Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 74, No 10, 2021

MATHEMATICS
Differential equations

CONSTRUCTION OF RADIAL AND NON-RADIAL SOLUTIONS FOR LOCAL AND NON-LOCAL EQUATIONS OF LIOUVILLE TYPE

Petar Popivanov, Angela Slavova

(*Submitted on September 17, 2021*)

Abstract

This paper deals with radial and non-radial solutions for local and non-local Liouville type equations. At first non-degenerate and degenerate mean field equations are studied and radially symmetric solutions to the Dirichlet problem for them are written into explicit form. Non-radial solution is constructed in the case of Blaschke type nonlinearity. The Cauchy boundary value problem for nonlinear Laplace equation with several exponential nonlinearities is considered and C^2 smooth monotonically decreasing radial solution $u(r)$ is found. Moreover, $u(r)$ has logarithmic growth at ∞. Our results are applied to the differential geometry, more precisely, minimal non-superconformal degenerate two dimensional surfaces are constructed in $\mathbf{R}^4$ and their Gaussian, respectively normal curvatures are written into explicit form. At the end of the paper several examples of local Liouville type PDE with radial coefficients which do not have radial solutions are given.

Key words: Liouville type equation, Dirichlet problem, mean field equation, radial and non-radial solutions

2000 Mathematics Subject Classification: 35J05, 35J65, 35B05, 35B40, 53A10

1. Introduction and formulation of the main results. In 1949 ONSAGER [7] considered in the frames of the statistical mechanics nonlocal elliptic equations with exponential type nonlinearities. Similar type equations arise in the mean field equations of hydrodynamic turbulence in equilibrium. Recently, a

DOI:10.7546/CRABS.2021.10.03

lot of papers have appeared on the subject. Below we propose several of them: [3, 6, 8, 11, 15, 17–20]. In studying the corresponding boundary value problem (bvp) very often variational methods are applied. This approach can be illustrated by the classical mean field equation (nonlocal elliptic equation of Liouville type with exponential nonlinearity):

$$\begin{aligned} &\Delta u + \lambda \frac{e^u}{\int_\Omega e^u\,dx} = 0 \\ &u|_{\partial\Omega} = 0 \end{aligned} \tag{1}$$

in the bounded smooth domain $\Omega \subset \mathbf{R}^2$, $\lambda > 0$ is the spectral parameter.

In [15] the deterministic Liouville type equation was studied:

$$\Delta u + \lambda \left(\tau \frac{e^u}{\int_\Omega e^u\,dx} + (1-\tau) \frac{\gamma e^{\gamma u}}{\int_\Omega e^{\gamma u}\,dx} \right) = 0, \Omega \in \mathbf{R}^2, \tag{2}$$

Ω – bounded, $0 < \tau < 1$, $\lambda > 0$,

$$u|_{\partial\Omega} = 0.$$

For $0 < \lambda < \lambda_{\tau,\gamma}$, $\lambda_{\tau,\gamma} > 0$ being appropriate constant there exists a solution of (2) corresponding to the minimum of some functional. Moreover, if $\Omega = B_1$ the solutions are radial. However, the authors of [15] establish a link between (2) and a special local bvp defined in $\mathbf{R}^2$ (this bvp is a special case of the cosmic string problem – see for example [9]). Our aim here is to find radial solutions of the Dirichlet problem for local and non-local PDE of Liouville type. To do this an approach coming from the classical theory of ODE will be applied (see [1, 10, 12, 19]). In fact, in many cases the solutions of our PDE with constant data on $S_1 = \partial B_1$ are radially symmetric (see for example [4, 14]). Other methods used in the investigation of elliptic PDE with exponential nonlinearities are the topological ones – see [5] as an illustrative example.

Below we shall write down the main objects of our considerations in this paper as follows:

$$\begin{aligned} &\Delta u + \lambda \frac{e^u}{\int_{B_1} e^u\,dx} = 0, \quad x \in B_1,\ \lambda > 0,\ B_1 \subset \mathbf{R}^2, \\ &u|_{\partial B_1} = C = \text{const}, \end{aligned} \tag{3}$$

$$\begin{aligned} &\Delta u + \lambda \frac{e^u |x|^n}{\int_{B_1} e^u\,dx} = 0, \quad x \in B_1,\ \lambda > 0, \\ &u|_{\partial B_1} = 0, \end{aligned} \tag{4}$$

$$\begin{aligned} &\Delta u + \lambda \frac{e^u |x|^2}{\int_{B_1} e^u |x|^2\,dx} = 0, \quad x \in B_1,\ \lambda > 0, \\ &u|_{\partial B_1} = 0, \end{aligned} \tag{5}$$

$$
(6)\qquad \begin{aligned}&\Delta u+\lambda\frac{e^u|x|^m}{(1+|x|^{m+2})^{2-\lambda B^2/(m+2)^2}}=0,\quad x\in B_1,\ \lambda>0,\\ &u|_{\partial B_1}=0,\end{aligned}
$$

