Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 74, No 10, 2021

MATHEMATICS
Coding theory

ON THE MAXIMAL CARDINALITY OF BINARY TWO-WEIGHT CODES

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(*Submitted by Academician V. Drensky on June 7, 2021*)

Abstract

In this note we prove a general upper bound on the size of a binary $(n, \{d_1, d_2\})$-code with $d_2 > 2d_1$. This bound is used to settle recent conjectures on the maximal cardinality of an $(n, \{2, d\})$-code. The special case of $d = 4$ is also resolved using a classical shifting technique introduced by Erdős, Ko and Rado.

Key words: two-weight codes, the Erdős–Ko–Rado theorem, non-linear codes, main problem of coding theory

2010 Mathematics Subject Classification: 05A05, 05A20, 94B25, 94B65

1. Introduction. A binary (nonlinear) code $C \subset \mathbb{F}_2^n$ is called a two-distance code, or, with a certain abuse of language, a two-weight code if the possible distances between two different words of the code take on two different values, i.e.

$$\{d(u,v) \mid u,v \in C, u \neq v\} = \{d_1, d_2\},$$

where $0 < d_1 < d_2 < n$. Here $d(u,v)$ denotes the Hamming distance between u and v, i.e. the number of positions in which the words u and v are different. A binary two-weight code of length n, cardinality M, and with distances d_1 and d_2

The research of the first author was supported by the Bulgarian National Science Fund under Contract KP-06-32/2 – 07.12.2019.

The research of the second author was supported by the Research Fund of Sofia University under Contract No 80-10-88/25.03.2021.

DOI:10.7546/CRABS.2021.10.01

is called an $(n, M, \{d_1, d_2\})$-code. If the cardinality is not specified we speak of an $(n, \{d_1, d_2\})$-code. A natural problem is to determine the maximal cardinality, denoted by $A_2(n, \{d_1, d_2\})$, of a binary two-weight code of fixed length n and given distances d_1 and d_2.

A systematic investigation of this problem for the non-linear case was made by Boyvalenkov et al. [1], where along with proving upper bounds and various facts about $A_2(n, \{d_1, d_2\})$, the authors state two conjectures:

(A) $A_2(n, \{2, 4\}) = \binom{n}{2} + 1$ for all $n \geq 6$, and that

(B) $A_2(n, \{2, d\}) = \begin{cases} n & \text{for } 5 \leq d \leq n-2, \\ n+1 & \text{for } d = n-1. \end{cases}$

Let us note that $A_2(4, \{2, 4\}) = 8$ and $A_2(5, \{2, 4\}) = 16$. The optimal codes are the even weight codes. In the case $n = 6$ we split the words of the even weight code into 16 pairs of complementary words. Any choice of a word from each of these pairs gives a $(6, \{2, 4\})$-code. For $n \geq 7$ the code consisting of the zero word and all words of weight two gives an $\left(\binom{n}{2} + 1, \{2, 4\}\right)$-code.

This paper is structured as follows. In Section 2, we prove a upper bound on the size of a binary $(n, \{d_1, d_2\})$-code which improves on the bound from Theorem 6 in [1]. In Section 3, we settle Conjecture (B) using ideas from Section 2, which deals with the special case of $d_1 = 2$. In Section 4, we prove Conjecture (A) using a classical shifting technique introduced by Erdős, Ko and Rado in [2].

2. A bound on the size of a code with two distances. In this section we consider binary codes with parameters $(n, \{d_1, d_2\})$ with $d_2 > 2d_1$. Without loss of generality we assume that the zero word $\mathbf{0} = (\underbrace{0, 0, \ldots, 0}_{n})$ is in C. The following observation is now straightforward:

