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BOUNDARY-VALUE PROBLEMS FOR ALMOST NONLINEAR SINGULARLY PERTURBED SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. A boundary-value problems for almost nonlinear singularly perturbed systems of ordinary differential equations are considered. An asymptotic solution is constructed under some assumption and using boundary functions and generalized inverse matrix and projectors.

1. Formulation of the problem. A construction of the solution of singularly perturbed systems of ordinary differential equations is connected with application of different asymptotic methods. The works of A. Tikhonov [11], [12], N. Levinson [2], [6], W. Wazov [17] are fundamental in this direction.

The method and results of A. B. Vasil'eva [13], [14] and A. B. Vasil'eva, V. F. Butuzov [15], [16] give a possibility to construct asymptotic solution of

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singularly perturbed systems using boundary functions. This method will use in a present paper.

Another asymptotic method for solving singularly perturbed systems is the method of the regularization, described from S. A. Lomov in [7]. Singularly perturbed systems of integro-differential equations are considered in [3].

Let it is given a system

$$\varepsilon \frac{dx}{dt} = Ax + \varepsilon f(t, x, \varepsilon) + \varphi(t), \quad t \in [a, b], \quad 0 < \varepsilon \ll 1, \tag{1}$$

with a boundary condition

$$l(x) = h, \quad h \in \mathbb{R}^m. \tag{2}$$

It is assumed that the coefficients of the boundary-value problem (1), (2) are satisfied the next conditions:

H1: A is $n \times n$ matrix with constant coefficients. Its eigenvalues have a negative real parts, $Re\lambda_i < 0$, $\lambda_i \in \sigma(A)$, $i = \overline{1, n}$.

H2: $f(t, x, \varepsilon) \in C^\infty(\Omega)$ is n-dimensional vector-function, where $\Omega \equiv \{(t, x, \varepsilon) | a \le t \le b, |x| \le \rho, \varepsilon \in (0, \varepsilon_0]\}$, i.e. there exist positive constants k_i such, that $\|f^{(i)}(t, x, \varepsilon)\| \le k_i$.

H3: $\varphi(t) \in C^\infty[a, b]$ is n-dimensional vector-function.

H4: l is m-dimensional linear bounded functional, $l = col(l_1, \ldots, l_m)$, $l \in (x : C[a, b] \to \mathbb{R}^n, \mathbb{R}^m)$, $\|l(\psi)\| \le \overline{b}\|\psi\|$, $\overline{b} = \text{const}$, $\overline{b} > 0$.

If $\varepsilon = 0$, from (1) is obtained the degenerate system $Ax_0(t) + \varphi(t) = 0$, which under conditions H1, H3 has an unique continuous solution $x_0(t) = -A^{-1}\varphi(t)$, for $\forall \varphi(t) \in C^\infty[a, b]$.

The asymptotic series of the solution of the nonlinear problem (1), (2) will be constructed basing on the conditions H1–H4, the method of boundary functions and some additional assumptions

If instead of function $f(t, x, \varepsilon)$ in (1) is placed $(n \times n)$ matrix $A_1(t)$, then it is obtained a boundary-value problem which is investigated in [5], [4].

The construction of the asymptotic solution of (1), (2) is based on generalized inverse matrices and projectors too. [1], [9], [10], [8].

2. Formally asymptotic expansion. The formally asymptotic expansion of the solution of the boundary-value problem (1), (2) is sought in the form

$$x(t, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i (x_i(t) + \Pi_i(\tau)), \quad \tau = \frac{t - a}{\varepsilon}. \tag{3}$$

The coefficients $x_i(t)$, $\Pi_i(\tau)$ of expansion (3) are unknown n-dimensional vector functions and its determination is accomplished by substitution of (3) in the system (1).

$$\begin{aligned}\varepsilon\sum_{i=0}^{\infty}\varepsilon^i\left(\frac{dx_i(t)}{dt}+\frac{1}{\varepsilon}\frac{d\Pi_i(\tau)}{d\tau}\right) &= A\sum_{i=0}^{\infty}\varepsilon^i(x_i(t)+\Pi_i(\tau))+\\ &+\varepsilon f\left(t,\sum_{i=0}^{\infty}\varepsilon^i(x_i(t)+\Pi_i(\tau)),\varepsilon\right)+\varphi(t)\end{aligned} \tag{4}$$

The function $f(t,\sum_{i=0}^{\infty}\varepsilon^i(x_i(t)+\Pi_i(\tau))$ is presented in the form [15].

$$f\left(t,\sum_{i=0}^{\infty}\varepsilon^i(x_i(t)+\Pi_i(\tau)),\varepsilon\right)=\overline{f}(t,\varepsilon)+\Pi f(\tau,\varepsilon), \tag{5}$$

where

$$\begin{aligned}\overline{f}(t,\varepsilon) &= f\left(t,\sum_{i=0}^{\infty}\varepsilon^i x_i(t),\varepsilon\right)\\ \Pi f(\tau,\varepsilon) &= f\left(\varepsilon\tau+a,\sum_{i=0}^{\infty}\varepsilon^i(x_i(\varepsilon\tau+a)+\Pi_i(\tau)),\varepsilon\right)-\\ &\quad - f\left(\varepsilon\tau+a,\sum_{i=0}^{\infty}\varepsilon^i x_i(\varepsilon\tau+a),\varepsilon\right)\end{aligned}$$

The function $\overline{f}(t,\varepsilon)$ is expanded in the series of Taylor in neighbourhood of points $(t,x_0(t),0)$. The coefficients before the same powers of ε are grouped and the last function takes on the form

$$\overline{f}(t,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i\overline{f}_i(t), \tag{6}$$

where

$$\overline{f}_i(t)=\begin{cases} f(t,x_0(t),0), & i=0,\\ f'_x(t,x_0(t),0)x_i(t)+g_i(t,x_0(t),\dots,x_{i-1}(t)), & i=1,2,\dots\end{cases} \tag{7}$$

Derivatives of $(i-1)$-th order with respect to ε and x of the function f take part in the functions g_i.

The function $\Pi f(\tau,\varepsilon)$ is expanded in the series of Taylor too in neighbourhood of the point $(a,x_0(a)+\Pi_0(\tau),0)$, and $x_i(t)$ – in neighbourhood of $t=a$. Then $\Pi f(\tau,\varepsilon)$ takes on the representation

$$\Pi f(\tau,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i\Pi_i f(\tau), \tag{8}$$

where

$$(9)\quad \Pi_i f(\tau)=\begin{cases} f(a,x_0(a)+\Pi_0(\tau),0)-f(a,x_0(a),0), & i=0,\\ f'_x(a,x_0(a)+\Pi_0(\tau),0)\Pi_i(\tau)+G_i(\tau,\Pi_0(\tau),\ldots,\Pi_{i-1}(\tau)), & i=1,2,3,\ldots \end{cases}$$

The expansions (6) and (8) are substituted in the system (4) through (5). It is separated the variables with respect to t and τ. The coefficients before identical powers of ε are equalized. Thus the elements of the regular series take on the form

$$(10)\qquad x_i(t)=\begin{cases} -A^{-1}\varphi(t), & i=0,\\ A^{-1}\left(\dfrac{dx_{i-1}(t)}{dt}-\overline{f}_{i-1}(t)\right), & i=1,2,3,\ldots \end{cases}$$

The boundary functions are obtained successively as solutions of the next linear differential equations

$$(11)\qquad \frac{d\Pi_i(\tau)}{d\tau}=A\Pi_i(\tau)+f_i(\tau),\quad \tau\in\left[0,\frac{b-a}{\varepsilon}\right]$$

where

$$(12)\qquad f_i(\tau)=\begin{cases} 0, & i=0,\\ \Pi_{i-1}f(\tau), & i=1,2,3,\ldots \end{cases}$$

The series (3) is substituted in the boundary condition (2), the coefficients before the same powers of ε are equalized and it is obtained the following equations

$$(13)\qquad l(x_i(\cdot))+l\left(\Pi_i\left(\frac{(\cdot)-a}{\varepsilon}\right)\right)=\begin{cases} h, & i=0,\\ 0, & i=1,2,3,\ldots \end{cases}$$

It is considered a linear system

$$\frac{dx}{d\tau}=Ax,\quad \tau\in\left[0,\frac{b-a}{\varepsilon}\right]$$

and let $X(\tau)=\exp(A\tau)$ is its normal fundamental matrix of solutions.

