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ON A NEW APPROACH TO WILLIAMSON'S GENERALIZATION OF PÓLYA'S ENUMERATION THEOREM

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Abstract. Pólya's fundamental enumeration theorem and some results from Williamson's generalized setup of it are proved in terms of Schur-Macdonald's theory (S-MT) of "invariant matrices". Given a permutation group $W \leq S_d$ and a one-dimensional character χ of W, the polynomial functor F_χ corresponding via S-MT to the induced monomial representation $U_\chi = ind_W^{S_d}(\chi)$ of S_d, is studied. It turns out that the characteristic $ch(F_\chi)$ is the weighted inventory of some set $J(\chi)$ of W-orbits in the integer-valued hypercube $[0, \infty)^d$. The elements of $J(\chi)$ can be distinguished among all W-orbits by a maximum property. The identity $ch(F_\chi) = ch(U_\chi)$ of both characteristics is a consequence of S-MT, and is equivalent to a result of Williamson. Pólya's theorem can be obtained from the above identity by the specialization $\chi = 1_W$, where 1_W is the unit character of W.

Introduction. The philosophy of the present paper is that Pólya's theory of enumeration, which was developed in [7], and some of Williamson's generalizations of it from [10], can be seen as subsumed by Schur-Macdonald's

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theory of "invariant matrices" (cf. [8, 6]). The cornerstone of the latter is the equivalence between the category of finite dimensional K-linear representations of the symmetric group S_d and the category of polynomial homogeneous degree d functors on the category of finite-dimensional K-linear spaces (K is a field of characteristic 0). A natural background for the generalization of Pólya's theory is the induced monomial representations of S_d, which correspond via that equivalence to the so called semi-symmetric powers (cf. [4]). Both objects are determined by fixing a permutation group $W \leq S_d$ and a one-dimensional K-valued character χ of W. Then the representation $ind_W^{S_d}(\chi)$ corresponds to the d th semi-symmetric power $[\chi]^d(-)$. In particular, when $W = S_d$ and χ is the alternating character or the unit character, we obtain the exterior power $\bigwedge^d(-)$ or the symmetric power $S^d(-)$, respectively. If W is the unit group, then the regular representation of S_d corresponds to the tensor power $\otimes^d(-)$.

A substantial part of problems of combinatorial analysis deals with a finite set of objects, often called figures, and the number of figures is usually irrelevant, provided that it is large enough. For convenience, we may suppose that the set of figures coincides with the set N_0 of non-negative integers. The figures form configurations, that is, elements $(j_1, \ldots, j_d)$ of the free monoid $Mo(N_0)$ generated by N_0. Let $(x_i)_{i\in N_0}$ be a family of independent variables. One introduces a homomorphism of monoids (weight function) $w\colon Mo(N_0) \to K[(x_i)_{i\in N_0}]$, where the target of w is the (commutative) algebra of polynomials in $x_0, x_1, x_2, \ldots$, over the field K, considered to be a monoid with respect to multiplication.

In the generic case, one may suppose without loss of generality that w is the canonical homomorphism of the free monoid onto the free commutative monoid, both generated by $N_0 : w(i) = x_i$, $i \in N_0$; the other cases can be obtained by an appropriate specialization of the variables x_i. Under this assumption, the monomial $x_{j_1} \ldots x_{j_d}$ is the weight of the configuration $(j_1, \ldots, j_d)$, the weight function w is S_d-compatible and induces a bijection between the orbit space $S_d\backslash N_0^d$ and the set of degree d monomials in $x_0, x_1, x_2, \ldots$.

