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# A CHARACTERIZATION OF VARIETIES OF ASSOCIATIVE ALGEBRAS OF EXPONENT TWO*

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Abstract. It was recently proved that any variety of associative algebras over a field of characteristic zero has an integral exponential growth. It is known that a variety $\mathcal{V}$ has polynomial growth if and only if $\mathcal{V}$ does not contain the Grassmann algebra and the algebra of $2 \times 2$ upper triangular matrices. It follows that any variety with overpolynomial growth has exponent at least 2. In this note we characterize varieties of exponent 2 by exhibiting a finite list of algebras playing a role similar to the one played by the two algebras above.

Let $F$ be a field of characteristic zero and $\mathcal{V}$ a variety of associative algebras over $F$. Let $F\langle X\rangle$ be the free algebra of countable rank over $F$ and $F\langle X\rangle/Id(\mathcal{V})$ the corresponding free algebra of the variety $\mathcal{V}$ where $Id(\mathcal{V})$ is the

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T-ideal of polynomial identities of $\mathcal{V}$. The exponent of a variety $\mathcal{V}$ is defined as follows: for every $n \geq 1$ let $P_n$ be the space of multilinear polynomials in the variables $x_1, \ldots, x_n$. If $c_n(\mathcal{V}) = \dim P_n/(P_n \cap Id(\mathcal{V}))$ is the $n$-th codimension of $\mathcal{V}$ and $\mathcal{V}$ has at least one non-trivial identity it is well known ([8]) that the sequence of codimensions is exponentially bounded. Then one defines the exponent of $\mathcal{V}$ as $\mathrm{Exp}(\mathcal{V}) = \lim_{n\to\infty} \sqrt[n]{c_n(\mathcal{V})}$. Hence if $\mathcal{V}$ is nilpotent, then $\mathrm{Exp}(\mathcal{V}) = 0$. It has been shown in [1] and [2] that for every non-nilpotent variety $\mathcal{V}$, $\mathrm{Exp}(\mathcal{V})$ exists and is a positive integer.

Kemer in [4] described in various ways the T-ideals (or varieties) of polynomial growth. Later [5] he proved that a variety $\mathcal{V}$ has a polynomially bounded codimension sequence if and only if $G \notin \mathcal{V}$ and $UT_2(F) \notin \mathcal{V}$ where $G$ is the infinite dimensional Grassmann algebra and $UT_2(F)$ is the algebra of $2 \times 2$ upper triangular matrices over $F$. From its characterization (see also [3]) it follows that if $\mathrm{Exp}(\mathcal{V}) = 1$ then the codimensions of $\mathcal{V}$ are polynomially bounded.

In this note we shall characterize the varieties $\mathcal{V}$ of exponent two. To this end, we view $G = G^{(0)} + G^{(1)}$ with its natural $\mathbf{Z}_2$-grading where $G^{(0)}$ and $G^{(1)}$ are the spaces generated by the monomials of even degree and odd degree respectively. We then define the following five algebras over $F$:

1) $A_1 = \begin{pmatrix} G & G \\ 0 & G^{(0)} \end{pmatrix}$;

2) $A_2 = \begin{pmatrix} G^{(0)} & G \\ 0 & G \end{pmatrix}$;

3) $A_3 = UT_3(F)$, the algebra of $3 \times 3$ upper triangular matrices over $F$;

4) $A_4 = M_2(F)$, the algebra of $2 \times 2$ matrices over $F$;

5) $A_5 = M_{1,1}(G) = \begin{pmatrix} G^{(0)} & G^{(1)} \\ G^{(1)} & G^{(0)} \end{pmatrix}$ equipped with the $\mathbf{Z}_2$-grading

$$M_{1,1}^{(0)} = \begin{pmatrix} G^{(0)} & 0 \\ 0 & G^{(0)} \end{pmatrix}, \quad M_{1,1}^{(1)} = \begin{pmatrix} O & G^{(1)} \\ G^{(1)} & 0 \end{pmatrix}.$$