Lemma 1 [15]. *If eigenvalues* $\lambda_i,\ \ i=\overline{1,n}$ *of* $n\times n$ *matrix* A *satisfy an inequality* $Re\lambda_i<-2\alpha_1,\ \alpha_1>0,\ \alpha_1=\mathrm{const}$, *then exists constant* $c_1,\ c_1>0$, *such that:*

$$\|\exp(At)\|\le c_1\exp(-\alpha_1 t),\quad t\ge 0,$$

where under norm of the matrix $B = [b_{ij}(t)]_{j=\overline{1,n}}^{i=\overline{1,n}}$, $t \in [a,b]$ *is understood*

$$\|B\| = \max_{1\le i\le n} \sum_{j=1}^{n} |b_{ij}(t)|, \quad t \in [a,b].$$

Let by $D(\varepsilon)$ is denoted the following $m \times n$ matrix:

$$D(\varepsilon) = l\left(X\left(\frac{(\cdot)-a}{\varepsilon}\right)\right) \tag{14}$$

In dependence of structure of the functional l will be considered two cases for the form of $D(\varepsilon)$.

2.1. $\boldsymbol{D(\varepsilon) = D_0 + O\left(\varepsilon^s \exp\left(-\frac{\alpha}{\varepsilon}\right)\right), \alpha > 0, s \in \mathbf{N}}$**.** In this case D_0 is constant $m \times n$ matrix. The elements $O(\varepsilon^s exp(-\frac{\alpha}{\varepsilon}))$ are exponentially small.

Let for the matrix D_0 is fulfilled the condition:

H5: $rankD_0 = n_1 < \min(m,n)$.

Then $rankP = n - n_1 = r$, and $rankP^* = m - n_1 = d$, here P and P^* are projectors

$$P : \mathbb{R}^n \to \ker(D_0), \quad P^* : \mathbb{R}^m \to \ker(D_0^*), \quad D_0^* = D^T.$$

Dimensions of P and P^* are $n \times n$ and $m \times m$ respectively. Let by P_r is denoted $n \times r$ matrix, which consist of r in number arbitrary linear independent columns of the matrix P, and by P_d^* – $d \times m$ matrix, consisting of d in number arbitrary linear independent rows of the matrix P^*.

The boundary-value problem with respect to $\Pi_0(\tau)$ is considered, i.e. the system (11) with boundary condition (13) when $i = 0$

$$\frac{d\Pi_0(\tau)}{d\tau} = A\Pi_0(\tau), \quad l\left(\Pi_0\left(\frac{(\cdot)-a}{\varepsilon}\right)\right) = h - l(x_0(\cdot)). \tag{15}$$

The general solution of the homogeneous system $\Pi_0(\tau) = X(\tau)c_0, \quad c_0 \in \mathbb{R}^n$ is substituted in the boundary condition (15) and the following system about c_0 is obtained.

$$D(\varepsilon)c_0 = h_0, \quad h_0 = h - l(x_0(\cdot)). \tag{16}$$

Because of ignoring the exponentially small elements in the matrix $D(\varepsilon)$ the last system takes on the form

$$D_0 c_0 = h_0. \tag{17}$$

In accordance with H5 this system has r-parametric solution

$$c_0 = P_r c_0^r + D_0^+ h_0, c_0^r \in \mathbb{R}^n \tag{18}$$

if and only if

$$P_d^* h_0 = 0.$$

By D_0^+ is denoted an unique Moore-Penrose inverse matrix of the matrix D_0. The equality (18) is substituted in the general solution of the problem (15) and the next expression for $\Pi_0(\tau)$ is obtained

$$\Pi_0(\tau) = X_r(\tau) c_0^r + q_0(\tau), \tag{19}$$

where the denotations $X_r(\tau) = X(\tau)P_r$ and $q_0(\tau) = X(\tau)D_0^+ h_0$ are introduced.

In (19) the vector c_0^r is unknown. It will be found from the condition for solvability when the next boundary function $\Pi_1(\tau)$ is defined from the system (11) and the condition (13) under $i = 1$, i.e.

$$\frac{d\Pi_1(\tau)}{d\tau} = A\Pi_1(\tau) + f_1(\tau), \quad l\left(\Pi_1\left(\frac{(\cdot) - a}{\varepsilon}\right)\right) = -l(x_1(\cdot)). \tag{20}$$

Here the function $f_1(\tau)$ has the form from (12) — $f_1(\tau) = f(a, x_0(a) + \Pi_0(\tau), 0) - f(a, x_0(a), 0)$. The function $f(a, x_0(a) + \Pi_0(\tau), 0)$ is expanded in the series of Taylor in neighbourhood of point $(a, x_0(a), 0)$, to the second order, for instance. In the obtained series is substituted $\Pi_0(\tau)$ from (19) and the following representation is received for $f_1(\tau)$

$$\begin{aligned} f_1(\tau) = f'_x(a, x_0(a), 0)(X_r(\tau)c_0^r + q_0(\tau)) + \frac{1}{2!} f''_x(a, x_0(a) + \theta(X_r(\tau)c_0^r + \\ + q_0(\tau)), 0)(X_r(\tau)c_0^r + q_0(\tau))^2, \quad 0 < \theta < 1 \end{aligned} \tag{21}$$

The general solution of nonhomogeneous system (20) is determinate by the formula of Cauchy

$$\Pi_1(\tau) = X(\tau)c_1 + \int_0^\tau X(\tau)X^{-1}(s) f_1(s) ds \tag{22}$$

The solution (22) is substituted in the boundary condition of (20) and for $c_1 \in \mathbb{R}^n$ is obtained the next system

$$D(\varepsilon)c_1 = h_1(\varepsilon), \; h_1(\varepsilon) = -l(x_1(\cdot)) - l\left(\int_0^{\frac{(\cdot)-a}{\varepsilon}} X\left(\frac{(\cdot) - a}{\varepsilon}\right) X^{-1}(s) f_1(s) ds\right). \tag{23}$$

If in $h_1(\varepsilon)$ is substituted the expression for $f_1(\tau)$ from (21), thus $h_1(\varepsilon)$ will depend on c_0^r nonlinearly.