((6) is a local bvp in B_1).

$$
(7)\qquad \Delta u+\lambda|F'(z)|^2\frac{e^u}{\int_{B_1}e^u}=0,\quad x\in B_1,\ \lambda>0,
$$

$F(z)$ is Blaschke function $F(z)=\prod_{j=1}^n\frac{z-\alpha_j}{1-\overline{\alpha}_jz}$, $0\le|\alpha_j|<1$.

$$
(8)\qquad \begin{aligned}&\Delta u+\textstyle\sum_{j=1}^n\lambda_j|x|^{\rho_j}e^{\kappa_ju}=0 \text{ in } \mathbf{R}^2\setminus\overline{B}_1,\\ &u|_{\partial B_1}=u_0=\text{const},\ \frac{\partial u}{\partial n}\Big|_{\partial B_1}=u_1=\text{const}\le 0,\\ &\kappa_1>\kappa_2>\cdots>\kappa_n>0.\end{aligned}
$$

If the solution $u=u(r)$, we shall consider also (8) in $r\ge 0$ with data $u(0)=u_0$, $\frac{\partial u}{\partial r}u(0)=0$, $\rho_{j'}\ge 1$, $\rho_{j''}=0$, $\rho=(\rho_{j'},\rho_{j''})$.

The bvp (1), (2), (3), (4), (5), (7) represent 2-dimensional elliptic eigenvalue problems with exponential nonlinearities (see [17]). This is the first result of our paper.

Proposition 1.

(a) *Consider the nondegenerate mean field equation* (3). *The solutions of* (3) *are radially symmetric and can be written explicitly as*

$$
u=\log\frac{8|a|^2}{\mu(1+|a|^2r^2)^2},\quad r\le 1,
$$

where $|a_\pm|=\frac{\sqrt{2}\pm\sqrt{2-p^2}}{p}$, $0<p\le\sqrt{2}$, $p=\sqrt{\mu}e^{C/2}$, $\mu=\lambda\frac{e^{-C}}{\pi}\left(1-\frac{\lambda}{8\pi}\right)$, $0<\lambda<8\pi$.

(b) *The radial solutions of the degenerate mean field equation* (4) *exist and can be written into explicit form.*

(c) *The radial solution of* (5) *is given into logarithmic explicit form and exists for* $0<\lambda<16\pi$. *There is no solution for* $\lambda>16\pi$.

(d) *Equation* (6) *possesses one parameter family of logarithmic solutions* $u_B(r)$, $B>0$. *The Dirichlet problem for* (6) *has two radially symmetric solutions if* $1<\frac{(m+2)2^{-\frac{1}{\log 4}}}{\sqrt{\lambda\log 2}}$, *only one radially symmetric solution for* $1=\frac{(m+2)2^{-\frac{1}{\log 4}}}{\sqrt{\lambda\log 2}}$ *and is nonsolvable in the opposite cases.*

(e) *The bvp* (7) *possesses a solution for each* $\lambda>0$ *if* $F(z)$ *has at least one multiple root. It is not radially symmetric in the general case. If* $F(z)$ *has only simple roots, then the spectrum of* (7) *is bounded.*

At the end of Section 2 we propose an application of Proposition 1 to differential geometry – more precisely, to the theory of minimal, non-superconformal degenerate two dimensional surfaces in $\mathbf{R}^4$. We propose also several examples of local Liouville type PDE with radial coefficients which have non-radial solutions. It is worth pointing out that the non-radial solutions are constructed via the radial ones.

