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SOME APPLICATIONS OF SIMONS' INEQUALITY

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Abstract. We survey several applications of Simons' inequality and state related open problems. We show that if a Banach space X has a strongly sub-differentiable norm, then every bounded weakly closed subset of X is an intersection of finite union of balls.

This work gathers several applications of Simons' inequality to Banach spaces, and in particular to smoothness of norms. Some of these applications have already been published, and therefore this article can partly be considered as a survey. Others, such as Theorem 10 and its corollaries, are published here for the first time. It seems appropriate to present in a single article results which are closely related, but scattered in various publications. Our goal is to stimulate research on these topics where many natural and simply stated questions are still open.

One of the most celebrated of James' theorems asserts that a subset A of a Banach space X is weakly compact if and only if every continuous linear form $y \in X^*$ attains its supremum on A, [17].The combinatorial principle that lies in

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the proof of James' theorem – or rather, of the separable case of James' theorem; see Appendix 1 – was found out by S. Simons [23]. Let us recall the statement of this beautiful result.

Theorem 1. *Let E be a set, and let (x_n) be a uniformly bounded sequence of functions on E. Let B be a subset of E such that for every sequence $\{\lambda_n; n \geq 1\}$ of positive real numbers with $\sum\limits_{n=1}^{\infty} \lambda_n = 1$, there exists $b \in B$ such that*

$$\sup\left\{\sum_{n=1}^{\infty} \lambda_n x_n(y); y \in E\right\} = \sum_{n=1}^{\infty} \lambda_n x_n(b)$$

Then

$$\sup_{b\in B}\left\{\overline{\lim_{n\to\infty}} x_n(b)\right\} \geq \inf\left\{\sup_E g; g \in \operatorname{conv}(x_n)\right\}$$

In these notes, we will refer to this statement as "Simons' inequality". We recall Simons' proof in the Appendix 2. One of the remarkable features of this inequality is that the left-hand side deals with the pointwise behaviour of the sequence (x_n) of the subset B, while the right-hand side provides information on the global behaviour on E of a convex combination of the $x_n's$. Moreover the statement is not refering to topological notions, although it is frequently applied to situations where there are some topologies around. Since the result is not topological, it will be applicable in problems where we have no information on the regularity of the set B. Let us illustrate this remark by an example, which is already figuring in Simons' original article [23].

Corollary 2. *Let X be a Banach space, and B be a subset of $S_{X^*} = \{y \in X^*; \|y\| = 1\}$ such that for every $x \in X$, there exists $y \in B$ with $\|x\| = y(x)$. Let (x_n) be a bounded sequence in X such that $x_n(y)$ converges for every $y \in B$. Then (x_n) is a weakly Cauchy sequence in X.*

P r o o f. Pick any $y_o \in B_{X^*} = \{y \in X^*; \|y\| \leq 1\}$. We have to show that $\{y_o(x_n); n \geq 1\}$ converges. If not, there exist $\varepsilon > 0$ and $n_1 < n_2 < n_3 < \cdots$ such that $|y_o(x_{n_{2k}} - x_{n_{2k-1}})| > \varepsilon$ for every $k \geq 1$. Let $x'_k = x_{n_{2k}} - x_{n_{2k-1}}$; we may and do assume that $y_o(x'_k) > \varepsilon$ for every $k \geq 1$. It follows from our assumptions that

$$\lim_{k\to\infty} x'_k(b) = 0$$

for every $b \in B$. Hence by Simons' inequality, there is $x \in \operatorname{conv}(x'_k)$ such that

$$\sup_{B_{X^*}}(x) = \|x\| < \varepsilon$$

but $y_o(x) > \varepsilon$, a contradiction. □

Corollary 2 generalizes Rainwater's theorem [21], where $B = \text{Ext}(B_{X^*})$, and Lebesgue's dominated convergence theorem for uniformly bounded sequences of continuous functions on a compact space. Rainwater's theorem can be shown through integral representation arguments; here we are bound to use a combinatorial argument since our only assumption on B is the norm-attainment, and no assumption is made on topological regularity.

In the last years, new proofs of Simons' inequality were provided by Simons himself [24] and E. Oja [20]. Several applications of Simons' inequality to smoothness in Banach spaces were found (see [6], where results from [22], [26] are reproved, [3], [13]). We refer to [7], or to the books [1] and [14], for some of these applications. In the present work we will display some recent progress where this technique plays the leading role.

We will use the following notation.

Definition 3. *Let X be a Banach space. A boundary is a subset B of $S_{X^*} = \{y \in X^*; \|y\| = 1\}$ such that for every $x \in X$, there exists $y \in B$ such that $y(x) = \|x\|$.*

With this notation we have the crucial:

Lemma 4. *Let X be a Banach space, and let $B \subseteq S_{X^*}$ be a boundary. If there exist $\varepsilon > 0$ and $\{y_n; n \geq 1\}$ in X^* such that $B \subseteq \bigcup_{n\geq 1}^{\infty} B(y_n; 1-\varepsilon)$ then $X^* = \overline{\text{span}}^{\|\|}(\{y_n; n \geq 1\})$. In particular X^* is separable.*

