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ON MINIMIZING $\|S-(AX-XB)\|_p^p$

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Abstract. In this paper, we minimize the map $F_p(X)=\|S-(AX-XB)\|_p^p$, where the pair (A,B) has the property $(FP)_{C_p}$, $S\in C_p$, X varies such that $AX-XB\in C_p$ and C_p denotes the von Neumann-Schatten class.

1. Introduction. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators acting on a complex Hilbert space H. For A and B in $\mathcal{L}(\mathcal{H})$, let $\delta_{A,B}$ denote the operator on $\mathcal{L}(\mathcal{H})$ defined by $\delta_{A,B}(X)=AX-XB$. If $A=B$, then δ_A is called the inner derivation induced by A. A well-known result of J. Anderson and C. Foias [1] says that if A and B are normal operators such that, $AS=SB$ then, for all $X\in\mathcal{L}(\mathcal{H})$,

$$\|S-(AX-XB)\|\geq\|S\|. \tag{1.1}$$

In this paper we obtain an inequality similar to (1.1), where the operator norm is replaced by the $\|\cdot\|_p$ norm on the von Neumann-Schatten class C_p, $1\leq p<\infty$. We prove that, if the pair (A,B) has the property $(FP)_{C_p}$, i.e. $(AT=TB$, where

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$T \in C_p$ implies $A^*T = TB^*$) $1 \leq p < \infty$ and $S \in \ker \delta_{A,B} \cap C_p$ then, the map F_p defined by $F_p(X) = \|S - (AX - XB)\|_p^p$ has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In other words, we have

$$\|S - (AX - XB)\|_p^p \geq \|T\|_p^p \tag{1.2}$$

if, and for $1 < p < \infty$ only if, $AV - VB = 0$. Thus in Halmos$'$ terminologie [5] the zero commutator is the commutator approximant in C_p of T. Additionally, we show that if, the pair (A, B) has the property $(FP)_{C_p}$ and $S \in \ker \delta_{A,B} \cap C_p$, $1<p<\infty$ then, the map F_p has a critical point at W if, and only if, $AW - WB = 0$, i.e. if $\mathcal{D}_W F_p$ is the Frechet derivative at W of F_p, the set

$$\{W \in \mathcal{L}(\mathcal{H}) : \mathcal{D}_W F_p = 0\} \tag{1.3}$$

coinsides with $\ker \delta_{A,B}$ (the kernel of $\delta_{A,B}$).

2. Preliminaries. For details of the von Neumann-Schatten class see [8].

Theorem 2.1 [2]. *If $1 < p < \infty$, then the map $F_p : C_p \longmapsto \mathbb{R}^+$ defined by $X \longmapsto \|X\|_p^p$, is differentiable at every $X \in C_p$ with derivative $\mathcal{D}_X F_p$ given by*

$$\mathcal{D}_X F_p(T) = p \operatorname{Re} tr(|X|^{p-1} U^* T), \tag{2.1}$$

where tr denotes trace, $\operatorname{Re} z$ is the real part of a complex number z and $X = U|X|$ is the polar decomposition of X. If $\dim \mathcal{H} < \infty$, then the same result holds for $0 < p \leq 1$ at every invertible X.

Theorem 2.2 [6]. *If $\mathcal{U}$ is a convex set of C_p, with $1 < p < \infty$, then the map $X \longmapsto \|X\|_p^p$, where $X \in \mathcal{U}$ has at most a global minimizer.*

3. Orthogonality. The following definition generalizes the idea of orthogonality in Hilbert space.

Definition 3.1. *Let $\mathbb{C}$ be a complex numbers and let E be a normed linear space. Let F and G be two subspaces of E. If $\|x + y\| \geq \|y\|$ for all $x \in F$ and for all $y \in G$, then F is said to be orthogonal to G.*

Definition 3.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$. The pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$ if, $AC = CB$, where $C \in \mathcal{L}(\mathcal{H})$ implies $A^*C = CB^*$.*