In general, the main characters of Pólya's play are the orbit spaces $W\backslash N_0^d$, where $W \leq S_d$ is a permutation group acting in a standard way on the integer-valued hypercube $N_0^d \subset Mo(N_0)$. It turns out that there exist bijections of certain orbit subspaces, including the whole orbit space, onto some index sets which arise naturally within the boundaries of the semi-symmetric powers. Given a K-linear space E with basis $(v_i)_{i\in N_0}$, we can construct a basis $(v_j)_{j\in J(N_0^d,\chi)}$ for $[\chi]^d(E)$. Here $J(N_0^d, \chi)$ is a system of distinct representatives of a set of W-orbits in N_0^d, and the corresponding orbit subspace depends on the character χ. In particular, if χ is the unit character 1_W, then $J(N_0^d, 1_W)$ is a transversal for all W-orbits in N_0^d.

The formal infinite sum of monomials

$$g(W; x_0, x_1, x_2, \ldots) = \sum_{j \in J\left(N_0^d, 1_W\right)} x_{j_1} \ldots x_{j_d}$$

is a homogeneous degree d symmetric function in a countable set of variables and it counts the W-orbits in N_0^d provided with weights. By the fundamental theorem on symmetric functions and by Newton's formulae

$$(P) \qquad g(W; x_0, x_1, x_2, \ldots) = Z(W; p_1, \ldots, p_d),$$

where $p_s = \sum_{i \in N_0} x_i^s$ are the power sums and $Z(W; p_1, \ldots, p_d)$ is an uniquely defined isobaric polynomial in $p_1, \ldots p_d$ with rational coefficients. Pólya's enumeration theorem asserts that this polynomial coincides with the cyclic index of the group W. By the specialization $x_{n+1} = x_{n+2} = \cdots = 0$ in (P) we establish the finite version of Pólya's theorem, where the symmetric polynomial $g(W; x_0, x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ in $n+1$ variables counts the weighted W-orbits in the integer-valued hypercube $[0, n]^d$.

The left-hand side of equality (P) is *a fortiori* the characteristic of the polynomial functor $[1_W]^d(-)$ while the right-hand side is the characteristic of the induced monomial representation $ind_W^{S_d}(1_W)$. The coincidence of these characteristics is a consequence of Schur-Macdonald's equivalence.

In general, the characteristic $ch([\chi]^d(-))$ is a symmetric function $g(\chi; x_0, x_1, x_2, \ldots)$ which inventories the weighted W-orbits O in N_0^d subject to the maximum condition "O contains $|W:H|$ in number H-orbits", where $H \leq W$ is the kernel of χ and $|W:H|$ is the index of H in W. On the other hand, the characteristic $ch(ind_W^{S_d}(\chi))$ is equal to the so called generalized cyclic index $Z(\chi; p_1, \ldots, p_d)$ of the group W with respect to the character χ. Schur-Macdonald's equivalence yields the identity

$$g(\chi; x_0, x_1, x_2, \ldots) = Z(\chi; p_1, \ldots, p_d),$$

which turns into Pólya's theorem in case $\chi = 1_W$.

For instance, let W be the whole symmetric group S_d and let χ be the alternating character ε_d. Then $g(\varepsilon; x_0, x_1, x_2, \ldots) = ch(\bigwedge^d(-))$ is the d th elementary symmetric function e_d which inventories the S_d-orbits in N_0^d consisting of sequences $(i_1, \ldots, i_d)$ with pairwise distinct components. By a tradition dating back to Pólya, the formal expression $Z(A_d - S_d; p_1, \ldots, p_d)$ is widely used in combinatorial literature for denoting the generalized cyclic index $Z(\varepsilon_d; p_1, \ldots, p_d)$. Since $\varepsilon_d = ind_{A_d}^{S_d}(1_{A_d}) - 1_{S_d}$ in the appropriate Grothendieck group,

$$Z(\varepsilon_d; p_1, \ldots, p_d) = Z(A_d; p_1, \ldots, p_d) - Z(S_d; p_1, \ldots, p_d).$$

Thus, in the present context, the "old" notation $Z(A_d - S_d; p_1, \dots, p_d)$ can be viewed as an archetype.