Proof. Indeed if not, there exists $z \in X^{**}$ with $\|z\| = 1$ and $z(y_n) = 0$ for every $n \geq 1$. Since $\|z\| = 1$ there exists $y \in B_{X^*}$ such that $z(y) > 1 - \frac{\varepsilon}{2}$. Since B_X is w^*-dense in $B_{X^{**}}$, there exists a sequence (x_k) in B_X such that

$$\lim_{k\to\infty} x_k(y_n) = z(y_n) = 0 \tag{1}$$

for every $n \geq 1$, and

$$\lim_{k\to\infty} x_k(y) = z(y) > 1 - \frac{\varepsilon}{2}. \tag{2}$$

By (2) we may assume without loss of generality that $x_k(y) > 1 - \frac{\varepsilon}{2}$ for every $k \geq 1$. It follows from (1) and our assumption that for every $b \in B$

$$\overline{\lim_{k\to\infty}} x_k(b) \leq 1 - \varepsilon.$$

Hence by Simons' inequality there exists $x \in \operatorname{conv}(x_k)$ such that

$$\|x\| < 1 - \frac{\varepsilon}{2}$$

but we have

$$\|x\| \geq x(y) > 1 - \frac{\varepsilon}{2}$$

and this is a contradiction. □

Remark 5. The conclusion fails if we replace in Lemma 4, $B(y_n; 1-\varepsilon)$ by $B(y_n; 1-\varepsilon_n)$ where $\lim \varepsilon_n = 0$. In fact, there are non-separable dual spaces X^* such that

$$B_{X^*} \subseteq \bigcup_{m=1}^{\infty} B(y_n; 1-\varepsilon_n) \tag{3}$$

with $\varepsilon_n > 0$ for every n.

Here is an example: we equip the space $Y = c_o(\mathbb{N})$ with the norm

$$|||(u_n)||| = \|u_n\|_\infty + \sum_{n=1}^{\infty} 2^{-n}|u_n|.$$

Clearly the bidual norm on $Y^{**} = \ell_\infty(\mathbb{N})$ is defined by the same formula. It is not difficult to check that for every $u \in \ell_\infty(\mathbb{N})$ we have

$$\inf\{\|u-v\|_\infty; v \in c_o(\mathbb{N})\} = \inf\{|||u-v|||; v \in c_o(\mathbb{N})\}$$

and moreover that the second infimum is never attained in the above equation. In particular

$$\inf\{|||u-v|||; v \in c_o(\mathbb{N})\} < |||u||| \tag{4}$$

for every $u \in \ell_\infty(\mathbb{N})$. Let now C be a countable norm-dense subset of $c_o(\mathbb{N})$, and $D = Q \cap (0,1)$. We enumerate $C \times D = \{(x_n, \rho_n); n \geq 1\}$. It follows from (4) that if $\varepsilon_n = 1 - \rho_n$

$$\{|||u||| \leq 1\} \subseteq \bigcup_{n=1}^{\infty} B_{|||.|||}(x_n; 1-\varepsilon_n)$$

and thus $Y^{**} = X^*$ provides us with an example.

Let us mention however, that if X is reflexive, (4) implies that X^* is separable. Indeed for any $x \in S_X$ we can pick $y \in S_{X^*}$ with $y(x) = 1$. Since $y \in B(y_n, 1-\varepsilon_n)$ for some n, we have $y_n(x) \neq 0$ for this n, and thus $\{y_n; n \geq 1\}$

separates X. It follows that the compact set (B_X, w) is metrizable, hence X^* is separable.

We recall now a definition from [4].

Definition 6. *Let X be a Banach space and $u \in S_X$. The norm $\|.\|$ of X is said to be strongly sub-differentiable (in short, s.s.d.) at u if*

$$\lim_{t \to 0^+} \frac{\|u + tx\| - 1}{t}$$

exists uniformly on $x \in S_X$. The norm of X is s.s.d. if it is s.s.d. at every $u \in S_X$.

The following lemma is shown in [4].

Lemma 7. *The following assertions are equivalent:*
1) *The norm of X is s.s.d. at $u \in S_X$.*

2) *The set*

$$J(u) = \{y \in X^*; y(u) = \|y\| = 1\}$$

is strongly exposed by u in B_{X^*}, that is: for any $\varepsilon > 0$, there is $\eta > 0$ such that $y(u) > 1 - \eta$ and $\|y\| \leq 1$ implies $\operatorname{dist}(y, J(u)) < \varepsilon$.

Proof. 2) $\Rightarrow$ 1): For any $h \in S_X$, let $\varphi_h(t) = \|u + th\|$. The function φ_h is convex, hence we have for any $t > 0$,

$$\varphi'_h(0) \leq t^{-1}[\varphi_h(t) - \varphi_h(0)] \leq \varphi'_h(t)$$

where φ'_h denotes the right derivative of φ_h. It is easily seen that

$$\varphi'_h(0) \geq \langle y, h \rangle$$

for every $y \in J(u)$ and

$$\varphi'_h(t) = \langle y_t, h \rangle$$

for some $y_t \in S_{X^*}$ with $\langle y_t, u + th \rangle = \|u + th\|$. Hence for every $y \in J(u)$

$$\langle y, h \rangle \leq t^{-1}[\|u + th\| - 1] \leq \langle y_t, u + th \rangle.$$

We have $\lim\limits_{t \to 0^+} \langle y_t, u \rangle = 1$ and thus by 2) we have $\lim\limits_{t \to 0^+} \operatorname{dist}(y_t, J(u)) = 0$. The result follows now from the above inequalities.

