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Strong Maximum Principle for Viscosity Solutions of Fully Nonlinear Cooperative Elliptic Systems

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Abstract: In this paper, we consider the validity of the strong maximum principle for weakly coupled, degenerate and cooperative elliptic systems in a bounded domain. In particular, we are interested in the viscosity solutions of elliptic systems with fully nonlinear degenerated principal symbol. Applying the method of viscosity solutions, introduced by Crandall, Ishii and Lions in 1992, we prove the validity of strong interior and boundary maximum principle for semi-continuous viscosity sub- and super-solutions of such nonlinear systems. For the first time in the literature, the strong maximum principle is considered for viscosity solutions to nonlinear elliptic systems. As a consequence of the strong interior maximum principle, we derive comparison principle for viscosity sub- and super-solutions in case when on of them is a classical one. The main novelty of this work is the reduction of the smoothness of the solution. In the literature the strong maximum principle is proved for classical C^2 or generalized C^1 solutions, while we prove it for semi-continuous ones.

Keywords: strong maximum principle; degenerate fully non-linear elliptic systems; viscosity solutions

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1. Introduction

In this paper, we give the latest result of research on the validity of Maximum Principle (MP) for fully nonlinear, weakly-coupled elliptic systems.

In 1927, the study on MP was started by E. Hopf with his notorious paper [1]. He studied a strictly elliptic operator

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u + \sum_{i=1}^{n} b^{i}(x)D^{i}u + c(x)u$$

in some domain $\Omega \subseteq R^n$. Hopf's maximum principle states that if $c = 0$ and $Lu \geq 0$ ($Lu \leq 0$) in Ω, then u is a constant if u it attains a maximum (minimum) at some interior point for Ω. Moreover, suppose $c \geq 0$ and $c/\lambda(x)$ is bounded, where $\lambda(x)$ is the function from the ellipticity condition

$$0 < \lambda(x) \cdot |\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq \Lambda(x) \cdot |\xi|^2, x \in \overline{\Omega}, \xi \neq 0,$$

Then, u does not attain non-negative maximum (non-positive minimum) at interior for Ω point, if u is not a constant. Earlier results on Hoph maximum principle under much more restrictive hypothesis are discussed in [2] page 156.

Following E. Hopf, the classical maximum principle was discussed in many works, in between them the famous books of M. Protter and H. Weinberger [2] and D. Gilbarg and N. Trudinger [3], as well in the survey paper of P. Pucci and Serrin J.P [4], etc. Analysis of the classical Hopf MP is given in [4] as well. The correlation between positivity, maximum and comparison principles for cooperative and non-cooperative elliptic systems is studied in [5]. Complete results for validity of the classical maximum principle for linear elliptic operators are proved by H. Berestycki, L. Nirenberg and S.R.S. Varadhan. In [6], the authors are given necessary and sufficient conditions for the validity of MP, namely, the positiveness of the first eigenvalue of the operator with null Dirichlet boundary data.

Despite the complete studies of MP for elliptic equations, it is still a matter of interest for systems of elliptic PDEs. For instance, MP for linear cooperative elliptic systems is proved in [7] under some structural conditions, in between them $\frac{|b_i^k(x)|}{\lambda(x)}$ to be bounded for all $x \in \Omega$, i and k. Here, λ is the function from the ellipticity condition. Remark 1.7 in the same paper concerns the validity of the strong maximum principle under the structural condition ψ (Definition 1.1 of the same paper), namely, the existence of a positive in $\overline{\Omega}$ function $\psi(x) \in C^2(\overline{\Omega})$ such that $L^k(\psi) - f^k \geq 0$ in Ω.

A further example for recent research on maximum principles can be found in [8]. The author introduces a rather restricting structural condition "c" on the inward unit normal vector ν. It states that ν is a left-eigenvector of $\{b_i^k\}$ at any point of $\partial\Omega$. Furthermore, the scalar product $(\nu.f)$ is non-negative. Under condition "c", the author proves the validity of MP. This way, the usual condition for the validity of MP—cooperativeness and non-cooperativeness—is replaced by condition "c". Although in [8] the proves are given for parabolic systems, they can be applied to elliptic ones as well.

Another interesting proof of MP for cooperative elliptic systems is given in [9], where a fixed point index property is used.

In [10], the MP is applied to the problem of the minimal matrix norm of a characteristic matrix. Under different conditions, it is proved that the norm of every C^2 smooth solution of an elliptic system has no positive local maximums in the domain.

