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ASYMPTOTIC BEHAVIOUR OF COLENGTH OF VARIETIES OF LIE ALGEBRAS

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Abstract. We study the asymptotic behaviour of numerical characteristics of polynomial identities of Lie algebras over a field of characteristic 0. In particular we investigate the colength for the cocharacters of polynilpotent varieties of Lie algebras. We prove that there exist polynilpotent Lie varieties with exponential and overexponential colength growth. We give the exact asymptotics for the colength of a product of two nilpotent varieties.

1. Introduction. We study numerical characteristics of varieties of Lie algebras. Among the most important characteristics for the polynomial identities of any variety of algebras are the codimension and the colength sequences. There are a lot of papers about the codimension growth of associativeand Lie

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algebras. The asymptotic behaviour of the colength sequence has not been studied so intensively. It is known that for any variety of associative algebras the colength is polynomially bounded, see Berele and Regev [3]. This result gives a wide class of examples of varieties of Lie algebras with a polynomially bounded colength (the so-called SPI Lie algebras). As it was shown by the authors [9], the same restriction for the colength can be obtained for a more general class of Lie algebras (the so-called API-algebras or Lie algebras of associative type) but, in general, the colength growth may be faster than any polynomial function. In [9] we constructed also an example of a Lie variety with a subexponential colength growth.

In the present paper we prove that there exist Lie varieties with exponential and overexponential colength growth. For a product of two nilpotent varieties we give the exact asymptotic for the colength. All these results were announced in [10].

We recall all essential notions. For more details on general theory of varieties of Lie algebras we refer to [2]. The characteristic of the ground field Φ is supposed to be equal to zero. We omit the Lie brackets in the monomials if the product is left-normed, i.e. $abc = ((ab)c)$.

Let $\mathbf{V}$ be a variety of Lie algebras over the field Φ. Denote by $F = F(X, \mathbf{V})$ the relatively free algebra of the variety $\mathbf{V}$ with a countable set of generators $X = \{x_1, x_2, \ldots\}$. Denote also by $P_n = P_n(\mathbf{V})$ the set of all multilinear Lie polynomials in $x_1, \ldots, x_n$ in F. The action $\sigma(x_i) = x_{\sigma(i)}$ of the symmetric group S_n can be naturally extended to the vector space P_n. The structure of P_n as an S_n-module is an important characterization of $\mathbf{V}$ and gives a lot of information about $\mathbf{V}$.

Denote by χ_λ the irreducible character of the symmetric group S_n corresponding to the partition λ of n and, for a variety $\mathbf{V}$, consider the decomposition of the character $\chi(P_n(\mathbf{V}))$ as a sum of irreducible components

$$\chi_n(\mathbf{V}) = \chi(P_n(\mathbf{V})) = \sum_{\lambda \vdash n,} m_\lambda \chi_\lambda. \tag{1}$$

The integer $c_n = c_n(\mathbf{V}) = \dim P_n(\mathbf{V})$ is called the n-th codimension of $\mathbf{V}$. The total number of summands

$$l_n = l_n(\mathbf{V}) = \sum_{\lambda \vdash n,} m_\lambda \tag{2}$$

in the sum (1) is called the *n-th colength* of the variety $\mathbf{V}$. Important numerical characteristics of $\mathbf{V}$ are also the multiplicities m_λ in (1).

Denote by d_λ the dimension of the irreducible S_n-module corresponding to λ. Then for the introduced above numerical characteristics the following relation holds:

$$c_n(\mathbf{V}) = \dim P_n(\mathbf{V}) = \sum_{\lambda \vdash n,} m_\lambda d_\lambda. \tag{3}$$

Recall that for associative algebras the colength and the multiplicities are bounded by a polynomial function n^q for a suitable q (see, for instance, [3, Theorem 16]). In the case of Lie algebras there are examples of varieties with non-polynomial but subexponential growth of multiplicities and colength [9]. In this paper we present examples of varieties with an overexponential colength growth. From the asymptotic estimates for the number of distinct partitions of n we derive the existence of multiplicities also with an overexponential growth. We also construct for any integer $b \geq 2$ an example of a variety with the colength of the type $b^{\frac{n}{2}}$.