$$h_1(\varepsilon) = \overline{D}_1(\varepsilon)c_0^r + b_1(\varepsilon, c_0^r),$$

where

$$\overline{D}_1(\varepsilon) = -l\left(\int_0^{\frac{(\cdot)-a}{\varepsilon}} X\left(\frac{(\cdot)-a}{\varepsilon}\right) X^{-1}(s) f'_x(a, x_0(a), 0) X_r(s) ds\right),$$

$$b_1(\varepsilon, c_0^r) = P(\varepsilon, c_0^r) + s_1(\varepsilon),$$

$$P(\varepsilon, c_0^r) = -l\left(\int_0^{\frac{(\cdot)-a}{\varepsilon}} X\left(\frac{(\cdot)-a}{\varepsilon}\right) X^{-1}(s) \frac{1}{2!} f''_x(a, x_0(a) + \right.$$

$$\left. + \theta(X_r(s)c_0^r + q_0(s)), 0)(X_r(s)c_0^r + q_0(s))^2 ds\right),$$

$$s_1(\varepsilon) = -l(x_1(\cdot)) - l\left(\int_0^{\frac{(\cdot)-a}{\varepsilon}} X\left(\frac{(\cdot)-a}{\varepsilon}\right) X^{-1}(s) f'_x(a, x_0(a), 0) q_0(s) ds\right),$$

Then the system (23) takes on the form

$$D(\varepsilon)c_1 = \overline{D}_1(\varepsilon)c_0^r + b_1(\varepsilon, c_0^r) \tag{24}$$

The exponentially small elements in the matrix $D(\varepsilon)$ are rejected. The system (24) becomes the following

$$D_0 c_1 = \overline{D}_1(\varepsilon)c_0^r + b_1(\varepsilon, c_0^r)$$

and has a solution

$$c_1 = P_r c_1^r + D_0^+(\overline{D}_1(\varepsilon)c_0^r + b_1(\varepsilon, c_0^r)) \tag{25}$$

if and only if

$$P_d^*(\overline{D}_1(\varepsilon)c_0^r + b_1(\varepsilon, c_0^r)) = 0. \tag{26}$$

Analysis of $\overline{D}_1(\varepsilon)$ and $b_1(\varepsilon, c_0^r)$ from nonlinear with respect to c_0^r equation (26) shows, that the coefficients before c_0^r are exponentially small. Because of this c_0^r will seek in the form

$$c_0^r = c_{00}^r + c_{01}^r\varepsilon^{-1} + c_{02}^r\varepsilon^{-2} + \cdots,$$

as the coefficients c^r_{0j}, $j = 0, 1, 2, \ldots$ are determinate by the method of indefinite coefficients from the equality (26) under some conditions. On this way the function $\Pi_0(\tau)$ is defined completely and has the form

$$\Pi_0(\tau) = X_r(\tau)(c^r_{00} + c^r_{01}\varepsilon^{-1} + c^r_{02}\varepsilon^{-2} + \cdots) + q_0(\tau).$$

The vector c^r_1 must be determined from (25) to define function $\Pi_1(\tau)$. For this purpose the boundary-value problem with respect to $\Pi_2(\tau)$ is considered, i.e. (11) and (13) under $i = 2$. Continuing this process for determining c^r_{i-1}, which participates in $\Pi_{i-1}(\tau)$ it is sufficiently to consider the problem

$$\frac{d\Pi_i(\tau)}{d\tau} = A\Pi_i(\tau) + f_i(\tau), \quad l\left(\Pi_i\left(\frac{(\cdot) - a}{\varepsilon}\right)\right) = -l\left(x_i(\cdot)\right).$$

It has general solution of the form $\Pi_i(\tau) = X(\tau)c_i + \int_0^\tau X(\tau)X^{-1}(s)f_i(s)ds$, which is substituted in the boundary condition and the system $D(\varepsilon)c_i = h_i(\varepsilon, c^r_{i-1})$ is obtained. In the case under consideration the last system takes on the form

$$D_0 c_i = h_i(\varepsilon, c^r_{i-1}) \tag{27}$$

In accordance with condition H5, the system (27) has solution

$$c_i = P_r c^r_i + D_0^+ h_i(\varepsilon, c^r_{i-1}),$$

if and only if

$$P^*_d h_i(\varepsilon, c^r_{i-1}) = 0 \tag{28}$$

The vector $h_i(\varepsilon, c^r_{i-1})$ has the representation

$$h_i(\varepsilon, c^r_{i-1}) = \overline{D}_2(\varepsilon)c^r_{i-1} - b_i(\varepsilon),$$

where

$$\overline{D}_2(\varepsilon) = -l\left(\int_0^{\frac{(\cdot)-a}{\varepsilon}} X\left(\frac{(\cdot) - a}{\varepsilon}\right) X^{-1}(s) f'_x(a, x_0(a) + \Pi_0(s), 0)X_r(s)ds\right),$$

$$\begin{aligned} b_i(\varepsilon) = l(x_i(\cdot)) + l\Bigg(\int_0^{\frac{(\cdot)-a}{\varepsilon}} X\left(\frac{(\cdot) - a}{\varepsilon}\right) X^{-1}(s)\Bigg(& f'_x(a, x_0(a) + \\ & + \Pi_0(s), 0)X(s)D_0^+ h_{i-1}(\varepsilon) + G_{i-1}(s, \Pi_0(s), \ldots, \Pi_{i-2}(s)) + \\ & + f'_x(a, x_0(a) + \Pi_0(s), 0) \int_0^s X(s)X^{-1}(p)f_{i-1}(p)dp\Bigg)ds\Bigg). \end{aligned}$$

The system (28) becomes the next

$$P_d^*\overline{D}_2(\varepsilon)c_{i-1}^r = P_d^* b_i(\varepsilon). \tag{29}$$

It is important to remark that the system (26) is nonlinear with respect to c_0^r, but the systems (29) are linear with respect to c_{i-1}^r. It is due to the form of $\Pi_{i-1}f(\tau)$, $i = 2, 3, \dots$ from (9), where $\Pi_{i-1}(\tau)$ do not participate in the argument of f_x' but participates as a multiplier. Availability of the only infinitely small functions in (29), shows that the vector c_{i-1}^r is sought in the form

$$c_{i-1}^r = c_{i-1,0}^r + c_{i-1,1}^r\varepsilon^{-1} + c_{i-1,2}^r\varepsilon^{-2} + \cdots,$$

as the coefficients $c_{i-1,j}$, $j = 0, 1, 2, \dots$ are defined from (29).

Thus the function $\Pi_{i-1}(\tau)$ is determinate completely and has the representation

$$\begin{aligned} \Pi_{i-1}(\tau) = X_r(\tau)(c_{i-1,0}^r + c_{i-1,1}^r\varepsilon^{-1} + c_{i-1,2}^r\varepsilon^{-2} + \cdots) + X(\tau)D_0^+ h_i(\varepsilon) + \\ + \int_0^\tau X(\tau)X^{-1}(s)f_{i-1}(s)ds, \end{aligned} \tag{30}$$

Remark 1. According to the form of functions $\Pi_i f(\tau)$ the matrices $\overline{D}_i(\varepsilon)$, $i = 3, 4, \dots$ are obtained $\overline{D}_2(\varepsilon) \equiv \overline{D}_i(\varepsilon)$, $i = 3, 4, \dots$.