Theorem 1. *Consider the local equation* (8) *in radial coordinates* (r, φ) *in the whole plane* $r \geq 0$ *equipped with the data* $u(0) = \alpha$, $u'(0) = 0$. *Then if* $u = u(r)$ *is radial solution of*

$$\begin{aligned} &u_{rr} + \tfrac{1}{r} u_r + \textstyle\sum_{j=1}^{n} \mu_j r^{\rho_j} e^{\kappa_j u} = 0, \ r > 0, \ \rho_1 \geq 1, \dots, \rho_{j'} \geq 1; \\ &\rho_{j'+1} = \dots = \rho_n = 0, \ \mu_j > 0, \ 1 \leq j \leq n, \ \kappa_1 > \dots > \kappa_n > 0, \\ &u(0) = \alpha, \ u'(0) = 0. \end{aligned} \tag{9}$$

(9) *has a* $C^2(r \geq 0)$ *smooth radial solution* $u = u(r, \mu)$, $\mu = (\mu_1, \dots, \mu_n)$ *which is strictly monotonically decreasing in* r. *Moreover, there exists*

$$\lim_{r \to \infty} \frac{u(r)}{\ln r} = -m = -\int_0^\infty \sum_{j=1}^{n} \mu_j r^{\rho_j + 1} e^{\kappa_j u(r)} \, dr > -\infty.$$

The above integral is convergent at $r = \infty$ iff $m > \max_{1 \leq j \leq n} \frac{\rho_j + 2}{\kappa_j} = M_0$. Put $C_j = \frac{\rho_j + 2}{\kappa_j}$. Then $M_0 < m \leq 2M_0$. If $C_j > C_1$ for $j \geq 2$ it follows that $m > 2C_1$, while $m > C_j$ for each $j = 1, \dots, n$.

Corollary 1. *Suppose that* $C_1 = \dots = C_n > 0 \Rightarrow M_0 = C_j$. *Then* $m = 2C_1 > M_0$ *and* $-\int_{\mathbf{R}^2} \Delta u = 4\pi C_1$.

The paper is organized as follows. Some additional results from the complex analysis, proof of Proposition 1 and several examples of radial and non-radial solutions of nonlinear PDE are proposed in Section 2. In Section 3 the proof of Theorem 1 is sketched.

2. Some additional results from the complex analysis, proof of Proposition 1 and several examples. Denote by $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, where $z = x + iy \in \mathbf{C}^1$. Then the two-dimensional Laplace operator $\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$. Assume that $f(z)$ is analytic function of z. According to the definition of analyticity $\frac{\partial z}{\partial \bar{z}} = 0$, while we denote $\frac{d}{dz} f(z) = f'(z)$. Evidently, $\frac{\partial}{\partial z} f(\bar{z}) = 0$ as $\frac{\partial z}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}}{\partial z} = 0$ and $\frac{\partial}{\partial \bar{z}} f(\bar{z}) = f'(\bar{z})$. Suppose that $f(z_0) \neq 0$. Then near z_0 in $\mathbf{C}^1$ there exists a single-valued branch of $\log f(z) = \log |f(z)| + i \arg f(z) \Rightarrow \Delta \log |f(z)| = 0$ near z_0.

On the other hand, $\frac{\partial}{\partial z} \log\left(1 + \left|\Phi(z)\overline{\Phi(z)}\right|\right) = \frac{\Phi'(z)\overline{\Phi(z)}}{1 + |\Phi|^2}$, $\Phi(z)$ being analytic function. There are no difficulties to check that $\Delta \log(1 + |\Phi|^2) = e^{\log \frac{4|\Phi'|^2}{(1+|\Phi|^2)^2}}$. In this way we conclude that if $F(z)$, $\Phi(z)$ are analytic functions near z_0, $F(z_0) \neq 0$, $\Phi'(z_0) \neq 0$, then the real function

$$u = \log \frac{8|\Phi'|^2}{|F(z)|^2(1+|\Phi|^2)^2} \tag{10}$$

satisfies near z_0 the Liouville equation

$$\Delta u + |F|^2 e^u = 0. \tag{11}$$

Conversely, if the C^4 solution u of (11) with $F(z_0) \neq 0$ satisfies (11) near z_0, then one can find analytic function $\Phi(z)$, $\Phi'(z_0) \neq 0$ for which (11) holds near z_0. Certainly, $F(z)$ is analytic.

We shall prove now Proposition 1 in the unit disc $B_1 \subset \mathbf{R}^2$.