- if $\mathrm{wt}(\boldsymbol{c}_1) = \mathrm{wt}(\boldsymbol{c}_2) = d_1$, then $\mathrm{wt}(\boldsymbol{c}_1 * \boldsymbol{c}_2) = d_1/2$ and $d(\boldsymbol{c}_1, \boldsymbol{c}_2) = d_1$;
- if $\mathrm{wt}(\boldsymbol{c}_1) = d_1, \mathrm{wt}(\boldsymbol{c}_2) = d_2$, then $\mathrm{wt}(\boldsymbol{c}_1 * \boldsymbol{c}_2) = d_1/2$ and $d(\boldsymbol{c}_1, \boldsymbol{c}_2) = d_2$;
- if $\mathrm{wt}(\boldsymbol{c}_1) = \mathrm{wt}(\boldsymbol{c}_2) = d_2$, then

  either $\mathrm{wt}(\boldsymbol{c}_1 * \boldsymbol{c}_2) = d_2 - d_1/2$, and $d(\boldsymbol{c}_1, \boldsymbol{c}_2) = d_1$,

  or else $\mathrm{wt}(\boldsymbol{c}_1 * \boldsymbol{c}_2) = d_2/2$, and $d(\boldsymbol{c}_1, \boldsymbol{c}_2) = d_2$.

Here we denote, as usual, by $\boldsymbol{c}_1 * \boldsymbol{c}_2$ the star of the vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$, i.e. if $\boldsymbol{c}_1 = (\alpha_1, \ldots, \alpha_n)$, $\boldsymbol{c}_2 = (\beta_1, \ldots, \beta_n)$, then

$$\boldsymbol{c}_1 * \boldsymbol{c}_2 = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_n\beta_n).$$

Lemma 1. *Let C be an $(n, \{d_1, d_2\})$-code with $\mathbf{0} \in C$, $d_2 > 2d_1$, and let $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3 \in C$ be words of weight d_2. If*

$$d(\boldsymbol{c}_1, \boldsymbol{c}_2) = d(\boldsymbol{c}_1, \boldsymbol{c}_3) = d_1,$$

then $d(\boldsymbol{c}_2, \boldsymbol{c}_3) = d_1$.

Proof. Assume for a contradiction that $d(\boldsymbol{c}_2, \boldsymbol{c}_3) = d_2$. Denote by α the number of positions in which all three words $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and $\boldsymbol{c}_3$ have 1's. Then by $\mathrm{wt}(\boldsymbol{c}_1 * \boldsymbol{c}_3) = d_2 - d_1/2$ we get $d_2 - d_1/2 - \alpha \leq d_1/2$, whence $\alpha > d_1$. On the other hand, $\mathrm{wt}(\boldsymbol{c}_2 * \boldsymbol{c}_3) = d_1/2$ implies $d_1/2 - \alpha \geq 0$, a contradiction. □

This lemma implies that the graph with vertices – the words of weight d_2, and neighbourhood between two vertices iff the corresponding words are at distance d_1, is a union of (possibly trivial) cliques.

Let us denote by A the $(M-1)$-by-n matrix having as rows the non-zero words of C. By Lemma 1, the words can be ordered in such way that

$$AA^T = \begin{pmatrix} \frac{d_1}{2}J + \frac{d_1}{2}I & \frac{d_1}{2}J & \frac{d_1}{2}J & \dots & \frac{d_1}{2}J \\ \frac{d_1}{2}J & (d_2 - \frac{d_1}{2})J + \frac{d_1}{2}I & \frac{d_2}{2}J & \dots & \frac{d_2}{2}J \\ \frac{d_1}{2}J & \frac{d_2}{2}J & (d_2 - \frac{d_1}{2})J + \frac{d_1}{2}I & \dots & \frac{d_2}{2}J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{d_1}{2}J & \frac{d_2}{2}J & \frac{d_2}{2}J & \dots & (d_2 - \frac{d_1}{2})J + \frac{d_1}{2}I \end{pmatrix},$$

where the diagonal matrices are of size $k_0 \times k_0, k_1 \times k_1, \dots, k_s \times k_s$, respectively, where $k_0 + k_1 + \dots + k_s = M - 1$.

Using standard techniques for computing determinants, one can verify that the determinant of the matrix B is not zero.