Theorem 1. *Let the conditions H1-H5, $P_d^* h_0 = 0$ are fulfilled and the matrix $D(\varepsilon)$ has the form $D(\varepsilon) = D_0 + O\left(\varepsilon^s exp\left(-\frac{\alpha}{\varepsilon}\right)\right)$, where $s \in \mathbb{N}$, $\alpha > 0$. Then the boundary-value problem* (1), (2) *has formally asymptotic expansion of the solution in the form* (3). *The elements of the regular series $x_i(t)$ have the form* (10) *and the coefficients of the singular series $\Pi_i(\tau)$, $\tau = \frac{t-a}{\varepsilon}$, $i = 0, 1, 2\dots$ have the representation* (30), *as c_i^r satisfy the equation* (26) *under $i = 0$ and the equation* (29) *under $i = 1, 2, 3, \dots$. The following inequalities are real for the boundary functions*

$$\|\Pi_i(\tau)\| \le c^* exp(-\alpha^*\tau), \quad \tau \in \left[0, \frac{b-a}{\varepsilon}\right], \quad i = 0, 1, \dots, \tag{31}$$

where c^* and α^* are positive constants.

Proof. The exposition above shows that it is sufficiently to prove the exponentially decreasing of the boundary functions.

From (19) is known that $\Pi_0(\tau) = X_r(\tau)c_0^r + q_0(\tau)$, where $X_r(\tau) = X(\tau)P_r$, and $q_0(\tau) = X(\tau)D_0^+ h_0$. From Lemma 1 and H1 is obtained

$$\|X(\tau)\| \le c_1 \exp(-\alpha_1\tau).$$

It is known that $\lim\limits_{\varepsilon\to 0}\dfrac{\exp\left(-\frac{t-a}{\varepsilon}\right)}{\varepsilon^n}=0$, when t is fixed in $[a,b]$. Therefore positive constants c_2 and α_2 exist such that the following bound is fulfilled

$$\|X_r(\tau)c_0^r\| \leq c_2\exp(-\alpha_2\tau).$$

Let $\|D_0^+h_0\| \leq c_3$, where c_3 is positive consrtant. Then

$$\|\Pi_0(\tau)\| \leq \|X(\tau)P_r c_0^r\| + \|X(\tau)\|\|D_0^+h_0\|) \leq c_0^*\exp(-\alpha\tau),$$

where $\alpha=\min(\alpha_1,\alpha_2), c_0^*=c_2+c_1c_3$. This shows that the exponential bound (31) is true for $\Pi_0(\tau)$

Further the proof is done inductively. Keeping in mind

$$f_l(s)=f_x'(a,x_0(a)+\Pi_0(s),0)\Pi_{l-1}(s)+G_l(s,\Pi_0(s),\dots,\Pi_{l-2}(s)),$$

$$0\leq s\leq\tau,\ \ t\in\left[0,\frac{b-a}{\varepsilon}\right],$$

done bound for $\Pi_0(\tau)$ and Lemma1 the bound (31) is proved for every i.

Corollary 1. *If $rankD_0=n_1=n$, then the boundary value problem* (1), (2) *has an unique formally asymptotic expansion in the form* (3). *The coefficients $x_i(t)$ have the form* (10) *and the boundary functions are the following*

$$\Pi_i(\tau)=X(\tau)D_0^+h_i(\varepsilon)+\int_0^\tau X(\tau)X^{-1}(s)f_i(s)ds,\ \ i=0,1,\dots,$$

if and only if the conditions (26) *and* (28) *are fulfilled. In this case $rankP=0\ \ {=}{>}\ \ rankP_r=0$ and $c_i=D_0^+h_i,\ \ i=0,1,\dots,\ \ h_0=-l(x_0)+h,\ \ h_i(\varepsilon)=b_i(\varepsilon)\ \ i=1,2,\dots$.*

Corollary 2. *If $m=n$ and $detD_0\neq 0$, then the boundary value problem* (1), (2) *has an unique formally asymptotic expansion in the form* (3). *The coefficients $x_i(t)$ have the form* (10) *and the boundary functions are the next*

$$\Pi_i(\tau)=X(\tau)D_0^{-1}h_i(\varepsilon)+\int_0^\tau X(\tau)X^{-1}(s)f_i(s)ds,\ \ i=0,1,\dots.$$

Remark 2. If $m\neq n$, but $rankD_0=n_1=m$, then $P^*=0$ and all systems of the form $D_0c_i^r=h_i,\ \ i=0,1,\dots$ are always solvable. A families of boundary functions is obtained in this case.

2.2. $\boldsymbol{D(\varepsilon) = D_0 + \varepsilon D_1 + \cdots + \varepsilon^s D_s + O\left(\varepsilon^q \exp\left(-\frac{\alpha}{\varepsilon}\right)\right), \alpha > 0, q \in \mathbf{N}}$**.** Here D_i, $i = 0, 1, \ldots, s$ are constant $m \times n$ matricies. The exponentially small elements in the matrix $D(\varepsilon)$ are ignored then the systems (17), (23) and (27) can rewritten as follows

$$(D_0 + \varepsilon D_1 + \cdots + \varepsilon^s D_s)\, c_i = h_i(\varepsilon), \quad i = 0, 1, 2 \ldots,$$

$$h_i(\varepsilon) = \begin{cases} h_0, & i = 0 \\ h_i(\varepsilon), & i = 1, 2, 3, \ldots \end{cases} \tag{32}$$

In this case the constants c_i are sought in the form $c_i = c_{i0} + \varepsilon c_{i1} + \cdots + \varepsilon^s c_{is}$, $c_{ij} \in \mathbb{R}^n$, $j = \overline{0, s}$. It is introduced denotations $c_0 = [c_{00}\ c_{01} \ldots c_{0s}]^T - (s+1)n$-dimensional vector, $b_0 = [h_0\ 0\ 0 \ldots 0]^T - (2s+1)m$-dimensional vector and

$$Q = \begin{bmatrix} D_0 & & & & \\ D_1 & D_0 & & \mathbf{0} & \\ D_2 & D_1 & D_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ D_s & D_{s-1} & D_{s-2} & \ldots & D_0 \\ & D_s & D_{s-1} & \ldots & D_1 \\ & & D_s & \ldots & D_2 \\ & \mathbf{0} & & \ddots & \vdots \\ & & & & D_s \end{bmatrix},$$

thus the system (32) becomes the following

$$Qc_0 = b_0, \quad b_0 = [h_0\ 0\ 0 \ldots 0]^T \tag{33}$$

If it is considered the system (32) under $i = 1, 2, \ldots,$ assumption that $h_i(\varepsilon)$ has the form

$$h_i(\varepsilon) = h_{i0} + \varepsilon h_{i1} + \cdots + \varepsilon^s h_{is} + O\left(\varepsilon^m exp\left(-\frac{\alpha}{\varepsilon}\right)\right), \quad i = 1, 2, \ldots,$$
$$m \in \mathbb{N}, \quad \alpha > 0, \quad \alpha = \text{const}$$

and the exponentially small elements in $h_i(\varepsilon)$ are rejected then systems analogous to systems from (33) is obtained, i.e.

$$Qc_i = b_i, \tag{34}$$

where

$$c_i = [c_{i0}\, c_{i1} \dots c_{is}]^T,\ \ c_{ij} \in \mathbb{R}^n,\ \ j = \overline{0,s},\ \ i = 1,2,\dots,$$

$$b_i = [h_{i0}\, h_{i1} \dots h_{is}\, 0\, 0 \dots 0]^T,\ \ i = 1,2,\dots$$

Let the following condition is fulfilled

H6: $rankQ = (s+1)n,\ \ (2s+1)m > (s+1)n.$

Then $rankP_1 = 0$, and $rankP_1^* = d_1 = (2s+1)m - (s+1)n$, where P_1 and P_1^* are projectors

$$P_1 : \mathbb{R}^{(s+1)n} \to ker(Q),\ \ P_1^* : \mathbb{R}^{(2s+1)m} \to ker(Q^*),\ \ Q^* = Q^T,$$

The algebraic systems (33), (34) have an unique solution

$$c_i = Q^+ b_i,\ \ i = 0,1,2,\dots, \tag{35}$$

if and only if

H7: $P_1^* b_i = 0 \ => \ P_{1d_1}^* b_i = 0,$

where Q^+ is the unique Moore-Penrose inverse matrix of the matrix Q and the matrix $P_{1d_1}^*$ is consisted of d_1 in number linear undependence rows of the matrix P_1^*.

