with principal symbol $(|\xi|^2 + 1)^{s/2}$.

Clearly, u and v satisfy the equation

$$\begin{cases} (\Delta_g + \lambda^2)u = v \quad \text{in} \quad \Omega_a, \\ Bu = 0 \quad \text{on} \quad \Gamma, \\ \lambda^{-1}\partial_\nu u|_S + N(\lambda)f = 0, \end{cases}$$

where $f = u|_S$ and $\nu = -\nu'$ denotes the inner unit normal to S. By Green's formula we have

$$-\mathrm{Im}\,\langle N(\lambda)f, f\rangle_{L^2(S)} = -\mathrm{Im}\,\langle u, c^{-2}v\rangle_{L^2(\Omega_{a_0})} \tag{6}$$
$$\leq e^{-\beta\lambda}\|u\|^2_{L^2(\Omega_{a_0})} + e^{\beta\lambda}\|v\|^2_{L^2(\Omega)},$$

$\forall\beta$. Given any $X > 0$ take a function $\rho_X(t) \in C_0^\infty(\mathbb{R})$, $0 \leq \rho_X(t) \leq 1$, $\rho_X(t) = 1$ for $|t| \leq X$, $\rho_X(t) = 0$ for $|t| \geq X + 1$. Denote by Δ_S the Laplace-Beltrami operator on S. We need the following

Lemma. *For every $X > 0$ there exists $\gamma_0 = \gamma_0(X) \geq 0$ so that*

$$-\mathrm{Im}\,\langle N(\lambda)f, f\rangle_{L^2(S)} \geq e^{-\gamma_0\lambda}\|\rho_X(\lambda^{-1}\sqrt{-\Delta_S})f\|^2_{L^2(S)}. \tag{7}$$

Proof. Without loss of generality we may suppose that S is of radius 1. It is well known that the outgoing Neumann operator can be expressed in terms of the Hankel functions of second type, $H_\nu^{(2)}(z)$. Let $\{\mu_j\}$ be the eigenvalues of $\sqrt{-\Delta_S}$ repeated according to multiplicity. We have the identities

$$-\mathrm{Im}\,\langle N(\lambda)f, f\rangle_{L^2(S)} = -\sum \mathrm{Im}\left(\frac{h_\nu'(\lambda)}{h_\nu(\lambda)}\right)\alpha_j^2, \tag{8}$$

$$\|\rho_X(\lambda^{-1}\sqrt{-\Delta_S})f\|^2_{L^2(S)} = \sum \rho_X^2(\lambda^{-1}\mu_j)\alpha_j^2, \tag{9}$$

where $\{\alpha_j\}$ are such that

$$\|f\|^2_{L^2(S)} = \sum \alpha_j^2,$$

and $h_\nu(z) = z^{1/2}H_\nu^{(2)}(z)$, $\nu = \sqrt{\mu_j^2 + (\frac{n}{2} - 1)^2}$, satisfies the equation

$$h_\nu''(z) = \left(\frac{\nu^2 - 1/4}{z^2} - 1\right)h_\nu(z). \tag{10}$$

For real $z > 0$, set $\psi_\nu(z) = -\mathrm{Im}\,\dfrac{h_\nu'(z)}{h_\nu(z)}$, $\eta_\nu(z) = -\mathrm{Re}\,\dfrac{h_\nu'(z)}{h_\nu(z)}$. In view of (10) we have

$$\psi_\nu'(z) = \mathrm{Im}\left(\left(\frac{h_\nu'(z)}{h_\nu(z)}\right)^2 - \frac{h_\nu''(z)}{h_\nu(z)}\right) = 2\eta_\nu\psi_\nu. \tag{11}$$

This implies

$$\frac{d}{dz}\left\{\psi_\nu(\nu z)\exp\left(-2\nu\int_{z_0}^{z}\eta_\nu(\nu y)dy\right)\right\}=0,$$

and hence

$$\psi_\nu(\nu z)=\psi_\nu(\nu z_0)\exp\left(2\nu\int_{z_0}^{z}\eta_\nu(\nu y)dy\right). \tag{12}$$

