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Subordination principle for generalized fractional evolution equations

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Preface

Fractional Calculus, as an extension of the classical Calculus, started with the ideas of Gottfried Leibniz by the end of the XVII century and had been developed progressively up to now. During the recent decades Fractional Calculus attracted the attention of many researchers in different areas, such as mathematics, physics, biology, chemistry, engineering, social sciences. The main reason is that the differential and integral equations of fractional order can model mathematically various natural and industrial processes more adequately than these restricted to integer order. In particular, Fractional Calculus has become a frequently used tool in hereditary physics due to the efficiency of fractional differential equations in the modelling of anomalous diffusion or wave phenomena when some memory mechanisms of power-law or logarithmic type are present.

Fractional partial differential equations are widely used to capture the power-law dependence on time of the mean squared displacement in anomalous diffusion processes. However, most of the complex systems do not show a mono-scaling behavior. Instead, transitions between different diffusion regimes in course of time are observed. One way to model such a multi-scaling behavior is by replacing the single operator of fractional derivative by more general integro-differential operators with specific memory kernels. This leads to the so-called generalized fractional diffusion and diffusion-wave equations.

The present dissertation is devoted to the study of subordination principle for generalized fractional evolution equations. By means of a subordination principle, it is possible to construct solutions of such evolution equations from the solutions of classical integer order equations, or simpler fractional order ones. It is a useful tool for establishing well-posedness, for deriving integral representations of the solutions, and for the study of their regularity, asymptotic behavior, and other properties. Moreover, the subordination principle defines a hierarchy in the variety of generalized fractional evolution equations, which is essential for the proper classification and understanding of the related mathematical models.

The main tools in the present study are the theory of Fractional Calculus operators and special functions, Laplace transform, and the properties of Bernstein functions and related classes of functions.

This dissertation is an outcome of the author's research work during the past seven years (2015-2021). It is based on 11 articles, published in this period: [10]-[15], [18]-[20], [22], and [25]. In the outline below for every chapter we indicate which of these publications have been used.

The dissertation is organized as follows. The Introduction contains a short overview on subordination principles and motivation for the present study. **Chapter 1** contains notations, definitions and basic properties of fractional integration and differentiation operators, Laplace transform, Mittag-Leffler functions and functions of Wright type. In **Chapter 2**, after an introduction to Bernstein functions and some background material on abstract Volterra equations, we prove two general subordination theorems. **Chapter 3**, [14, 20], is devoted to a detailed study of subordination principle for space-time fractional evolution equations. As an application, a number of explicit expressions in terms of special functions and integral representations for the solutions are derived. The rest of the thesis is concerned with generalized time-fractional evolution equations. To demonstrate the crucial role of the subordination principle in the study of this class of equations, the fractional Jeffreys' heat conduction equation is considered as a model problem in **Chapter 4**, [22]. In **Chapter 5**, [10, 11, 12], we establish subordination results for the subdiffusion equation of distributed order in time and for more general subdiffusion equation with memory kernel. As an application, useful estimates are derived in the scalar case. **Chapter 6**, [15], is concerned with a multinomial generalization of the Mittag-Leffler function, which is related to relaxation equations with multiple time-derivatives. The last two chapters consider equations describing phenomena intermediate between diffusion and wave propagation. In **Chapter 7**, [13, 18], an open problem concerning positivity of the fundamental solution for distributed-order time-fractional diffusion-wave equations is discussed and partly answered. Positivity of the fundamental solution is necessary for physical acceptability of the model, as well as for the proof of subordination principle. **Chapter 8**, [13, 19, 25], is concerned with equations governing wave propagation in viscoelastic media with completely monotone relaxation moduli. The particular case of fractional Jeffreys' fluid is studied in detail and the physical meaning of the subordination formula is discussed. The dissertation ends with concluding remarks.

Contents

Introduction

During the recent decades Fractional Calculus attracted the attention of many researchers [73]. Evolution equations with fractional derivatives are extensively used for modelling of materials and processes with memory. In the attempt to find more adequate models, linear equations involving discrete or continuous distribution of fractional derivatives, or more general integro-differential operators of convolutional type, are introduced [62, 97, 99]. This raises the need of methods for study and proper classification of the variety of generalized fractional evolution equations. One useful tool to achieve this goal is the so-called *principle of subordination*.

The original subordination principle for stochastic processes in connection with diffusion equations and semigroups was introduced by S. Bochner in 1949 [28]. A detailed study of stochastic processes, their transition semigroups, generators, and subordination results can be found in [29], Chapters 4.3 and 4.4.