We do not consider the case (a) as it is similar to the considerations of (b), (c), (e). As we mentioned above the classical solutions of (3) are radially symmetric (see [11]). The Laplace operator Δ in polar coordinates in the plane (r, φ) is given by $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$. These solutions are not uniquely determined. Via Pohozaev type identity one can prove that if classical solution of (3) exists, then $\lambda < 8\pi$.

Proof of Proposition 1(b). Our first step is to construct radial solutions of equation

$$\Delta u + \mu e^u |x|^n = 0, \quad x \in B_1, \tag{12}$$

where $\mu = \frac{\lambda}{\int_{B_1} e^u \, dx}$.

Then according to (10) the solution $u(r)$ of (12) takes the form

$$u = \log \frac{2(n+2)^2 C}{\mu (C + r^{n+2})^2}, \tag{13}$$

$C > 0$ being a parameter. The condition $u|_{\partial B_1} = 0 \iff 2(n+2)^2 C = \mu (C+1)^2$. Thus,

$$0 < C_{\pm}(\mu) = \frac{(n+2)^2 - \mu \pm (n+2)\sqrt{(n+2)^2 - 2\mu}}{\mu}. \tag{14}$$

Certainly, $\frac{(n+2)^2}{2} \geq \mu > 0$ is a necessary condition for the existence of classical solution to the Dirichlet problem for (12). Then in polar coordinates

$$\frac{\lambda}{\mu} = \int_{B_1} e^u = \frac{4\pi C (n+2)^2}{\mu} \int_0^1 \frac{r \, dr}{(C + r^{n+2})^2}.$$

Therefore,

$$\lambda = 4\pi C (n+2)^2 F(C), \quad F(C_{\pm}) = \int_0^1 \frac{r \, dr}{(C_{\pm}(\mu) + r^{n+2})^2} > 0. \tag{15}$$

Put $G(C) = CF(C) = \int_0^{C^{-\frac{1}{n+2}}} \frac{s\,ds}{(1+s^{n+2})} C^{\frac{2}{n+2}-1}$, $n \geq 1$. Therefore, $C \to +\infty \Rightarrow G(C) \sim \frac{1}{2C}$, $G(C) \sim KC^{\frac{2}{n+2}-1}$, $C \to 0$, $K = \text{const}$, i.e. the continuous mapping $G(C)$ is onto $(0, \infty)$. On the other hand, one can check that $C_+(\mu) \sim \frac{2(n+2)^2}{\mu}$, $\mu \to 0$, $C_\pm\left(\frac{(n+2)^2}{2}\right) = 1$, $C_-(\mu) \sim \frac{\mu^2}{(n+2)^2}$, $\mu \to 0$. So $C_\pm$ is a continuous mapping from $\left(0, \frac{(n+2)^2}{2}\right)$ onto $(0, \infty)$. As $G'(C) < 0$, G is monotonically decreasing, etc.

Consequently, we can solve the transcendental equation (15) finding $\mu = \mu(\lambda)$, $\mu > 0$. Formula (13) with $\mu = \mu(\lambda)$ gives us the solution of (4). □

Proof of Proposition 1(c). Again $\frac{\lambda}{\mu} = \int_{B_1} e^u |x|^2\,dx$, $u(r)$ is a solution of the Dirichlet problem for (12) with $u|_{\partial B_1} = 0$, $C_\pm(\mu)$ is given by (14). The only difference from the previous case is that

$$0 < \frac{\lambda}{\mu} = \frac{64\pi C}{\mu} \int_0^1 \frac{r^3\,dr}{(C + r^4)^2} = \frac{16\pi}{\mu} \frac{1}{C+1},$$

i.e.

$$\lambda = \frac{\pi\mu}{1 \pm \sqrt{1 - \mu/8}} = F_{1\pm}(\mu), \quad 0 < \mu \leq 8. \tag{16}$$

Evidently, $F_{1\pm}(8) = 8\pi$, $F_{1+}(0) = 0$, $F_{1-}(0) = 16\pi$. Therefore, for each $\lambda \in (0, 8\pi]$ there exists $\mu \in (0, 8]$ such that $F_{1+}(\mu) = \lambda$ and for each $\lambda \in [8\pi, 16\pi)$ there exists $\mu \in (0, 8]$ such that $F_{1-}(\mu) = \lambda$. Obviously, $\lambda \in (0, 16\pi)$, F_{1+} is monotonically increasing, F_{1-} is monotonically decreasing and (16) implies that $\mu = \frac{\lambda}{\pi^2}\left(2\pi - \frac{\lambda}{8}\right)$. □