Fix $z_0 = 2$. We are going to show that for $\nu \geq \nu_0 \gg 1$ we have: $\forall\delta > 0$, $\exists c = c(\delta) \geq 0$ so that

$$\psi_\nu(\nu z) \geq e^{-c\nu}, \qquad \forall z \geq \delta, \tag{13}$$

and

$$\psi_\nu(z) > 0, \qquad \forall z > 0. \tag{14}$$

By Olver's expansions

$$\psi_\nu(\nu z_0)=\frac{\sqrt{z_0^2-1}}{z_0}+O(\nu^{-1}).$$

Clearly, this together with (12) imply (14). To prove (13) we will first consider the case when $z \geq 2$. Again by Olver's expansions

$$\eta_\nu(\nu z)=\frac{4z^2-3}{2z(z^2-1)}\nu^{-1}+O(\nu^{-2}),$$

uniformly for $z \geq 2$, and hence $\eta_\nu(\nu z) > 0$. This together with (12) yield

$$\psi_\nu(\nu z) \geq \psi_\nu(\nu z_0) \geq Const > 0,$$

which proves (13) in this case. Furthermore, still by Olver's expansions we have $\eta_\nu(\nu z) = O(1)$ uniformly in $\delta \leq z \leq 2$. Hence, by (12), for $\delta \leq z \leq 2$,

$$\begin{aligned}\psi_\nu(\nu z) &\geq \psi_\nu(\nu z_0)\exp\left(-2\nu\int_{\delta}^{2}|\eta_\nu(\nu y)|dy\right)\\ &\geq \psi_\nu(\nu z_0)\exp\left(-C\nu\right), \qquad C>0,\end{aligned}$$

which implies (13) in this case.

Let now $1/2 < \nu \leq \nu_0$. Using the well known asymptotics of the Hankel functions as $z \to +\infty$, $\nu > 1/2$ fixed, we get

$$\psi_\nu(z) = 1 + O(z^{-1}), \qquad 1/2 < \nu \leq \nu_0. \tag{15}$$

Since $\nu = O(\lambda)$ on $\operatorname{supp} \rho_X$, it is easy to see that (7) follows from (8) and (9) combinned with (13), (14) and (15).

Let $\chi \in C_0^\infty(\mathbb{R}^n)$, $\chi = 1$ for $|x| \le a_0 + 2$, $\chi = 0$ for $|x| \ge a_0 + 3$. Applying the Carleman estimates of Lebeau-Robbiano [4], [5] to the function χu leads to

$$\begin{aligned} &\int_{\Omega_{a_0+2}} \left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) dx \\ &\le e^{2\gamma_1\lambda} \int_{a_0+2\le|x|\le a_0+3} \left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) dx + e^{2\gamma_1\lambda}\|v\|^2_{L^2(\Omega)}, \end{aligned} \tag{16}$$

with some $\gamma_1 > 0$. To eliminate the first term in the RHS of (16) we will use the Carleman estimates up to S. Set $P = -\lambda^{-2}\Delta - 1$. If $\varphi \in C^\infty(\Omega_a)$, then $P_\varphi := e^{\lambda\varphi} P e^{-\lambda\varphi}$ is again a $\lambda - \Psi DO$ with principal symbol $p_\varphi(x,\xi) = p(x, \xi + i\nabla_x\varphi)$, p being the principal symbol of P considered as a $\lambda - \Psi DO$. We will construct a real-valued C^∞ function φ defined in a neighbourhood of $a_0 \le |x| \le a$ such that $\nabla\varphi \ne 0$ on $a_0 \le |x| \le a$, $\varphi = -1$ on $|x| = a_0$, $\varphi \ge \gamma_1 + 1$ on $a_0 + 2 \le |x| \le a_0 + 3$ and satisfying the condition

$$p_\varphi(x,\xi) = 0 \Rightarrow \{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} > 0. \tag{17}$$

We will be looking for φ in the form $\varphi(r)$, $r = |x|$. It is easy to see that (17) is equivalent to

$$\varphi'\left(\varphi''\varphi' + \frac{1+\varphi'^2}{r}\right) > 0 \qquad \text{for} \quad a_0 \le r \le a. \tag{18}$$