1) $\Rightarrow$ 2): If 2) fails there is a sequence y_n in B_{X^*} with $\lim y_n(u) = 1$ and $\operatorname{dist}(y_n, J(u)) > \alpha > 0$ for every n. By Hahn-Banach we find $h_n \in S_X$ such that

$$\langle y_n, h_n\rangle - \langle y, h_n\rangle > \alpha$$

for every $y \in J(u)$. Pick $y \in J(u)$ such that

$$y(h_n) = \lim_{t\to 0^+} t^{-1}[\|u + th_n\| - 1]$$

We have

$$\begin{aligned} \|u + th_n\| - 1 &\geq \langle y_n, u + th_n\rangle - y(u) \\ &= \langle y_n - y, u\rangle + t\langle y_n - y, h_n\rangle + ty(h_n) \end{aligned}$$

hence

$$\begin{aligned} t^{-1}[\|u + th_n\| - 1] - y(h_n) &\geq \langle y_n - y, h_n\rangle + t^{-1}\langle y_n - y, u\rangle \\ &\geq \alpha + t^{-1}\langle y_n - y, u\rangle \\ &\geq \frac{\alpha}{2} \end{aligned}$$

provided that

$$t \geq \frac{2}{\alpha}\langle y - y_n, u\rangle$$

but since $\lim_{n\to\infty} \langle y - y_n, u\rangle = 0$, this clearly shows that

$$\lim_{t\to o^+} t^{-1}[\|u + th\| - 1]$$

is not uniform on $h \in S_X$. $\square$

The following proposition answered a question of R. Paya.

Proposition 8. *Let X be a Banach space with a strongly sub-differentiable norm. Then X is an Asplund space.*

Proof. We may and do assume that X is separable; we have to show that X^* is separable. By Mazur's theorem (see [1, Chapter I]), the norm of X is Gâteaux-differentiable on a dense G_δ subset Ω of S_X. Let D be a countable dense subset of S_X consisting of such points. Pick any $x \in S_X$; there exist $x_n \in D$ with $\lim \|x - x_n\| = 0$. If $\|y_n\| = y_n(x_n) = 1$, we have $\lim y_n(x) = 1$ and thus by Lemma 7

$$\lim_{n\to\infty} \operatorname{dist}(y_n, J(x)) = 1.$$

It follows that

$$B = S_{X^*} \cap (\bigcup_{n=1}^{\infty} B(y_n, 1/2))$$

is a boundary; and now Lemma 4 concludes the proof. □

We refer to R. Haydon's fundamental work [15, 16] for examples of Asplund spaces without Gâteaux-differentiable norms. M. Jimenez and J. P. Moreno [19] have considered different examples of Asplund spaces without equivalent Fréchet-smooth norms. These examples enjoy in fact stronger properties: namely, they don't even have an equivalent s.s.d. norm, since every equivalent norm is such that the dual space contains a proper closed norming subspace (see Corollary 13 below). Very recently, S. Todorcevic [27] has been able to construct a similar space with no Gâteaux-smooth equivalent norm. However the construction of such spaces, which have the so-called Kunen property, relies crucially on the Continuum Hypothesis, or at least on some weakenings of this property which still do not belong to the classical set theory ZFC. The following question is still open:

Question 9. *Does there exist an example in ZFC of an Asplund space with no equivalent strongly sub-differentiable norm?*

If the norm of X is Fréchet-differentiable on $X\backslash\{0\}$ then every closed convex bounded subset of X is an intersection of balls (see [5]). Clearly this cannot be true for s.s.d. norms, since for instance any norm on a finite-dimensional space is s.s.d. However a natural weakening of this intersection property holds true:

Theorem 10. *Let X be a Banach space with a s.s.d. norm. Then every bounded and weakly closed subset of X is an intersection of finite unions of balls.*

P r o o f. The main concept here is the "ball topology" b_X, which is defined and studied in [10]: given a Banach space X (and its norm $\|.\|$), the ball topology is the coarsest topology for which the closed balls (in the given norm) are closed. In other words, a typical b_X-neighbourhood V of 0 has the form

$$V = X\backslash \bigcup_{i=1}^{n} B(x_i, \rho_i)$$

where $\rho_i < \|x_i\|$ for $i = 1, 2, \ldots, n$. What we have to show is that the ball topology coincides with the weak topology on the bounded subsets of X.