Proof of Proposition 1(d). The considerations for (6) are similar to the previous ones. For the unknown constants $m > 0$, $n > 0$, $p > 0$ we consider the Dirichlet problem

$$\begin{aligned} &\Delta u + \lambda e^u \tfrac{|x|^m}{(1+|x|^n)^p} = 0 \text{ in } B_1 \\ &u|_{\partial B_1} = 0. \end{aligned} \tag{17}$$

We are looking for a radial solution of (17) having the form

$$u = \log(\varphi^{2A} B^2), \quad \varphi(r) > 0, \quad B = \text{const} > 0. \tag{18}$$

For $\varphi(r)$ we obtain the following ODE

$$\varphi\varphi'' + \frac{\varphi\varphi'}{r} - (\varphi')^2 = -\lambda \frac{B^2}{2A} \varphi^{2A+2} \frac{r^m}{(1 + r^n)^p}. \tag{19}$$

Assume that $\varphi = 1 + r^\nu$. Then

$$\nu^2 r^{n-2} = -\frac{\lambda B^2}{2A} r^m$$

for $\nu = n$, $2A+2=p$, $n=m+2$, $\nu^2 = -\frac{\lambda B^2}{2A} < 0 \iff A < 0$.

Evidently, $A = \frac{p-2}{2} < 0$. Thus,

$$\nu = n = m+2 \Rightarrow \frac{(m+2)^2}{B^2} = -\frac{\lambda}{2A} = \frac{\lambda}{2-p},$$

i.e. $p = 2 - \frac{\lambda B^2}{n^2} < 2$; $B > 0$ is a parameter. Then equation (17) takes the form

$$\Delta u + \lambda e^u \frac{|x|^m}{(1+|x|^{m+2})^{2-\lambda B^2/n^2}} = 0 \text{ in } B_1,$$

i.e. we obtain (6). Dirichlet problem (6) possesses the radial solution

$$u(r) = \log\left(B^2\left(1+r^{m+2}\right)^{-\frac{\lambda B^2}{(m+2)^2}}\right), \tag{20}$$

where $B > 0$ is arbitrary.

To solve the Dirichlet problem (6) we must have

$$\frac{B^2}{2^{\lambda B^2/(m+2)^2}} = 1 \text{ for some } B > 0 \iff \frac{B}{2^{\lambda B^2/2(m+1)^2}} = 1. \tag{21}$$

Consider the function $f(B) = \frac{B}{2^{\lambda B^2/2(m+2)^2}} > 0$, $B > 0$.

One can check that

$$f(\infty) = 0,\ f(0) = 0,\ f'(B) = 0 \iff B = B_* = \frac{m+2}{\sqrt{\lambda \log 2}}$$
$$\Rightarrow f_{\max} = f(B_*) = \frac{(m+2)2^{-\frac{1}{\log 4}}}{\sqrt{\lambda \log 2}}.$$

In this way (21) possesses two solutions for $1 < f_{\max}$, only one solution for $1 = f_{\max}$ and does not have any solution for $1 > f_{\max}$. Proposition 1(d) is proved. □

Sketch of the proof of Proposition 1e). We look for solution of (7) having the form $u = \log \frac{8a^2}{\mu(1+a^2|F(z)|^2)^2}$ and such that $e^{C/2} = \frac{2\sqrt{2}a}{\sqrt{\mu}(1+a^2)}$, as $|F(z)||_{S_1} = 1$. The number a satisfies a quadratic equation, possessing two positive roots a_+, a_- for $0 < \mu \le 2e^{-C}$, namely, $a_+ = \sqrt{2}e^{-C/2}\mu^{-1/2}\left(1+\sqrt{1-\frac{\mu}{2}e^C}\right)$, $a_- = \frac{\sqrt{2}}{2}e^{C/2}\frac{\mu^{1/2}}{1+\sqrt{1-\frac{\mu}{2}e^C}}$. Moreover, $\lambda_\pm = \frac{8}{a_\pm^2}\int_{B_1}\frac{dx\,dy}{\left(a_\pm^{-2}+|F(z)|^2\right)^2}$, $z = x+iy$. The remaining part of the proof is standard but rather technical and we omit it. □

We shall illustrate some of the results of Proposition 1 with an application in geometry. It concerns the model example of de Azevedo Tribuzy and Guadalupe [16].