MP for problems with non-Dirichlet boundary conditions is studied as well. The validity of MP for degenerate oblique derivative problem for elliptic equations is proved in [11] (Lemma 2.1.2, p. 71), for the particular case when boundary vector field violates Schapiro–Lopatinski condition. In the same paper, the uniqueness of the solutions is proved by MP, as well for estimates of max. norm of the solutions. MP for nonlinear cooperative elliptic systems with mixed boundary conditions is proved in [12]. Furthermore, the strong MP is proved in [13] for vector bundles on Riemannian manifolds.

The strong MP is considered in [14] for weak solutions of quasi-linear elliptic equations on Lorentzian and Riemannian manifolds.

The authors studied the validity of MP for cooperative elliptic systems in several papers. In [15], the strong interior and boundary MP is proved for the classical sub- and super-solutions of linear elliptic system

$$L^k u^k = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^k(x)\frac{\partial u^k}{\partial x_j}\right) + \sum_{i=1}^{n} b_i^k(x)\frac{\partial u^k}{\partial x_i} + \sum_{l=1}^{N} m_{kl}(x)u^l + f^k(x)$$

in a bounded domain, $k = 1, \dots, N$. For linear systems, if the maximum is attained at some interior point for one component of the solution $u = (u^1, \dots, u^N)$, then at the same point is attained the maximum for all components of the vector u.

In [16], strong interior and boundary MP is proved for the classical sub- and super-solutions of quasi-linear systems of the type

$$L^k u^k = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^k(x)\frac{\partial u^k}{\partial x_j}\right) + F^k(x, u^1, \dots, u^N, \frac{\partial u^k}{\partial x_1}, \dots, \frac{\partial u^k}{\partial x_n}).$$

As a natural development of the works above, in this article we study MP for viscosity solutions of fully nonlinear quasi-monotone elliptic systems. In their pioneering work of 1991 [17], Ishii and Koike consider viscosity solutions for systems of fully nonlinear second order elliptic equations. In particular, they generalize the Peron's method for existence of viscosity solutions for quasi-monotone systems. Moreover, the authors prove uniqueness and comparison principle for semi-continuous viscosity sub- and super-solutions. Note that the quasi-monotone systems are more general than the cooperative ones, see Example 2.3 in [17]. The work in [17] is essential for our research inspiring the authors to consider viscosity solutions of nonlinear elliptic systems.

In the present paper, we prove strong interior and boundary MP for semi-continuous viscosity sub- and super-solutions of fully nonlinear, degenerate and cooperative elliptic systems. Viscosity solutions have applications in some real-world and financial processes, for instance, in the theory of the optimal control and the theory of differential games, where the value functions are viscosity solutions of the associated systems, see in [18–21]. Let us recall that the main advantage of the notion of viscosity solutions is the minimal smoothness of the sub-and super-solutions, which are only semi-continuous functions. Therefore, the value function is only continuous one. Finally, the strong interior and boundary MP for viscosity sub- and super-solutions shed light on the qualitative properties of the solutions to system (1) as uniqueness, perturbation and asymptotic questions, etc.

Furthermore, comparison principle for viscosity sub-and-super solutions to (1), when on of them is classical sub- or super-solution is also proved in Theorem 2 under the same conditions for the validity of the strong interior MP.

The study of the validity of strong MP for quasi-linear systems with non-linear principal symbol is a matter of future research.

Let $\Omega \subset R^n$ be a bounded domain. Let us consider in Ω the weakly coupled nonlinear system

$$F^k(x, u^1(x), \ldots, u^N(x), Du^k(x), D^2u^k(x)) = 0 \tag{1}$$

for $k = 1, \ldots, N$ and $x \in \Omega$, where

$$F^k(x, u^1(x), \ldots, u^N(x), Du^k(x), D^2u^k(x)) = G^k(x, u^k(x), Du^k(x), D^2u^k(x)) + \sum_{j=1}^{N} c_{kj}(x)u^j(x)$$

Here, $G^k(x, z^k, p^k, X^k) \in C(\Omega \times R \times R^n \times S^n)$, where S^n denotes the set of all real symmetric matrices of order n, and $c_{kj}(x) \in C(\overline{\Omega})$ for $k, j = 1, \ldots, N$

We suppose that (1) is a quasi-monotone system, i.e.,

$$c_{kj} \leq 0 \text{ for } k \neq j, \sum_{j=1}^{n} c_{kj}(x) \geq 0 \text{ in } \Omega \tag{2}$$

(see in [17]) as well as stronger condition

$$c_{kj} \leq 0 \text{ for } k \neq j, \sum_{j=1}^{n} c_{kj}(x) \geq \lambda > 0 \text{ in } \Omega \text{ and } k = 1, \ldots, N \tag{3}$$

Condition (3) is similar to condition (A3) in [17] for weakly coupled system (1).