In order to compare the asymptotic behaviour of functions we use the following notions. Let $f(n)$ and $g(n)$ be two functions of natural argument. Then $f(n) \ll g(n)$ means that $f(n)$ does not exceed $g(n)$ starting from some value of n. Now, let $g_\alpha(n)$ be a function of natural argument depending also on the positive real parameter α and let $g_\alpha(n) \leq g_\beta(n)$ for all n if $\alpha < \beta$. Denote

$$f(n) \simeq g_\gamma(n) \Longleftrightarrow \gamma = \inf\{\alpha \mid f(n) \ll g_\alpha(n)\} = \sup\{\beta \mid g_\beta(n) \ll f(n)\}.$$

Note also that

$$f(n) \simeq g_\gamma(n) \Longleftrightarrow \forall \varepsilon\ \exists N : g_{\gamma-\varepsilon}(n) \ll f(n) \ll g_{\gamma+\varepsilon}(n) \text{ for } n \geq N.$$

The next two statements are the main results of the paper.

Theorem 1. *The varieties of Lie algebras $\mathbf{A}^3$ and $\mathbf{AN}_3$ have an overexponential colength growth.*

Theorem 2. *The colength of the variety $\mathbf{N}_b\mathbf{N}_2$, $b \geq 2$, is asymptotically equal to the exponential function $b^{\frac{n}{2}}$, i.e. $l_n(\mathbf{N}_b\mathbf{N}_2) \simeq (\sqrt{b})^n$.*

2. Examples of varieties with overexponential colength. First we prove an easy statement based on a simple estimation of dimensions of irreducible S_n-modules.

Proposition. *The colength and codimension of any variety* $\mathbf{V}$ *of linear algebras satisfy the inequality*

$$l_n(\mathbf{V}) \geq \frac{c_n(\mathbf{V})}{\sqrt{n!}}.$$

P r o o f. It is well-known that

$$n! = \sum_{\lambda \vdash n,} d_\lambda^2.$$

Using this relation we obtain the restriction $d_\lambda < \sqrt{n!}$ for dimension of any irreducible module. Now we apply formulas (2) and (3), and the proof is complete. $\square$

Now we give examples of varieties of Lie algebras with colength growth asymptotically faster than any b^n, where b is an arbitrary constant.

Petrogradsky [5] found the asymptotics of the codimension growth of polynilpotent varieties of Lie algebras. Using his result and the above Proposition we obtain lower bounds for the colength.

First let $\mathbf{V} = \mathbf{A}^3$ be the variety of all solvable Lie algebras with a solvability length not greater than 3. Then by [5]

$$c_n(\mathbf{A}^3) \simeq \frac{n!}{(\ln n)^n}, \tag{4}$$

and we get

$$l_n(\mathbf{A}^3) \geq \frac{\sqrt{n!}}{(\ln n)^n}. \tag{5}$$

We can rewrite (4) in the form $c_n(\mathbf{A}^3) \simeq g_1(n)$ where

$$g_\gamma(n) = \frac{n!}{(\ln n)^{\frac{n}{\gamma}}}.$$

The inequality (5) completes the proof of Theorem 1 for $\mathbf{A}^3$. In particular, it follows that any variety which contains $\mathbf{A}^3$ has an overexponential colength growth.

Any polynilpotent variety $\mathbf{V} = \mathbf{N}_{s_q}\mathbf{N}_{s_{q-1}}\cdots\mathbf{N}_{s_1}$ has a faster colength growth if $q \geq 3$. Denote by $\ln^{(k)} x$ the k-multiple composition of the natural logarithm, i.e. $\ln^{(k)} x = \ln(\ln(\ldots x)\ldots)$. For polynilpotent varieties the following asymptotics of codimension growth was found in [5].

Theorem ([5, Theorem 2.2])**.** *Let* $\mathbf{V} = \mathbf{N}_{s_q}\mathbf{N}_{s_{q-1}}\cdots\mathbf{N}_{s_1}$. *If* $q \geq 3$, *then there exists an infinitely small value* $o(1)$ *such that*

$$c_n(\mathbf{V}) = \frac{n!}{(\ln^{(q-2)} n)^{n/s_1}} \left(\frac{s_2 + o(1)}{s_1}\right)^{\frac{n}{s_1}}.$$