Theorem 2. *Let C be a binary $(n, M, \{d_1, d_2\})$-code. Then $M \leq n + 1$.*

Proof. By the above argument $\det AA^T \neq 0$ and hence the matrix AA^T is of full rank over $\mathbb{Q}$. Now using the Sylvester inequality, we get

$$M - 1 = \operatorname{rank} AA^T \leq \operatorname{rank} A \leq n,$$

which proves the theorem. □

Corollary 3. *If $d_2 > 2d_1$, we have $A_2(n, \{d_1, d_2\}) \leq n + 1$.*

Let us note that this theorem improves significantly on the bound given in Theorem 6 from [1], which for the case $q = 2$ gives $A_2(n, \{d_1, d_2\}) \leq 2n + 1$. Equality in Corollary 3 can be achieved, for instance, if $d_1 = 2$, $d_2 = n - 1$. As we shall prove in the next section, there exist pairs (d_1, d_2) for which this bound can be improved.

3. On $(n, \{2, d\})$-codes. Using the idea from the previous section, we can tackle the second part of Conjecture 1 from [1]. The authors conjecture there that $A_2(n, \{2, d\}) = n$ for $n \geq 6$, $5 \leq d \leq n - 1$, and $A_2(n, \{2, n-1\}) = n + 1$.

The construction of an $(n, M = n, \{2, d\})$-code, as well as of a code of cardinality $n + 1$ for $d = n - 1$ is given in [1]. The upper bound is easily verified in the cases of d odd, as demonstrated in [1]. For $d = n - 1$ this bound follows from Theorem 2. Below we consider the most interesting case where d is even.

Let us assume that $n \geq 8$ and $6 \leq d \leq n - 2$, d even. Furthermore, let C be an $(n, M, \{2, d\})$-code with d and n satisfying the above restrictions.

Without loss of generality $\mathbf{0} \in C$. All the remaining words are of weight 2 or d. We denote by a the number of words in C that are of Hamming weight 2. We have $1 \leq a \leq M - 2$ since there exist two words at distance 2 and not all distances between different words are equal to 2. As before, denote by A the $(M-1)$-by-n matrix that has as rows the non-zero words of C. Then up to a row and column permutation A has the following form:

$$
(1) \qquad A = \left(\begin{array}{c|cccc|ccccc}
1 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
\vdots & & & \ddots & & & & \ddots & & \\
1 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 \\
\hline
0 & 1 & 1 & \ldots & 1 & & & & & \\
\vdots & & & \ddots & & & & & & \\
0 & 1 & 1 & \ldots & 1 & & & & & \\
\cline{1-5}
1 & 0 & 0 & \ldots & 0 & & & B & & \\
\vdots & & & \ddots & & & & & & \\
1 & 0 & 0 & \ldots & 0 & & & & &
\end{array} \right).
$$

The matrix B is the $(M - a - 1)$-by-$(n - a - 1)$ matrix formed by the bottom $M - a - 1$ rows (corresponding to the words of weight d) and the rightmost $n - a - 1$ columns. We denote by C_i, $i = 0, 1$, the set of all words of C that are of weight d and have i in the first coordinate.

By Lemma 1 the graph with vertices – the words of weight d, and edges – the pairs of words of weight d that are at distance 2, is a union of (possibly trivial) cliques.

1) Let us first assume that $a > \frac{d}{2}$. Assume that both C_0 and C_1 are non-empty. For $\boldsymbol{c}_0 \in C_0$ and $\boldsymbol{c}_1 \in C_1$, we have

$$d(\boldsymbol{c}_0, \boldsymbol{c}_1) \geq a + 1 + (d - 1) - (d - a) = 2a > d,$$

a contradiction. Thus we have either $C_0 = \varnothing$, or $C_1 = \varnothing$.

(a) Assume $C_1 = \varnothing$. Since every two words from C_0 are obviously at distance 2, we have

$$BB^T = (d - a - 1)J + I,$$

and it is easily checked that $\det BB^T \neq 0$. Now we have

$$M - a - 1 = \operatorname{rank} BB^T \leq \operatorname{rank} B \leq n - a - 1,$$

whence $M \leq n$.