Bernstein functions play an essential role in the definition of Bochner subordination. A function $\phi : (0, \infty) \to \mathbb{R}$ is a Bernstein function if ϕ is of class C^∞, $\phi(s) \geq 0$ for all $s > 0$ and

$$(-1)^{n-1}\phi^{(n)}(s) \geq 0 \ \text{ for all } \ n \in \mathbb{N} \ \text{ and } \ s > 0.$$

If ϕ is a Bernstein function then it admits a continuous extension to the half-plane $\Re s \geq 0$, which is holomorphic for $\Re s > 0$, and satisfies $\Re\phi(s) > 0$ for all $\Re s > 0$. A basic example of a Bernstein function is $\phi(s) = s^\alpha$, $0 \leq \alpha \leq 1$.

Consider a family of functions $\{p_t(\tau)\}$ indexed by $t \geq 0$ and defined on $\tau \geq 0$, and such that for each fixed $t > 0$, $p_t(\tau)$ is a probability density function on $\tau \geq 0$, that is,

$$p_t(\tau) \geq 0 \quad \text{and} \quad \int_0^\infty p_t(\tau)\, d\tau = 1. \tag{0.1}$$

For fixed $t > 0$ the Laplace transform of $p_t(\tau)$ is defined by

$$\mathcal{L}\{p_t\}(s) = \int_0^\infty e^{-s\tau} p_t(\tau)\, d\tau, \quad \Re s > 0.$$

A Bochner subordinator is a family $\{p_t\}_{t>0}$ as defined above, such that

$$\mathcal{L}\{p_t\}(s) = e^{-t\phi(s)}, \quad \Re s > 0,$$

where ϕ is a Bernstein function.

A subordinator example which yields a closed form expression is the following

$$p_t(\tau) = \frac{te^{-t^2/4\tau}}{2\sqrt{\pi}\tau^{3/2}}, \qquad \mathcal{L}\{p_t\}(s) = e^{-t\sqrt{s}}.$$

It is the special case $\alpha = 1/2$ of the important Lévy subordinator family of index α with corresponding Bernstein function $\phi(s) = s^\alpha$, where $0 < \alpha \leq 1$.

Let A be a closed linear operator densely defined on a Banach space X, which generates a C_0-semigroup $S_1(t)$. Then the first order abstract Cauchy problem

$$u'(t) = Au(t), \quad t > 0; \quad u(0) = v \in X, \tag{0.2}$$

is well posed and the solution $u(t)$ is given by $u(t) = S_1(t)v$. Let $\{p_t\}_{t>0}$ be a Bochner subordinator with corresponding Bernstein function ϕ. Then the Bochner integral

$$S_1^\phi(t)v = \int_0^\infty p_t(\tau)S_1(\tau)v\,d\tau, \quad v \in X, \quad t > 0,$$

defines again a C_0-semigroup on X. The semigroup $S_1^\phi(t)$ is called subordinate (in the sense of Bochner) to the semigroup $S_1(t)$ with respect to the Bernstein function ϕ. The semigroup $S_1^\phi(t)$ is generated by the operator $-\phi(-A)$ [88]. In particular, if $\phi(s) = s^\alpha$, then $S_1^\phi(t)$ is an analytic semigroup generated by the operator $-(-A)^\alpha$, where the fractional power is understood in the sense of Balakrishnan [8] (for more details see [108], Chapter 9). In this way, the subordination principle in the sense of Bochner gives the possibility to construct from $S_1(t)$ new semigroups, which define the solutions of the Cauchy problem (0.2) when the operator A is replaced by a new operator $-\phi(-A)$. For details on subordination and Bochner's functional calculus we refer to [101], Chapter 13.

Another type of subordination formulae establishes a relation between the solutions of two Cauchy problems with the same operator A, but different operators acting with respect to the time variable t. For instance, there is always a simple way to go from the second order Cauchy problem to the first order one. Assume the operator A generates a strongly continuous cosine family

$S_2(t)$, $t > 0$, on a Banach space X (see e.g. [2], Section 3.14). This is equivalent to well-posedness of the second order Cauchy problem

$$u''(t) = Au(t), \quad t > 0; \quad u(0) = v \in X, \; u'(0) = 0, \tag{0.3}$$

which solution is given by $u(t) = S_2(t)v$ for $t > 0$. If A generates a cosine family $S_2(t)$ then A generates a holomorphic C_0-semigroup $S_1(t)$ of angle $\pi/2$, which is related to the cosine family $S_2(t)$ by the abstract Weierstrass formula ([2], Theorem 3.14.17)

$$S_1(t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\tau^2/(4t)} S_2(\tau)\, d\tau, \quad t > 0. \tag{0.4}$$

This formula allows one to compute the solution of the first order Cauchy problem (0.2) from the solution of the second order one (0.3) and shows that the subordinate solution possesses better regularity. The subordination relation (0.4) was generalized in [9] to the case of fractional evolution equations.