It is known that the vector c_i has the form $c_i = [c_{i0}\ c_{i1} \dots c_{is}]^T$, $i = 0,1,2,\dots$, from (35) is obtained that the first n components of the vector $Q^+ b_i$ are components of the vector c_{i0} in effect, the next n components of the vector $Q^+ b_i$ are components of the vector c_{i1} and etc., the last n components of the vector $Q^+ b_i$ are components of c_{is}. Then constants c_i, $i = 0,1,2,\dots$ take on the form

$$c_i = \sum_{j=0}^{s} \varepsilon^j \left[Q^+ b_i\right]_{n_j},$$

where the index n_j shows which n in number components of the vector $Q^+ b_i$ are taken. Then the systems (11) have the next solutions

$$\Pi_i(\tau) = X(\tau) \sum_{j=0}^{s} \varepsilon^j \left[Q^+ b_i\right]_{n_j} + \int_0^\tau X(\tau) X^{-1}(s) f_i(s) ds,\ \ i = 0,1,2\dots \tag{36}$$

According to Lemma 1 it is followed

$$\|X(\tau)\| = \|\exp(A\tau)\| \le c_1 \exp(-\alpha_1 \tau),\ \ \tau \in \left[0, \frac{b-a}{\varepsilon}\right],$$

$$\|X(\tau) X^{-1}(s)\| \le c_1 \exp(-\alpha_1(\tau - s)),\ \ 0 \le s \le \tau,\ \ \tau \in \left[0, \frac{b-a}{\varepsilon}\right].$$

Keeping in mind Theorem 1 it is obtained

$$\|f_i(s)\| \le c_{f_i} \exp(-\alpha_{f_i} s), \;\; 0 \le s \le \tau, \;\; \tau \in \left[0, \frac{b-a}{\varepsilon}\right].$$

The vectors $\sum_{j=0}^{s} \varepsilon^j \left[Q^+ b_i\right]_{n_j}, \;\; j=\overline{0,s}$, $i=0,1,\dots$ under $\varepsilon \to 0$ are limited. Let

$$\| \sum_{j=0}^{s} \varepsilon^j \left[Q^+ b_i\right]_{n_j} \| \le a_1, \;\; a_1 = const, \;\; a_1 > 0.$$

Thus

$$\|\Pi_i(\tau)\| \le a_1 c_1 \exp(-\alpha_1 \tau) + \int_0^\tau c_1 \exp(-\alpha_1(\tau - s)) c_{f_1} \exp(-\alpha_{f_1} s) ds =$$

$$= a_1 c_1 \exp(-\alpha_1 \tau) + c_1 c_{f_1} \exp(-\alpha_1 \tau) \int_0^\tau \exp(-(\alpha_{f_1} - \alpha_1) s) ds =$$

$$= a_1 c_1 \exp(-\alpha_1 \tau) + \frac{c_1 c_{f_1}}{(\alpha_{f_1} - \alpha_1)} \exp(-\alpha_1 \tau) \left(1 - \exp(-(\alpha_{f_1} - \alpha_1)\tau\right)$$

or

$$\|\Pi_i(\tau)\| \le c^* exp(-\alpha^* \tau), \;\; \tau \in \left[0, \frac{b-a}{\varepsilon}\right], \;\; i = 0,1,2,\dots,$$

where

$$c^* = \max\left(c_1 a_1 + \frac{c_1 c_{f_1}}{\alpha_{f_1} - \alpha_1}, \frac{c_1 c_{f_1}}{\alpha_1 - \alpha_{f_1}}\right), \;\; \alpha^* = \min(\alpha_1, \; \alpha_{f_1}),$$

are positive constants.

On this way the following theorem is proved.

Theorem 2. *Let the conditions H1-H4, H6, H7 are fulfilled and the matrix* $D(\varepsilon) = \sum_{i=0}^{s} D_i \varepsilon^i$. *Then the boundary-value problem* (1), (2) *has an unuque formally asymptotic expansion of the solution in the form* (3). *The coefficients* $x_i(t)$ *have the form* (10) *and the boundary functions* $\Pi_i(\tau)$*- the form* (36). *For the last are real the inequalities* (31).

Remark 3. If $rankQ < (s+1)n$, then c_i is obtained under defining of the boundary function $\Pi_{i+1}(\tau)$ from the condition for solvability of the system with respect to c_{i+1}, analogously to the case 2.1.

3. Bound of the remaider term of the asymptotic series. Let exact solution of the problem (1), (2) has the form

$$x(t,\varepsilon) = X_n(t,\varepsilon) + \varepsilon^{n+1}\xi(t,\varepsilon), \tag{37}$$

where $X_n(t,\varepsilon) = \sum_{i=0}^{n}\varepsilon^i(x_i(t) + \Pi_i(\tau))$, $\tau = \dfrac{t-a}{\varepsilon}$, $t \in [a,b]$, $\xi(t,\varepsilon)$ is the remainder term of the asymptotic series and for this function will prove an inequality $\|\xi(t,\varepsilon)\| \le K$, where K is positive constant, when $t \in [a,b]$ and $\varepsilon \in (0,\varepsilon_0]$. It is substituted (37) in the system (1) and the boundary condition (2) and for the remainder term is obtained the problem

$$\varepsilon\frac{d\xi(t,\varepsilon)}{dt} = A\xi(t,\varepsilon) + \frac{1}{\varepsilon^{n+1}}H(t,\xi(t,\varepsilon),\varepsilon), \quad l(\xi(\cdot,\varepsilon)) = 0, \tag{38}$$

where

$$H(t,\xi(t,\varepsilon),\varepsilon) = AX_n(t,\varepsilon) + \varepsilon f(t, X_n(t,\varepsilon) + \varepsilon^{n+1}\xi(t,\varepsilon),\varepsilon) + \varphi(t) - \varepsilon\frac{dX_n(t,\varepsilon)}{dt}$$

The function $f(t, X_n(t,\varepsilon) + \varepsilon^{n+1}\xi(t,\varepsilon),\varepsilon)$ is expanded in the series of Taylor

$$f(t, X_n(t,\varepsilon) + \varepsilon^{n+1}\xi(t,\varepsilon), 0) = f(t, X_n(t,\varepsilon),\varepsilon) + R_0(t,\xi(t,\varepsilon),\varepsilon)$$

where

$$R_0(t,\xi(t,\varepsilon),\varepsilon) = \varepsilon^{n+1}f'_x(t, X_n(t,\varepsilon) + \theta\varepsilon^{n+1}\xi(t,\varepsilon),\varepsilon), \quad 0 < \theta < 1$$

and the functtion $f(t, X_n(t,\varepsilon),\varepsilon)$ is represented on the next way

$$f(t, X_n(t,\varepsilon),\varepsilon) = f\left(t, \sum_{i=0}^{n}\varepsilon^i x_i(t),\varepsilon\right) + f\left(\varepsilon\tau + a, \sum_{i=0}^{n}\varepsilon^i(x_i(\varepsilon\tau + a) + \Pi(\tau)),\varepsilon\right) -$$

$$-f\left(\varepsilon\tau + a, \sum_{i=0}^{n}\varepsilon^n x_i(\varepsilon\tau + a),\varepsilon\right)$$