The main result of this note is the following

**Theorem 1.** *Let $F$ be a field of characteristic zero and $\mathcal{V}$ a variety of associative $F$-algebras. Then* $\mathrm{Exp}(\mathcal{V}) > 2$ *if and only if $A_i \in \mathcal{V}$ for some $i \in \{1, \ldots, 5\}$.*

For every $i = 1, \ldots, 5$ let $\mathcal{V}_i = \text{var}(A_i)$ be the variety generated by the algebra $A_i$. The above list of algebras cannot be reduced; in fact we shall prove the following

**Proposition 1.** *For all $i \neq j$, $\mathcal{V}_i \not\subseteq \mathcal{V}_j$.*

Hence $\mathcal{V}_1, \ldots, \mathcal{V}_5$ are the only minimal varieties of exponent $> 2$ in the sense that, for every $i$, $\text{Exp}(\mathcal{V}_i) > 2$ and for every subvariety $\mathcal{W}$ of $\mathcal{V}_i, \text{Exp}(\mathcal{W}) \leq 2$. From the proof of Theorem 1 it will be clear that $\text{Exp}(\mathcal{V}_1) = \text{Exp}(\mathcal{V}_2) = \text{Exp}(\mathcal{V}_3) = 3$ and $\text{Exp}(\mathcal{V}_4) = \text{Exp}(\mathcal{V}_5) = 4$.

Invoking the result of Kemer mentioned above we get

**Corollary 1.** *Let $\mathcal{V}$ be a variety of algebras over a field of characteristic zero. Then $\text{Exp}(\mathcal{V}) = 2$ if and only if $A_1, \ldots, A_5 \notin \mathcal{V}$ and either $G \in \mathcal{V}$ or $UT_2(F) \in \mathcal{V}$.*

Proof of Theorem 1. Suppose $\text{Exp}(\mathcal{V}) = p > 2$. By a result of Kemer ([6]) there exists a finite dimensional $\mathbf{Z}_2$-graded algebra $B = B^{(0)} + B^{(1)}$ such that $\mathcal{V} = \text{var}(G(B))$ where $G(B) = G^{(0)} \otimes B^{(0)} + G^{(1)} \otimes B^{(1)}$ is the Grassmann envelope of $B$. Let $B = B_1 \oplus \cdots \oplus B_k + J$ be the Wedderburn-Malcev decomposition of $B$ where $J$ is the Jacobson radical of $B$ and $B_1, \ldots, B_k$ are simple subalgebras that are homogeneous in the $\mathbf{Z}_2$-grading. For each $i = 1, \ldots k$, let $B_i = B_i^{(0)} + B_i^{(1)}$ and $J = J^{(0)} + J^{(1)}$ be the induced $\mathbf{Z}_2$-grading (see [6, p. 21]).

Let now $\overline{F}$ be the algebraic closure of the field $F$ and $\overline{B} = B \otimes_F \overline{F}$. Then $G(B) \otimes_F \overline{F} \cong G(B \otimes_F \overline{F}) = G(\overline{B})$ and the $n$-th codimension of $G(\overline{B})$ over $\overline{F}$ equals the $n$-th codimension of $G(B)$ over $F$, for all $n$. It follows that the exponent of $G(B)$ over $F$ coincides with the exponent of $G(\overline{B})$ over $\overline{F}$. Since $G(\overline{B}) \in \text{var}(G(B)) = \mathcal{V}$, in order to prove that $A_i \in \mathcal{V}$ for some $i$, it is enough to show that $G(\overline{B})$ contains a copy of $A_i$ for some $i$. In particular we may assume that $F$ is algebraically closed.

From [2] we obtain that $\text{Exp}(\mathcal{V})$ is computed as follows: consider all possible products of the form

$$C_1 J C_2 J \cdots J C_t \neq 0 \tag{1}$$

where $C_1, \ldots, C_t \in \{B_1, \ldots, B_k\}$ are distinct and define

$$p^{(0)} = \dim(C_1^{(0)} \oplus \cdots \oplus C_t^{(0)}), \quad p^{(1)} = \dim(C_1^{(1)} \oplus \cdots \oplus C_t^{(1)}).$$

Then $p = \text{Exp}(\mathcal{V})$ is the maximal value of $p^{(0)} + p^{(1)}$ where $C_1, \ldots, C_t$ satisfy (1).