In Section 1 we give a construction of a basis for the semi-symmetric power $[\chi]^d(E)$, starting from a basis for the K-linear space E. Section 2 is devoted to the main result of that paper – Theorem 2.1.2, which generalizes Pólya's Hauptsatz from [7, Ch. 1, No 16]. Moreover, Propositions 2.2.1 and 2.2.2 show that the set of semi-symmetric powers is stable with respect to the tensor product and the composition of polynomial functors (cf. [6]). The corresponding canonical isomorphisms of functors yield generalizations of Pólya's product and insertion rules (cf. [7, Ch. 1, No 27]). We note that Williamson's generalization [10, (13)] of Pólya's enumeration theorem, and one of his generalizations of insertion rule (cf. [10, (25)]) are equivalent to 2.1.2, and 2.2.2, (iii), respectively, but our approach seems to have conceptual advantages. In Section 3 we give an interpretation of Theorem 2.1.2 in combinatorial terms.

1. Basis of a semi-symmetric power. Throughout the rest of the paper we fix a field K of characteristic 0. All linear spaces and linear maps under consideration are assumed to be K-linear. Given a finite group G, by a (linear) representation of G we mean a K-linear representation. Moreover, if a one-dimensional character of G is given, then, by default, it is K-valued. By N_0 we denote the set of non-negative integers. For any finite set S the number of its elements is denoted by $|S|$.

The results from this section have been announced in [3].

1.1. Let G be a finite group. We call a G-module any linear representation M of the group G. Any element a of the group ring KG defines a linear endomorphism $a\colon M \to M$ by the formula $z \mapsto az$.

Let M be a G-module. Given a one-dimensional character α of G, we set:
$a_\alpha = |G|^{-1} \sum_{g \in G} \alpha(g) g$, $a_\alpha \in KG$,
${}_\alpha M =$ the G-submodule of M generated by all differences $\alpha^{-1}(g)z - gz$, where $g \in G$ and $z \in M$,
M_α = the G-submodule of M consisting of all $z \in M$ such that $gz = \alpha^{-1}(g)z$, for any $g \in G$.

Clearly, M_α is the isotypical component of M, afforded by the character α^{-1}.

The following lemma is easily verified and well-known (cf. [9, Ch. I, sec. 2, No 6, Theorem 8]).

Lemma 1.1.1. *Let M be a G-module. Then*
(i) The G-submodule ${}_\alpha M$ of M is the kernel of a_α;

(ii) The G-submodule M_α of M is the image of a_α.

1.2. Let M be a linear space and let G be a finite group. The structure of a *monomial* G-module on M can be defined by the following data: (a) A basis $(v_i)_{i\in I}$ for M; (b) An action of G on the index set I; (c) A family $(\gamma_i)_{i\in I}$ of maps $G \to K\backslash\{0\}$ such that $\gamma_i(gh) = \gamma_{hi}(g)\gamma_i(h)$ for $i \in I$ and $g, h \in G$. Then the corresponding (monomial) action of G on M is defined by the rule

$$gv_i = \gamma_i(g)v_{gi}. \tag{1.2.1}$$

We write G_i for the stabilizer of $i \in I$ in the group G and $G^{(i)}$ for a left transversal of G_i in G. Note that the restriction of γ_i on G_i is a one-dimensional character of the group G_i.

Let α be a one-dimensional character of the group G. Let $I(M,\alpha)$ be the set of all $i \in I$ such that the maps γ_i and α^{-1} coincide on the subgroup G_i.

Lemma 1.2.2. *(i) The set $I(M,\alpha)$ is a G-stable subset of I. (ii) One has $a_\alpha(v_i) = 0$ for $i \in I\backslash I(M,\alpha)$.*

Proof. (i) Given $i \in I$, suppose $g \in G$ and $h \in G_i$. Then $G_{gi} = gG_ig^{-1}$ and $\alpha(ghg^{-1}) = \alpha(h)$. Moreover, $\gamma_{gi}(ghg^{-1}) = \gamma_{g^{-1}gi}(gh)\gamma_{gi}(g^{-1}) = \gamma_{gi}(g^{-1})\gamma_i(gh) = \gamma_{ghi}(g^{-1})\gamma_i(gh) = \gamma_i(g^{-1}gh) = \gamma_i(h)$.

