Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 74, No 3, 2021

MATHEMATICS
Probability theory

CONTINUOUS-TIME CONTROLLED BRANCHING PROCESSES

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(*Submitted by Academician V. Drensky on November 24, 2020*)

Abstract

Controlled branching processes with continuous time are introduced and limiting distributions are obtained in the critical case. An extension of this class as regenerative controlled branching processes with continuous time is proposed and some asymptotic properties are considered.

Key words: controlled branching processes, renewal processes, regenerative processes, random time change, limiting distributions

2020 Mathematics Subject Classification: Primary 60J80; Secondary 60K05, 60J85, 60K10

1. Introduction. The main models and results about *Controlled Branching Processes* (CBPs) are presented in the recent monograph [1]. Notice that CBPs are integer-valued discrete-time Markov processes where the population size in every generation could be randomly regulated before the reproduction by an emigration of a part of the population or after the reproduction by an immigration of individuals. It is interesting to point out that in the CBP the evolutions of the individuals are not independent although they reproduce independently of each other. A general definition of CBP with continuous time (CT) does not exist up

This research has been supported by grants KP-6-H22/3 (Bulgarian National Science Fund at the Ministry of Education and Science), the Junta de Extremadura (grant GR18103 and IB16099), the Ministerio de Economía y Competitividad of Spain (grant MTM2015-70522-P), the Ministerio de Ciencia e Innovación of Spain (grant PID2019-108211GB-I00/AEI/10.13039/501100011033) and the Fondo Europeo de Desarrollo Regional.

DOI:10.7546/CRABS.2021.03.04

to now. The main motivation behind the present paper is to introduce a new class of continuous-time controlled branching processes.

Let $\{Z_n,\ n = 0, 1, \ldots\}$ be a CBP (see (2) and (3) in Section 2) and let $\{N(t),\ t \geq 0\}$ be a renewal process. We study the process $\{Y(t),\ t \geq 0\}$, defined by $Y(t) = Z_{N(t)}$, which is a first attempt to introduce a CBP with CT. If $N(t) = n$ then, at time t, the population size is $Y(t) = Z_n$. We can consider the renewal period as a common lifespan of all individuals. It is clear that $\{Y(t),\ t \geq 0\}$ is not a Markov process unless $\{N(t),\ t \geq 0\}$ is a homogeneous Poisson process. In all cases, the evolutions of the individuals are not independent. The proposed new process could be also referred to as CBP *subordinated by a renewal process* or as *randomly indexed* CBP.

A randomly indexed *Bienayme–Galton–Watson* (BGW) branching process was introduced in [2] as an alternative of the geometric Brownian motion for modelling daily stock prices. The BGW branching process subordinated by a general renewal process were studied in [3,4] (critical case) and also in [5,6] (non-critical cases).

The proposed here general CBP with CT is an essential generalization of the processes mentioned above. In this work, we investigate $\{Y(t),\ t \geq 0\}$ in the critical case. The paper is organized as follows. In Section 2 we define and discuss the CBP with CT and with single and multitype control functions. In Section 3 we present two limit theorems when the mean of the renewal periods is either finite (Theorem 1) or infinite (Theorem 2) for a CBP with CT and a single control function. Towards the goal of studying the general CBP with CT and multitype control functions, in Section 4 we investigate the particular case with three specific control functions. That is the CBP with random migration. We obtain limiting distributions by considering again finite and infinite mean of the renewal periods (Theorems 3 and 4, respectively). Finally, in Section 5 we propose an extension of the process $\{Y(t),\ t \geq 0\}$, namely, the regenerative process $\{U(t),\ t \geq 0\}$. It coincides with $\{Y(t),\ t \geq 0\}$ until it hits zero, then upon staying at zero for a random time period, the process regenerates. The basic definition of the alternating regenerative processes was proposed in [7] where the so-called *Basic Regenerative Theorem (BRT)* was proved. We apply this theorem as well as the limit theorems from Sections 3 and 4 to obtain limiting distributions. The proofs of all theorems will appear later in JAP.

2. Description of models. On a certain probability space $(\Omega, \mathcal{A}, P)$, we consider the following three independent sets of random variables.