The function $f\left(t, \sum_{i=0}^{n}\varepsilon^i x_i(t),\varepsilon\right)$ is expanded in the series of Taylor in neighbourhood of points $(t, x_0(t),\varepsilon)$ and keeping in mind (7) for the last is obtained

$$f\left(t, \sum_{i=0}^{n}\varepsilon^i x_i(t),\varepsilon\right) = \sum_{i=0}^{n}\varepsilon^i\overline{f}_i(t) + \varepsilon^{n+1}\overline{g}(t, x_0(t),\ldots,x_n(t),\varepsilon)$$

The function $f\left(\varepsilon\tau+a, \sum\limits_{i=0}^{n} \varepsilon^i(x_i(\varepsilon\tau+a)+\Pi(\tau)), \varepsilon\right) - f\left(\varepsilon\tau+a, \sum\limits_{i=0}^{n} \varepsilon^n x_i(\varepsilon\tau+a), \varepsilon\right)$ is expanded in the series of Taylor in neighbourhood of points $(a, x_0(a)+\Pi_0(\tau), \varepsilon)$, as the functions $x_i(\varepsilon\tau + a)$ are expanded in the series of Taylor too, but in the neighbuorhood of point $t = a$ and keeping in mind the equalities (9) it is obtained

$$f\left(\varepsilon\tau + a, \sum_{i=0}^{n} \varepsilon^i(x_i(\varepsilon\tau + a) + \Pi(\tau)), \varepsilon\right) - f\left(\varepsilon\tau + a, \sum_{i=0}^{n} \varepsilon^n x_i(\varepsilon\tau + a), \varepsilon\right) =$$
$$= \sum_{i=0}^{n} \varepsilon^i \Pi_i f(\tau) + \varepsilon^{n+1} \Pi G(\tau, \Pi_0(\tau), \ldots, \Pi_n(\tau), \varepsilon).$$

Then

$$f(t, X_n(t, \varepsilon) + \varepsilon^{n+1}\xi(t, \varepsilon), \varepsilon) = \sum_{i=0}^{n} \varepsilon^i \left[\overline{f}_i(t) + \Pi_i f(\tau)\right] +$$
$$+\varepsilon^{n+1}\left[\overline{g}(t, x_0(t), \ldots, x_n(t), \varepsilon) + \Pi G(\tau, \Pi_0(\tau), \ldots, \Pi_n(\tau), \varepsilon)\right] +$$
$$+\varepsilon^{n+1}\xi(t, \varepsilon) f'_x(t, X_n(t, \varepsilon) + \theta\varepsilon^{n+1}\xi(t, \varepsilon), \varepsilon), \quad 0 < \theta < 1,$$

and the function $H(t, \varepsilon)$ takes on the form

$$H(t, \varepsilon) = \varepsilon^{n+1}\left[\varepsilon\xi(t, \varepsilon) f'_x(t, X_n(t, \varepsilon) + \theta\varepsilon^{n+1}\xi(t, \varepsilon), \varepsilon) + H_1(t, \varepsilon)\right], \tag{39}$$

where $H_1(t, \varepsilon) = \varepsilon\left[\overline{g}(t, x_0(t), \ldots, x_n(t), \varepsilon) + \Pi G(\tau, \Pi_0(\tau), \ldots, \Pi_n(\tau), \varepsilon)\right] + Ax_{n+1}(t)$ $+\Pi_n f(\tau)$. The equality (39) is substituted in (38) and the problem for the remainder term becomes the next

$$\varepsilon\frac{d\xi}{dt} = A\xi + \varepsilon\xi(t, \varepsilon) f'_x(t, X_n + \theta\varepsilon^{n+1}\xi, \varepsilon) + H_1(t, \varepsilon), \quad l\left(\xi(\cdot, \varepsilon)\right) = 0 \tag{40}$$

It is considered the function $H_1(t, \varepsilon)$. It consists of the functions $\overline{g}$, ΠG, Ax_{n+1}, $\Pi_n f$. The function $\overline{g}$ consists of continuous and bounded in the domain Ω functions, i.e it is bounded too. Let $\|\overline{g}(t, x_0(t), \ldots, x_n(t), \varepsilon)\| \le \eta_1$. About function ΠG is obtained analogously $\|\Pi G(\tau, \Pi_0(\tau), \ldots, \Pi_n(\tau), \varepsilon)\| \le \eta_2$ under $(t, x, \varepsilon) \in \Omega$.

Keeping in mind that $x_n(t)$ is countinuous function in the interval $[a, b]$, an inequality $\|Ax_n(t)\| \le \eta_3$ is obtained. According to Theorem 1 the function $\Pi_n f(\tau)$ is exponentially small therefore positive constant η_4 exists such that $\|\Pi_n f(\tau)\| \le \eta_4$. Then

$$\|H_1(t, \varepsilon)\| \le \eta, \quad \eta = \varepsilon(\eta_1 + \eta_2) + \eta_3 + \eta_4$$

Let $W(t,s,\varepsilon)$ is normal fundamental matrix of the solutions of the homogeneous system

$$\varepsilon\frac{d\xi}{dt} = A\xi, \quad W(s,s,\varepsilon) = E_n.$$

Then the followig contentions are fulfilled [15, 3, 2, 6]:

Lemma 2. *For the matrix $W(t,s,\varepsilon)$ when $a \le s \le t \le b$, $0 < \varepsilon \le \varepsilon_0$ is fulfiled the next equality*

$$\|W(t,s,\varepsilon)\| \le \beta \exp\left(-\alpha\frac{t-s}{\varepsilon}\right),$$

where α and β are positive constants.

Lemma 3. *Every continuous solution of the system* (40) *is solution of the integral equation*

$$\begin{aligned}\xi(t,\varepsilon) &= W(t,a,\varepsilon)\xi(a,\varepsilon)+\\ &+\int_a^t W(t,s,\varepsilon)\frac{1}{\varepsilon}\left[\varepsilon f_x'(s,X_n+\theta\varepsilon^{n+1}\xi,\varepsilon)\xi(s,\varepsilon)+H_1(s,\varepsilon)\right]ds.\end{aligned} \tag{41}$$

Lemma 4. *When $\varepsilon \to 0$ the integral $\int_a^t \left\|\frac{1}{\varepsilon}W(t,s,\varepsilon)\right\| ds$ is uniformly bounded in the interval $[a,b]$, i.e. a positive constant M exists, such that when $\varepsilon \to 0$ and $t \in [a,b]$ is fulfilled*

$$\int_a^t \left\|\frac{1}{\varepsilon}W(t,s,\varepsilon)\right\| ds \le M.$$

From the condition H2 is obtained that $f_x'(t,X_n+\theta\varepsilon^{n+1},\varepsilon)$ is bounded in the domain Ω, i.e.

$$\|f_x'(t,X_n+\theta\varepsilon^{n+1},\varepsilon)\| \le k_1, \quad 0<\theta<1$$

Let $W(t,a,\varepsilon)\xi(a,\varepsilon) = F(t,\varepsilon)$. The system (40) is solved by the method of successive approximations, i.e.