(b) Assume $C_0 = \varnothing$. Now we have

$$BB^T = \left(\begin{array}{cccc|ccc|ccc|c} d-1 & d-2 & \dots & d-2 & & & & & & & \\ d-2 & d-1 & \dots & d-2 & & & & & & & \\ \vdots & \vdots & \ddots & \vdots & & \frac{d}{2}J & & & \frac{d}{2}J & & \dots \\ d-2 & d-2 & \dots & d-1 & & & & & & & \\ \hline & & & & d-1 & \dots & d-2 & & & & \\ & \frac{d}{2}J & & & \vdots & \ddots & \vdots & & \frac{d}{2}J & & \dots \\ & & & & d-2 & \dots & d-1 & & & & \\ \hline & & & & & & & d-1 & \dots & d-2 & \\ & \frac{d}{2}J & & & & \frac{d}{2}J & & \vdots & \ddots & \vdots & \dots \\ & & & & & & & d-2 & \dots & d-1 & \\ \hline & \vdots & & & & \vdots & & & \vdots & & \ddots \end{array}\right).$$

It can be proved again that $\det BB^T \neq 0$, and we can repeat the above argument:

$$M - a - 1 = \operatorname{rank} BB^T \leq \operatorname{rank} B \leq n - a - 1.$$

2) Now we consider the case where $1 \leq a \leq \frac{d}{2}$. The structure of C is again the same as in (1). We keep the notation form 1), i.e. C_0 is the set of all words of weight d that start with 0, and C_1 is the set of all words of weight d that start with 1.

Let us note that if $\boldsymbol{c}_0 \in C_0$, and $\boldsymbol{c}_1 \in C_1$, then $d(\boldsymbol{c}_0, \boldsymbol{c}_1) = d$. This is obvious if $a \geq 2$. Assume that $a = 1$. Set

$$\boldsymbol{c}_0 = (0, 1, \underbrace{1, \dots, 1}_{d-1}, 0, \dots, 0),$$
$$\boldsymbol{c}_1 = (1, 0, \underbrace{1, \dots, 1}_{d-1}, 0, \dots, 0).$$

Consider a word $\boldsymbol{c}' \in C$. Obviously, $\boldsymbol{c}' = (0, 1, *, *, \dots, *)$, or $\boldsymbol{c}' = (1, 0, *, *, \dots, *)$. In both cases, we have

$$|d(\boldsymbol{c}_0, \boldsymbol{c}') - d(\boldsymbol{c}_1, \boldsymbol{c}')| = 2$$

which is impossible since this difference can take on only the values 0 and $d - 2$.

Now we compute again $\det BB^T$ which turns out to be not 0. Hence using the chain of inequalities

$$M - a - 1 = \operatorname{rank} BB^T \leq \operatorname{rank} B \leq n - a - 1$$

we get again $M \leq n$. Thus we have proved the following theorem.

Theorem 4. *If C is an $(n, M, \{2, d\})$-code with $n \geq 8$, $6 \leq d \leq n - 2$, d even, then $M \leq n$.*

This implies the validity of Conjecture 1(b) from [1]:

Corollary 5. *If $d \geq 5$ then*

$$A_2(n, \{2, d\}) = \begin{cases} n & \text{for } 5 \leq d \leq n-2, \\ n+1 & \text{for } d = n-1. \end{cases}$$

4. The shifting technique. In this section, we consider the case $d_1 = 2$, $d_2 = 4$. The following definition goes back to Erdős, Ko and Rado [2] (see also Frankl [3]) and is introduced here for binary vectors and binary codes.

Let $C \subset \mathbb{F}_2^n$ and let $\boldsymbol{v} \in \mathbb{F}_2^n$. We denote by $\text{supp}(\boldsymbol{v})$ the set of non-zero coordinate positions of $\boldsymbol{v}$. So $\text{supp}(\boldsymbol{v})$ can be thought of as a subset of $\{1, \ldots, n\}$. The (i,j)-shift of $\boldsymbol{v}$ is defined by

$$s_{i,j}(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} + \boldsymbol{e}_i + \boldsymbol{e}_j & \text{if } i \notin \text{supp}(\boldsymbol{v}),\ j \in \text{supp}(\boldsymbol{v}),\ \boldsymbol{v} + \boldsymbol{e}_i + \boldsymbol{e}_j \notin C; \\ \boldsymbol{v} & \text{otherwise.} \end{cases} \tag{2}$$

Here $\boldsymbol{e}_i$ is the unit vector with 1 in position i. The (i,j)-shift of a binary code C is defined by

$$S_{i,j}(C) = \{s_{i,j}(\boldsymbol{v}) \mid \boldsymbol{v} \in C\}. \tag{3}$$

Our proof is based on the following lemma.