Moreover, suppose the system (1) is a degenerate elliptic one, i.e.,

$$G^k(x, z^k, p^k, X^k) \leq G^k(c, z^k, p^k, Y^k) \text{ whenever } X^k \geq Y^k \tag{4}$$

and monotone increasing one w.r.t. z variable, i.e.,

$$G^k(x, z^k, p^k, X^k) \geq G^k(c, y^k, p^k, X^k) \text{ whenever } z^k \geq y^k \tag{5}$$

for $k = 1, \ldots, N$, $x \in \Omega$, $p^k \in R^n$, $X^k, Y^k \in S^n$.

As the principal symbols in (1) are nonlinear ones, one expects low smoothness of the solution. That is why the class of viscosity solutions is a proper choice of functional space to work in.

Let us recall the definition of viscosity sub- and super-solution to (1) (Definition 2.1, page 1997, [17]):

Definition 1. *Let $u = (u^1, \ldots, u^N) : \overline{\Omega} \to R^N$ be a locally bounded function.*

(i) We call u a viscosity sub-solution to (1) if whenever $\psi \in C^2(\Omega))$, $1 \leq k \leq N$ and $u^{k*} - \psi$ attains its local maximum at $x \in \Omega$, then

$$F^k_*(x, u^*(x), D\psi, D^2\psi) \leq 0.$$

(ii) We call u a viscosity super-solution to (1) if whenever $\psi \in C^2(\Omega))$, $1 \leq k \leq N$ and $u^k_* - \psi$ attains its local minimum at $x \in \Omega$, then

$$F^{k*}(x, u_*(x), D\psi, D^2\psi) \geq 0.$$

(iii) If u is both viscosity sub- and super-solution to (1) the we call it a viscosity solution to (1). Here,

$$u^{k*} = \lim_{\epsilon \to 0} \sup\{u^{k*}(y) : |x - y| < \epsilon, y \in \overline{\Omega}\}$$

and

$$u^k_* = \lim_{\epsilon \to 0} \inf\{u^{k*}(y) : |x - y| < \epsilon, y \in \overline{\Omega}\}.$$

Note that $u^{k*} = u^k$ for $u^k \in USC(\Omega)$, $u^k_* = u^k$ for $u^k \in LSC(\Omega)$ and $F^{k*} = F^k_* = F^k$ for $F^k \in C(\overline{\Omega})$.

Further in the text, $USC(\Omega)$ is the set of upper semi-continuous functions $u = (u^1, \ldots, u^N) : \overline{\Omega} \to R^N$. We use the notion "absolute maximum" as well.

Definition 2. *If $\sup_{\overline{\Omega}} u^k(x) = M_k$ then $M = \max_{1 \leq k \leq N}\{M_k\}$, we call the absolute maximum of $u(x)$.*

2. Strong Interior Maximum Principle

The strong interior MP for viscosity sub-solutions of the nonlinear, weakly coupled and cooperative system (1) is formulated in the following theorem:

Theorem 1. *(Strong interior maximum principle) Suppose conditions (3)–(5) hold. If $u(x) \in USC(\Omega)$, $u = (u^1, \ldots, u^N)$, is a viscosity sub-solution to (1) and*

$$F^k(x, 0, 0, 0) = G^k(x, 0, 0) \geq 0 \tag{6}$$

for $x \in \overline{\Omega}$ and $k = 1, 2, \ldots, N$, then $u(x)$ does not attain absolute positive maximum at an interior point of Ω.