We propose here a slight generalization of [16] and we find two-parametric family of its radially symmetric solutions into explicit form. Consider the system

$$\begin{array}{c}(K^2-\kappa^2)^{1/4}\Delta\log|\kappa-K| = 2|x|^n(2K-\kappa)\\ (K^2-\kappa^2)^{1/4}\Delta\log|\kappa+K| = 2|x|^n(2K+\kappa)\end{array}, \tag{22}$$

where $K^2 > \kappa^2$, $K < 0$ and K denotes the Gaussian curvature of some two-dimensional non-super conformal minimal surface $M_2 \subset \mathbf{R}^4$, while κ stands for the normal curvature of M_2. The classical case is (22) with $n = 0$. As it is shown in [16] (K, κ), $n = 0$ defines minimal non-super conformal surface $M_2 \subset \mathbf{R}^2$. Our generalization (22) admits a power degeneration of the righthand side of (22). To find solutions of the system (22) we put

$$\begin{array}{l}0 > K-\kappa = -e^u\\ 0 > K+\kappa = -e^v\end{array} \Rightarrow |K-\kappa| = e^u,\ |K+\kappa| = e^v \Rightarrow \begin{array}{l}K = -\dfrac{e^u+e^v}{2}\\ \kappa = \dfrac{e^u-e^v}{2}\end{array}$$

$$\Rightarrow (K^2-\kappa^2)^{1/4} = e^{(u+v)/4}.$$

Then (22) takes the form

$$\begin{array}{c}\Delta u = -|x|^n(3e^p+e^q)\\ \Delta v = -|x|^n(e^p+3e^q)\end{array}, \tag{23}$$

where $p = (3u-v)/4$, $q = (3v-u)/4$.

One can easily see that

$$\begin{array}{c}\Delta p + 2|x|^n e^p = 0\\ \Delta q + 2|x|^n e^q = 0.\end{array} \tag{24}$$

System (23) reduces to two scalar equations (24). According to formula (13) for equation (12) with $\mu = 2$ we have that

$$p = \log\frac{16C_1}{(C_1+r^4)^2},\ C_1 > 0,\ C_1 - \text{arbitrary},$$
$$q = \log\frac{16C_2}{(C_2+r^4)^2},\ C_2 > 0,\ C_2 - \text{arbitrary}.$$

Having in mind that $u = (q+3p)/2$, $v = (p+3q)/2$ we get two-parametric family of solutions of (22)

$$\begin{array}{l}K = -\dfrac{1}{2}\dfrac{16^2C_1^{1/2}C_2^{1/2}}{(C_1+r^4)(C_2+r^4)}\left(\dfrac{C_1^{1/2}}{(C_1+r^4)^2}+\dfrac{C_2^{1/2}}{(C_2+r^4)^2}\right)\\ \kappa = \dfrac{1}{2}\dfrac{16^2C_1^{1/2}C_2^{1/2}}{(C_1+r^4)(C_2+r^4)}\left(\dfrac{C_1^{1/2}}{(C_1+r^4)^2}-\dfrac{C_2^{1/2}}{(C_2+r^4)^2}\right).\end{array} \tag{25}$$

At the end of this section we shall discuss the problem for existence of non-radial solutions of Liouville type PDE with radially symmetric coefficients. Starting from radial solutions of some classes of Liouville type PDE we shall construct several classes of non-radial solutions of other PDE with radially symmetric coefficients. It is proved in [4] that if $\Delta u + f(u) = 0$ in B_1, $u \in C^2(\overline{B}_1)$, $f \in C^1$ and $f(u) \geq 0$ everywhere, then any non-trivial solution of the Dirichlet problem $u|_{\partial B_1} = 0$ is positive in B_1 and radially symmetric. No unicity holds for the radial solutions $u(r)$. Moreover, $\frac{\partial u}{\partial r} < 0$ for $0 < r < 1$. Let $\Delta u + f(r,u) = 0$ in B_1 and $u|_{\partial B_1} = 0$, $u \in C^2(\overline{B}_1)$, $f \in C^1$ and $f(r,u)$ is decreasing in r, i.e. $f'_r(r,u) \leq 0$. Then each positive solution $u > 0$ is radially symmetric and $\frac{\partial u}{\partial r} < 0$ for $0 < r < 1$. This result does not hold if f is not decreasing in r (there are counter examples). Consider the polynomial $P(z)$ in $\mathbf{C}^1$. Then $P(z)$ is radially symmetric, i.e. $|P(re^{i\varphi})| = g(r)$ iff P is a monomial: $P(z) = az^m$.