(i) Define the set $X = \{X_n(i),\ n, i = 1, 2, \ldots\}$ of non-negative integer-valued i.i.d. random variables with probability generating function (p.g.f.) $f(s) = E\left[s^{X_1(1)}\right]$. Let also I_0 be a positive integer-valued random variable independent of X with p.g.f. $\Delta(s) = E\left[s^{I_0}\right]$. Recall that the classical BGW branching process with I_0 ancestors is defined as follows:

$$Z_0 = I_0, \quad Z_n = \sum_{i=1}^{Z_{n-1}} X_n(i), \; n = 1, 2, \dots, \tag{1}$$

where $\sum_{i=1}^{0} = 0$. Taking into account the independence of the individual evolutions, (1) implies $E[s^{Z_n} \mid Z_0 = I_0] = \Delta(f_n(s))$, where $f_n(s) = E[s^{Z_n} \mid Z_0 = 1]$ and $f_n(s) = f_{n-1}(f(s))$, $n = 1, 2, \dots$ Clearly, zero is an absorbing state.

(*ii*) Define the set $\phi = \{\phi_n(k),\; n = 1, 2, \dots; k = 0, 1, \dots\}$ of non-negative integer-valued random variables, independent of X, where for every fixed k the subset $\phi(k) = \{\phi_n(k),\; n = 1, 2, \dots\}$ consists of i.i.d. random variables with p.g.f. $g_k(s) = E[s^{\phi_1(k)}]$. The CBP is defined as follows:

$$Z_0 = I_0, \quad Z_n = \sum_{i=1}^{\phi_n(Z_{n-1})} X_n(i), \quad n = 1, 2, \dots, \tag{2}$$

with ϕ referred as the set of random control functions. Notice that if $\phi_n(k) \equiv k$ a.s. for every n, then $\{Z_n,\; n = 0, 1, \dots\}$ is a BGW branching process defined by (1). If this is not the case, let us point out that since $h_n(s) = E[s^{Z_n} \mid Z_0 = I_0] \neq \Delta(\psi_n(s))$, where $\psi_n(s) = E[s^{Z_n} \mid Z_0 = 1]$, we have that the evolutions of the individuals are not independent. It follows from (2) that the state zero will be absorbing if and only if $\phi_n(0) = 0$, a.s. for every n. For more details see [[1]]. Definition (2) can be generalized introducing the set of random control functions $\phi_D = \{\phi_{n,d}(k),\; n = 1, 2, \dots;\; k = 0, 1, \dots;\; d \in D\}$ and the set of random variables $X_D = \{X_{n,d}(i),\; n, i = 1, 2, \dots;\; d \in D\}$, where D is an index set. Then, the CBP with multitype control functions is defined as follows:

$$Z_0 = I_0, \quad Z_n = \left(\sum_{d \in D} \sum_{i=1}^{\phi_{n,d}(Z_{n-1})} X_{n,d}(i) \right)^+, \quad n = 1, 2, \dots, \tag{3}$$

where as usual $a^+ = \max\{0, a\}$. In some branching models it is assumed that for every fixed d the random variables $\{X_{n,d}(i),\; n, i = 1, 2, \dots\}$ are integer-valued i.i.d. and, for every fixed k and d, the subset $\phi_d(k) = \{\phi_{n,d}(k),\; n = 1, 2, \dots\}$ consists of non-negative integer-valued i.i.d. random variables. Notice that the random variable $X_{n,d}(i)$ can be negative, allowing individual emigration in the model. We will consider a particular case of (3) in Section 4 which admits a random migration component.

(*iii*) Finally, define the set $J = \{J_n,\; n = 1, 2, \dots\}$ of positive i.i.d. random variables, independent of X and ϕ, with cumulative distribution function (c.d.f.) $G(x) = P(J_1 \leq x),\; x > 0; G(0) = 0$, and the corresponding renewal process

$N(t) = \max\{n \geq 0 : \ S_n \leq t\}, \ t \geq 0$,where $S_0 = 0, \ S_n = \sum_{i=1}^{n} J_i, \ n = 1, 2, \ldots$ We will consider two cases: $\mu = E[J_1] = \int_0^{\infty} x dG(x) < \infty$ and $\mu = \infty$ with

$$1 - G(t) \sim t^{-\rho} \mathcal{L}(t)/\Gamma(1-\rho), \ 0 < \rho < 1, \ t \to \infty, \tag{4}$$

where $\mathcal{L}(t)$ is a slowly varying function $(s.v.f.)$.