$$\xi_0(t,\varepsilon) = 0$$

$$\begin{aligned}\xi_n(t,\varepsilon) &= F(t,\varepsilon)+\\ &+\int_a^t W(t,s,\varepsilon)\frac{1}{\varepsilon}\left[\varepsilon f_x'(s,X_n+\theta\varepsilon^{n+1}\xi,\varepsilon)\xi_{n-1}(s,\varepsilon)+H_1(s,\varepsilon)\right]ds.\end{aligned} \tag{42}$$

Theorem 3. *Let h, h_1, k_1, h_3, $\overline{b}$, $\overline{\beta}$, ε_0 and M are positive constants such that*

$$\|W(t,a,\varepsilon)\| \le \overline{\beta},\ \ \|F(t,\varepsilon)\| \le h_1,\ \ h_1 = 2\overline{\beta}h,\ \ 0 < 2\overline{\beta} < 1,$$

$$\|f'_x(t, X_n + \theta\varepsilon^{n+1}, \varepsilon)\| \le k_1,\ \ \int_a^t \left\| \frac{1}{\varepsilon} W(t,s,\varepsilon) \right\| ds \le M,$$

$$\|R_0^+\| \le h_3,\ \ \|l(\psi)\| \le \overline{b}\|\psi\|,\ \ h_3\overline{b} < 2,\ \ \varepsilon_0 < \frac{1}{2Mk_1}.$$

If $\dfrac{M\eta}{1-2\overline{\beta}} \le h$*, then the asymptotic representation of the solution of the boundary value problem* (1), (2) *has the form* (37)*, where* $\xi(t,\varepsilon)$ *satisfies the condition*

$$\|\xi(t,\varepsilon)\| \le 2h,$$

and the vector $\xi(a,\varepsilon)$ *is defined from algebraic system*

$$R(\varepsilon)\xi(a,\varepsilon) = \overline{\overline{g}}(\varepsilon), \tag{43}$$

where $R(\varepsilon) = l\left(W(\cdot,a,\varepsilon)\right)$ *is* $m \times n$ *matrix and*

$$\overline{\overline{g}}(\varepsilon) = -l\left(\int_a^{(\cdot)} W(\cdot,s,\varepsilon)\frac{1}{\varepsilon}\left(\varepsilon f'_x(s, X_n + \theta\varepsilon^{n+1}\xi(s,\varepsilon),\varepsilon)\xi(s,\varepsilon) + H_1(s,\varepsilon)\right) ds \right).$$

P r o o f. By (42) is proved (see [4]), that the system (40) has an unique continuous solution which do not leave the domain Ω_1, where $\Omega_1 \equiv \{(t,\xi) | a \le t \le b, \|\xi\| \le 2h\}$, i.e. $\|\xi_k\| \le 2h$.

Let a limit of the sequence of the successive approximations is $\xi(t,\varepsilon)$, i.e. $\lim\limits_{n\to\infty} \xi_n(t,\varepsilon) = \xi(t,\varepsilon)$. It satisfies the integral equation (41). This shows that when $\varepsilon \to 0$ and $t \in [a,b]$ it is fulfilled $\|\xi(t,\varepsilon)\| \le 2h$. Thus the system (40) has an unique solution which do not leave the domain Ω_1 and depends on arbitrary vector $\xi(a,\varepsilon)$. Finally, it must to showed that this vector do not leave the domain Ω_1. For this purpose the integral equation (41) is substituted in the boundary condition $l(\xi(\cdot,\varepsilon)) = 0$ and the system (43) is obtained. Let the matrix $R(\varepsilon)$ has the form

$$R(\varepsilon) = R_0 + O\left(\varepsilon^s \exp(-\frac{\gamma}{\varepsilon})\right),$$

where $s \in \mathbb{N}$, γ is positive constant, R_0 is $m \times n$ constant matrix. The exponentially small elements in the matrix $R(\varepsilon)$ are ignored and the system (43) takes on the form

$$R_0\xi(a,\varepsilon) = \overline{\overline{g}}(\varepsilon)$$

It is assumed that $rankR_0 = n$, i.e. the matrix R_0 has a full rank then the last algebraic system has an unique solution

$$\xi(a,\varepsilon) = R_0^+\overline{\overline{g}}(\varepsilon),$$

if and only if

$$P_3^*\overline{\overline{g}}(\varepsilon) = 0.$$

Here by R_0^+ is denoted the unique Moore-Penrose inverse matrix of the matrix R_0 and by P_3^* - matrix projector $P_3^* : \mathbb{R}^m \to \ker(R_0^*)$.

Then

$$\begin{aligned}
&\|\xi(a,\varepsilon)\| \leq \|R_0^+\|\|\overline{\overline{g}}(\varepsilon)\| \leq \\
&\leq h_3h_4\int_a^t \left\|W(t,s,\varepsilon)\frac{1}{\varepsilon}\right\| \left(\|\varepsilon f'_x(s, X_n + \theta\varepsilon^{n+1}\xi,\varepsilon)\|\|\xi(s,\varepsilon)\| + \|H_1(s,\varepsilon)\|\right) ds \leq \\
&\leq h_3h_4M\left[\varepsilon k_1 2h + \eta\right] \leq h_3h_4M\left[2\varepsilon_0k_1h + \frac{h(1-\overline{\beta})}{M}\right] \leq \\
&\leq 2M\left[\frac{2k_1h}{2Mk_1} + \frac{h}{M} - \frac{2h\overline{\beta}}{M}\right] \leq 4h[1-\overline{\beta}] \leq 4\frac{1}{2}h = 2h,
\end{aligned}$$

i.e. $\|\xi(a,\varepsilon)\| \leq 2h$, which shows that the vector $\xi(a,\varepsilon)$ do not leave the domain Ω_1. Thus the theorem is proved.

The asymptotic series of the nonlinear problem (1), (2) satisfies

$$\lim_{\varepsilon\to 0} x(t,\varepsilon) = x_0(t), \quad t \in (a,b].$$

4. Example. Let $t \in [0,1]$, $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$ and the problem (1), (2) has the next coefficients

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad f(t,x,\varepsilon) = \begin{pmatrix} x_1^2+1 \\ x_2^2 \end{pmatrix}, \quad \varphi(t) = \begin{pmatrix} 2t-1 \\ 2t+1 \end{pmatrix},$$

$$lx(\cdot) \equiv Mx(0)+Nx(1) = h, \ M = \begin{pmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix},$$

From (10) for $x_0(t)$ and $x_1(t)$ is obtained

$$x_0(t) = \begin{pmatrix} 2t-3 \\ 2t-2 \end{pmatrix}, \quad x_1(t) = \begin{pmatrix} 4t^2-16t+14 \\ 4t^2-14t+11 \end{pmatrix}$$

For the fundamental matrix of solutions of the system $\dfrac{dx}{dt} = Ax$ is found

$$X(t) = \begin{pmatrix} 3-2e^{-t} & 2e^{-t}-2 \\ 3-3e^{-t} & 3e^{-t}-2 \end{pmatrix} e^{-t}, \ \ X^{-1}(t) = \begin{pmatrix} 3-2e^{t} & 2e^{t}-2 \\ 3-3e^{t} & 3e^{t}-2 \end{pmatrix} e^{t}.$$

The matrix $D(\varepsilon)$ from (14) has the form $D(\varepsilon) = MX(0) + NX\left(\dfrac{1}{\varepsilon}\right) = D_0 + O\left(\exp\left(-\dfrac{\alpha}{\varepsilon}\right)\right)$, where

$$D_0 \equiv M = \begin{pmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \end{pmatrix}, \text{ then } D_0^+ = \frac{1}{30}\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \end{pmatrix}, \ P = \frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix},$$

$$P^* = \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}.$$ It is clearly that $rankP = 1$ and $rankP^* = 2$,

$$\text{then } P_r \equiv P_1 = \frac{1}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}, P_d^* \equiv P_2^* = \frac{1}{6}\begin{pmatrix} 5 & 1 & -2 \\ 1 & 5 & 2 \end{pmatrix}.$$