(ii) The complement of $I(M,\alpha)$ in I also is G-stable; let $i \in I\backslash I(M,\alpha)$. We have

$$\begin{aligned} a_\alpha(v_i) &= |G|^{-1} \sum_{g\in G^{(i)}} \sum_{h\in G_i} \alpha(gh)\gamma_i(gh)v_{ghi} \\ &= |G|^{-1} \sum_{g\in G^{(i)}} \alpha(g)\gamma_i(g)(\sum_{h\in G_i} \alpha(h)\gamma_i(h))v_{gi}, \end{aligned}$$

and the equality $a_\alpha(v_i) = 0$ holds because the product $\alpha\gamma_i$ is not the unit character of the group G_i. □

We fix a system I^* of distinct representatives of all G-orbits in I. Moreover, we set $J(M,\alpha) = I^* \cap I(M,\alpha)$ and $J_0(M,\alpha) = I^*\backslash J(M,\alpha$. Following [1, Ch. III, sec. 5, No 4], we obtain a basis for the linear space M consisting of

$$v_j, \quad j \in J(M,\alpha), \tag{1.2.3}$$

$$v_i - \alpha(g)\gamma_i(g)v_{gi}, \quad i \in I^*,\ g \in G^{(i)},\ g \notin G_i, \tag{1.2.4}$$

$$v_i, \quad i \in J_0(M,\alpha). \tag{1.2.5}$$

Proposition 1.2.6. *Let G be a finite group and let α be a one-dimensional character of G. Then the following three statements hold for every monomial G-module M defined by the formula* (1.2.1)*:*

(i) The union of the families (1.2.4) *and* (1.2.5) *is a basis for* ${}_{\alpha}M$*;*

(ii) The family $a_{\alpha}(v_j)$*,* $j \in J(M, \alpha)$*, is a basis for* M_{α}*;*

(iii) The family v_j *mod*$({}_{\alpha}M)$*,* $j \in J(M, \alpha)$*, is a basis for the factor-space* $M/{}_{\alpha}M$*.*

Proof. (i) The family (1.2.4) is in ${}_{\alpha}M$ by definition. Lemma 1.2.2, (ii), and Lemma 1.1.1, (i), imply that the family (1.2.5) is contained in ${}_{\alpha}M$. Now, set $J = J(M, \alpha)$ and suppose that $\sum_{j \in J} k_j a_{\alpha}(v_j) = 0$ for some $k_j \in K$ such that $k_j = 0$ for all but a finite number of indices $j \in J$. We have

$$\sum_{j \in J} k_j a_{\alpha}(v_j) = |G|^{-1} \sum_{j \in J} \sum_{g \in G^{(j)}} k_j |G_j| \alpha(g) \gamma_j(g) v_{gj},$$

hence $k_j = 0$ for all $j \in J$, which proves part (i). In addition, we have proved that the elements $a_{\alpha}(v_j)$, $j \in J(M, \alpha)$, are linearly independent.

(ii) Lemma 1.1.1, (ii), yields that the elements $a_{\alpha}(v_j)$, $j \in J(M, \alpha)$, are in M_{α} and, moreover, that each element of M_{α} has the form $a_{\alpha}(z)$ for some $z \in M$. Since the union of families (1.2.3) – (1.2.5) is a basis for M and since the endomorphism a_{α} annihilates (1.2.4) and (1.2.5), part (ii) holds.