If $q = 2$, *then*

$$c_n(\mathbf{V}) = (n!)^{\frac{s_1-1}{s_1}} (s_2 + o(1))^{\frac{n}{s_1}}.$$

From the above stated Proposition we obtain a lower bound for the colength growth if $q \geq 3$:

$$l_n(\mathbf{V}) \geq \frac{\sqrt{n!}}{(\ln^{(q-2)} n)^{n/s_1}}.$$

If we have only two factors, i.e. $q = 2$ and $\mathbf{V} = \mathbf{N}_b\mathbf{N}_a$, then we find

$$l_n(\mathbf{N}_b\mathbf{N}_a) \geq b^{\frac{n}{a}} (n!)^{\frac{a-2}{2a}}.$$

Hence, if $a \geq 3$, then the colength is growing as an overexponential function, and the proof of Theorem 1 is complete. $\square$

In the case $a = 2$ these formulas give us an exponential lower bound for the colength. In other words, we have the following.

Lemma. *Let* $\mathbf{V} = \mathbf{N}_b\mathbf{N}_2$. *Then for any positive* ε *there exists an integer*

N such that the condition $n \geq N$ implies the inequality

$$l_n(\mathbf{N}_b\mathbf{N}_2) \geq (b-\varepsilon)^{\frac{n}{2}}.$$

We shall use this bound in the proof of Theorem 2.

Remark. There exist varieties of Lie algebras with overexponential multiplicities for some of the irreducible S_n-modules.

The remark follows from the well-known asymptotics for the number $p(n)$ of distinct partitions of n, see for example [1]. This value has an intermediate growth between polynomial and exponential. More precisely, $p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$.

Note also that in [4] the following estimate for the colength of the variety $\mathbf{AN}_2$ was found:

$$l_n(\mathbf{AN}_2) \sim \exp\left(\pi\sqrt{\frac{2}{3}n}\right). \tag{6}$$

For this variety the colength growth is intermediate between polynomial and exponential.

3. Proof of Theorem 2. In this section we complete the proof of Theorem 2. By the Lemma from Section 2 it is sufficient to verify the upper bound of the form

$$l_n(\mathbf{N}_b\mathbf{N}_2) \leq (b+\varepsilon)^{\frac{n}{2}}. \tag{7}$$

for any $\varepsilon > 0$.

We shall prove (7) by induction on b. If $b = 1$, then (7) follows from (6) since $a^{\sqrt{n}}$ is asymptotically less than $(\sqrt{1+\varepsilon})^n$ for any a. Now, let $b > 1$ and let $L = F(X, \mathbf{N}_b\mathbf{N}_2)$ be a free algebra of the variety $\mathbf{N}_b\mathbf{N}_2$. Then $(L^3)^{b+1} = 0$ and $(L^3)^b \cap P_N$ is a non-zero S_N-submodule of $P_N = P_N(\mathbf{N}_b\mathbf{N}_2)$. Obvilously, the length $l_N(\mathbf{N}_b\mathbf{N}_2)$ of the S_N-module P_N is equal to the sum of $l_N(\mathbf{N}_{b-1}\mathbf{N}_2)$ and the length of module $P_N \cap (L^3)^b$. Since $l_N(\mathbf{N}_{b-1}\mathbf{N}_2) \simeq (\sqrt{b-1})^N$, it is sufficient to restrict the length of $K = P_N \cap (L^3)^b$ by $(\sqrt{b+\varepsilon})^N$.

For convenience we shall denote some of the free generators as x's and some as y's. In particular, let $Y_i = y_{i_1}y_{i_2}y_{i_3}$ be a left-normed product of y_{i_1}, y_{i_2} and y_{i_3}. We denote by $X_i = \text{ad } x_i$ the adjoint action of x_i on L.

Consider the multilinear monomial on generators $\{x_1, x_2, \ldots, x_N\}$ of the type

$$g_t = (Y_1 X_{11} \ldots X_{1t_1}) \cdots (Y_b X_{b1} \ldots X_{bt_b}),$$

where $N = 3b + n$, $n = \sum_{i=1}^{b} t_i$, and $t = (t_1, \ldots, t_b)$ is a multiindex.

For any multiindex t we generate by the element g_t the submodule K_t in K. Clearly, K is the sum of all such submodules, i.e.

$$K = \sum_t K_t. \tag{8}$$

First we shall establish an upper bound for the multiplicity of an arbitrary irreducible summand in some K_t.