Definition 3.3. *Let $\mathcal{U}(A, B) = \{X \in \mathcal{L}(\mathcal{H}) : AX - XB \in C_p\}$ and $F_p : \mathcal{U} \longmapsto \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (AX - XB)\|_p^p$, where $T \in \ker \delta_{A,B} \cap C_p$, $1 \leq p < \infty$.*

Theorem 3.1 [7]. *Let $A \in \mathcal{L}(\mathcal{H})$, if A is normal and $S \in C_p, 1 \leq p < \infty$ such that $AS = SA$, then*

$$\|S - (AX - XA)\|_p^p \geq \|S\|_p^p, \tag{3.1}$$

for every $X \in \mathcal{L}(\mathcal{H})$.

Theorem 3.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$, if A and B are normal operators and $T \in C_p, 1 \leq p < \infty$ such that $AT = TB$, then*

$$\|T - (AX - XB)\|_p^p \geq \|T\|_p^p, \tag{3.2}$$

for every $X \in \mathcal{L}(\mathcal{H})$.

P r o o f. Taking on $\mathcal{H} \oplus \mathcal{H}$,

$$Q = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix},$$

then Q is normal on $\mathcal{H} \oplus \mathcal{H}$ and $S \in C_p$. Since $AT = TB$, then it results that $QS = SQ$. Since

$$QY - YQ = \begin{bmatrix} 0 & AX - XB \\ 0 & 0 \end{bmatrix},$$

then it follows from Teorem 3.1 that,

$$\|S - (QY - YQ)\|_p^p \geq \|S\|_p^p, \ \forall S \in C_p$$

consequently we obtain,

$$\|T - (AX - XB)\|_p^p = \|S - (QY - YQ)\|_p^p \geq \|S\|_p^p = \|T\|_p^p. \qquad \square$$

Lemma 3.3. *Let $A, B \in \mathcal{L}(\mathcal{H})$. The following statements are equivalent:*

(1) *The pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$*

(2) *If $AT = TB$ where $T \in \mathcal{L}(\mathcal{H})$, then $\overline{R(T)}$ reduces A, $\ker(T)^{\perp}$ reduces B, and $A \mid_{\overline{R(T)}}$ and $B \mid_{\ker(T)^{\perp}}$ are normal operators, where R and* ker *denote the range and the kernel, respectively.*

P r o o f. (1) $\Rightarrow$ (2): Since $AT = TB$ and the pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$, $A^*T = TB^*$ and so $\overline{R(T)}$ and $\ker(T)^{\perp}$ are reducing subspaces for A and B, respectively. Since $A(AT) = (AT)B$, we obtain $A^*(AT) = (AT)B^*$ by $(FP)_{L(H)}$, and the identity $A^*T = TB^*$ implies that $A^*AT = AA^*T$. Thus we see that $A \mid_{\overline{R(T)}}$ is normal. Clearly (B^*, A^*) satisfies $(FP)_{\mathcal{L}(\mathcal{H})}$, and $B^*T^* = T^*A^*$.

Therefore it follows from the above argument that $B^* \mid_{\overline{R(T^*)}} = (B \mid_{\ker(T)^\perp})^*$ is normal.

(2) $\Rightarrow$ (1): Let $T \in \mathcal{L}(\mathcal{H})$ such that $AT = TB$. Taking the two decompositions of $\mathcal{H}$, $\mathcal{H}_1 = \mathcal{H} = \overline{R(T)} \oplus \overline{R(T)}^\perp$, $\mathcal{H}_2 = \mathcal{H} = \ker T \oplus \ker(T)^\perp$. Then we can write A and B on $\mathcal{H}_1$ into $\mathcal{H}_2$ respectively,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where A_1, B_1 are normal operators. Also we can write T and X on $\mathcal{H}_2$ into $\mathcal{H}_1$

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \quad , X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

It follows from $AT = TB$ that $A_1T_1 = T_1B_1$. Since A_1 and B_1 are normal operators, then by applying the Putnam Fuglede′s theorem, we obtain $A_1^*T_1 = T_1B_1^*$, that is, $A^*T = TB^*$. $\square$

Theorem 3.4. *Let $A, B \in \mathcal{L}(\mathcal{H})$. If the pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$, then*

(3.3) $$\|C - (AX - XB)\| \geq \|C\|,$$

for every operator $C \in \ker \delta_{A,B}$ and for every $X \in \mathcal{L}(\mathcal{H})$.