Denote by ${}^C D_t^\alpha$ the Caputo fractional derivative of order $\alpha > 0$

$${}^C D_t^\alpha u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(\tau)\, d\tau,$$

where m is a positive integer, such that $m - 1 < \alpha \leq m$. Consider the abstract Cauchy problem for the fractional evolution equation with a general linear closed operator A densely defined on a Banach space X

$${}^C D_t^\alpha u(t) = Au(t), \quad t > 0, \quad 0 < \alpha \leq 2, \tag{0.5}$$

supplemented with the initial conditions $u(0) = v \in X$ for $0 < \alpha \leq 1$ and $u(0) = v \in X$, $u'(0) = 0$ for $1 < \alpha \leq 2$. Denote by $S_\alpha(t)$ the solution operator corresponding to problem (0.5), i.e. the solution of (0.5) is given by $u(t) = S_\alpha(t)v$. For $\alpha = 1$ and $\alpha = 2$ the corresponding solution operators are respectively the C_0-semigroup of operators $S_1(t)$ and the strongly continuous cosine family $S_2(t)$, generated by the operator A.

The subordination principle for the abstract Cauchy problem (0.5) states that if problem (0.5) is well posed for some $\alpha^* \in (0, 2]$ then it is well posed for all $\alpha \in (0, \alpha^*)$ and the corresponding solution operators S_α and S_{α^*} are related via the subordination identity

$$S_\alpha(t) = \int_0^\infty \varphi_{\alpha/\alpha^*}(t, \tau) S_{\alpha^*}(\tau)\, d\tau, \quad t > 0, \tag{0.6}$$

where $\varphi_\gamma(t,\tau) = t^{-\gamma}M_\gamma(\tau t^{-\gamma})$ with $M_\gamma(\cdot)$ being a function of Wright type defined by the series

$$M_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1, \quad z \in \mathbb{C}. \tag{0.7}$$

For any fixed $t > 0$ the subordination kernel $\varphi_\gamma(t,\tau)$ is a probability density function on $\tau \geq 0$, i.e.

$$\varphi_\gamma(t,\tau) \geq 0, \quad \int_0^\infty \varphi_\gamma(t,\tau)\, d\tau = 1, \qquad t,\tau > 0.$$

If S_{α^*} is a bounded solution operator, then S_α is a bounded analytic solution operator in some sector of the complex plane.

Therefore, the main idea of the subordination principle for problem (0.5) is that one and the same operator A guarantees better properties of the solution when α is smaller and that the set of operators A, for which (0.5) is well posed, shrinks when α increases. In particular, if there exists an exponentially bounded solution operator for $\alpha > 2$, then A is necessarily a bounded operator [9]. For this reason we consider only $\alpha \in (0,2]$.

Subordination principle for fractional evolution equations has found various applications, e.g. in the study of inverse problems [84], for asymptotic analysis of diffusion wave equations [63], for the study of stochastic solutions [80], semilinear equations of fractional order [57], systems of fractional order equations [54]. Based on the subordination principles for space- and time-fractional diffusion equations and the dominated convergence theorem, exact asymptotic expressions for the fundamental solution of the multi-dimensional space-time fractional diffusion equation are established recently in [34]. Stochastic interpretation of the subordination principle for fractional evolution equations is discussed in [48, 77, 80, 97]. Other useful applications can be found in [110].

It is worth noting that the subordination relation (0.6), considered as an integral transform

$$\mathcal{S}f(t) = \int_0^\infty \varphi_\gamma(t,\tau) f(\tau)\, d\tau, \quad t > 0,$$

appears to be a particular case of an integral transform introduced by Stanković in [104]. For a recent study on related classes of integral transforms we refer to [94]. Moreover, the subordination relations for time-fractional evolution equations can be placed in the general context of parameter-shift formulas and integral transforms composition method as in [40].

Subordination principle for abstract Volterra evolution equations

$$u(t) = v + \int_0^t k(t-\tau)Au(\tau)\,d\tau + f(t), \qquad t > 0, \quad v \in X, \tag{0.8}$$

is studied in [93], Chapter 4, by employing the notion of completely positive kernels.

In general, a subordination principle consists of the following: Given two Cauchy problems, (P) and (P_*), problem (P) is called subordinated to problem (P_*) if and only if well-posedness of problem (P_*) implies well-posedness of problem (P) and the solution operator $S(t)$ of problem (P) admits the integral representation

$$S(t) = \int_0^\infty \varphi(t,\tau)S_*(\tau)\,d\tau, \quad t > 0,$$

where $S_*(t)$ is the solution operator of problem (P_*) and $\varphi(t,\tau)$ is a probability density function (PDF) in $\tau \geq 0$ when $t > 0$ is considered as a parameter, that is

$$\varphi(t,\tau) \geq 0, \quad \int_0^\infty \varphi(t,\tau)\,d\tau = 1. \tag{0.9}$$

An important particular case of (0.5) is the time-fractional diffusion equation, where $0 < \alpha < 1$ and A is some realization of the Laplace operator. It was derived via the framework of a continuous time random walk under the assumption that the mean waiting time has a power-law decaying tail proportional to t^α, $\alpha \in (0,1)$. The solution of this equation accurately describes the power-law decaying behavior in a large number of anomalous diffusion processes. To improve the modeling accuracy, evolution equations with multiple time-derivatives or time-derivatives of distributed order are proposed, which permit to describe also processes whose scaling law changes with time [97, 98]. Generalized diffusion equations with different memory kernels are popular mathematical tools for description of a variety of non-Fickian diffusion processes. The relation between generalized diffusion equations and subordination schemes is recently discussed in [33]. On the other hand, generalized diffusion-wave equations emerge in the modeling of wave propagation in viscoelastic media [75, 109]. As a result, various linear generalizations of the single-order diffusion-wave equation (0.5), which involve fractional time derivatives in their formulation, have been proposed in the literature.