Let Q_i, $i = 1, \ldots, b$, be the permutation group on the set $\{y_{i1}, y_{i2}, y_{i3}\}$. Obviously, Q_i is isomorphic to S_3. Similarly, let R_i, $i = 1, \ldots, b$, be the permutation group on the set $\{x_{i1}, \ldots, x_{it_i}\}$; clearly, $R_i \simeq S_{t_i}$. Consider the subgroup $H = Q_1 \times R_1 \times \ldots \times Q_b \times R_b$ of the group $G = S_N$. Clearly, the S_N-module K is also an H-module.

Note, that for any $i = 1, \ldots, b$, the character of the ΦQ_i-module $\Phi Q_i g_t$ is irreducible and corresponds to the partition $\lambda = (2, 1)$. (It is a well-known fact from the theory of varieties of Lie algebras that this is the character of the multilinear component of degree 3 in the free Lie algebra.)

Applying the same arguments as in [7] Proposition 2, one can see that the character of the ΦR_i-module $M = \Phi R_i g_t$ is a sum of distinct irreducible characters with multiplicities 1, i.e.

$$\chi(M) = \sum_{\lambda \vdash t_i} \varepsilon_\lambda \chi_\lambda, \tag{9}$$

where $\varepsilon_\lambda \leq 1$. Therefore the total number of summands in (9) does not exceed the number $p(t_i)$ of distinct partitions.

It follows that in the decomposition of the H-module $\Phi H g_t$ as a sum of irreducible components the number of summands will be not greater than $p(t_1)\cdots p(t_b)$. This latter value has a subexponential growth.

Now we consider one irreducible with respect to H summand U in $\Phi H g_t$ and generate a G-module W. Our nearest goal is to estimate the multiplicities of the irreducible summands in W.

Let φ be the H-character of U and let ψ be some irreducible G-character of W. Denote by $m = (G : H)$ the index of H in G. First we shall find a bound for the multiplicity in the induced from φ representation. Let $\langle \mathrm{Ind}\ \varphi, \psi\rangle_G = q$. Then, as it is known (see, for example, [6]), $\langle \varphi, \mathrm{Res}\ \psi\rangle_H = q$, and we obtain:

$$m\varphi(1) = \mathrm{Ind}\ \varphi(1) \geq q\psi(1), \quad \mathrm{Res}\ \psi(1) = \psi(1) \geq q\varphi(1).$$

Hence $q \leq \sqrt{m}$.

On the other hand

$$(G : H) = \frac{N!}{t_1!\cdots t_b!(3!)^b} = \binom{N}{t_1, \dots, t_b}\frac{(3b)!}{(3!)^b},$$

where $\binom{N}{t_1, \dots, t_b}$ is a generalized binomial coefficient. Hence

$$(G : H) \leq \frac{b^N(3b)!}{(3!)^b} \leq (b+\varepsilon)^N$$

and the multiplicity q is bounded from above by $(\sqrt{b+\varepsilon})^N$.

The total number of summands in (8) can be bounded by a polynomial function on N in the following way. This number is equal to the number $\binom{N+b-1}{b-1}$ of monomials of degree N on b commuting variables, which is a function on N of the type $(N+b-1)^{b-1}$. Now the number of summands in (8) is polynomially bounded and any fixed summand can be decomposed to at most $T = p(t_1)\cdots p(t_b)$ irreducible H-modules U, where $t_1 + \cdots + t_b = N - 3b$ and $p(m)$ is the number of distinct partitions of the integer m. Hence, $p(m) \sim \exp\left(\pi\sqrt{\frac{2m}{3}}\right)$ and $T \leq p(N)^b \sim C^{\sqrt{N}}$ for some constant C. Finally, any irreducible H-module U generates a G-module W with no more than $p(N)$ non-isomorphic summands with multiplicities not greater than $(b+\varepsilon)^{\frac{N}{2}}$. This completes the proof of Theorem 2. $\square$

In conclusion we formulate the following conjecture. Is it true that a variety $\mathbf{V}$ has an exponential colength if and only if it does not contain the subvarieties $\mathbf{A}^3$ and $\mathbf{AN}_3$?

In particular, is it true that any proper subvariety of $\mathbf{A}^3$ or of $\mathbf{AN}_3$ has an exponentially bounded colength? The latter conjecture is true for any subvariety of finite basic rank in $\mathbf{A}^3$ as follows from [8] and for any proper subvariety $\mathbf{V}$ of $\mathbf{AN}_2$ as it was shown in [4].

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