Definition 1. Let $\{Z_n, \ n = 0, 1, \ldots\}$ be a CBP defined by (2) or (3) and let $\{N(t), \ t \geq 0\}$ be a renewal process given in (iii). Then, the continuous time process $\{Y(t), \ t \geq 0\}$, defined by $Y(t) = Z_{N(t)}$, is called *continuous time controlled branching process* $(CTCBP)$ or *CBP with CT and with multitype control functions*, respectively.

The process $\{Y(t), \ t \geq 0\}$, can be called *randomly indexed CBP with CT* or *CBP subordinated by a renewal process.* If $S_n \leq t < S_{n+1}$ then $N(t) = n$ and $Y(t) = Z_n$, $n = 0, 1, \ldots$ Then $Y(t)$ could be also considered as an age-dependent branching process, in which all individuals in a given generation have the same life-time and they give birth to their offspring simultaneously. If $Y(t) = Z_n$, then the lifespan of the individuals from the n-th generation is equal to J_n. In general, $\{Y(t), \ t \geq 0\}$ is not a Markov process excepting the particular case when $\{N(t), \ t \geq 0\}$ is a homogeneous Poisson process. Note that, $\{Y(t), \ t \geq 0\}$ is an example of branching process with dependent evolutions of the individuals in contrast with Bellman–Harris or Markov branching processes, where the evolutions of the individuals are independent. Finally, let us point out that the discrete-time CBP $\{Z_n, \ n = 0, 1, \ldots\}$ is embedded in the continuous-time CBP process $\{Y(t), \ t \geq 0\}$. In fact, $Z_n = Y(S_n), \ n = 0, 1, \ldots$ Moreover if $J_n \equiv 1$ a.s. for every n, then $Y(n) = Z_n, \ n = 0, 1, \ldots$

3. Critical controlled branching processes subordinated by a renewal process. In this section we will consider the process $Y(t) = Z_{N(t)}$, where $\{Z_n, n = 0, 1, \ldots\}$ is the CBP defined by (2) with zero as an absorbing state and $\{N(t), \ t \geq 0\}$ is the renewal process (iii). Let us introduce the following notations for $k \geq 0 : \varepsilon(k) = E[\phi_1(k)]$, $\nu^2(k) = \mathrm{Var}[\phi_1(k)]$, $\tau_m(k) = k^{-1} E[Z_{n+1} | Z_n = k] = k^{-1} \varepsilon(k) m, m = E[X_1(1)]$, $l_2(k) = \mathrm{Var}[Z_{n+1} | Z_n = k] = m^2 \nu^2(k) + \sigma^2 \varepsilon(k), \sigma^2 = \mathrm{Var}[X_1(1)]$.

Further on the following Condition A is assumed to hold:
(a) $\tau_m(k) = 1 + \dfrac{c}{k}$, $k = 1, 2, \ldots, \ 0 < c < \infty$; (b) $l_2(k) = \nu k + O(1), \ k \to \infty$, $0 < \nu < \infty$; (c) $\sup_{k \geq 1} (g_k^{1/k})'''(1) < \infty$, where recall $g_k(s) = E[s^{\phi_1(k)}]$; (d) $\{\phi_1(k)$, $k = 1, 2, \ldots\}$ have infinite divisible distributions.

It is worth pointing out that for a CBP, the threshold parameter determining the criticality of the process is the asymptotic mean growth rate, i.e. $\lim_{k \to \infty} \tau_m(k)$, when it exists (notice that the asymptotic mean growth rate for a BGW branching process coincides with m). We will investigate the continuous time process $Y(t) = Z_{N(t)}$ presented by Definition 1 with (2).