A subspace N of a dual space X^* is said to be norming if

$$\|x\| = \sup\{y(x); y \in N \cap B_{X^*}\}$$

for every $x \in X$. The following claim was shown in [10]

Claim 11. *Let X be a separable space and $y \in X^*$. If y belongs to all norming subspaces of X^*, then y is b_X-continuous on B_X.*

Proof. Let $\{x_n; n \geq 1\}$ be a sequence in B_X with $b_X - \lim_{n\to\infty} x_n = 0$. Then [10, Lemma 2.1] there exists a subsequence $\{x'_n\}$ of $\{x_n\}$ such that if

$$v_n \in \operatorname{conv}\{x'_i; i \geq n\}$$

then $b_X - \lim_{n\to\infty} v_n = 0$. The construction of the subsequence $\{x'_n\}$ is completed by induction (see [10]). We say that $\{x'_n\}$ is convex-clustering at 0. If y is not b_X-continuous on B_X, there exists a sequence $\{x_n\}$ which is convex-clustering at 0, and such that $\lim y(x_n) = \alpha \neq 0$. Since $b_X - \lim v_n = 0$ for any $v_n \in \operatorname{conv}\{x_i; i \geq n\}$, we have for every $x \in X$,

$$\lim_{n\to\infty} \operatorname{dist}(x, \operatorname{conv}\{x_i; i \geq n\}) \geq \|x\|.$$

Let $z \in X^{**}$ be a ω^*-cluster point of $\{x_n\}$. Given $\varepsilon > 0$, there is N such that

$$\operatorname{dist}(x, \operatorname{conv}\{x_i; i \geq N\}) \geq \|x\| - \frac{\varepsilon}{2}$$

and thus there is $y \in S_{X^*}$ such that

$$y(x_i - x) \geq \|x\| - \varepsilon$$

for any $i \geq N$, and thus

$$\|z - x\| \geq y(z - x) \geq \|x\| - \varepsilon$$

it follows that $\|z - x\| \geq \|x\|$ for any $x \in X$. Thus by homogeneity

$$\|\lambda z - x\| \geq \|x\|$$

for any $x \in X$ and $\lambda \in \mathbb{R}$. We have therefore for any $x \in X$.

$$\|x\| = \operatorname{dist}(x, \mathbb{R}z) \tag{5}$$

For any subspace Y of X^* and any $x \in X$, we have

$$\|x\|_{Y^*} = \|x\|_{X^{**}/Y^\perp} = \operatorname{dist}(x, Y^\perp). \tag{6}$$

Since $\operatorname{span} z = \operatorname{Ker}(z)^{\perp}$, (5) means that

$$\|x\| = \|x\|_{\operatorname{Ker}(z)^*}$$

that is, $\operatorname{Ker}(z)$ is norming. But $\lim y(x_n) = y(z) = \alpha \neq 0$, hence we have found a norming subspace, namely $\operatorname{Ker}(z)$, to which y does not belong. □

Now [10, Proposition 2.5] provides us with a separable reduction argument.

Claim 12. *Let X be a Banach space, and $y \in X^*$. If for any $S \subseteq X$ separable, $Y_{|S}$ is b_S-continuous on B_S, then y is b_X-continuous on B_X.*

Proof. If it is not so, there is $\varepsilon > 0$ such that $A = B_X \cap y^{-1}((-\varepsilon, \varepsilon))$ is not a b_X- neighbourhood of 0 in B_X. We define by induction a sequence x_n in B_X. If $E_n = \operatorname{span}\{x_1, \ldots, x_n\}$, let D_n be a countable dense subset of E_n. We enumerate $\{V_n^k; k \geq 1\}$ the sets of the form

$$X \setminus \bigcup_{j=1}^{l} B(u_j, l_j)$$

with $u_j \in D_n, l_j \in Q \cap (0, \|u_j\|)$. The construction is done as follows: pick any $x_1 \in B_X \setminus A$. If $x_1, x_2, \ldots, x_{n-1}$ have been determined, pick

$$x_n \in (B_X \cap V_1^{n-1} \cap V_2^{n-2} \cap \ldots \cap V_{n-1}^1) \setminus A.$$

This can be done since A is not a b_X-neighbourhood of 0 in B_X. We now let $S = \overline{\operatorname{span}}^{\|\cdot\|}(\{x_n; n \geq 1\})$. The sets

$$B_X \cap V_1^{n-1} \cap V_2^{n-2} \cap \ldots \cap V_{n-1}^1 \qquad (n \geq 1)$$

form a base of b_S-neighbourhoods of 0. Hence

$$b_S - \lim_{n \to \infty} x_n = 0$$

and since $|y(x_n)| > \varepsilon$ for every n, $y_{|S}$ is not b_S-continuous on B_S. This concludes the proof of the claim. □

For proving Theorem 10, we now have to show that every $y \in X^*$ is b_X-continuous on B_X. By claim 11 and 12, it suffices to show that for every $S \subseteq X$ separable, S^* contains no proper norming subspace. Of course the norm is s.s.d. on S. If $N \subseteq S^*$ is norming then $N \cap B_{S^*}$ is w^*-dense in B_{S^*}. Since (B_{S^*}, w^*) is compact metrizable we may pick a sequence $\{y_n\}$ in $N \cap B_{S^*}$ which is w^*-dense in B_{S^*}.