Let $u_0(r)$ be some non-trivial radial solution of $\Delta u + f(u) = 0$, $f(u) \geq 0$, $u|_{\partial B_1} = 0$. Assume that $g(z)$ is a non-trivial analytic function for $|z| < 1 + \varepsilon_0$, $\varepsilon_0 > 0$ and consider the smooth function $U(z) = u_0(|g(z)|)$, $|g(z)| = \sqrt{\overline{g(z)}g(z)}$. Certainly

$$\Delta U = 4\frac{\partial^2}{\partial z \partial \overline{z}} U(z), \quad \frac{\partial U}{\partial z} = u_0'(|g(z)|)\frac{g'(z)}{2}\sqrt{\overline{g(z)}(g(z))^{-1}}.$$

Thus,

$$\Delta U + |g'(z)|^2 f(U) = 0, \quad U|_{|g(z)|=1} = u_0(1) = 0$$

but $U(z)$ is not radially symmetric if $|g(z)|$ is not radially symmetric function. For example, let $g(z) = z^m + C$, $m \geq 1$, $C \neq 0$. Then $|g(z)|$ is not radially symmetric, $|g(z)|^2 = r^{2m} + |C|^2 + 2r^m|C|\cos(m\varphi - \psi)$, where $C = |C|e^{i\psi}$, $z = re^{i\varphi}$. Moreover, $|g(z)| = 1 \iff r^{2m} + |C|^2 - 1 + 2r^m|C|\cos(m\varphi - \psi) = 0 \Rightarrow |C| \leq 1$.

In this way we get that the PDE with radial coefficients

$$\Delta U(z) + m^2|z|^{2m-2} f(U) = 0$$

possesses the non-radial solution $U(z) = u_0(|z^m + C|)$.

Example 1. Let $f(u) = \sum_{j=1}^n A_j e^{\kappa_j u}$, $A_j > 0$, $\kappa_1 > \cdots > \kappa_n > 0$ and $u_0(r)$ be non-trivial radial solution of the Dirichlet problem $\Delta u + f(u) = 0$, $u|_{\partial B_1} = 0$. Then $U(z) = u_0(|z^m + C|)$, $C \neq 0$ is positive nonradial solution of $\Delta U(z) + m^2|z|^{2m-2} f(U) = 0$.

3. Existence results for global radial solutions of some local Liouville type equations. Short proof of Theorem 1. The initial value problem (9) can be written as

$$\begin{aligned} & ru'' + u' = -\sum_{j=1}^n \mu_j r^{\rho_j+1} e^{\kappa_j u} = -f(r,u) < 0, \ r > 0, \\ & u(0) = \alpha, \\ & u'(0) = 0. \end{aligned} \tag{26}$$

Thus, $\frac{d}{dt}(ru') = -rf(r, u(r))$, $r \geq 0$. Therefore,

$$ru' = -\int_0^r sf(s, u(s))\, ds = g(r, u(r)) = \tilde{g}(r) < 0, \quad r \geq 0, \tag{27}$$

$\tilde{g}'(r) < 0$ for $r > 0$.

We rewrite the Cauchy problem (26) as

$$u' = -\begin{cases} \frac{1}{r}\int_0^r sf(s, u(s))\, ds, & r \geq 0, \\ 0, & r < 0. \end{cases} \tag{28}$$

If we put

$$F(r, u) = -\begin{cases} \frac{1}{r}\int_0^r sf(s, u(s))\, ds, & r \geq 0, \\ 0, & r < 0, \end{cases}$$

then $F(r, u)$ is Lipschitz continuous function with respect to u and (26) takes the form

$$\begin{aligned} &u'(r) = F(r, u(r)),\ r \geq 0, \\ &u(0) = \alpha. \end{aligned} \tag{29}$$

A classical result (Picard E.) gives us existence and uniqueness of $u \in C^1$ solution near $r = 0$, $r > 0$ of (29). The idea of the proof of Theorem 1 can be found in [19]. The rest of the proof is technical and we omit it due to lack of space.