Also recall that a simple finite dimensional $\mathbf{Z}_2$-graded algebra over $F$ is isomorphic to one of the following algebras:

i) $M_{a,b}(F) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where $A_{11}, A_{12}, A_{21}, A_{22}$ are $a \times a, a \times b, b \times a$ and $b \times b$ matrices respectively, $a > 0, b \geq 0$, with grading

$$M^{(0)}_{a,b}(F) = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad M^{(1)}_{a,b}(F) = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix}.$$

ii) $M_N(F) \oplus cM_N(F)$ where $c^2 = 1$, with grading $M_N(F)^{(0)} = M_N(F)$, $M_N(F)^{(1)} = cM_N(F)$.

Now, if $B$ contains one simple component of type $i)$ with $a + b \geq 2$ or of type $ii)$ with $N \geq 2$, then we will get that $G(B)$ contains an algebra isomorphic to either $A_4$ or $A_5$ and in this case we will be done.

Therefore, since $p > 2$, we may assume that one of the following possibilities occurs:

1) for some $i \neq l$, $B_iJB_l \neq 0$ where $B_i \cong F + cF, c^2 = 1$ and $B_l \cong F$;

2) for some $i \neq l$, $B_iJB_l \neq 0$ where $B_i \cong F$ and $B_l \cong F + cF, c^2 = 1$;

3) there exist distinct $B_i, B_l, B_m$ such that $B_iJB_lJB_m \neq 0$ and $B_i \cong B_l \cong B_m \cong F$.

Suppose 1) holds. Then there exists $a + cb \in B_i$ such that $(a + cb)j1_3 \neq 0$ where $1_3$ is the unit element of $B_l$ and $j \in J$ is homogeneous. By eventually multiplying by $c$ on the left, we may assume that $(a + cb)j_01_3 \neq 0$ for some $j_0 \in J^{(0)}$. Write $a+cb = u_{11}(a+b)+u_{22}(a-b)$ where $u_{11} = (1+c)/2, u_{22} = (1-c)/2$ and $1 = 1_{B_i}$ is the unit element of $B_i$. Set $u_{33} = 1_3$.

First consider the case when $j_0u_{33}$ and $cj_0u_{33}$ are linearly dependent over $F$. Since $c^2 = 1$ it follows that $cj_0u_{33} = \pm j_0u_{33}$.

Suppose $cj_0u_{33} = j_0u_{33}$. Then $u_{11}j_0u_{33} = j_0u_{33}$ and $u_{22}j_0u_{33} = 0$. If we set $u_{13} = j_0u_{33}$, then the $u_{hk}$'s behave like the corresponding matrix units of $3 \times 3$ matrices and the algebra generated by $u_{11}, u_{22}, u_{33}, u_{13}$ over $F$ is isomorphic to the $\mathbf{Z}_2$-graded algebra $D = \begin{pmatrix} F & 0 & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$ with grading

$$D^{(0)} = \left\{ \begin{pmatrix} \lambda & 0 & \nu \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \right\}, \quad D^{(1)} = \left\{ \begin{pmatrix} \lambda & 0 & \nu \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Clearly,

$$G(D) = \left\{ \begin{pmatrix} a+b & 0 & z \\ 0 & a-b & 0 \\ 0 & 0 & t \end{pmatrix} \right\}$$

where $a, t \in G_0, b \in G_1, z \in G$. It is easy to check that $G(D) \cong \begin{pmatrix} G & G \\ 0 & G^{(0)} \end{pmatrix} = A_1$ and the map

$$\begin{pmatrix} a+b & 0 & z \\ 0 & a-b & 0 \\ 0 & 0 & t \end{pmatrix} \mapsto \begin{pmatrix} a+b & z \\ 0 & t \end{pmatrix}$$

is an isomorphism. Hence $A_1 \in \mathcal{V}$ and we are done.