The useful applications of the subordination principle for fractional evolution equations (0.5), mentioned above, give the author the motivation to study the subordination principle for different types of generalized linear fractional evolution equations. In the present dissertation we develop a methodology for establishing subordination relations, which are helpful for the classification and understanding of a variety of mathematical models, which use fractional derivatives in their formulation. Several equations of this type are analyzed with the subordination principle as the unifying theme. The considered equations can be divided in three classes:

(I) space-time fractional evolution equations;

(II) evolution equations, subordinated to the first-order equation (0.2);

(III) evolution equations, subordinated to the second-order equation (0.3), which do not belong to the class (II).

Class (I) consists of fractional evolution equations (0.5) with $\alpha \in (0, 1)$, in which the operator A is replaced by the operator $-(-A)^{\beta}$, where $\beta \in (0, 1)$, i.e. they are fractional equations simultaneously in space and in time. When the operator A is the second-order space-derivative, or a multidimensional Laplace operator, or a more general elliptic operator, then the classical first-order problem (0.2) is a mathematical model of diffusion and the classical second-order problem (0.3) is a mathematical model of wave propagation. Therefore, for convenience, in this dissertation we use the notions generalized subdiffusion equations and generalized diffusion-wave equations for the equations of groups (II) and (III), respectively.

For a unified approach, the considered equations are represented in the form of Volterra integral equation (0.8). As in the case of the original Bochner subordination, the proofs are based on the theory of Bernstein functions [101] and Laplace transform. Various applications of the derived subordination relations are presented: integral representations for the solutions of the considered problems, closed-form solutions in particular cases, analysis of regularity, asymptotic behaviour, monotonicity; visualization of the solution behaviour; estimates for the solutions of the scalar equations, which are useful when boundary-value problems are studied applying eigenfunction expansion technique.

Chapter 1

Fractional calculus operators and special functions

This chapter contains preliminaries used throughout the whole dissertation. The operators of fractional integration and differentiation of Riemann-Liouville and Caputo type are introduced, as well as some special functions intimately related to fractional calculus: Mittag-Leffler function and its Prabhakar generalization, Mainardi function and the Lévy extremal stable density.

1.1 Some notations and definitions

The sets of positive integers, real, and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{C}_+ = \{z \in \mathbb{C},\ \Re z > 0\}$.

By $\Sigma(\theta)$ we denote the open sector in $\mathbb{C}$

$$\Sigma(\theta) = \{z \in \mathbb{C} \backslash \{0\}; |\arg z| < \theta\}, \quad \theta \in (0, \pi).$$

For the multivalued complex functions, considered in this dissertation, such as $\log z$ or $z^\alpha = \exp(\alpha \log z)$, we take the principal branch.

Let X be a Banach space with norm $\|.\|$. Assume $-\infty \le a < b \le +\infty$ and $1 \le p < \infty$. Then $L^p((a, b); X)$ denotes the space of all (equivalent classes of) Bochner-measurable functions $f : (a, b) \to X$, such that $\|f(t)\|^p$ is integrable for $t \in (a, b)$. It is a Banach space with norm

$$\|f\|_{L^p((a,b);X)} = \left(\int_a^b \|f(\tau)\|^p \, d\tau \right)^{1/p}.$$

Let $m \in \mathbb{N}$. We denote by $C([a,b];X)$ and $C^m([a,b];X)$ the spaces of functions $f : [a,b] \to X$, which are continuous, resp. m-times continuously differentiable, endowed respectively with the norms

$$\|f\|_C = \sup_{t\in[a,b]} \|f\|, \quad \|f\|_{C^m} = \sup_{t\in[a,b]} \sum_{k=0}^{m} \|f^{(k)}(t)\|.$$

Let $I = (0,T)$, $I = \mathbb{R}_+$, or $I = \mathbb{R}$, $m \in \mathbb{N}$, $1 \le p < \infty$. The Sobolev spaces can be defined in the following way

$$W^{m,p}(I;X) = \left\{ f|\exists\varphi \in L^p(I;X) : f(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + \frac{t^{m-1}}{(m-1)!} * g(t), t \in I \right\},$$

where $*$ is the Laplace convolution

$$(f_1 * f_2)(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau.$$

Note that in the defining representation of functions $f \in W^{m,p}(I;X)$ it holds $c_k = f^{(k)}(0)$ and $g(t) = f^{(m)}(t)$. Denote

$$W_0^{m,p}(I;X) = \left\{ f \in W^{m,p}(I;X)|\ f^{(k)}(0) = 0,\ k = 0,1,\dots,m-1 \right\}.$$

If X is the scalar field $\mathbb{R}$ or $\mathbb{C}$, then the image space in the notations of the function spaces defined above will be dropped.