Given any constant $C > 2(a_0 + 3)$, it is easy to check that the function $\varphi'(r) = \sqrt{\dfrac{C}{r} - 1}$ satisfies (18) with $a = C/2$. Define $\varphi(r)$ as follows

$$\varphi(r) = -1 + \int_{a_0}^r \sqrt{Ct^{-1} - 1}\, dt.$$

Clearly, if we take $C \ge C_1(a_0, \gamma_1)$ we can arrange $\varphi(a_0 + 2) \ge \gamma_1 + 1$ and hence $\varphi(r) \ge \gamma_1 + 1$ for $a_0 + 2 \le r \le a$. Fix $C = \max\{2(a_0+3), C_1(a_0,\gamma_1)\}$ and $a = C/2$. Since $\varphi(a_0) = -1$, there exist $a_0 < a_1 < a_2 < a_0 + 1$ so that $\varphi(r) < 0$ for $a_1 \le r \le a_2$. Choose a function $\chi_1 \in C^\infty(\mathbb{R}^n)$, $\chi_1 = 0$ for $|x| \le a_1$, $\chi_1 = 1$ for $|x| \ge a_2$. We would like to apply the Carleman estimates up to S to the function $\chi_1 u$. Set $w = e^{\lambda\varphi}\chi_1 u$. We are going to prove the estimate

$$\begin{aligned} &\|w\|_{H^1(a_0\le|x|\le a)} + \|w|_S\|_{H^1(S)} \\ &\le O(\lambda^{1/2})\|P_\varphi w\|_{L^2(a_0\le|x|\le a)} + O(1)\|\mathrm{Op}_\lambda(\eta) w|_S\|_{L^2(S)}, \end{aligned} \tag{19}$$

where $\eta(x',\xi') \in C_0^\infty(T^*S)$, $\eta = 1$ for $r_0(x',\xi') \le 3$, $\eta = 0$ for $r_0(x',\xi') \ge 4$, $r_0(x',\xi')$ denotes the principal symbol of $-\Delta_S$. Before proceeding to the proof of (19) we will complete the proof of (4). Since $P_\varphi w = -\lambda^{-2}e^{\lambda\varphi}[\Delta,\chi_1]u$ and $w|_S = e^{\varphi(a)\lambda}f$, (19) implies

$$\int_{a_2\le|x|\le a} \left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) e^{2\lambda\varphi}dx \le \int_{a_1\le|x|\le a_2} \left(|u|^2 + |\lambda^{-1}\nabla u|^2\right) e^{2\lambda\varphi}dx$$

$$+O(1)e^{2\lambda\varphi(a)}\|\mathrm{Op}_\lambda(\eta)f\|^2_{L^2(S)} - e^{2\lambda\varphi(a)}\|f\|^2_{L^2(S)}. \tag{20}$$

Since $\gamma_1 < \varphi$ on $a_0+2 \le |x| \le a_0+3$, the first term in the RHS of (16) is estimated from above by the LHS of (20) times a factor $e^{-\delta_1\lambda}$, $\delta_1 > 0$. On the other hand, since $\varphi < 0$ on $a_1 \le |x| \le a_2$, the first term in the RHS of (20) is estimated from above by the LHS of (16) times a factor $e^{-\delta_2\lambda}$, $\delta_2 > 0$. Therefore, we have

$$e^{-2\gamma_2\lambda}\|u\|^2_{L^2(\Omega_{a_0+2})} + \|f\|^2_{L^2(S)} \le e^{2\gamma_3\lambda}\|v\|^2_{L^2(\Omega)} + O(1)\|\mathrm{Op}_\lambda(\eta)f\|^2_{L^2(S)}, \tag{21}$$

with some constants γ_2 and γ_3. On the other hand, taking $\eta(x',\xi') = \rho_X(\sqrt{r_0(x',\xi')})$, applying (7) with $X = \sqrt{3}$ and combining with (6) give

$$\|\mathrm{Op}_\lambda(\eta)f\|^2_{L^2(S)} \le o(1)\|f\|^2_{L^2(S)} + e^{-(\beta-\gamma_0)\lambda}\|u\|^2_{L^2(\Omega_{a_0})} + e^{(\beta+\gamma_0)\lambda}\|v\|^2_{L^2(\Omega)}, \tag{22}$$

$\forall\beta$. Clearly, taking $\beta > 2\gamma_2 + \gamma_0$, (4) follows from (21) and (22).