The system (16), where $h_0 = (8 \ -8 \ 16)^T$, is solvable (in this case $P_2^* h_0 = 0$) and its solution is

$$c_0 = P_1 c_0^1 + D_0^+ h_0 = \frac{1}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix} c_0^1 + \frac{8}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Then $\Pi_0(\tau)$ from (19) takes on the form

$$\Pi_0(\tau) = \frac{e^{-\tau}}{5}\begin{pmatrix} -8-8c_0^1+2e^{-\tau}(8+3c_0^1) \\ -8-8c_0^1+3e^{-\tau}(8+3c_0^1) \end{pmatrix} = \begin{pmatrix} ae^{-\tau}+2be^{-2\tau} \\ ae^{-\tau}+3be^{-2\tau} \end{pmatrix},$$

$$\text{where } \ a = \frac{-8-8c_0^1}{5}, \ \ b = \frac{8+3c_0^1}{5}.$$

Through (26) and ignoring the exponentially small elements $O\left(\dfrac{e^{-\frac{1}{\varepsilon}}}{\varepsilon}\right)$, $O\left(e^{-\frac{2}{\varepsilon}}\right)$, $O\left(e^{-\frac{3}{\varepsilon}}\right)$, $O\left(e^{-\frac{4}{\varepsilon}}\right)$ the nonlinear system for defining c_0^r is obtained

$$-\frac{2}{15}e^{-\frac{1}{\varepsilon}}\begin{pmatrix} -460 \\ -224 \end{pmatrix} c_0^1 = \frac{1}{6}\begin{pmatrix} 35 \\ 19 \end{pmatrix} + \frac{8}{15}e^{-\frac{1}{\varepsilon}}\begin{pmatrix} -205 \\ -101 \end{pmatrix} + \frac{1}{6}e^{-\frac{1}{\varepsilon}}\begin{pmatrix} 24a^2-48b^2 \\ 12a^2-24b^2 \end{pmatrix},$$

(44) and it is found $c_0^1 = -1,66888 + 0,05843\varepsilon^{-1}$.

Thus $\Pi_0(\tau)$ is defined completely.

By the linear system

$$\frac{e^{-\frac{1}{\varepsilon}}}{30}\begin{pmatrix} 384a + 288b + 1840 \\ 192a + 144b + 896 \end{pmatrix} c_1^1 = \frac{1}{6}\begin{pmatrix} 61 \\ 29 \end{pmatrix} + \frac{1}{6}e^{\frac{1}{\varepsilon}}\begin{pmatrix} \frac{223}{15}(48b + 24a + 42) - \\ \frac{223}{15}(24b + 12a + 101) - \end{pmatrix}$$

$$\begin{matrix} -96b^3 - 120ab^2 - 884b^2 - 672ab + 72a^3 + 362a^2 - 3370b + 210a \\ -48b^3 - 60ab^2 - 436b^2 - 33ab + 36a^3 + 178a^2 - 1478b + 108a \end{matrix}\Bigg),$$

where a and b are the expression indicated above after c_0^1 from (44) is substituted and c_1^1 is determined

$$c_1^1 = -0,905432 - 0,082926\varepsilon^{-1}.$$

This shows that for $\Pi_1(\tau)$ is found

$$\Pi_1(\tau) = \frac{1}{5}\begin{pmatrix} 6e^{-\tau} - 8 \\ 9e^{-\tau} - 8 \end{pmatrix} e^{-\tau}(-0,905432 - 0,082926\varepsilon^{-1}) + \begin{pmatrix} 2e^{-\tau} - 1 \\ 3e^{-\tau} - 1 \end{pmatrix} e^{-\tau}B +$$

$$+\begin{pmatrix} -3b^2e^{-3\tau} - 4abe^{-2\tau} + (5b^2 - a^2 + 12b + 4ab - 4a)e^{-\tau} - \\ -\frac{11}{2}b^2e^{-3\tau} - 6abe^{-2\tau} + (\frac{15}{2}b^2 - a^2 + 12b + 6ab - 6a)e^{-\tau} - \end{pmatrix}$$

$$\begin{matrix} -10a\tau + a^2 - 2b^2 - 12b + 4a \\ -10a\tau + a^2 - 2b^2 - 12b + 6a \end{matrix}\Bigg) e^{-\tau},$$

where

$$B = \frac{1}{30}\left(-223 + \left(\frac{61}{2}b^2e^{-\frac{3}{\varepsilon}} + 34abe^{-\frac{2}{\varepsilon}} - \left(\frac{85}{2}b^2 - 6a^2 + 72b + 34ab - 34a\right)e^{-\frac{1}{\varepsilon}} + \right.\right.$$

$$\left.\left. + \frac{60a}{\varepsilon} - 6a^2 + 12b^2 + 72b - 34a\right) e^{-\frac{1}{\varepsilon}}\right).$$

For the solution is obtained

$$x(t,\varepsilon) = \begin{pmatrix} 2t - 3 \\ 2t - 2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 6e^{-\frac{t}{\varepsilon}} - 8 \\ 9e^{-\frac{t}{\varepsilon}} - 8 \end{pmatrix} e^{-\frac{t}{\varepsilon}}(-1,66888 + 0,05843\varepsilon^{-1}) +$$

$$+\frac{8}{5}\begin{pmatrix} 2e^{-\frac{t}{\varepsilon}}-1 \\ 3e^{-\frac{t}{\varepsilon}}-1 \end{pmatrix} e^{-\frac{t}{\varepsilon}} + \varepsilon\left(\begin{pmatrix} 4t^2-16t+14 \\ 4t^2-14t+11 \end{pmatrix}\right.$$

$$+\frac{1}{5}\begin{pmatrix} 6e^{-\frac{t}{\varepsilon}}-8 \\ 9e^{-\frac{t}{\varepsilon}}-8 \end{pmatrix} e^{-\frac{t}{\varepsilon}}(-0,905432-0,082926\varepsilon^{-1}) + \begin{pmatrix} 2e^{-\frac{t}{\varepsilon}}-1 \\ 3e^{-\frac{t}{\varepsilon}}-1 \end{pmatrix} e^{-\frac{t}{\varepsilon}}B+$$

$$+\left(\begin{array}{c} -3b^2e^{-3\frac{t}{\varepsilon}}-4abe^{-2\frac{t}{\varepsilon}}+(5b^2-a^2+12b+4ab-4a)e^{-\frac{t}{\varepsilon}}- \\ -\frac{11}{2}b^2e^{-3\frac{t}{\varepsilon}}-6abe^{-2\frac{t}{\varepsilon}}+(\frac{15}{2}b^2-a^2+12b+6ab-6a)e^{-\frac{t}{\varepsilon}}- \end{array}\right. +$$

$$\left.\left.\begin{array}{c} -10a\tau+a^2-2b^2-12b+4a \\ -10a\tau+a^2-2b^2-12b+6a \end{array}\right) e^{-\frac{t}{\varepsilon}}\right) + O(\varepsilon^2),$$

where $a = 1,0702 - 0,0934928\varepsilon^{-1}$, $b = 0,5986864 + 0,0350598\varepsilon^{-1}$.

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