Theorem 1. *Assume Condition A,* $0 < \delta = 2c/\nu < 1, 0 < \mu < \infty$. *Then*
(*i*) $P(Y(t) > 0) \sim K\mu^{1-\delta}t^{-(1-\delta)}$, *as* $t \to \infty$, *with* $K > 0$ *(well determined).*
(*ii*) $E[Y(t)] \sim (K\nu/2\mu^{\delta})t^{\delta}$, $\mathrm{Var}[Y(t)] \sim (K\nu^2/2(\delta+1))\mu^{-(\delta+1)}t^{\delta+1}$, $t \to \infty$.
(*iii*) $\lim_{t\to\infty} P\left(\dfrac{Y(t)}{t/\mu} \le x | Y(t) > 0\right) = \Gamma_{\nu/2,1}(x),\ x \ge 0.$

Remark 1. By Theorem 1 we can conclude that in the case $0 < \delta < 1$ the asymptotic behaviour of the continuous-time CBP $\{Y(t), t \ge 0\}$ with $0 < \mu < \infty$ is similar to that of the embedded discrete-time CBP $\{Z_n, n = 0, 1, \ldots\}$. Note that the case $\delta = 1$ is an open problem.

Now we consider the case $\mu = \infty$. The appropriate normalization factor of $\{Y(t),\ t \ge 0\}$ is $a(t) = (\Gamma(1-\rho)(1-G(t)))^{-1}$.

Theorem 2. *Assume Condition A,* $0 < \delta = 2c/\nu < 1$, $\mu = \infty$ *and* (4). *Then*
(*i*) $P(Y(t) > 0) \sim L_1^*(t)t^{-(1-\delta)\rho}$, *as* $t \to \infty$, *with* $L_1^*(t) = K\mathcal{L}^{1-\delta}(t)\dfrac{\Gamma(\delta)}{\Gamma(1-\rho+\delta\rho)}$.
(*ii*) $E[Y(t)] \sim L_2^*(t)t^{\rho\delta}$, *as* $t \to \infty$, *with* $L_2^*(t) = \dfrac{K\nu\Gamma(\delta)}{2\rho\Gamma(\delta\rho)\mathcal{L}^{\delta}(t)}$.
(*iii*) $\lim_{t\to\infty} P\left(\dfrac{Y(t)}{a(t)} \le x | Y(t) > 0\right) = \Psi(x),\ x \ge 0,$ *where*

$$\Psi(x) = \frac{\Gamma(1-\rho+\delta\rho)}{\Gamma(\delta)} \int_0^{\infty} u^{-(1-\delta)}\Gamma_{\nu/2,1}\left(\frac{x}{u}\right) dG_{\rho}(u), \quad x \ge 0,$$

and $G_\rho(x)$ *is the c.d.f. of the Mittag-Leffler distribution of order* ρ.

Remark 2. Recall that $G_\rho(x) = P(\Lambda_\rho \le x) = P(\xi_\rho^{-\rho} \le x)$, where ξ_ρ is a ρ-stable r.v. For the limiting c.d.f. $\Psi(x)$ we have that $\Psi(x) = P(\xi\eta \le x)$, where ξ and η are independent random variables with c.d.f.s $\Gamma_{\nu/2,1}(x)$ and $V_{\rho,1-\delta}(x) = \int_0^x u^{-(1-\delta)} dG_\rho(u)$, respectively. Assuming $0 < \delta < 1$, $\mu = \infty$ and (4), the asymptotic behaviour of the continuous-time CBP $\{Y(t),\ t \ge 0\}$ is different from that of the embedded discrete-time CBP $\{Z_n, n = 0, 1, \ldots\}$ due to the heavy tail of the lifespan c.d.f. $G(x)$. As before the case $\delta = 1$ is an open problem as well as other critical subclasses considered in [1,8].

4. Critical branching processes with random migration and continuous time. In this section we will consider a particular case of the CBP with multitype control functions (3). Let $X = \{X_n(i),\ n, i = 1, 2, \ldots\}$ be non-negative integer-valued i.i.d. random variables and let $\eta = \{(\eta_{n,1}, \eta_{n,2})\}$ and $I = \{I_n\}$ be two independent sets of non-negative integer-valued i.i.d. random variables, which are independent from X. Let $\{\xi_n,\ n = 1, 2,\}$ be i.i.d. random variable with $P(\xi_n = -1) = p$, $P(\xi_n = 0) = q$, $P(\xi_n = 1) = r$, being $p + q + r = 1$, and independent from X, η and I.