It follows from Lemma 7 that

$$J(u) \cap (\bigcup_{n=1}^{\infty} B(y_n; 1/2)) \neq \emptyset$$

for every $u \in S_X$. This means that

$$B = S_{X^*} \cap (\bigcup_{n=1}^{\infty} B(y_n; 1/2))$$

is a boundary. Hence by Lemma 4,

$$X^* = \overline{\text{span}}^{\|\cdot\|}\{y_n; n \geq 1\}$$

and thus $X^* = N$. This concludes the proof of Theorem 10. □

Corollary 13. *If a Banach space X has a s.s.d.norm, then X^* contains no proper norming subspace.*

P r o o f. It suffices to show that for any $z \in S_{X^{**}}$ the space $\text{Ker}(z)$ is not a norming subspace of X^*. If it were norming, then by the equation (6) we would have

$$\|z - x\| \geq \|x\| \tag{7}$$

for every $x \in X$. Pick $y \in B_{X^*}$ such that $z(y) > 0$. Let $\{x_\alpha\}$ be a net in B_X such that $w^* - \lim x_\alpha = z$. Since the norm is w^*-l.s.c. we have by (7) that

$$\underline{\lim}_{\alpha} \|x_\alpha - x\| \geq \|x\|$$

and thus

$$b_X - \lim_{\alpha} x_\alpha = 0.$$

On the other hand

$$\lim_{\alpha} y(x_\alpha) = z(y) > 0$$

and thus y is not b_X-continuous on B_X. This contradicts Theorem 10. □

We say that a space X has the finite-infinite intersection property (in short, $I.P._{f,\infty}$) if for any collection $\{B_\alpha; \alpha \in J\}$ of closed balls such that

$$\cap\{B_\alpha; \alpha \in J\} = \emptyset$$

there exists a finite subset F of J such that

$$\cap\{B_\alpha; \alpha \in F\} = \emptyset.$$

With this notation we have

Corollary 14. *Let X be a Banach space with a s.s.d.norm. If X has the $I.P._{f,\infty}$ then X is reflexive.*

Proof. Pick any $z \in S_{X^{**}}$. We consider the collection $\mathcal{B}$ of all balls

$$B_{x,\rho} = \{x' \in X : \|x - x'\| \leq \rho\}$$

where $x \in X$ and $\rho > \|x - z\|$. It follows from the local reflexivity principle that any finite subcollection of $\mathcal{B}$ has a non-empty intersection. By the $I.P._{f,\infty}$ we have $\bigcap \mathcal{B} \neq \emptyset$; if $x' \in \bigcap \mathcal{B}$ we have $\|x - x'\| \leq \|x - z\|$ for any $x \in X$. this can be written $\|x''\| \leq \|(z - x') - x''\|$ for any $x'' \in X$, and this means by (6)

$$\|x''\| = \operatorname{dist}(x'', \mathbb{R}(z - x')) = \|x''\|_{N^*}$$

if $N = \operatorname{Ker}(z - x')$. Therefore N is norming; but by Corollary 13 this forces $N = X^*$ and thus $z = x' \in X$. Therefore $X = X^{**}$. □

Example 15. If there is a norm-one projection Π from X^{**} onto X, then X has the $I.P._{f,\infty}$ (use w^*-compactness in X^{**} and then Π). In particular if X is a dual space with a s.s.d. norm then X is reflexive. This latter statement was shown in [4] with a different proof.

Remark 16. By Corollary 13 and [12, Lemma 2.4], if the norm of X is s.s.d. then for every $z \in X^{**}$ we have

$$\bigcap_{x \in X} B_{X^{**}}(x, \|z - x\|) = \{z\}.$$

It follows for instance that X has the "unique extension property", that is, $Id_{X^{**}}$ is the only contractive map from X^{**} to X^{**} whose restriction to X is Id_X.

Following [13], we will now describe applications of Simons' inequality in a different direction. It is known (see [1, Chapter I]) that non-Asplund spaces have equivalent norm which are "rough", that is, uniformly non-Fréchet differentiable. However, given norms may have many points of Fréchet-smoothness: for instance if a compact set K is the closure of the set of its isolated points then the canonical norm of $C(K)$ is Frechet smooth at every point of a norm-dense open set.

Nevertheless, the following statement shows that every equivalent norm on a non-Asplund space has "some kind" of roughness property. In what follows, for any $x \in S_X$ we denote

$$J(x) = \{y \in S_{X^*}; y(x) = 1\}.$$

Proposition 17. *Let X be a non-Asplund space, and $\varepsilon > 0$. There exists a norm convergent sequence $\{x_n\}$ in S_X such that for any $n \neq k$,*

$$\operatorname{dist}(J(x_n), J(x_k)) > 1 - \varepsilon.$$

Proof. We first assume that X is separable. By Mazur's theorem we may pick a countable subset D of S_X, consisting of points of Gâteaux-smoothness of the norm of X, and norm-dense in S_X. By Lemma 4, the set

$$B = S_{X^*} \cap (\cup_{x \in D} B(J(x); 1 - \varepsilon))$$

is not a boundary, hence there is $x_o \in S_X$ such that

$$J(x_o) \cap B(J(x); 1 - \varepsilon) = \emptyset \tag{8}$$

for every $x \in D$. We pick a sequence $\{x_n\}$ in D such that $\lim_n \|x_n - x_o\| = 0$. Observe that any w^* -cluster point to the sequence $J(x_n)$ belongs to $J(x_o)$. By (8) and the w^* — lower semi-continuity of the norm, it follows that there exists a subsequence $\{x'_n\}$ of $\{x_n\}$ with $\operatorname{dist}(J(x'_n), J(x'_k)) > 1 - \varepsilon$ for any $k \neq n$. If X is not separable, we may find a separable subspace S of X such that S^* is not separable. By the above there exists a norm-convergent sequence $\{x_n\}$ in the unit sphere of S such that

$$\operatorname{dist}(J_S(x_n), J_S(x_k)) > 1 - \varepsilon$$

for every $n \neq k$. The result follows since the canonical quotient map from X^* onto S^* has norm one. □