1.2 Laplace transform

Denote by $L^1_{loc}(\mathbb{R}_+;X)$ the space of functions $f : \mathbb{R}_+ \to X$, integrable in the sense of Bochner on any interval $[0,\tau]$, $\tau > 0$.

The Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+;X)$ is defined by

$$\mathcal{L}\{f(t)\}(s) = \widehat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad \Re s > 0,$$

if the integral is absolutely convergent for $\Re s > 0$.

A real-valued infinitely differentiable on $\mathbb{R}_+$ function $f(t)$ is said to be a completely monotone function ($\mathcal{CMF}$) if

$$(-1)^n f^{(n)}(t) \geq 0, \quad t > 0, \quad n \in \mathbb{N}_0. \tag{1.1}$$

The characterization of the class $\mathcal{CMF}$ is given by the Bernstein's theorem which states that a function is completely monotone if and only if it can be represented as the Laplace transform of a non-negative measure (non-negative function or generalized function).

Next the Post-Widder inversion formula for the Laplace transform is formulated in the general case of X valued functions (see e.g. [2]):

Theorem 1.1. *Let $f(t)$, $t \geq 0$, be a X valued continuous function, such that $f(t) = O(e^{\gamma t})$ as $t \to \infty$ for some real γ. Then*

$$f(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} \left(\frac{d^n \widehat{f}}{ds^n}\right) \left(\frac{n}{t}\right) \tag{1.2}$$

uniformly on compact subsets of $\mathbb{R}_+$.

The asymptotic behaviour of a function $f(t)$ as $t \to \infty$ can be determined by looking at the behaviour of its Laplace transform $\widehat{f}(s)$ as $s \to 0$. The following version of the Karamata-Feller Tauberian theorem establishes such a correspondence, see [38], Chapter XIII.

Denote by $\omega_\alpha(t)$ the function

$$\omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha > 0, \quad t > 0. \tag{1.3}$$

The Laplace transform of this function satisfies the identity

$$\mathcal{L}\{\omega_\alpha(t)\}(s) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-st} t^{\alpha-1}\, dt = s^{-\alpha}, \quad \alpha > 0, \quad \Re s > 0. \tag{1.4}$$

Theorem 1.2. [38] *Let $L : \mathbb{R}_+ \to \mathbb{R}_+$ be a function that is slowly varying at ∞, that is, for every fixed $x > 0$ we have $L(tx)/L(t) \to 1$ as $t \to \infty$. Let $\alpha > 0$ and $f : \mathbb{R}_+ \to \mathbb{R}$ be a nonnegative function, which Laplace transform $\widehat{f}(s)$ exists for all $s \in \mathbb{C}_+$. Then*

$$\widehat{f}(s) \sim \frac{1}{s^\alpha} L\left(\frac{1}{s}\right) \quad \textit{as} \quad s \to 0$$

if and only if

$$f(t) \sim \omega_\alpha(t)L(t) \quad as \quad t \to \infty.$$

Here the function $\omega_\alpha(t)$ is defined in (1.3) and the approaches are on the positive real axis.

Here and in what follows the notation $f(t) \sim g(t)$ as $t \to t_*$ means that $\lim_{t\to t_*} f(t)/g(t) = 1$.

The following characterization of functions which are holomorphic and bounded in a sector of the complex plane is useful, see [93], Theorem 0.1.

Theorem 1.3. [93] *Let F be a function defined on $(0, \infty)$ and $\theta_0 \in (0, \pi/2]$. Then the assertions* **(i)** *and* **(ii)** *are equivalent:*
(i) *$F(s)$ admits holomorphic extension to the sector $|\arg s| < \pi/2 + \theta_0$ and $sF(s)$ is bounded on each sector $|\arg s| \le \pi/2 + \theta$, $\theta < \theta_0$;*
(ii) *there is a function $f(t)$ holomorphic for $|\arg t| < \theta_0$ and bounded on each sector $|\arg t| \le \theta < \theta_0$, such that $F(s) = \widehat{f}(s)$ for each $s > 0$.*

1.3 Fractional integration and differentiation

The Euler gamma function $\Gamma(z)$ is defined by the identity

$$\Gamma(z) = \int_z^\infty \xi^{a-1} e^{-\xi}\, d\xi, \quad \Re z > 0.$$

For this function the reduction formula $\Gamma(z+1) = z\Gamma(z)$ holds. In particular, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. The formula for the n-fold iterated integral ($n \in \mathbb{N}$, $t > 0$) reads

$$J_t^n f(t) = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} f(\tau_n)\, d\tau_n = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau.$$