(iii) Lemma 1.1.1 and part (ii) imply (iii). $\square$

1.3. Let $W \leq S_d$ be a permutation group and let χ be a one-dimensional character of W. We recall several definitions from [2]. Let E be a linear space with basis $(v_i)_{i \in L}$. Then the d th tensor power $\otimes^d E$ has a standard structure of a monomial W-module via the rule

$$\sigma(v_{i_1} \otimes \cdots \otimes v_{i_d}) = v_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes v_{i_{\sigma^{-1}(d)}}$$

for $\sigma \in W$ and $(i_1, \ldots, i_d) \in L^d$.

The linear factor-space $[\chi]^d(E) = \otimes^d E/{}_{\chi}(\otimes^d E)$ is called *d-th semi-symmetric power of weight* χ *of* E. The image of the tensor $x_1 \otimes \cdots \otimes x_d$ via the canonical homomorphism $\otimes^d E \to [\chi]^d(E)$ is denoted by $x_1 \chi \ldots \chi x_d$. Clearly, the vectors of type $x_1 \chi \ldots \chi x_d$, $x_s \in E$, generate the semi-symmetric power $[\chi]^d(E)$ as a linear space. The *d-th semi-symmetric power of weight* χ of a linear map $l \colon E \to E'$ is defined by the formula

(1.3.1) $$[\chi]^d(l) \colon [\chi]^d(E) \to [\chi]^d(E'),$$

$$([\chi]^d(l))(x_1 \chi \ldots \chi x_d) = l(x_1) \chi \ldots \chi l(x_d).$$

Varying the arguments E and l, we obtain a polynomial homogeneous degree d functor $[\chi]^d(-)$ on the category of (finite-dimensional) linear spaces (cf. [2, 6]).

Proposition 1.2.6 for $G = W$, $\alpha = \chi$ and $M = \otimes^d E$ yields the following

Proposition 1.3.2. *Let $W \leq S_d$ be a permutation group and let χ be a one-dimensional character of W. Let E be a linear space with basis $(v_i)_{i \in L}$. Then the family $(v_j)_{j \in J(\otimes^d E, \chi)}$, where $v_j = v_{j_1}\chi \dots \chi v_{j_d}$, is a basis for the d-th semi-symmetric power $[\chi]^d(E)$.*

Remark 1.3.3. When the set L is well-ordered, the Cartesian product $I = L^d$ is lexicographically well-ordered. If the opposite is not stated, we suppose that the elements of I^* are lexicographically minimal in their W-orbits. In this case we write $J(L^d, \chi)$ for $J\left(\otimes^d E, \chi\right)$. In particular, if L is the integer-valued interval $[0, n]$, where $n \in N_0$, then we denote the index set $J(L^d, \chi)$ by $J(n, d, \chi)$.

2. The theorem.

2.1. Again let $W \leq S_d$ be a permutation group and let χ be a one-dimensional character of W. Let $x_0, x_1, x_2, \dots$ be an infinite sequence of independent variables. We set

$$g_n(\chi; x_0, \dots, x_n) = \sum_{j \in J(n,d,\chi)} x_{j_1} \dots x_{j_d}$$

and

$$g(\chi; x_0, x_1, x_2, \dots) = \sum_{j \in J\left(N_0^d, \chi\right)} x_{j_1} \dots x_{j_d},$$

where $J(n, d, \chi)$ and $J(N_0^d, \chi)$ are the index sets from (1.3.3).

Lemma 2.1.1. *The sequence $(g_n)_{n \geq 0}$ determines a symmetric function in a countable set of variables $x_0, x_1, x_2, \dots$, which coincides both with the characteristic of the polynomial functor $[\chi]^d(-)$ and with the formal infinite sum of monomials $g(\chi; x_0, x_1, x_2, \dots)$.*