Now let $cj_0u_{33} = -j_0u_{33}$. Then $u_{11}j_0u_{33} = 0, u_{22}j_0u_{33} = j_0u_{33}$ and the elements $u_{11}, u_{22}, u_{33}, u_{23} = j_0u_{33}$ generate a $\mathbf{Z}_2$-graded algebra isomorphic to $D' = \begin{pmatrix} F & 0 & 0 \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$ with $\mathbf{Z}_2$-grading

$$D'^{(0)} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \nu \\ 0 & 0 & \mu \end{pmatrix} \right\}, \quad D'^{(1)} = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & \nu \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

In this case the isomorphism of algebras $G(D') \cong A_1$ is given by the map

$$\begin{pmatrix} a+b & 0 & 0 \\ 0 & a-b & z \\ 0 & 0 & t \end{pmatrix} \mapsto \begin{pmatrix} a-b & z \\ 0 & t \end{pmatrix}$$

where $a, t \in G_0, b \in G_1$ and $z \in G$.

Now consider the case when $j_0u_{33}$ and $cj_0u_{33}$ are linearly independent over $F$. In this case $u_{11}, u_{22}, u_{33}, u_{13} = u_{11}j_0u_{33}$ and $u_{23} = u_{22}j_0u_{33}$ are linearly independent and form a subalgebra in $B$ isomorphic to $D'' = \begin{pmatrix} F & 0 & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$ with $\mathbf{Z}_2$-grading

$$D''^{(0)} = \left\{ \begin{pmatrix} \lambda & 0 & \nu \\ 0 & \lambda & \nu \\ 0 & 0 & \mu \end{pmatrix} \right\}, \quad D''^{(1)} = \left\{ \begin{pmatrix} \lambda & 0 & \nu \\ 0 & -\lambda & -\nu \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Hence $G(D'') \in \mathcal{V}$ and

$$G(D'') = \left\{ \begin{pmatrix} a+b & 0 & z+w \\ 0 & a-b & z-w \\ 0 & 0 & t \end{pmatrix} \right\}$$

where $a, z, t \in G_0, b, w \in G_1$.

As before we construct an algebra isomorphism $G(D'') \cong A_1$ by setting

$$\begin{pmatrix} a+b & 0 & z+w \\ 0 & a-b & z-w \\ 0 & 0 & t \end{pmatrix} \mapsto \begin{pmatrix} a-b & z-w \\ 0 & t \end{pmatrix} \quad \text{where } a, z, t \in G_0, b, w \in G_1.$$

In case 2) holds then the same procedure as above shows that $A_2 \in \mathcal{V}$.

Finally suppose that 3) holds. Then there exist $j_0, j_0' \in J^{(0)}, j_1, j_1' \in J^{(1)}$ such that $1_1(j_0 + j_1)1_2(j_0' + j_1')1_3 \neq 0$ where $1_1, 1_2, 1_3$ are the unit elements of $B_i, B_l, B_m$ respectively. In this case at least one of the products $1_1 j_r 1_2 j_s' 1_3,\ r, s \in \{0, 1\}$ is non-zero. Then, for fixed $r$ and $s$ set $u_{11} = 1_1, u_{22} = 1_2, u_{33} = 1_3, u_{12} = 1_1 j_r 1_2, u_{23} = 1_2 j_s' 1_3, u_{13} = 1_1 j_r 1_2 j_s' 1_3$ and let $D_{rs}$ be the $\mathbf{Z}_2$-graded subalgebra of $B$ generated by $u_{11}, u_{22}, u_{33}, u_{12}, u_{23}, u_{13}$. By taking the Grassmann envelope of the algebra $D_{rs}$, we get that $\mathcal{V}$ must contain at least one of the following four algebras denoted $E_1, E_2, E_3, E_4$ respectively

$$UT_3(F), \begin{pmatrix} G^{(0)} & G^{(0)} & G^{(1)} \\ 0 & G^{(0)} & G^{(1)} \\ 0 & 0 & G^{(0)} \end{pmatrix}, \begin{pmatrix} G^{(0)} & G^{(1)} & G^{(1)} \\ 0 & G^{(0)} & G^{(0)} \\ 0 & 0 & G^{(0)} \end{pmatrix}, \begin{pmatrix} G^{(0)} & G^{(1)} & G^{(0)} \\ 0 & G^{(0)} & G^{(1)} \\ 0 & 0 & G^{(0)} \end{pmatrix}.$$