In the proof of Theorem 1 is used the notion of super- and sub-jet of second order. For the sake of completeness, the definition follows:

Definition 3. *Superjet of second order $J^{2,+}u(x)$ of function $u : \Omega \to R$ at point $x \in \Omega$ is defined as*

$$J^{2,+}u(x) = \left\{(p, X) \in R^n \times S^n : u(x+h) \leq u(x) + \langle p, h\rangle + \frac{1}{2}\langle Xh, h\rangle + \sigma(|h|^2) \text{ as } h \to 0\right\},$$

$$\bar{J}^{2,+}u(x) = \Big\{(p, X) \in R^n \times S^n : \text{ for some sequence } (x^k, p^k, X^k) \in \Omega \times R^n \times S^n,$$
$$(p^k, X^k) \in J^{2,+}u(x) \text{ we have } (x^k, v(x^k), p^k, X^k) \to (x, v(x), p, X) \text{ as } k \to \infty\Big\}.$$

Subjet of second order $J^{2,-}u(x)$ of function $u : \Omega \to R$ at point $x \in \Omega$ is defined as

$$J^{2,-}u(x) = \left\{(p, X) \in R^n \times S^n : u(x+h) \geq u(x) + \langle p, h\rangle + \frac{1}{2}\langle Xh, h\rangle + \sigma(|h|^2) \text{ as } h \to 0\right\}.$$

$$\bar{J}^{2,-}u(x) = \Big\{(p,X) \in R^n \times S^n : \textit{ for some sequence } (x^k, p^k, X^k) \in \Omega \times R^n \times S^n,$$

$$(p^k, X^k) \in J^{2,-}u(x) \textit{ we have } (x^k, v(x^k), p^k, X^k) \to (x, v(x), p, X) \textit{ as } k \to \infty\Big\}.$$

The following proposition (Proposition 2.3 in [17]) states that Definition 3 is equivalent to Definition 1 as a definition of viscosity solution:

Proposition 1. *Let $u : \bar{\Omega} \to R^m$ is locally bounded function. Then,*
(i) u is a sub-solution to (1) if and only if for every $(p,X) \in J^{2,+}u^{k*}(x)$

$$F^k_*(x, u^*(x), p, X) \leq 0;$$

u is a super-solution to (1) if and only if for every $(p,X) \in J^{2,-}u^k_*(x))$

$$(F^{k*}(x, u_*(x), p, X) \geq 0.$$

(ii) Suppose that F_* (F^*) is quasi-monotone. Then, u is a sub-solution (super-solution) to (1) if and only if

$$F^k_*(x, u^*(x), p, X) \leq 0 \textit{ for all } (p,X) \in \bar{J}^{2,+}u^{k*}(x)$$
$$(F^{k*}(x, u_*(x), p, X) \geq 0 \textit{ for all } (p,X) \in \bar{J}^{2,-}u^k_*(x)$$

Proof of Theorem 1. Without loss of generality let us suppose that the absolute maximum is attained for $u^1(x_1)$, i.e., $u^1(x_1) = M$ for some $x_1 \in \Omega$. As $u^1(x) \leq M$ and $u^1(x_1) = M$, then $(0,0) \in J^{2,+}u^1(x_1)$. From Definition 1, (3) and (5), we get the following impossible chain of inequalities:

$$0 \geq G^1(x_1, u^1(x_1), 0, 0) + \sum_{j=1}^{N} c_{1j}(x_1)u^j(x_1)$$

$$= G^1(x_1, M, 0, 0) + M\sum_{j=1}^{N} c_{1j}(x_1) + \sum_{j=2}^{N} c_{1j}(x_1)\Big(u^j(x_1) - M\Big)$$

$$\geq G^1(x_1, 0, 0, 0) + M\lambda \geq M\lambda > 0.$$

Theorem 1 is proved. □

As a consequence of Theorem 1, we obtain the following comparison principle for viscosity sub-and super-solutions to (1) when one of them is a classical one:

Theorem 2. *Suppose conditions (3)–(5) hold, $u = (u^1, \dots, u^N)$ and $u(x) \in USC(\Omega)$ is a viscosity sub-solution to (1) and $v(x)$, $v^k(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, $k = 1, \dots, N$ is a classical super-solution to (1). If $u^k(x) \leq v^k(x)$ for $k = 1, \dots, N$ and $x \in \partial\Omega$, then $u^k(x) \leq v^k(x)$ for $x \in \Omega$ and $k = 1, \dots, N$.*