Proof. Let $v_0, \dots, v_n$ be a basis for the linear space $E = K^{n+1}$. Given an element $x = (x_0, \dots, x_n) \in K^{n+1}$ let (x) denote the linear endomorphism of E defined by the diagonal matrix $diag(x_0, \dots, x_n)$ with respect to that basis. Then, according to (1.3.1) and (1.3.2), the linear endomorphism $[\chi]^d((x))$ of the semi-symmetric power $[\chi]^d(E)$ is defined by the diagonal matrix $diag(x_{j_1} \dots x_{j_d})$ with respect to the basis $(v_j)_{j \in J(n,d,\chi)}$. In particular, the polynomial g_n is the trace of the endomorphism $[\chi]^d((x))$. Therefore g_n is a symmetric polynomial in the $n+1$ variables $x_0, \dots, x_n$ and, moreover, the sequence $(g_n)_{n \geq 0}$ is a symmetric function $g(x_0, x_1, x_2, \ldots)$ in a countable set of variables, which coincides with the characteristic of the polynomial functor $[\chi]^d(-)$ (cf. [6, Ch. I, Appendix, A7]). Since $J(n, d, \chi) = J(N_0^d, \chi) \cap [0, n]^d$, then $g(\chi; x_0, \ldots x_n, 0, 0, \ldots) = g_n(x_0, \ldots, x_n)$ for any $n \in N_0$. In other words, we have $g(\chi; x_0, x_1, x_2, \ldots) = g(x_0, x_1, x_2, \ldots)$. □

Let $p_1, \ldots, p_d$ be independent variables. We set

$$Z(\chi; p_1, \ldots, p_d) = |W|^{-1} \sum_{\sigma \in W} \chi(\sigma) p_1^{c_1(\sigma)} \ldots p_d^{c_d(\sigma)},$$

where $c_s(\sigma)$ is the number of cycles of length s in the cyclic decomposition of the permutation $\sigma \in W$. If χ is the unit character, then $Z(1_W; p_1, \ldots, p_d)$ is the standard cyclic index $Z(W; p_1, \ldots, p_d)$ of the group W.

Theorem 2.1.2. *Let $W \leq S_d$ be a permutation group and let χ be a one-dimensional character of W. Then one has*

$$g(\chi; x_0, x_1, x_2, \ldots) = Z(\chi; p_1, \ldots, p_d), \tag{2.1.3}$$

where p_s are the power sums in the variables $x_0, x_1, x_2, \ldots$.

P r o o f. By Frobenius reciprocity, the right-hand side of equality (2.1.3) is the characteristic of the induced monomial representation $ind_W^{S_d}(\chi)$ of S_d. Lemma 2.1.1 yields that the left-hand side is the characteristic of the polynomial functor $[\chi]^d(-)$. Due to [4], $ind_W^{S_d}(\chi)$ and $[\chi]^d(-)$ correspond each other via Schur-Macdonald's equivalence. Then [6, Ch. I, Appendix, A7] implies that both characteristics coincide. □

Letting $x_{n+1} = x_{n+2} = \cdots = 0$ we obtain

Corollary 2.1.4. *For any $n \in N_0$ one has*

$$g_n(\chi; x_0, \ldots, x_n) = Z(\chi; p_1, \ldots, p_d),$$

where p_s are the power sums in the $n + 1$ variables $x_0, \ldots, x_n$.

Remark 2.1.5. If χ in Theorem 2.1.2 is the unit character, then we establish Pólya's classical theorem.

Remark 2.1.6. Corollary 2.1.4 for $n = 0$ turns into the so called "orthogonality relations" among the one-dimensional characters of the group W. (cf. [9, Ch. I, sec. 2, No 3, Theorem 3]).

Remark 2.1.7. The coefficients of $|W|$ times the generalized cyclic index $Z(\chi; p_1, \dots, p_d)$, considered to be a polynomial in the variables $p_1, \dots, p_d$, are rational integers.

2.2. Now, we shall prove two statements concerning tensor product and composition of semi-symmetric powers as polynomial functors (cf. [6, Ch. I, Appendix, A7]).