Consider $D = \{1, 2, 3\}$, $X_{n,1}(i) = X_n(i)$, $X_{n,2}(i) = -\eta_{n,2}$, $X_{n,3}(i) = I_n$,

$\phi_{n,1}(k) = \min\{k, k + \xi_n \eta_{n,1}\}^+$, $\phi_{n,2}(k) = \xi_n^- \mathbf{1}_{\{k>0\}}$, $\phi_{n,3}(k) = \xi_n^+ \mathbf{1}_{\{k>0\}}$, where $a^+ = \max\{0, a\}$ and $a^- = \max\{0, -a\}$. The CBP (3) with the previous specifications of the control functions can be rewritten as

$$(5) \qquad Z_0 > 0, \quad Z_n = \left(\sum_{i=1}^{Z_{n-1}} X_n(i) + M_n \mathbf{1}_{\{Z_{n-1}>0\}} \right)^+, n = 1, 2, \ldots,$$

where $M_n = \begin{cases} -\sum_{j=1}^{\eta_{n,1}} X_n(j) - \eta_{n,2}, & \text{with probability } p, \\ 0, & \text{with probability } q, \\ I_n, & \text{with probability } r, \end{cases}$

with $\sum_{k=1}^{x} = 0$, if $x \leq 0$. This model can be interpreted as follows: in each generation a random number of families and individuals can emigrate with probability p, or new immigrants appear with probability r, or there is no migration with probability q. Zero is an absorbing state. The particular case with $\eta_{n,1} \equiv 1$ a.s. and $\eta_{n,2} \equiv 0$ a.s., for every n, is considered in detail in the monograph [1]. The CBP defined by (5) is studied in [9] and [10].

Denote $m = E[X_1(1)]$, $2b = \mathrm{Var}[X_1(1)]$, $\epsilon_1 = E[\eta_{1,1}]$, $\epsilon_2 = E[\eta_{1,2}]$, $a = E[I_1]$ and $\theta = \dfrac{2E[M_n]}{\mathrm{Var}[X_1(1)]} = \dfrac{ra - p(\epsilon_1 + \epsilon_2)}{b}$. Notice that for this model the asymptotic mean growth rate, namely,

$$\lim_{k\to\infty} k^{-1} E[Z_n \mid Z_{n-1} = k] = \lim_{k\to\infty} k^{-1} \sum_{i=1}^{3} E[X_{1,i}(1)] E[\phi_{1,i}(k)]$$
$$= \lim_{k\to\infty} k^{-1}[(km - \epsilon_1 m - \epsilon_2)p + kmq + (km + a)r] = m.$$

Thus the criticality parameter is the offspring mean, as in a BGW branching process. We consider the *critical case* under the following Condition B:
(a) $m = 1$ and $0 < 2b < \infty$; (b) $0 < a < \infty$, $0 \leq \eta_{n,1} \leq N_1 < \infty$ a.s. and $0 \leq \eta_{n,2} \leq N_2 < \infty$ a.s., where N_1 and N_2 are some constants.

Note that in this case $\{Y(t),\ t \geq 0\}$ is a CBP with CT which admits two types of emigration (family and individual) and an immigration component in the non-zero states. Then the following two theorems hold.

Theorem 3. *Assume Condition B,* $0 < \theta < 1$ *and* $0 < \mu < \infty$.
(*i*) *If* $\Delta'(1) = E[Y(0)] < \infty$, *then as* $t \to \infty$.
(*a*) $P(Y(t) > 0) \sim K_1(t)\mu^{1-\theta} t^{-(1-\theta)}$, *with* $K_1(t)$ *a s.v.f.*
(*b*) $E[Y(t)] \sim (b\theta/\mu^\theta) K_1(t) t^\theta, \mathrm{Var}[Y(t)] \sim (b^2\theta/\mu^{1+\theta})(\theta + 1) K_1(t) t^{1+\theta}$.
(*c*) $P\left(\dfrac{Y(t)}{bt/\mu} \leq x | Y(t) > 0\right) \to 1 - e^{-x} = \mathcal{E}(x),\ x \geq 0.$

(*ii*) *If* $\Delta(s) = 1 - (1-s)^{\kappa} L_0\left(\frac{1}{1-s}\right)$, $L_0(x)$ *is a s.v.f. and* $0 < \kappa < 1 - \theta$, *then*

(*a*) $P(Y(t) > 0) \sim K_2(t)\mu^{\kappa} t^{-\kappa}$, *as* $t \to \infty$, *with* $K_2(t)$ *a s.v.f.*

(*b*) $\lim_{t\to\infty} P\left(\frac{Y(t)}{bt/\mu} \le x | Y(t) > 0\right) = F(x)$, *where* $F(x)$ *is the c.d.f. with a Laplace transform*

$$\varphi(\lambda) = 1 - \frac{C\lambda^{\kappa}}{(1+\theta)^{\theta+\kappa}} - \lambda\theta \int_0^1 (1-x)^{-\kappa}(1+\lambda x)^{-\theta-1} dx, \tag{6}$$

where $C = \Gamma(1-\kappa)\Gamma(1-\theta)/\Gamma(1-\theta-\kappa)$.