Proof of Theorem 2. Let us consider the system

$$f^k(x, w^1(x), \dots, w^N(x), Dw^k(x), D^2w^k(x)) = 0 \tag{7}$$

for $k = 1, \dots, N$ and $x \in \Omega$, where

$$f^k(x, w^1(x), \dots, w^N(x), Dw^k(x), D^2w^k(x))$$
$$= G^k(x, w^k(x) + v^k(x), Dw^k(x) + Dv^k(x), D^2w^k(x) + D^2v^k(x))$$

$$+\sum_{j=1}^{N} c_{kj}(x)(w^k(x) + v^k(x))$$

The function $w(x) = u(x) - v(x)$, $w^k(x) \in USC(\Omega)$, $k = 1, \ldots, N$ is a viscosity sub-solution to (7). Indeed, if $(p^k, X^k) \in J^{2,+}w^k(x)$, we get by Remark 2.7 in [22] that $(p^k + Dv^k(x), X^k + D^2v^k(x)) \in J^{2,+}u^k(x)$. Thus, the following inequality holds:

$$\begin{aligned} &f^k(x, w^1(x), \ldots, w^N(x), p^k, X^k) \\ &= G^k(x, u^k(x), p^k + Dv^k(x), X^k + D^2v^k(x)) \\ &+ \sum_{j=1}^{N} c_{kj}(x)u^k(x) \le 0, \end{aligned}$$

because $u(x)$ is a viscosity sub-solution to (1).

As for $w = 0$ we get

$$f^k(x, 0, 0, 0) = G^k(x, v^k(x), Dv^k(x), D^2v^k(x)) + \sum_{j=1}^{N} c_{kj}(x)v^k(x) \ge 0,$$

because $v(x)$ is a classical super-solution and therefore condition (6) is satisfied.

From the strong interior maximum principle it follows that $w(x)$ does not attain positive absolute maximum in Ω. Thus either the absolute maximum is attained on $\partial\Omega$, i.e., $M \le 0$ because $w^k(x) \le 0$ on $\partial\Omega$, or the absolute maximum is attained at some interior point of Ω and hence $M \le 0$ again. As $u^k(x) - v^k(x) \le M_k \le M \le 0$ for every $x \in \Omega$ and $k = 1, \ldots, N$, Theorem 2 is proved. □

3. Strong Boundary Maximum Principle

Having the strong interior MP at hand, one can easily derive the strong boundary MP for the viscosity subs-solutions of the nonlinear cooperative elliptic system (1).

Theorem 3. *(Strong boundary MP) Assume that conditions (3)–(6) hold and Ω satisfies the interior sphere condition. Let $u(x) = (u^1(x), \ldots, u^N(x))$, $u^k(x) \in USC(\bar{\Omega})$ be a viscosity sub-solution to (1). If $u(x)$ attains an absolute positive maximum M at some boundary point $x_0 \in \partial\Omega$, i.e., $u^k(x_0) = M$ for some $1 \le k \le N$, then for every non-tangential direction ρ pointing into Ω the following inequality holds:*

$$\lim_{t \to +0} \frac{u^k(x_0 + \rho t) - u^k(x_0)}{t} < 0 \tag{8}$$

Proof of Theorem 3. Without loss of generality we suppose that $k = 1$, i.e., $u^1(x_0) = M$.

It follows from Theorem 1 that $u^1(x) < M$ for every $x \in \Omega$. After shifting the origin, if necessary, let $B_R = \{|x| < R\}$ be an interior ball touching the boundary of Ω only at the point x_0. Let us consider function $v(x)$ such that $v^1(x) = M - e^{-\beta|x|^2/2} + e^{-\beta R^2/2}$, $v^k(x) = M$ for $k = 2, \ldots, N$, where constant β satisfies conditions (9)–(12). In the annulus $U = \{x \in \Omega : r < |x| < R\}$ the function $v(x)$ is a classical super-solution to (1).

Indeed, for $2 \le k \le N$ we have

$$F^k(x, v^1(x), \ldots, v^N(x), Dv^k(x), D^2v^k(x)) = G^k(x, M, 0, 0) + \sum_{j=1}^{N} c_{kj}(x)v^j(x)$$

$$\ge G^k(x, 0, 0, 0) + M\sum_{j=1}^{N} c_{kj}(x) - c_{k1}\left[e^{-\beta|x|^2/2} - e^{-\beta R^2/2}\right] \ge M\lambda > 0,$$

because $c_{k1}(x) \le 0$ for $k = 2, \ldots, N$ from (3).