Let $I = (0, T)$ for some $T > 0$ and let X be a Banach space. The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}_+$ is defined as a generalization of the above formula for the n-fold iterated integral as follows

$$J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau = (\omega_\alpha * f)(t), \qquad \alpha > 0, \quad t > 0,$$

for $f \in L^1(I; X)$. Let us set $J_t^0 f(t) = f(t)$. The fractional order integral operators obey the semigroup property

$$J_t^\alpha J_t^\beta = J_t^{\alpha+\beta}, \quad \alpha \ge 0, \ \beta \ge 0. \tag{1.5}$$

Let $\alpha > 0$ and $0 \leq m-1 < \alpha \leq m$, $m \in \mathbb{N}$, and use the notation $D_t^m = \frac{\mathrm{d}^m}{\mathrm{d}t^m}$. The fractional Riemann-Liouville derivative D_t^α of order $\alpha > 0$ is defined for functions $f \in L^1(I;X)$, such that $J_t^{m-\alpha} f \in W^{m,1}(I;X)$ as follows

$$D_t^\alpha f(t) = D_t^m J_t^{m-\alpha} f(t) = D_t^m (\omega_{m-\alpha} * f)(t).$$

The fractional Caputo derivative ${}^C D_t^\alpha$ of order $\alpha > 0$ is defined in the same class of functions by the relation

$${}^C D_t^\alpha f(t) = D_t^\alpha \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \omega_{k+1}(t) \right). \tag{1.6}$$

For functions $f \in W^{m,1}$ the fractional Caputo derivative admits the alternative representation

$${}^C D_t^\alpha f(t) = J_t^{m-\alpha} D_t^m f(t) = (\omega_{m-\alpha} * D_t^m f)(t). \tag{1.7}$$

Note that the subscript t in the notations of the fractional integration and differentiation operators defined above emphasizes that the operators act with respect to the time variable.

The following basic identities are satisfied

$$\begin{aligned} {}^C D_t^\alpha J_t^\alpha &= D_t^\alpha J_t^\alpha = I, &(1.8)\\ J_t^\alpha D_t^\alpha f(t) &= f(t) - \sum_{k=0}^{m-1} (\omega_{m-\alpha} * f)^{(k)}(0)\, \omega_{\alpha+k+1-m}(t), &(1.9)\\ J_t^\alpha\, {}^C D_t^\alpha f(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0)\, \omega_{k+1}(t). &(1.10) \end{aligned}$$

Some simple but relevant results valid for $\alpha, \beta, t > 0$ are

$$J_t^\alpha \omega_\beta(t) = \omega_{\alpha+\beta}(t); \quad D_t^\alpha \omega_\beta(t) = \omega_{\beta-\alpha}(t),\ \beta > \alpha.$$

In particular, $D_t^\alpha 1 = \omega_{1-\alpha}(t)$ for $0 < \alpha < 1$, while ${}^C D_t^\alpha 1 = 0$ for any $\alpha > 0$.

The Laplace transform of fractional order operators obeys the identities

$$\begin{aligned} \mathcal{L}\{J_t^\alpha f\}(s) &= s^{-\alpha} \mathcal{L}\{f\}(s), &(1.11)\\ \mathcal{L}\{D_t^\alpha f\}(s) &= s^\alpha \mathcal{L}\{f\}(s) - \sum_{k=0}^{m-1} (\omega_{m-\alpha} * f)^{(k)}(0)\, s^{m-1-k}, &(1.12)\\ \mathcal{L}\{{}^C D_t^\alpha f\}(s) &= s^\alpha \mathcal{L}\{f\}(s) - \sum_{k=0}^{m-1} f^{(k)}(0)\, s^{\alpha-1-k}. &(1.13) \end{aligned}$$

In their derivation the Laplace transform pair (1.4) is used.

Let $0 < \alpha < 1$. Then identity (1.6) reads

$$ {}^{C}D_t^{\alpha} f(t) = D_t^{\alpha} \left(f(t) - f(0)\right) = D_t^{\alpha} f(t) - f(0)\omega_{1-\alpha}(t). \tag{1.14}$$

Moreover, (1.9) and (1.10) imply

$$\begin{aligned} J_t^{\alpha} D_t^{\alpha} f(t) &= f(t) - \left(\omega_{1-\alpha} * f\right)(0), & (1.15)\\ J_t^{\alpha}\, {}^{C}D_t^{\alpha} f(t) &= f(t) - f(0), & (1.16)\end{aligned}$$

and (1.12) and (1.13) read

$$\begin{aligned} \mathcal{L}\{D_t^{\alpha} f\}(s) &= s^{\alpha}\mathcal{L}\{f\}(s) - \left(\omega_{1-\alpha} * f\right)(0), & (1.17)\\ \mathcal{L}\{{}^{C}D_t^{\alpha} f\}(s) &= s^{\alpha}\mathcal{L}\{f\}(s) - f(0)s^{\alpha-1}. & (1.18)\end{aligned}$$