If X is any separable Banach space, the support mapping J has a selector σ which is of the first Baire class from $(S_X, \|.\|)$ to (B_{X^*}, w^*); that is, there exist a sequence $\{\sigma_n\}$ of continuous map from $(S_X, \|.\|)$ to (B_{X^*}, w^*) such that for every $x \in S_X$

$$w^* - \lim_{n \to \infty} \sigma_n(x) = \sigma(x)$$

exists, and moreover $\langle \sigma(x), x\rangle = 1$ ([18]; see [1, Chapter 1]). We will use this "natural" selector for characterizing separable non-Asplund spaces. Let us point out that the following property is valid for one equivalent norm if and only if it is satisfied by every equivalent norm.

Propositon 18. *Let X be a separable Banach space, and let $\sigma(x) \in J(x)$ be a $(\|.\| - w^*)$ first Baire class selector of the support mapping J from S_X to S_{X^*}. The following assertions are equivalent:*

(i) X^* is not separable.

(ii) For any $\varepsilon > 0$, there exists a subset K of $(S_X, \|.\|)$ which is homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$, and such that $\|\sigma(x) - \sigma(x')\| > 1 - \varepsilon$ for every $x \neq x'$ in K.

Proof. $(ii) \Rightarrow (i)$ is clear since $\{0,1\}^{\mathbb{N}}$ is not countable.

$(i) \Rightarrow (ii)$: Since $(S_X, \|.\|)$ is a Polish space and σ is of the first Baire class, $B = \sigma(s_X)$ is an analytic subset of (B_{X^*}, w^*). Therefore there exists a continuous map Ψ from the Polish space $\sum = \mathbb{N}^{\mathbb{N}}$ onto (B, w^*). We fix $\varepsilon > 0$. Let us say that a subset A of $\sum$ has (*) if there exists a countable subset D of X^* such that

$$\Psi(A) \subseteq \bigcup_{x\in D} B(x, 1-\varepsilon).$$

Since $B = \sigma(S_X)$ is clearly a boundary and X^* is not separable, Lemma 4 shows that $\sum$ fails property (*). We denote

$$O = \bigcup \{V \subseteq \sum ; V open, V has\ (*)\}.$$

Since $\sum$ is metrizable and separable it is hereditarily Lindelof; therefore, O has (*), and thus $\sum \setminus O = F \neq \emptyset$. By definition of F, $(V \cap F)$ fails (*) for any open set V which intersects F. In particular

$$\|.\| - \mathrm{diam}(\Psi(F)) > 1 - \varepsilon$$

and we can pick x_1, x_2 in F such that $\|\Psi(x_1) - \Psi(x_2)\| > 1 - \varepsilon$. Since Ψ is continuous onto (B, w^*) and $\|.\|$ is w^* l.s.c., we can find V_0, V_1 open with $x_i \in V_i$ $(i = 0, 1)$ and such that

$$\|\Psi(x) - \Psi(x')\| > 1 - \varepsilon$$

for any $x \in V_0$ and $x' \in V_1$. The sets $(V_i \cap F)(i = 0, 1)$ both fail $(*)$ and therefore we can apply again the previous argument, but this time to $(V_0 \cap F)$ and $(V_1 \cap F)$ instead of $\sum$. We receive open sets V_{00}, V_{01}, V_{10} and V_{11} and iterate the construction. We can clearly assume that the diameter (in $\sum$) of the open sets of level n is less than n^{-1}. In this way we construct a subset C of $\sum$, which is homeomorphic to $\{0,1\}^{\mathbb{N}}$, and such that

$$\|\Psi(x) - \Psi(x'\| > 1 - \varepsilon \tag{9}$$

for every $x \neq x'$ in C. The set $K_0 = \Psi(C)$ is a subset of (B, w^*) which is w^*-homeomorphic to $\{0,1\}^{\mathbb{N}}$ and $\|y - y'\| > 1 - \varepsilon$ for any $y \neq y'$ in K_0.

To conclude the proof it suffices to observe that since σ is of the first Baire class, $\sigma^{-1}(K_0) = \Omega_0$ is an uncountable G_δ subset of S_X. Since Ω_0 is an uncountable G_δ in the Polish space S_X it contains a compact set K which is homeomorphic to $\{0,1\}^{\mathbb{N}}$. Clearly this set K works. □

Note that it follows from the proof that we may also require that the restriction of σ to K be $(\|.\| - w^*)$ continuous.

Remark 19. Using Stegall's construction [25], the above result may be refined. In fact it can be shown that any w^*-analytic boundary contains a biorthogonal system which is w^*-homeomorphic to $\{0,1\}^{\mathbb{N}}$. That is, instead of merely obtaining that our set $K_0 = \Psi(C)$ satisfies (9) we may construct K_0 in such a way that for every $y \in K_0$, there is z in X^{**} with $\|z\| = 1, z(y) > 1 - \varepsilon$ and $z(y') = 0$ for any $y' \neq y$ in K_0. This result is in fact implicit in Stegall's original paper [25].