For $k = 1$, we get

$$F^1(x, v, Dv^1, D^2v^1) = G^1\left(x, M - e^{-\beta|x|^2/2} + e^{-\beta R^2/2}, -De^{-\beta|x|^2}, -D^2e^{-\beta|x|^2}\right)$$

$$+M\sum_{j=1}^{N} c_{1j}(x) - c_{11}\left[e^{-\beta|x|^2/2} - e^{-\beta R^2/2}\right]$$

$$\geq M\lambda - c_{11}e^{-\beta|x|^2/2} + G^1\left(x, 0, \beta x e^{-\beta|x|^2/2}, (\beta I - \beta^2 x \otimes x)e^{-\beta|x|^2/2}\right)$$

$$\geq M\lambda - c_{11}e^{-\beta r^2/2} + G^1\left(x, 0, \beta x e^{-\beta|x|^2/2}, \beta I e^{-\beta|x|^2/2}\right)$$

If $\left|G^1(x, 0, q, T) - G^1(x, 0, 0, 0)\right| < \frac{M\lambda}{2}$ for $|q| + ||T|| < \delta$ and $x \in \bar{U}$, then

$$G^1\left(x, 0, \beta x e^{-\beta|x|^2/2}, \beta I e^{-\beta|x|^2/2}\right) \geq G^1(x, 0, 0, 0) - \frac{M\lambda}{2}$$

whenever

$$\left|\beta x e^{-\beta|x|^2/2}\right| \leq \beta R e^{-\beta r^2/2} < \frac{\delta}{2} \tag{9}$$

and

$$\|\beta I e^{-\beta|x|^2/2}\| < \frac{\delta}{2}. \tag{10}$$

Finally, we get

$$F^1(x, v, Dv^1, D^2v^1) \geq M\lambda - c_{11}(x)e^{-\beta r^2} - \frac{M\lambda}{2} > 0$$

when

$$\sup_{\bar{\Omega}}\left(c_{11}(x)e^{-\beta r^2/2}\right) < \frac{M\lambda}{2}. \tag{11}$$

As $\sup\limits_{x\in\partial B_r} u^1(x) = m_1 < M$, if

$$e^{-\beta r^2/2} < M - m_1, \tag{12}$$

then

$$\sup_{x\in\partial B_r} u^1(x) = m_1 < M - e^{-\beta r^2/2} \leq M - e^{-\beta|x|^2/2} + e^{-\beta R^2/2} = v^1(x)$$

for $x \in \partial B_r$.

Thus, $u^1(x) - v^1(x) < 0$ on ∂B_r and trivially $u^k(x) - v^k(x) = u^k(x) - M < 0$ on ∂B_r.

By the strong interior maximum principle, the function $u(x) - v(x)$ does not attain a positive absolute maximum at an interior point of U. As $u^k(x) - v^k(x) = u^k(x) - M \leq 0$ on ∂B_R for $k = 1, \ldots, N$, it follows that $u^k(x) \leq v^k(x)$ for $x \in \overline{U}$ and $k = 1, \ldots, N$.

For $k = 1$, we get

$$u^1(x) \leq M - e^{-\beta|x|^2/2} + e^{-\beta R^2/2} = v^1(x),$$

$$u^1(x_0) = v^1(x_0) = M.$$

Thus, for every direction ρ such that $(x_0, \rho) < 0$ we obtain the inequality

$$\lim_{x\to+0} \frac{u^1(x_0 + \rho t) - u^1(x_0)}{t} \leq \limsup_{x\to+0} \frac{e^{-\beta R^2/2} - e^{-\beta|x_0+t\rho|^2/2}}{t} = \beta(x_0, \rho) \cdot e^{-\beta R^2/2} < 0.$$

The proof is complete. $\square$

4. Conclusions

MP is a useful tool in studying the quantitative properties of the solution as uniqueness and some a-priori estimates.

Conditions (3)–(6) are sufficient ones for validity of the interior MP for the viscosity solutions of elliptic system (1) with fully nonlinear degenerated principal symbol. Furthermore, if one of viscosity sub- and super solutions is a classical one then comparison principle holds as well.

If conditions (3)–(6) and (8) hold, then the boundary MP holds for system (1).

The main novelty of this work is the reduction of the smoothness of the solution. In the literature, the strong maximum principle is proved for classical C^2 or generalized C^1 solutions, while we prove it for semi-continuous ones.