Let us note that if $f \in C([0,T])$ then $\left(\omega_{1-\alpha} * f\right)(0) = 0$ and the Laplace transform pair for the Riemann-Liouville derivative reduces to:

$$\mathcal{L}\{D_t^{\alpha} f\}(s) = s^{\alpha}\mathcal{L}\{f(t)\}(s). \tag{1.19}$$

For more details on fractional calculus operators we refer to [47, 58, 75, 89].

1.4 Mittag-Leffler functions

The classical Mittag-Leffler function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, z \in \mathbb{C},$$

is an entire function, introduced and studied by Gösta M. Mittag-Leffler at the beginning of 20-th century. The Mittag-Leffler function provides a simple generalization of the exponential function, $E_1(z) = e^z$. Other notable particular cases are

$$E_2(-z^2) = \cos z, \quad E_2(z^2) = \cosh z, \quad E_{1/2}\left(\pm z^{1/2}\right) = e^z \mathrm{erfc}\left(\mp z^{1/2}\right),$$

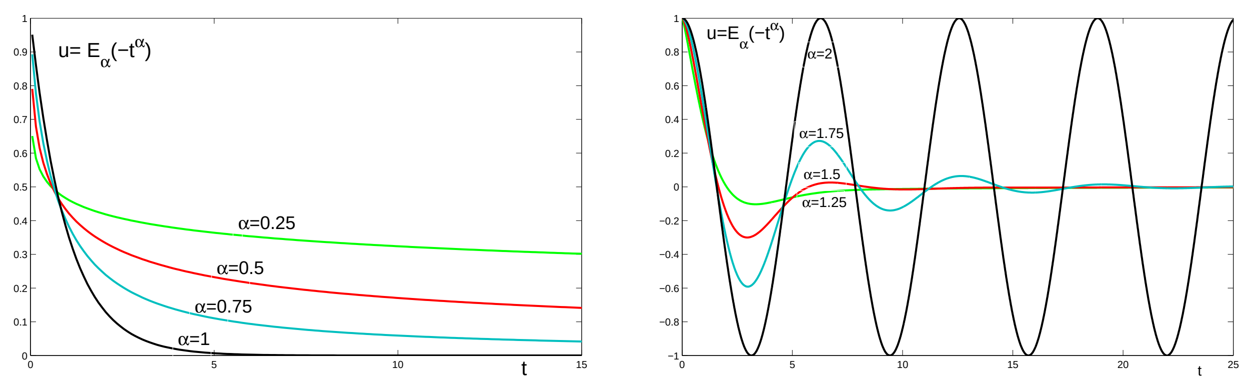
where $\mathrm{erfc}(\cdot)$ denotes the complementary error function

$$\mathrm{erfc}(z) = 1 - \mathrm{erf}(z),\ \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2}\, d\xi, \quad z \in \mathbb{C},$$

The function $u(t) = E_\alpha(-\lambda t^\alpha)$ is the solution of the ordinary fractional differential equation

$$ {}^C D_t^\alpha u(t) = -\lambda u(t), \quad \lambda > 0,\ t > 0, \tag{1.20}$$

with initial conditions $u(0) = 1$, $u^{(k)}(0) = 0$, $k = 1, \ldots, m-1$, where $m-1 < \alpha \le m$. Equation (1.20) is referred to as fractional relaxation equation for $\alpha \in (0,1)$ and fractional relaxation-oscillation equation for $\alpha \in (1,2)$. In comparison to ordinary relaxation ($\alpha = 1$), fractional relaxation exhibits a slower decay for large times (algebraic decay in comparison to exponential decay). Compared to the ordinary oscillation ($\alpha = 2$), the solution of the fractional relaxation-oscillation equation does not exhibit permanent oscillations, but an asymptotic algbraic decay. There are some attenuated oscillations, whose number increases with α. Therefore, we bserve features intermediate between relaxation and oscillation. For illustration of this behavior see Figure 1.1.



(**a**) (**b**)

Figure 1.1: Function $E_\alpha(-t^\alpha)$: (**a**) $0 < \alpha \le 1$; (**b**) $1 < \alpha \le 2$.