Using determinacy axioms this can be extended to every boundary B such that (B, w^*) belongs to the projective hierarchy (see [11]). It seems to be unknown whether every boundary B in the unit sphere of a non-separable dual X^* contains an uncountable biorthogonal system.

Remark 20. Let us mention some very recent work related with the subject of this note. In [2], a version of Simons' inequality for convex functions is shown, with some of its applications. In [8, 9], strong sub-differentiability of dual norms is applied to proximinality questions.

Let us conclude these notes with some open problems.

Question A. Let X be a separable space with non-separable dual. Does there exist a subset K of $(S_X, \|.\|)$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, and such that for some $\eta > 0, \|y - y'\| > \eta$ if $y \in J(x), y' \in J(x')$ with $x \neq x'$ in K?

Clearly this would improve on propositions 17 and 18. Let us state a few comments around this question. For a subset A of S_X, we define

$$\delta(A) = \sup\{d(J(x), J(x')); x, x' \in A\}$$

where

$$d(J(x), J(x')) = \inf\{\|y - y'\|; y \in J(x), y' \in J(x')\}.$$

For $\eta > 0$ and F a subset of S_X, we define

$$D_\eta(F) = F \backslash \bigcup \{ \ V open; \ \delta(V \cap F) < \eta \}.$$

The operation D_η is called a "derivation" in descriptive set theory; its prototype is the Cantor derivation which consists into removing isolated points from compact sets. We define now $F_0 = S_X$, $F_1 = D_\eta(F_0), \dots$ and for any ordinal $\alpha, F_{\alpha+1} = D_\eta(F_\alpha)$; for limit ordinals β, we let of course

$$F_\beta = \bigcap_{\alpha<\beta} F_\alpha.$$

Since S_X is a Polish space and the F_α's are closed, there is a smallest countable ordinal $\alpha < \omega_1$ such that $D_\eta(F_\alpha) = F_\alpha$. The set F_α is then the largest "perfect" subsect (with respect to the derivation D_η) contained in S_X.

It is not difficult to show (along the lines of the proof of proposition 18) that our question A has a positive answer if and only if this "perfect" subset F_α is non empty. It follows of course that if X^* is separable, D_η "stops" on the empty set, after a countable number of steps, for any $\eta > 0$. We are left with the question of the converse.

The spaces $C(K)$, equipped with their canonical norm, provide an instructive class of examples: indeed the set $B = J(\mathbb{1}_K) \cup J(-\mathbb{1}_K)$ is a boundary although $C(K)^*$ is not separable if K is not countable, and this shows that there is no hope of extending Lemma 4 to $(1-\varepsilon)$-neighbourhoods of sets $J(x_n)$. If $X = (C(K), \|.\|_\infty)$ we have $d(J(x), J(\mathbb{1}_K)) = 0$ or $d(J(x), J(-\mathbb{1}_K)) = 0$ for every $x \in X$. However it is easily seen that $\delta(V) = 2$ for any non-empty open subset V of S_X and thus $D_\eta(S_X) = S_X$ for any $\eta < 2$.

Let us also observe that if S_X contains no D_η-perfect non empty subset for every $\eta > 0$, then the set $\mathrm{Exp}_*(B_{X^*})$ of points of B_{X^*} which are exposed in B_{X^*} by points of X- or equivalently, the set of derivatives of the norm at points of Gateaux-smoothness — is norm-separable. We are therefore led to

Question B. Let X be a separable space such that the set $\mathrm{Exp}_*(B_{X^*})$ is norm-separable. Is X^* separable?

Note that no example is known of a separable Banach space X not containing $l_1(\mathbb{N})$, and such that the norm closed linear span of $\mathrm{Exp}_*(B_{X^*})$ is different from X^*. A positive answer to question B looks quite unlikely. But we can strengthen its assumptions and ask

Question C. Let X be a separable space. Suppose that there exists a countable subset $E = \{x_n; n \geq 1\}$ of S_{X^*} such that for every $x \in S_X$ such that

$J(x)$ is norm-separable, $\inf\{d(J(x), x_n); n \geq 1\} = 0$. Does it follow that X^* is separable?

It is not difficult to check that if the derivations D_η stop on the empty set for every $\eta > 0$ then the assumption of Question C is satisfied. Therefore a positive answer to Question C, and *a fortiori* to Question B, would provide a positive solution to Question A. If X is not assumed to be separable, Questions B and C have negative answers: the space $X = (l_\infty(\mathbb{N}), \|.\|_\infty)$ provides an example.

Appendix 1. *James' theorem in the separable case.*

Let C be a closed convex separable subset of a Banach space X such that $\sup\{y(x); x \in C\}$ is attained for every $y \in X^*$. Then C is weakly compact. Let us show it through Simons' inequality: first, the uniform boundedness principle shows that C is bounded. If C is not weakly compact, $\overline{C}^* \setminus C \neq \emptyset$, where $\overline{C}^*$ denotes the closure of C in (X^{**}, w^*). There is therefore $t \in X^{***}$ with $t \equiv 0$ on C but $t(z_o) > \varepsilon > 0$ for some $z_o \in \overline{C}^*$. Since C is norm- separable, there is a sequence $\{y_n\}$ in X^* with

$$\lim_{n\to\infty} y_n(z) = t(z)$$

for every $z \in \text{span}(C \cup \{z_o\})$. We may and do assume that $y_n(z_o) > \varepsilon$ for all n. Our assumptions allow us to apply Theorem 1 with $E = \overline{C}^*, B = C$ and $y_n = x_n$. We find therefore $g \in \text{conv}(y_n)$ with

$$\sup\{g(z); z \in \overline{C}^*\} < \varepsilon$$

but this contradicts $y(z_o) > \varepsilon$ and concludes the proof.