It is easy to check that each $E_i$ satisfies the identity $[x_1, x_2][x_3, x_4][x_5, x_6] \equiv 0$ and, according to [7], all the identities of $UT_3(F)$ . On the other hand, each one of the algebras $E_2, E_3, E_4$ has a subalgebra isomorphic to $UT_3(F)$. In the case of $E_4$ this subalgebra is generated by $e_{11}, e_{22}, e_{33}, xe_{12}, ye_{23}$ and $xye_{13}$ where $x$ and $y$ are two distinct generators of $G$. For $E_3$ it is the subalgebra generated by $e_{11}, e_{22}, e_{33}, e_{23}, xe_{12}$ and $xe_{13}$. For $E_2$ we take $e_{11}, e_{22}, e_{33}, e_{12}, xe_{13}$ and $xe_{23}$. Hence $UT_3(F) \in \mathcal{V}$ and we are done. From [1] and [2] it follows that $\mathrm{Exp}(\mathcal{V}_1) = \mathrm{Exp}(\mathcal{V}) = \mathrm{Exp}(\mathcal{V}_3) = 3$ and $\mathrm{Exp}(\mathcal{V}_4) = \mathrm{Exp}(\mathcal{V}_5) = 4$. Hence if $\mathcal{W} \ni A_i$ for some $i \in \{1, \dots, 5\}$ then $\mathrm{Exp}(\mathcal{W}) > 2$. $\square$

Proof of Proposition 1. It is clear that if $\mathcal{W} \subseteq \mathcal{V}$ are varieties, then $\mathrm{Exp}(\mathcal{W}) \le \mathrm{Exp}(\mathcal{V})$; hence $\mathcal{V}_4 \not\subseteq \mathcal{V}_i$ and $\mathcal{V}_5 \not\subseteq \mathcal{V}_i$ for all $i = 1, 2, 3$.

Since $UT_3(F)$ and $M_2(F)$ are the only two algebras among the $A_i$'s satisfying a standard identity, we get that $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_5 \not\subseteq \mathcal{V}_i,\ i = 3, 4$. Also, the algebra $M_2(F)$ satisfies the standard identity $S_4 \equiv 0$ but $S_4 \not\equiv 0$ on $UT_3(F)$, hence $\mathcal{V}_3 \not\subseteq \mathcal{V}_4$.

The algebra $M_{1,1}(F) \cong G \otimes G$ is the only algebra among the $A_i$'s satisfying the identity $[[x_1, x_2], [x_3, x_4], x_5] \equiv 0$; hence $\mathcal{V}_i \not\subseteq \mathcal{V}_5$ for $i = 1, 2, 3, 4$.

The algebra $A_1$ satisfies the identity $f_1 = [x_1, x_2, x_3][x_4, x_5] \equiv 0$ and the algebra $A_2$ satisfies the identity $f_2 = [x_1, x_2][x_3, x_4, x_5] \equiv 0$. Since $f_1 \not\equiv 0$ on $A_2$ and $f_2 \not\equiv 0$ on $A_1$, we get that $\mathcal{V}_1 \not\subseteq \mathcal{V}_2$ and $\mathcal{V}_2 \not\subseteq \mathcal{V}_1$. Moreover since $f_1$ and $f_2$ do not vanish on $UT_3(F)$ we get that $\mathcal{V}_3 \not\subseteq \mathcal{V}_1, \mathcal{V}_2$. $\square$

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