Remark 3. Theorem 3 shows that in the case $0 < \theta < 1$, the asymptotic behaviour of the continuous-time CBP $\{Y(t),\ t \ge 0\}$ with $0 < \mu < \infty$ is quite similar to that of the discrete-time process $\{Z_n,\ n = 0, 1, \ldots\}$. In this case we can conclude that the limiting properties of the embedded discrete-time process are transferred to the continuous-time process. Note that the asymptotic behaviour of $\{Y(t),\ t \ge 0\}$ in the cases $\theta \le 0$ and $\theta \ge 1$ is an open problem.

We consider now the case $\mu = \infty$. Notice that the appropriate normalization factor of $\{Y(t),\ t \ge 0\}$ is defined by $a(t) = (\Gamma(1-\rho)(1-G(t)))^{-1}$. Recall also that $\mathcal{L}$ is the s.v.f. introduced in (4).

Theorem 4. *Assume Condition B holds,* $0 < \theta < 1$, $\mu = \infty$ *and* (4).

(*i*) *If* $\Delta'(1) = E[Y(0)] < \infty$ *then as* $t \to \infty$

(*a*) $P(Y(t) > 0) \sim L_3^*(t) t^{-\rho(1-\theta)}$, *where*

$$L_3^*(t) = \mathcal{L}^{1-\theta}(t) K_1\left(\frac{t^{\rho}}{\mathcal{L}(t)}\right) \frac{\Gamma(\theta)}{\Gamma(1-\rho+\theta\rho)}, \quad \text{with } K_1(t) \ \text{ a s.v.f.}$$

(*b*) $E[Y(t)] \sim L_4^*(t) t^{\rho\theta}$, *as* $t \to \infty$, *with* $L_4^*(t) = \frac{b\theta\Gamma(1+\theta)K_1(t^{\rho}/\mathcal{L}(t))}{\Gamma(1+\theta\rho)\mathcal{L}^{\theta}(t)}$.

(*c*) $\lim_{t\to\infty} P\left(\frac{Y(t)}{ba(t)} \le x | Y(t) > 0\right) = \Phi(x)$, $x \ge 0$, *where*

$$\Phi(x) = \frac{\Gamma(1-\rho+\theta\rho)}{\Gamma(\theta)} \int_0^{\infty} \mathcal{E}\left(\frac{x}{u}\right) u^{-(1-\theta)} dG_{\rho}(u),$$

with $G_{\rho}(x)$ *the c.d.f. of the Mittag-Leffler distribution of order* ρ.

(*ii*) *If* $\Delta(s) = 1 - (1-s)^{\kappa} L_0\left(\frac{1}{1-s}\right)$, $0 < \kappa < 1 - \theta$ *and* $L_0(t)$ – *s.v.f., then*

(*a*) $P(Y(t) > 0) \sim L_5^*(t) t^{-\kappa\rho}$, *as* $t \to \infty$, *with*

$$L_5^*(t) = \mathcal{L}^{\kappa}(t) K_2\left(\frac{t^{\rho}}{\mathcal{L}(t)}\right) \frac{\Gamma(1-\kappa)}{\Gamma(1-\kappa\rho)}, \quad \text{where } K_2(x) \text{ is a s.v.f.}$$

(*b*) $\lim_{t\to\infty} P\left(\frac{Y(t)}{ba(t)} \le x \middle| Y(t) > 0\right) = \tilde{\Phi}(x)$, $x \ge 0$, *where*

$$\tilde{\Phi}(x) = \frac{\Gamma(1-\rho+\theta\rho)}{\Gamma(\theta)} \int_0^{\infty} F\left(\frac{x}{u}\right) u^{-(1-\theta)} dG_{\rho}(u),$$

with $F(x)$ the c.d.f. with a Laplace transform presented by (6).