Lemma 6. *Let C be a $(n, \{2,4\})$-code. Then $S_{i,j}(C)$ is also an $(n, \{2,4\})$-code.*

Proof. Let us consider two words $\boldsymbol{u}, \boldsymbol{v} \in C$, $\boldsymbol{u} \neq \boldsymbol{v}$. We have to show that $d(s_{i,j}(\boldsymbol{u}), s_{i,j}(\boldsymbol{v})) \in \{2,4\}$. We have four possibilities:

(1) $s_{i,j}(\boldsymbol{u}) = \boldsymbol{u}$, $s_{i,j}(\boldsymbol{v}) = \boldsymbol{v}$;

(2) $s_{i,j}(\boldsymbol{u}) = \boldsymbol{u}$, $s_{i,j}(\boldsymbol{v}) \neq \boldsymbol{v}$;

(3) $s_{i,j}(\boldsymbol{u}) \neq \boldsymbol{u}$, $s_{i,j}(\boldsymbol{v}) = \boldsymbol{v}$;

(4) $s_{i,j}(\boldsymbol{u}) \neq \boldsymbol{u}$, $s_{i,j}(\boldsymbol{v}) \neq \boldsymbol{v}$.

It is clear that in cases (1) and (4), we have $d(s_{i,j}(\boldsymbol{u}), s_{i,j}(\boldsymbol{v})) = d(\boldsymbol{u}, \boldsymbol{v})$. Cases (2) and (3) are similar and are treated in the same way. Hence we shall consider just case (2). Since $s_{i,j}(\boldsymbol{u}) = \boldsymbol{u}$ one of the following must take place:

(i) $i \notin \text{supp}(\boldsymbol{u})$, $j \notin \text{supp}(\boldsymbol{u})$;

(ii) $i \in \text{supp}(\boldsymbol{u})$, $j \in \text{supp}(\boldsymbol{u})$;

(iii) $i \in \text{supp}(\boldsymbol{u})$, $j \notin \text{supp}(\boldsymbol{u})$;

(iv) $i \notin \text{supp}(\boldsymbol{u})$, $j \in \text{supp}(\boldsymbol{u})$, $\boldsymbol{u} + \boldsymbol{e}_i + \boldsymbol{e}_j \in C$.

The vectors $\boldsymbol{v}$ and $s_{i,j}(\boldsymbol{v})$ have exactly one unit in positions i and j in all cases. So, in cases (i) and (ii) $d(\boldsymbol{u},\boldsymbol{v}) = d(\boldsymbol{u}, s_{i,j}(\boldsymbol{v}))$. Similarly, in case (iii), we get

$$d(\boldsymbol{u}, s_{i,j}(\boldsymbol{v})) = d(\boldsymbol{u},\boldsymbol{v}) - 2.$$

If $d(\boldsymbol{u},\boldsymbol{v}) = 4$, then $d(\boldsymbol{u}, s_{i,j}(\boldsymbol{v})) = 2$. If $d(\boldsymbol{u},\boldsymbol{v}) = 2$, then $d(\boldsymbol{u}, s_{i,j}(\boldsymbol{v})) = 0$, i.e. $\boldsymbol{u} = s_{i,j}(\boldsymbol{v})$ which contradicts the definition of an (i,j)-shift.

Finally, in case (iv)

$$d(\boldsymbol{u}, s_{i,j}(\boldsymbol{v})) = d(\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{e}_i + \boldsymbol{e}_j) = d(\boldsymbol{u} + \boldsymbol{e}_i + \boldsymbol{e}_j, \boldsymbol{v}) \in \{2,4\},$$

since $\boldsymbol{u} + \boldsymbol{e}_i + \boldsymbol{e}_j \in C$. □

A code C with the property

$$S_{i,j}(C) = C$$

for all $i < j$ is called stable. Clearly, every code can be transformed to a stable code by performing at most $\binom{n}{2}$ shifts, e.g. the shifts $S_{i,j}$ for all pairs i, j with $i < j$.