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Abbreviations

The following abbreviations are used in this manuscript:

MP	Maximum principle
USC	Upper semicontinuous functions

References

1. Hopf, E. Elementare bemerkung uber die losung partieller differentialgleichungen zweiter ordnung vom elliptischen typus. *Sitzungsberichte Preuss. Amademie Wiss.* **1927**, *19*, 147–152.
2. Protter, M.; Weinberger, H. *Maximum Principle in Differential Equations*; Prentice Hall: Upper Anhe, NJ, USA, 1976.
3. Gilbarg, D.; Trudinger, N. *Elliptic Partial Differential Equations of Second Order*; Springer: New York, NY, USA, 1998.
4. Pucci, P.; Serrin, J. The strong maximum principle revisited. *J. Differ. Equ.* **2004**, *196*, 1–66. [CrossRef]
5. Mitidieri, E.; Sweers, G. Weakly coupled elliptic systems and positivity. *Math. Nachr.* **1995**, *173*, 259–286. [CrossRef]
6. Berestycki, H.; Nirenberg, L.; Varadhan, S.R.S. The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Commun. Pure Appl. Math.* **1994**, *47*, 47–92. [CrossRef]
7. De Figueiredo, D.; Mitidieri, E. Maximum Principles for Linear Elliptic Systems. In *Selected Papers*; Costa, D.G., Ed.; Springer: Berlin, Germany, 2013; pp. 36–66.
8. Wang, X. A Remark on Strong Maximum Principle for Parabolic and Elliptic Systems. *Proc. Am. Math. Soc.* **1990**, *109*, 343–348. [CrossRef]
9. Corrêa, F.J.S.A.; Souto, M.A.S. On maximum principles for cooperative elliptic systems via fixed point index. *Nonlinear Anal. Theory Methods Appl.* **1996**, *26*, 997–1006. [CrossRef]
10. Rus, I.A. Maximum principles for elliptic systems and the problem of the minimum matrix norm of a characteristic matrix, revisited. *Stud. Univ. Babes-Bolyai Math.* **2013**, *58*, 199–211.
11. Popivanov, P.; Palagachev, D. *The Degenerate Oblique Derivative Problem for Elliptic and Parabolic Equations*; Academie Verlag: Berlin, Germany, 1997.

12. Karatson, J. A Maximum Principle for Some Nonlinear Cooperative Elliptic PDE Systems with Fixed Boundary Conditions. 2016. Available online: https://pdfs.semanticscholar.org/fe86/5e7f4682e0ce07abed3c0c2f9b52617b74db.pdf (accessed on 13 October 2021).
13. Sava-Halilaj, A.; Smoszyk, K. The strong elliptic maximum principle for vector bundles and application to minimal maps. *arXiv* **2012**, arXiv:1205.2379v1.
14. Andersson, L.; Galloway, G.J.; Howard, R. A strong maximum principle for weak solutions of quasi-linear elliptic equations with applications to Lorentzian and Riemannian geometry. *Commun. Pure Appl. Math.* **1998**, *51*, 581–624. [CrossRef]
15. Boyadjiev, G.; Kutev, N. Strong interior and boundary maximum principle for weakly coupled linear cooperative elliptic systems. *Comptes Rendus L'Acade'Mie Bulg. Des Sci.* **2019**, *72*, 861–870.
16. Boyadzhiev, G.; Kutev, N. Strong maximum principle for nonlinear cooperative elliptic systems. *AIP Conf. Proc.* **2019**, *2159*, 030005.
17. Ishii, S. Koike: Viscosity solutions for monotone systems of second order elliptic PDEs. *Commun. Part. Diff. Equ.* **1991**, *16*, 1095–1128. [CrossRef]
18. Barron, E.N. Differential games with maximum cost. *Nonlinear Anal. Theory Methods Appl.* **1990**, *14*, 971–989. [CrossRef]
19. Barron, E.N.; Ishii, H. The Bellman equation for minimizing the maximum cost. *Nonlinear Anal. Theory Methods Appl.* **1989**, *13*, 1067–1090. [CrossRef]
20. Dolcetta, I.C.; Evans, C. Optimal Switching for Ordinary Differential Equations. *SIAM J. Control Optim.* **1984**, *22*, 143–161. [CrossRef]
21. Fleming, W.H.; Sougandis, P.E. On The Existence of Value Functions of Two-Player, Zero-Sum Stochastic Differential Games. *Indiana Univ. Math. J.* **1989**, *38*, 293–314. [CrossRef]
22. Crandall, M.G.; Ishii, H.; Lions, P.-L. User's guide to viscosity solutions of second order partial differential equations. *Bul. Am. Math. Soc.* **1992**, *27*, 1–67. [CrossRef]