Remark 4. Notice that the parameter θ plays in Theorems 3 and 4 a similar role as the one played by δ in the case of a CBP with CT and single control function. Each of these two parameters appears in the rate of convergence of $P(Z_n > 0)$. This rate of convergence is needed to apply the renewal theory and obtain the behaviour of the process in continuous time.

It is also interesting to point out that for the limiting c.d.f. $\Phi(x)$ we have $\Phi(x) = P(\xi\eta \leq x)$, where ξ and η are independent random variables with c.d.f. $\mathcal{E}(x)$ and $V_{\rho,1-\theta}(x)$, respectively. Theorem 4 shows that for $0 < \theta < 1$, $\mu = \infty$ and (4), the asymptotic behaviour of the continuous-time CBP $\{Y(t),\ t \geq 0\}$ is quite different from that of the embedded discrete-time process $\{Z_n,\ n = 0, 1, \ldots\}$. An explanation is that the c.d.f. of the individual lifespan $G(x)$ has a heavy tail with $\mu = \infty$. Investigating the process $\{Y(t),\ t \geq 0\}$ in the cases $\theta \leq 0$ and $\theta \geq 1$ are open problems.

5. Regenerative controlled branching processes with continuous time. So far we studied models of branching processes absorbed at zero. In this section we will extend these models allowing an immigration component at zero.

Let $Y = \{Y(t),\ t \geq 0\}$ be the CBP with CT investigated in Sections 3 or 5 where $\tau = \inf\{t : Y(t) = 0\}$ is the life-period with $P(\tau < \infty) = 1$ and c.d.f. $B(t) = P(\tau \leq t) = P(Y(t) = 0)$. Assume also that $\zeta = \{\zeta_i,\ i = 1, 2, \ldots\}$ are non-negative i.i.d. random variables with c.d.f. $A(x) = P(\zeta_1 \leq x)$. The sets ζ and Y are assumed independent.

Let $Y_k = \{Y_k(t),\ t \geq 0\}$ be the i.i.d. copies of $Y = \{Y(t),\ t \geq 0\}$ with corresponding life-periods τ_k and c.d.f. $B(x)$.

We will use the sequence of the random vectors $\{(\zeta_i, \tau_i),\ i = 1, 2, \ldots\}$ to define the renewal epochs $S_0 = 0,\ S_n = S_{n-1} + \eta_n$, where $\eta_n = \zeta_n + \tau_n$, $n = 1, 2, \ldots$ Let $\varkappa(t) = \max\{n :\ S_n \leq t\}$ be the corresponding renewal process. Consider also the alternating renewal epochs $\{(S_n,\ S^*_{n+1}) :\ S^*_{n+1} = S_n + \zeta_{n+1},\ n = 0, 1, \ldots\}$ and introduce the process $\{\sigma(t),\ t \geq 0\}$, $\sigma(t) = t - S^*_{\varkappa(t)+1}$. Then, the regenerative branching process $U = \{U(t),\ t \geq 0\}$ is defined as follows:

$$U(t) = Y_{\varkappa(t)+1}(\sigma(t))1_{\{\sigma(t)\geq 0\}}, \quad t \geq 0. \tag{7}$$

Note that ζ is interpreted as a set of waiting periods. If $\sigma(t) \geq 0$, then it is called a *spent lifetime* and if $\sigma(t) < 0$, then $|\sigma(t)|$ is called a *rest waiting time.*

The process $\{U(t),\ t \geq 0\}$ develops as follows: $U(t)$ is defined as zero during the waiting periods $S_{n-1} \leq t < S^*_n$ and $U(t)$ coincides with the process $Y_n(t - S^*_n)$ during the life periods $S^*_n \leq t < S_n,\ n = 1, 2, \ldots$

Recall that $\Delta(s) = E[s^{I_0}] = E[s^{Y(0)}]$. Then, $Y_k(0)$ can be interpreted as immigration components at state zero. If $Y(n) = Z_n,\ n = 0, 1, \ldots$, and $\{Z_n,\ n = 0, 1, \ldots\}$ is a BGW branching process, then $\{U(n),\ n = 0, 1, \ldots\}$ is a branching process with state-dependent immigration (i.e. immigration at zero only) well-known as Foster–Pakes model (see [7] for more details and further generalizations).