It should be mentioned that the non-separable case of James' theorem seems to be much harder to handle. It would be extremely interesting to reduce it to a "simple" combinatorial principle.

Appendix 2. *A proof of Simons' inequality.*

We present here the original proof (from [23]) of Theorem 1.

We let

$$m = \inf\{\sup_E(g); g \in \text{conv}(x_n)\}$$

and for $b \in B$

$$u(b) = \overline{\lim}_{n\to\infty} x_n(b).$$

We have to show that $\sup_B u \geq m$. Let $\delta > 0$ be arbitrary. Choose $\lambda \in (0,1)$ such that

$$m - \delta(1+\lambda) - M\lambda \geq (m - 2\delta)(1-\lambda)$$

where

$$M = \sup\{\|x_n\|_\infty ; n \geq 1\}.$$

We will show that $\sup_B u \geq m - 2\delta$. Put $C_n = \operatorname{conv}\{x_p ; p \geq n\}$. By induction we choose $y_n \in C_n$ such that

$$\sup_B \left(\sum_{p \leq n} \lambda^{p-1} y_p \right) \leq \inf \left\{ \sup_B \left(\sum_{p \leq n-1} \lambda^{p-1} y_p + \lambda^{n-1} y \right) ; y \in C_n \right\} + \delta \left(\frac{\lambda}{2} \right)^n .$$

Since for all $n \geq 1$,

$$\frac{y_n + \lambda y_{n+1}}{1+\lambda} \in C_n$$

we have

$$\sup_B \left(\sum_{p \leq n} \lambda^{p-1} y_p \right) \leq \sup_B \left(\sum_{p \leq n-1} \lambda^{p-1} y_p + \lambda^{n-1} \left(\frac{y_n + \lambda y_{n+1}}{1+\lambda} \right) \right) + \delta \left(\frac{\lambda}{2} \right)^n .$$

Let $z_o = 0, z_n = \sum_{p \leq n} \lambda^{p-1} y_p$ for $n \geq 1$, and $z = \sum_{n \geq 1} \lambda^{n-1} y_n$. Multiplying the above inequality by $(1+\lambda)$ leads to

$$\begin{aligned} (1+\lambda) \sup_B z_n &\leq \sup_B (\lambda z_{n-1} + z_{n+1}) + \delta(1+\lambda) \left(\frac{\lambda}{2} \right)^n \\ &\leq \lambda \sup_B z_{n-1} + \sup_B z_{n+1} + \delta(1+\lambda) \left(\frac{\lambda}{2} \right)^n \end{aligned}$$

for $n \geq 1$. Thus

$$\lambda^{-n} (\sup_B z_{n+1} - \sup_B z_n) \geq \lambda^{-n+1} (\sup_B z_n - \sup_B z_{n-1}) - \frac{\delta(1+\lambda)}{2^n}.$$

Since $\sup_B z_1 - \sup_B z_o = \sup_B z_1 \geq m$, the above inequality and an easy induction yield

$$\begin{aligned} \lambda^{-n+1} (\sup_B z_n - \sup_B z_{n-1}) &\geq m - \delta(1+\lambda) \left(\sum_{i=1}^{n-1} 2^{-i} \right) \\ &\geq m - \delta(1+\lambda). \end{aligned}$$

Therefore

$$\begin{aligned}\sup_B z - \sup_B z_{n-1} &= \sum_{p\geq n}(\sup_B z_p - \sup_B z_{p-1})\\ &\geq \sum_{p\geq n}\lambda^{p-1}(m-\delta(1+\lambda)).\end{aligned}$$

Hence

$$\sup_B z - \sup_B z_{n-1} \geq \frac{\lambda^{n-1}}{1-\lambda}(m-\delta(1+\lambda)).$$

Since

$$(1-\lambda)\sum_{n\geq 1}\lambda^{n-1} = 1,$$

we may apply our assumption to $(1-\lambda)z$, and find $b \in B$ such that

$$z(b) = \sup_B z.$$

For all $n \geq 1$, we get

$$\begin{aligned}\lambda^{n-1}y_n(b) &= z(b) - z_{n-1}(b) - \sum_{p\geq n+1}\lambda^{p-1}y_p(b)\\ &\geq \sup_B z - \sup_B z_{n-1} - \sum_{p\geq n+1}\lambda^{p-1}M\\ &\geq \frac{\lambda^{n-1}}{1-\lambda}(m-\delta(1+\lambda)) - \frac{\lambda^n}{1-\lambda}M.\end{aligned}$$

Hence By the choice of λ, we have $y_n(b) \geq m - 2\delta$ for each $n \geq 1$. Since $y_n \in C_n$, it follows that

$$u(b) = \overline{\lim_{n\to\infty}} x_n(b) \geq m - 2\delta$$

and thus $\sup_B u \geq m - 2\delta$. The result follows since $\delta > 0$ is arbitrary.

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