We shall formulate only two formulas, crucial for the proof of Theorem 1. They are:

$$\lim_{r\to\infty} \frac{u(r)}{\log r} = \lim_{r\to\infty} ru'(r) = -m = -\int_0^\infty \sum_{j=1}^n r^{\rho_j+1} e^{\kappa_j u(r)}\, dr < 0,$$

$m = \text{const} > \max_j \frac{\rho_j+2}{\kappa_j}$ and

$$\frac{m^2}{2} - mC_1 = \int_0^\infty \sum_{j=2}^n \mu_j (C_j - C_1) r^{\rho_j+1} e^{\kappa_j u(r)}\, dr,$$

where $C_j = (\rho_j + 2)/\kappa_j$, $j = 1, 2, \ldots, n$.

REFERENCES

[1] Arnold V. (1992) Ordinary Differential Equations, Berlin, Heidelberg, Springer-Verlag.

[2] Dwight H. (1961) Tables of Integrals and Other Mathematical Data, 4th ed., NY, Macmillan Company, 336 pp.

[3] Eyink G., K. Sreenivasan (2006) Onsager and the theory of hydrodynamic turbulence, Rev. Modern Phys., **78**(1), 87–135.

[4] Gidas B., W.-M. Ni, L. Nirenberg (1979) Symmetry and related properties via the maximum principle, Commun. Math. Phys., **68**(3), 209–243.

[5] Malchiodi A. (2008) Topological methods for an elliptic equation with exponential nonlinearities, Discrete Contin. Dyn. Syst., **21**(1), 277–294.

[6] De Marchis F., T. Ricciardi (2017) Existence of stationary turbulent flows with variable positive vortex intensity, Nonlinear Anal.: Real World Appl., **38**, 222–244.

[7] Onsager L. (1949) Statistical hydrodynamics, Nuovo Cim., **6**, 279–287.

[8] Pistoia A., T. Ricciardi (2016) Concentrating solutions for a Liouville type equation with variable intensities in 2-D turbulence, Nonlinearity, **29**(2), 271–297.

[9] Poliakovsky A., G. Tarantello (2012) On a planar Liouville-type problem in the study of selfgravitating strings, J. Differ. Equ., **252**(5), 3668–3693.

[10] Popivanov P., A. Slavova (2011) Nonlinear waves. An introduction, Series of Analysis, Applications and Computation, **4**, Singapore, World Scientific.

[11] Popivanov P., N. Nikolov (2016) On the Dirichlet problem to the Liouville equation. Explicit formulas, C. R. Acad. Bulg. Sci., **69**(11), 1389–1400.

[12] Popivanov P., A. Slavova (2019) Nonlinear waves. A geometrical approach, Series of Analysis, Applications and Computation, **9**, Singapore, World Scientific.

[13] Popivanov P., A. Slavova, P. Zecca (2010) Regularizing property of the solutions of a dissipative semilinear wave equation, C. R. Acad. Bulg. Sci., **63**(7), 861–870.

[14] Prajapat J., G. Tarantello (2001) On a class of elliptic problems in $\mathbf{R}^2$: symmetry and uniqueness results, Proc. Royal Soc. Edinburgh: Sect. A Math., **131**(4), 967–985.

[15] Ricciardi T., R. Takahashi (2019) On radial two-species Onsager vortices near the critical temperature, Indiana Univ. Math. J., **68**(6), 1903–1926.

[16] Tribuzy, R. A., I. V. Guadalupe (1985) Minimal immersions of surfaces into 4-dimensional space form, Rendiconti del Seminario Matematico della Università di Padova, **73**, 1–13.

[17] Suzuki T. (1992) Global analysis for a 2-dimensional elliptic eigenvalue problem with the exponential nonlinearity, Annales de l'I.H.P. Analyse non linéaire, **9**(4), 367–397.

[18] Suzuki T. (2015) Mean Field Theories and Dual Variations: Mathematical Structures of the Mesoscopic Model, 2nd ed., Atlantis Studies in Math. for Engineering and Science, **11**, Paris, Atlantis Press.

[19] Yang Y. (1994) Self-duality of the gauge filed equations and the cosmological constant, Comm. Math. Phys., **162**(3), 481–498.

[20] Zhou C. (2011) Existence result for mean field equation on the equilibrium turbulence in the super critical case, Comm. Contemp. Math., **13**(4), 659–673.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Akad. G. Bonchev St, Bl. 8
1113 Sofia, Bulgaria
e-mails: `popivano@math.bas.bg`
`slavova@math.bas.bg`