Let $W \leq S_d$ and $V \leq S_r$ be permutation groups and let χ and θ be one-dimensional characters of W and V, respectively. We consider the Cartesian product $S_d \times S_r$ as a subgroup of S_{d+r} by identifying $(\sigma, \tau) \in S_d \times S_r$ with the permutation $s \mapsto \sigma(s)$, $1 \leq s \leq d$, $d + t \mapsto d + \tau(t)$, $1 \leq t \leq r$. The tensor product $\lambda = \chi \otimes \theta$ is a one-dimensional character of the subgroup $W \times V \leq S_d \times S_r$. We identify the wreath product $S_r \sim S_d$ with its natural faithful permutation representation in the symmetric group S_{dr} (cf. [5, Ch. 4, sec. 1, 4.1.18]). Then the wreath product $V \sim W$ is a permutation subgroup of $S_r \sim S_d$ and the tensor product $\mu = \theta^{\otimes d} \otimes \chi$ is a one-dimensional character of $V \sim W$.

Proposition 2.2.1. *(i) For any linear space E the formulae*

$$\Pi_E \colon [\chi]^d(E) \otimes [\theta]^r(E) \to [\lambda]^{d+r}(E),$$

$$(y_1 \chi \dots \chi y_d) \otimes (z_1 \theta \dots \theta z_r) \mapsto y_1 \lambda \dots \lambda y_d \lambda z_1 \lambda \dots \lambda z_r$$

where $y_s,\ z_t \in E$, give rise to a canonical isomorphism of linear spaces;

(ii) The family $\Pi = (\Pi_E)$, where E runs through all finite-dimensional linear spaces, establishes a canonical isomorphism

$$\Pi \colon [\chi]^d(-) \otimes [\theta]^r(-) \to [\lambda]^{d+r}(-)$$

of polynomial functors;

(iii) One has

$$g(\chi \otimes \theta; x_0, x_1, x_2, \dots) = Z(\chi; p_1, \dots, p_d) Z(\theta; p_1, \dots, p_r),$$

where p_s are the power sums in the variables $x_0, x_1, x_2, \dots$.

P r o o f. (i) This is proved in [2, sec. 2, Corollary 2.1.4].

(ii) It follows directly from (1.3.1) that the family Π is a morphism of functors. Then part (i) implies that Π is an isomorphism.

(iii) Since the trace is multiplicative with respect to tensor products, for each $n \in N_0$ we have $g_n(\chi \otimes \theta; x_0, \dots, x_n) = g_n(\chi; x_0, \dots, x_n) g_n(\theta; x_0, \dots, x_n)$. Therefore,

$$g(\chi \otimes \theta; x_0, x_1, x_2, \dots) = g(\chi; x_0, x_1, x_2, \dots) g(\theta; x_0, x_1, x_2, \dots)$$

and Theorem 2.1.2 completes the proof. $\square$

Proposition 2.2.2 (Pólya's insertion). *(i) For any linear space E the formulae*

$$\Delta_E\colon [\chi]^d([\theta]^r(E)) \to [\mu]^{dr}(E),$$

$$(z_1\theta\dots\theta z_r)\chi\dots\chi(z_{(d-1)r+1}\theta\dots\theta z_{dr}) \mapsto z_1\mu\dots\mu z_r\mu\dots\mu z_{(d-1)r+1}\mu\dots\mu z_{dr},$$

where $z_t \in E$, give rise to a canonical isomorphism of linear spaces;

(ii) The family $\Delta = (\Delta_E)$, where E runs through all finite-dimensional linear spaces, is a canonical isomorphism

$$\Delta\colon [\chi]^d(-) \circ [\theta]^r(-) \to [\mu]^{dr}(-)$$

of polynomial functors;

(iii) One has

$$g(\theta^{\otimes d} \otimes \chi; x_0, x_1, x_2, \dots) = Z(\chi; P_1, \dots, P_d),$$

where $P_s = Z(\theta; p_s, p_{2s}, \dots, p_{rs})$, $1 \le s \le d$, and p_k, $k \ge 1$, are the power sums in the variables $x_0, x_1, x_2, \dots$.