P r o o f o f (19). Since $\partial_\nu\varphi|_S = -1$, the boundary conditions on S become $\lambda^{-1}\partial_\nu w|_S = -(N(\lambda)+1)f_1$, where $f_1 := w|_S$. By the Carleman estimates of Lebeau-Robbiano [4], in view of (5), we have

$$\|w\|_{H^1(a_0\le|x|\le a)} \le O(\lambda^{1/2})\|P_\varphi w\|_{L^2(a_0\le|x|\le a)} + O(1)\|f_1\|_{H^1(S)}. \tag{23}$$

It is easy to see that (19) would follow from (23) and the estimate

$$\begin{aligned}&\|\mathrm{Op}_\lambda(1-\eta)f_1\|_{H^1(S)}\\ &\le O(\lambda^{1/2})\|P_\varphi w\|_{L^2(a_0\le|x|\le a)} + o(1)\|w\|_{H^1(a_0\le|x|\le a)} + o(1)\|f_1\|_{H^1(S)}.\end{aligned} \tag{24}$$

To prove (24) we will use that $1-\eta$ is supported in the elliptic region of the corresponding boundary value problem. Clearly, it suffices to prove (24) locally and then conclude by a partition of the unity on S. Given a $x_0 \in S$ take a small neighbourhood in $\mathbb{R}^n$, V, of x_0, and denote $U = V\cap S$, $V_+ = V\cap\{|x| < a\}$. Take in

V_+ the so called normal to the boundary local coordinates $x = (x', x_n) \in U \times [0, \delta]$, $0 < \delta \ll 1$. In these coordinates the principal symbols of P and P_φ write as follows

$$p = \xi_n^2 + r(x, \xi') - 1 = \xi_n^2 + r_0(x, \xi') - 1 + O(x_n|\xi'|^2),$$

$$\operatorname{Re} p_\varphi = \xi_n^2 + r(x, \xi') - 1 - (\varphi'_{x_n})^2 = \xi_n^2 + r_0(x, \xi') - 2 + O(x_n(|\xi'|^2 + 1)),$$

$$\operatorname{Im} p_\varphi = 2\varphi'_{x_n}\xi_n = -2\xi_n(1 + O(x_n)),$$

where $r_0(x', \xi')$ is the principal symbol of $-\Delta_S$ written in the coordinates $(x', \xi') \in T^*U$. Hence, the restriction of $p_\varphi = 0$ on T^*S is given by $r_0 = 2$. In what follows $\|\cdot\|_s$ and $\|\cdot\|_{s,+}$ will denote the norms in $H^s(\mathbb{R}^{n-1})$ and $H^s(\mathbb{R}^{n-1} \times \mathbb{R}^+)$, respectively, while $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_+$ will denote the scalar products in $L^2(\mathbb{R}^{n-1})$ and $L^2(\mathbb{R}^{n-1} \times \mathbb{R}^+)$, respectively. By $L^{s,k}_{cl}$ we will denote the space of $\lambda - \Psi DO$'s with symbols $a \sim \lambda^k \sum \lambda^{-j} a_j$ with a_j independent of λ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a_j| \le C_{\alpha\beta}(1 + |\xi|)^{s-j-|\beta|}.$$

We will also denote $\mathcal{D}_j := (i\lambda)^{-1}\partial_{x_j}$, $\mathcal{D} = (\mathcal{D}', \mathcal{D}_n)$. Let $\phi(t) \in C_0^\infty(\mathbb{R})$, $\phi = 1$ for $|t| \le \delta/2$, $\phi = 0$ for $|t| \ge \delta$. Let also $\zeta(x') \in C_0^\infty(U)$, $\zeta = 1$ in a small neighbourhood of $x_0 \in U$. Set

$$g = \operatorname{Op}_\lambda((1-\eta)|\xi'|)\phi(x_n)\zeta(x')w, \quad h := g|_{x_n=0} = \operatorname{Op}_\lambda((1-\eta)|\xi'|)\zeta(x')f_1.$$