Now we are going to prove our main result that implies the exact value of $A_2(n, \{2,4\})$.

Theorem 7. *Let C be a binary $(n,\{2,4\})$-code with $n \geq 6$. Then*

$$|C| \leq \binom{n}{2} + 1.$$

Proof. Assume for a contradiction that C is an $(n,\{2,4\})$-code of cardinality

$$|C| > \binom{n}{2} + 1.$$

Because of Lemma 6 we can assume that C is a stable code. Since the case $n = 6$ was settled in the introduction, we can assume that $n \geq 7$. Since the Hamming metric is translation invariant we can also assume that the zero word is in C. Hence all words in C are of weight 2 or 4.

Denote

$$C_i = \{v = (v_1, \ldots, v_n) \in C \mid v_n = i\},\ i = 0, 1.$$

We shall use induction on the length of C. Therefore we can assume that $|C_0| \leq \binom{n-1}{2}$ which in turn implies that $|C_1| > n - 1$.

Assume that $\boldsymbol{e}_i + \boldsymbol{e}_n \in C$, $i \neq n$. Since C is stable it contains also all vectors $\boldsymbol{e}_i + \boldsymbol{e}_j$ for all $j \in \{1, \ldots, n\} \setminus \{i\}$. This implies that all words in C of weight 4 have 1 in position i. Otherwise, such a word of weight 4 is at distance 6 from at least one of $\boldsymbol{e}_i + \boldsymbol{e}_j$. This uses the fact that $n \geq 7$. This observation implies immediately that there are at most four words of weight 2 in C_1. If there are exactly four words of weight 2 in C_1, then C_1 cannot contain a word of weight 4 and hence $|C_1| \le n-1$. This implies

$$|C| = |C_0| + |C_1| \le 1 + \binom{n-1}{2} + \binom{n-1}{1} = 1 + \binom{n}{2}.$$

Now let there exist exactly three words of weight 2 in C_1: $\boldsymbol{e}_{i_j} + \boldsymbol{e}_n$, $j = 1, 2, 3$. Then the only possible word of weight 4 in C_1 is $\boldsymbol{e}_{i_1} + \boldsymbol{e}_{i_2} + \boldsymbol{e}_{i_3} + \boldsymbol{e}_n$ and

$$|C| = |C_0| + |C_1| \le 1 + \binom{n-1}{2} + (1+3) < \binom{n}{2} + 1.$$

Now assume C_1 has two words of weight 4, $\boldsymbol{u}$ and $\boldsymbol{v}$ say. We consider the case where $d(\boldsymbol{u}, \boldsymbol{v}) = 4$, i.e.

$$\boldsymbol{u} = \boldsymbol{e}_{i_1} + \boldsymbol{e}_{i_2} + \boldsymbol{e}_{i_3} + \boldsymbol{e}_n, \quad \boldsymbol{v} = \boldsymbol{e}_{i_1} + \boldsymbol{e}_{j_2} + \boldsymbol{e}_{j_3} + \boldsymbol{e}_n,$$

where i_1, i_2, i_3, j_2, j_3 are all different. Let $k \in \{1, \ldots, n-1\} \setminus \{i_1, i_2, i_3, j_2, j_3\}$ ($n \ge 7$). Since C is stable we have that $\boldsymbol{w} = \boldsymbol{e}_{i_1} + \boldsymbol{e}_{i_2} + \boldsymbol{e}_{i_3} + \boldsymbol{e}_k \in C$. Now $d(\boldsymbol{w}, \boldsymbol{v}) = 6$, a contradiction. Thus we have proved that if $\boldsymbol{u}, \boldsymbol{v} \in C_1$ and they are both of weight 4, then $d(\boldsymbol{u}, \boldsymbol{v}) = 2$. Now by the Erdős–Ko–Rado theorem the number of the words of weight 4 in C_1 is at most $\binom{n-1-2}{3-2} = n-3$. Hence

$$|C| = |C_0| + |C_1| \le \binom{n-1}{2} + 1 + (n-3) \le \binom{n}{2} + 1. \qquad \square$$

Corollary 8. $A_2(n, \{2, 4\}) = \binom{n}{2} + 1$ *for all* $n \ge 6$.

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