Proof. Since the pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$, it follows from Lemma 3.3 that, $\overline{R(C)}$ reduces A, $\ker(C)^\perp$ reduces B, and $A \mid_{\overline{R(C)}}$ and $B \mid_{\ker(C)^\perp}$ are normal operators. Let,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

It follows from

$$AC - CB = \begin{bmatrix} A_1C_1 - C_1B_1 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

that $A_1C_1 = C_1A_1$ and we have

$$\|C - (AX - XB)\| = \left\| \begin{bmatrix} C_1 - (A_1X_1 - X_1B_1) & * \\ * & * \end{bmatrix} \right\|.$$

Since A_1 and B_1 are two normal operators, then the result of J. H. Anderson and C. Foias [1] guarentees that,

$$\|C_1 - (A_1X_1 - X_1B_1)\| \geq \|C_1\|,$$

so

$$\|C + AX - XB\| \geq \|C_1 - (A_1X_1 - X_1B_1)\| \geq \|C_1\| = \|C\| . \qquad \square$$

Remark 3.1. If A and B are two normal operators, then inequality (3.3) holds for every $C \in \ker \delta_{A,B}$. Hence, Theorem 3.4 generalizes the result given by J. H. Anderson and C. Foias [1]. In particular we have

$$R(\delta_{A,B}) \cap \ker \delta_{A,B} = \{0\} . \tag{3.4}$$

Corollary 3.5. *Let $A, B \in L(H)$ and $C \in \ker \delta_{A,B}$, then*

$$\|C + AX - XB\| \geq \|C\| , \forall X \in \mathcal{L}(\mathcal{H}).$$

In each of the following cases:
(1) A dominant and B^* $M-$hyponormal
*(2) A dominant and B^* $k-$**quasi**hyponormal*
*(3) A $k-$**quasi**hyponormal and B^* $k-$**quasi**hyponormal injective*
*(4) A $k-$**quasi**hyponormal and B^* dominant.*

Proof. Adapted from B. P. Duggal [3] if we have (1), (2), (3)and (4) the pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$. $\square$

Lemma 3.6. *Let $A, B \in \mathcal{L}(\mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H})$ such that the pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$. If $A\,|S|^{p-1}\,U^* = |S|^{p-1}\,U^*B$, where $p > 1$ and $S = U\,|S|$ is the polar decomposition of S, then $A\,|S|\,U^* = |S|\,U^*B$.*

Proof. If $T = |S|^{p-1}$, then

$$ATU^* = TU^*B. \tag{3.5}$$

We prove that

$$AT^nU^* = T^nU^*B, \tag{3.6}$$

for all $n \geq 1$. If $S = U\,|S|$, then $\ker U = \ker |S| = \ker |S|^{p-1} = \ker T$ and $(\ker U)^\perp = (\ker T)^\perp = \overline{R(T)}$. This shows that the projection U^*U onto $(\ker T)^\perp$ satifies $U^*UT = T$ and $TU^*UT = T^2$. By taking adjoints of (3.5) and since the pair (A, B) has the property $(FP)_{\mathcal{L}(\mathcal{H})}$, we get $BUT = UTA$ and $AT^2 = ATU^*UT = TU^*BUT = TU^*UTA = T^2A$. Since A commutes with the positive operator T^2, then A commutes with its square roots, that is,

$$AT = TA \tag{3.7}$$

By (3.5) and (3.7) we obtain (3.6). Let $f(t)$ be the map defined on $\sigma(T) \subset \mathbb{R}^+$ by $f(t) = t^{\frac{1}{p-1}}$; $1 < p < \infty$. Since f is a uniform limits of a sequence (P_i) of polynomials without constant term (since $f(0) = 0$), then it follows from (3.6) that $AP_i(T)U^* = P_i(T)U^*B$. Therefore $AT^{\frac{1}{p-1}}U^* = U^*T^{\frac{1}{p-1}}B$. □

Theorem 3.7. *Let $A, B \in \mathcal{L}(\mathcal{H})$. If the pair (A, B) has the property $(FP)_{C_p}$ and $S \in C_p$ such that $AS = SB$, then*

1) For $1 \leq p < \infty$, the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $AW - WB = 0$.