We have

$$i\mathcal{D}_n g|_{x_n=0} = -(N(\lambda) + 1)h + [N(\lambda), \operatorname{Op}_\lambda((1-\eta)|\xi'|)\zeta(x')]f_1.$$

Since $N(\lambda)$ has a parametrix of class $L^{1,0}_{cl}$ on $\operatorname{supp}(1-\eta)$ with principal symbol $-\sqrt{r_0 - 1}$, we have that the commutator above (which will be denoted by A) is of class $L^{1,-1}_{cl}$. Let P_φ^* be the formal adjoint to P_φ and denote $Q_1 = \dfrac{P_\varphi + P_\varphi^*}{2}$, $Q_2 = \dfrac{P_\varphi - P_\varphi^*}{2i}$ with principal symbols $\operatorname{Re} p_\varphi$ and $\operatorname{Im} p_\varphi$, respectively. Using the identities

$$\int_0^\infty \mathcal{D}_n^2 g \cdot \overline{g} dx_n = \int_0^\infty |\mathcal{D}_n g|^2 dx_n + i\lambda^{-1}\mathcal{D}_n g|_{x_n=0} \cdot \overline{g}|_{x_n=0},$$

$$\operatorname{Im} \langle Q_2 g, g\rangle_+ = -\lambda^{-1}\|h\|_0^2 + e(g),$$

where

$$|e(g)| \le o(1)\|g\|_{1,+}^2,$$

it is easy to get

$$\mathrm{Re}\,\langle (Q_1 - \mathcal{D}_n^2)g, g\rangle_+ + \|\mathcal{D}_n g\|_{0,+}^2 = \mathrm{Re}\,\langle P_\varphi g, g\rangle_+ + \lambda^{-1}\mathrm{Re}\,\langle N(\lambda)h + Af_1, h\rangle + e(g)$$

$$(25) \qquad \leq \varepsilon^{-1}\int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{-1}^2 dx_n + \varepsilon\|g\|_{1,+}^2 + O(\lambda^{-2})\|f_1\|_{H^1(S)}^2,$$

$\forall\varepsilon > 0$. On the other hand, the principal symbol of $Q_1 - \mathcal{D}_n^2$ is $\geq C|\xi'|^2$, $C > 0$, on $\mathrm{supp}(1-\eta)$, $0 \leq x_n \leq \delta$, $0 < \delta \ll 1$. Therefore, by Gärding's inequality we get

$$0 < C'\|g\|_{1,+}^2 \leq \varepsilon^{-1}\int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{-1}^2 dx_n + \varepsilon\|g\|_{1,+}^2 + O(\lambda^{-2})\|f_1\|_{H^1(S)}^2,$$

and hence

$$(26) \qquad \|g\|_{1,+}^2 \leq O(1)\int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{-1}^2 dx_n + O(\lambda^{-2})\|f_1\|_{H^1(S)}^2.$$

On the other hand,

$$\begin{aligned}\|h\|_0^2 &= -\int_0^\infty \frac{d}{dx_n}\|g(\cdot, x_n)\|_0^2 dx_n \\ &= -2\lambda\int_0^\infty \mathrm{Re}\,\langle g(\cdot, x_n), i\mathcal{D}_n g(\cdot, x_n)\rangle dx_n \leq O(\lambda)\|g\|_{1,+}^2,\end{aligned}$$

which combinned with (26) gives

$$\begin{aligned}\|h\|_0 &\leq O(\lambda^{1/2})\left(\int_0^\infty \|P_\varphi g(\cdot, x_n)\|_{-1}^2 dx_n\right)^{1/2} + O(\lambda^{-1/2})\|f_1\|_{H^1(S)} \\ &\leq O(\lambda^{1/2})\|P_\varphi w\|_{0,+} + O(\lambda^{-1/2})\|w\|_{1,+} + O(\lambda^{-1/2})\|f_1\|_{H^1(S)},\end{aligned}$$

which in turn implies (24) by making a partition of the unity on S.

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