Further on we will consider the case when $A(x)$ and $B(x)$ are non-lattice c.d.f.s, $A(0) = B(0) = 0$, and there exists

$$\lim_{t\to\infty} \frac{1 - A(t)}{1 - B(t)} = C, \quad 0 \le C \le \infty.$$

For the distribution of the waiting periods, $A(x)$ say, we will assume the following conditions:

$$m_A = E[\zeta_1] < \infty, \tag{8}$$

$$m_A = \infty,\ 1 - A(x) \sim x^{-\alpha} L_A(x), x \to \infty, \tag{9}$$

where $1/2 < \alpha \le 1$, $L_A(x)$ is a s.v.f., and $A(t) - A(t-h) = O(1/t)$, as $t \to \infty$ for every $h > 0$ fixed.

The asymptotic behaviour of $U(t)$ is related to the asymptotic behaviour of the regeneration period assuming that

$$\lim_{t\to\infty} P\left(\frac{Y_k(t)}{M(t)} \le x | \tau_k > t\right) = D(x),\ x \ge 0,$$

where $M(t)$ is a positive r.v.f. with exponent $\varsigma \ge 0$ and $D(x)$ is a proper c.d.f.

Let $\{U(t),\ t \ge 0\}$ be the regenerative branching process defined by (7) where $\{Y(t),\ t \ge 0\}$ is the CBP with CT investigated in Section 3. The following results hold by Theorems 1 and 2 applying also *Basic Regeneration Theorem* [7].

Theorem 5. *Assume Condition A,* $0 < \mu < \infty$ *and* $0 < \delta = 2c/\nu < 1/2$.
(i) *If, additionally,* (8) *or* (9) *holds and* $0 \le C < \infty$, *then for* $x \ge 0$

$$\lim_{t\to\infty} P\left(\frac{U(t)}{t/\mu} \le x\right) = \frac{\digamma_1(x) + C}{1 + C},$$

where $\digamma_1(x) = \pi^{-1} \sin \pi(1-\delta) \int_0^1 \Gamma_{\nu/2,1}\left(\frac{x}{u}\right)(1-u)^{-\delta} u^{-(1-\delta)} du.$
(ii) *If* (9) *holds and* $C = \infty$, *then for* $x \ge 0$

$$\lim_{t\to\infty} P\left(\frac{U(t)}{t/\mu} \le x | U(t) > 0\right) = \digamma_2(x),$$

where $\digamma_2(x) = \dfrac{1}{B(\delta, \alpha)} \displaystyle\int_0^1 \Gamma_{\nu/2,1}\left(\frac{x}{u}\right)(1-u)^{\alpha-1} u^{-(1-\delta)} du.$

Theorem 6. *Assume Condition A, $\mu = \infty$ with* (4) *and* $1/2 < (1-\delta)\rho < 1$.
(*i*) *If, additionally,* (8) *or* (9) *holds and* $0 \le C < \infty$, *then for* $x > 0$

$$\lim_{t\to\infty} P\left(\frac{U(t)}{a(t)} \le x\right) = \frac{G_1(x) + C}{1 + C},$$

where $G_1(x) = \pi^{-1} \sin \pi(1-\delta)\rho \int_0^1 \Psi(x/u^\rho)(1-u)^{(1-\delta)\rho-1} u^{-(1-\delta)\rho} du$, *with* $\Psi(x)$ *defined in Theorem* 2.
(*ii*) *If* (9) *holds and* $C = \infty$, *then for* $x \ge 0$

$$\lim_{t\to\infty} P\left(\frac{U(t)}{a(t)} \le x | U(t) > 0\right) = G_2(x),$$

where $G_2(x) = \dfrac{1}{B(1-(1-\delta)\rho,\ \alpha)} \displaystyle\int_0^1 \Psi(x/u^\rho)(1-u)^{\alpha-1} u^{-(1-\delta)\rho} du.$

We can consider also the process $U(t)$ defined by (1), where $Y(t)$ is the CBP with CT given by Definition 1 with (3), investigated in Section 4. Then applying BRT with Theorems 3 and 4 we can obtain counterparts of Theorems 5 and 6.

Acknowledgements. The authors would like to thank to the referees for the useful remarks.

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