P r o o f. (i) Due to [2, sec. 1 and sec. 2, Lemma 2.1.2], it is enough to note that:

(a) The expression

$$(z_1\theta\dots\theta z_r)\chi\dots\chi(z_{(d-1)r+1}\theta\dots\theta z_{dr}) \in [\chi]^d([\theta]^r(E))$$

is multilinear and semi-symmetric of weight μ;

(b) The expression

$$z_1\mu\dots\mu z_r\mu\dots\mu z_{(d-1)r+1}\mu\dots\mu z_{dr} \in [\mu]^{dr}(E)$$

is multilinear and semi-symmetric of weight θ with respect any group

$$Z_s = (z_{(d-s)r+1}, \dots, z_{(d-s+1)r}),$$

$1 \le s \le d$, of variables as well as semi-symmetric of weight χ with respect to $Z_1, \dots, Z_d$.

(ii) We take into account part (i) and (1.3.1).

(iii) Part (ii), [6, Ch. I, Appendix, (A7.3)] and Theorem 2.1.2 yield

$$\begin{aligned} g(\theta^{\otimes d} \otimes \chi; x_0, x_1, x_2, \dots) &= g(\chi; x_0, x_1, x_2, \dots) \circ g(\theta; x_0, x_1, x_2, \dots) \\ &= Z(\chi; p_1, \dots, p_d) \circ Z(\theta; p_1, \dots, p_r), \end{aligned}$$

where in the last two rows the symbol $\circ$ denotes the plethysm of symmetric functions. Now [6, Ch. I, sec. 8, (8.4)] implies part (iii). $\square$

3. Combinatorial interpretation.

3.1. Let G be a finite group which acts on a set I. Let α be a one-dimensional character of the group G with kernel $H \leq G$.

Lemma 3.1.1. *The following statements hold:*
(i) If O is a G-orbit in I, then all H-orbits in O have equal lengths;
(ii) The equalities $\alpha_{|G_i} = 1$ and $|G_i : H_i| = 1$ are equivalent for any $i \in I$.

Proof. (i) Let $i \in O$. Since H is a normal subgroup of G, then $\sigma H_i \sigma^{-1} \leq H$ for $\sigma \in G$. Therefore $|H : H_{\sigma i}| = |H : \sigma H_i \sigma^{-1}| = |H : H_i|$, that is, each H-orbit in O has the same number of elements.

(ii) We have $H_i = H \cap G_i$ and $\alpha_{|G_i} = 1$ is equivalent to $G_i \subset H$. $\square$

Given a G-orbit O in I, we denote by $\tau_H(O)$ the number of H-orbits in O. If one has $\alpha_{|G_i} = 1$ for some $i \in O$ (and, hence, for all $i \in O$), then O is said to be an *α-orbit.* Then Lemma 3.1.1 and the equality

$$|G : H||H : H_i| = |G : G_i||G_i : H_i|, \tag{3.1.2}$$

where $i \in I$, imply the next lemma.

Lemma 3.1.3. *The following two statements are equivalent:*
(i) The G-orbit O is an α-orbit;
(ii) One has $\tau_H(O) = |G : H|$.

3.2. As a consequence of Lemma 3.1.3, we establish a bijection between the index set $J(M, \alpha)$ from Proposition 1.2.6, under the additional condition $\gamma_i(g) = 1$ for all $i \in I$, $g \in G$, and the set of all α-orbits. Moreover, the equality (3.1.2) and Lemma 3.1.1, (i), yield that $\tau_H(O)$ is a divisor of $|G : H|$ for any G-orbit O in the set I. Hence the G-orbits O which satisfy the equivalent statements of Lemma 3.1.3, contain the maximum possible number $|G : H|$ of H-orbits. Thus we obtain a combinatorial interpretation of the bases for the isomorphic linear spaces $M/_\alpha M \simeq M_\alpha$ in terms of that maximum property. In particular, Proposition 1.3.2 and the main Theorem 2.1.2 can be restated in combinatorial terms.

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