2) For $1 < p < \infty$, the map F_p has a critical point at W if, and only if, $AW - WB = 0$.

3) For $0 < p \leq 1$, $\dim \mathcal{H} < \infty$ and $S - (AW - WB)$ is invertible, then F_p has a critical point at W, if $AW - WB = 0$.

Before proving this theorem we need the following lemma.

Lemma 3.8. *Let $A, B \in \mathcal{L}(\mathcal{H})$. The following statements are equivalent.*

(1) The pair (A, B) has the property $(FP)_{C_p}$

(2) If $AT = TB$ where $T \in C_p$, then $\overline{R(T)}$ reduces $A, \ker(T)^{\perp}$ reduces B, and $A\mid_{\overline{R(T)}}$ and $B\mid_{\ker(T)^{\perp}}$ are normal operators.

Proof. Since C_p is a bilateral ideal and $T \in C_p$, then $AT \in C_p$. It suffices to remark that $A(AT) = (AT)B$ implies $A^*(AT) = (AT)B^*$ by $(FP)_{C_p}$, and the identity $A^*T = TB^*$ implies that $A^*AT = AA^*T$. By the same arguments as in the proof of Lemma 3.3, the proof of this Lemma can be finished. □

Proof of Theorem 3.8. Since the pair (A, B) has the property $(FP)_{C_p}$, it follows from the above lemma that, $\overline{R(S)}$ reduces A, $\ker(S)^{\perp}$ reduces B, and $A\mid_{\overline{R(S)}}$ and $B\mid_{\ker(S)^{\perp}}$ are normal operators. Let,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

It follows from

$$AS - SB = \begin{bmatrix} A_1S_1 - S_1B_1 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

that $A_1S_1 = S_1B_1$ and we have

$$\|S - (AX - XB)\|_p^p = \left\| \begin{bmatrix} S_1 - (A_1X_1 - X_1B_1) & * \\ * & * \end{bmatrix} \right\|_p^p.$$

Since A_1 and B_1 are two normal operators, then it results from Theorem 3.2 that,

$$\|S_1 - (A_1X_1 - X_1B_1)\|_p^p \geq \|S_1\|_p^p,$$

hence it follows from [4, p. 82] that,

$$\|S - (AX - XB)\|_p^p \geq \|S_1 - (A_1X_1 - X_1B_1)\|_p^p \geq \|S_1\|_p^p = \|S\|_p^p,$$

i.e. $F_p(X) \geq F_p(W)$. Conversely, if F_p has a minimum then $\|S - (AW - WB)\|_p^p = \|S\|_p^p$. Since $\mathcal{U}$ is convex then, the set $\mathcal{V} = \{S - (AX - XB);\ X \in \mathcal{U}\}$ is also convex. Thus Theorem 2.2 implies that $S - (AW - WB) = S$.

2) Let $W, S \in \mathcal{U}$ and ϕ, φ be two maps defined respectively by

$$\phi : X \longmapsto S - (AX - XB);\ \varphi : X \longmapsto \|X\|_p^p.$$

Since the Frechet derivative of F_p is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \longrightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h},$$

it follows that $\mathcal{D}_W F_p(T) = [\mathcal{D}_{S-(AW-WB)}](TB - AT)$. If W is a critical point of F_p, then $\mathcal{D}_W F_p(T) = 0$, $\forall T \in \mathcal{U}$, by applyig Theorem 2.1 we get,

$$\mathcal{D}_W F_p(T) = p \operatorname{Re} tr[|S - (AW - WB)|^{p-1} W^*(TB - AT) =$$

$$p \operatorname{Re} tr[Y(TB - AT)] = 0,$$

where $S - (AW - WB) = W\,|S - (AW - WB)|$ is the polar decomposition of the operator $S - (AW - WB)$ and $Y = |S - (AW - WB)|^{p-1} W^*$. An easy calculation shows that $BY - YA = 0$, that is,

$$A\,|S - (AW - WB)|^{p-1} W^* = |S - (AW - WB)|^{p-1} W^* B.$$

It follows from Lemma 3.6 that

$$A\,|S - (AW - WB)|\,W^* = |S - (AW - WB)|\,W^* B.$$

By taking adjoints and since the pair (A, B) has the property $(FP)_{C_p}$, we get $A(T-(AW-WB)) = (T-(AW-WB))B$, then $A(AW-WB) = (AW-WB)B$. Hence $AW - WB \in R(\delta_{A,B}) \cap \ker \delta_{A,B}$, by applying the equality(3.4) it results that $AW - WB = 0$.

Conversely, if $AW = WB$, then W is a minimum and since F_p is differentible, then W is a critical point.

3) Suppose that $\dim \mathcal{H} < \infty$. If $AW - WB = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$. Taking $Y = |S|^{p-1}U^*$, where $S = U|S|$ is the polar decomposition of S. Since $AS = SB$ implies $S^*A = BS^*$, then $S^*AS = BS^*S$ and this implies that $|S|^2B = B|S|^2$ and $|S|B = B|S|$. Since $S^*A = BS^*$, i.e. $|S|U^*A = B|S|U^*$, then $|S|(U^*A - BU^*) = 0$ and since $B|S|^{p-1} = |S|^{p-1}B$, so $BY - YA = B|S|^{p-1}U^* - |S|^{p-1}U^*A = |S|^{p-1}(BU^* - U^*A)$. So, that $BY - YA = 0$ and $tr[(BY - YA)T] = 0$ for every $T \in \mathcal{L}(\mathcal{H})$. Since $S = S-(AW-WB)$, that is $0 = tr[YTB-YAT] = tr[Y(TB-AT)] = p\,\mathrm{Re}\,tr[Y(TB - AT)] = p\,\mathrm{Re}\,tr[|S|^{p-1}U^*(TB - AT)] = (\mathcal{D}_T\phi)(TB - AT) = (\mathcal{D}_W F_p)(T)$. $\square$

Remark 3.2. In Theorem 3.7 the implication W is a critical point $\Rightarrow$ $AW - WB = 0$, does not hold in the case $0 < p \leq 1$ because the functional calculus argument involving the function $t \longmapsto t^{\frac{1}{p-1}}$, where $0 \leq t < \infty$, is only valid for $1 < p < \infty$.

REFERENCES

[1] J. H. Anderson, C. Foias. Properties which normal operator share with normal derivation and related operators. *Pacific. J. Math.* (1975), 313–325.

[2] A. H. Almoadjil. The commutants of relatively prime powers in Banach algebra. *Proc. Amer. Math. Soc.* **57** (1976), 243–251.

[3] B. P. Duggal. On generalized Putnam-Fuglede theorem. *Monatshe. Math.* **107** (1989), 309–332.

[4] I. C. Gohberg, M. G. Krein. Introduction to the theory of linear nonselfadjoint operators. Transl. Math. Monographs vol. **18**, Amer. Math. Soc., Providence, 99–110.

[5] P. R. Halmos. Positive approximants of operators. *Indiana Univ. Math. J.* **21** (1972), 99–110.

[6] P. J. Maher. Some norm inequalities concerning generalized inverses. *Linear Algebra Appl.* **174** (1992), 951–960.

[7] P. J. Maher. Commutator approximant. *Proc. Amer. Math. Soc.* **115** (1992), 995–1000.

[8] J. R. Ringrose. Compact Nonselfadjoint Operators. Van Nostrand, Rheinhold, London, 1971

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