1.4.1 Mittag-Leffler function with two parameters

The two-parameter Mittag-Leffler function is an entire function defined by the series representation

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta \in \mathbb{R},\ z \in \mathbb{C}. \tag{1.21}$$

It is a generalization of the one parameter Mittag-Leffler function

$$E_\alpha(z) = E_{\alpha,1}(z),$$

and

$$E_{1,2}(z) = \frac{e^z - 1}{z}, \quad E_{2,2}(-z^2) = \frac{\sin z}{z}, \quad E_{2,2}(z^2) = \frac{\sinh z}{z}.$$

For $0 < \alpha < 2$ and $\beta > 0$ the following asymptotic expansions hold as $|z| \to \infty$

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), & |\arg z| \le \mu, \\ \varepsilon_{\alpha,\beta}(z), & \mu \le |\arg z| \le \pi, \end{cases} \tag{1.22}$$

where μ is such that $\alpha\pi/2 < \mu < \min\{\pi, \alpha\pi\}$ and

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{k=1}^{N-1} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O(|z|^{-N}), \quad |z| \to \infty.$$

The asymptotic expansion (1.22) implies the estimate ([89], Theorem 1.6)

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1 + |z|}, \quad \mu \le |\arg z| \le \pi. \tag{1.23}$$

Moreover, taking into account the identity $\Gamma(-n)^{-1} = 0$ for $n \in \mathbb{N}_0$, we derive from (1.22) two useful asymptotic expressions for $|z| \to \infty$ and $|\arg z| < (1 - \alpha/2)\pi$

$$\begin{aligned} E_\alpha(-z) &\sim \frac{z^{-1}}{\Gamma(1-\alpha)}, \quad \alpha > 0, \\ E_{\alpha,\beta}(-z) &\sim -\frac{z^{-2}}{\Gamma(\beta - 2\alpha)}, \quad \alpha > 0, \ \beta - \alpha = 0, -1, -2, ... \end{aligned} \tag{1.24}$$

The relations

$$\frac{d}{dz} E_\alpha(-z^\alpha) = -z^{\alpha-1} E_{\alpha,\alpha}(-z^\alpha), \quad z \in \mathbb{C} \backslash (-\infty, 0], \tag{1.25}$$

can be derived directly from the definition (1.21) of the Mittag-Leffler function.

The Mittag-Leffler function of real negative argument is completely monotone under some restrictions of the parameters. More precisely, it satisfies the property [83]:

$$E_{\alpha,\beta}(-t) \in \mathcal{CMF} \text{ for } t > 0 \quad \text{iff} \quad 0 < \alpha \leq 1,\ \beta \geq \alpha. \tag{1.26}$$

The Laplace transform of the function of Mittag-Leffler type $t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})$, $t > 0$, is given by

$$\mathcal{L}\left\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})\right\}(s) = \frac{s^{\alpha-\beta}}{s^{\alpha}+\lambda}, \quad t > 0. \tag{1.27}$$

It is relevant to point out the following representation of the functions of Mittag-Leffler type $t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})$ (excluding the case $\alpha = \beta = 1$) as Laplace transform [47]

$$t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha}) = \int_0^{\infty} e^{-rt}P_{\alpha,\beta}(r;\lambda)\,dr, \tag{1.28}$$

where

$$P_{\alpha,\beta}(r;\lambda) = \frac{r^{\alpha}\sin\beta\pi + \lambda\sin(\beta-\alpha)\pi}{\pi\left(r^{2\alpha}+2\lambda r^{\alpha}\cos\alpha\pi+\lambda^2\right)}r^{\alpha-\beta}.$$

This representation can be derived by inversion of (1.27). Let $\lambda > 0$ and $0 < \alpha \leq \beta \leq 1$. Then $P_{\alpha,\beta}(r;\lambda) \geq 0$ and representation (1.28) implies the complete monotonicity of the function $t^{\beta-1}E_{\alpha,\beta}(-\lambda t^{\alpha})$.

Consider the ordinary differential equation of fractional order $\alpha > 0$

$$({}^{C}D_t^{\alpha}u)(t) + \lambda u(t) = f(t), \quad t > 0, \quad \lambda \in \mathbb{R}, \tag{1.29}$$

with initial conditions $u(0) = a$ and $u^{(k)}(0) = 0$, $k = 1, 2, \ldots, m-1$, where $0 < m-1 < \alpha \leq m$. The solution can be obtained by applying Laplace transform and is given by:

$$u(t) = aE_{\alpha}(-\lambda t^{\alpha}) + \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda\tau^{\alpha})f(t-\tau)\,d\tau. \tag{1.30}$$

In the case $\alpha \in (0,1)$ (slow relaxation) the following estimates turn out to be useful: for any λ satisfying $\lambda \geq \lambda_0 > 0$ and any $T > 0$ there exists a constant $C > 0$, depending on α, T, λ_0, such that

$$0 < C \leq \lambda\int_0^T t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^{\alpha})\,dt < 1. \tag{1.31}$$

These bounds are useful in the study of direct and inverse problems for inhomogeneous time-fractional diffusion equations, based on eigenfunction expansion [96].

1.4.2 Prabhakar function

The Prabhakar function (or three parameter Mittag-Leffler function) is defined as follows [43, 91]

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C},\ \alpha \in \mathbb{R}_+,\ \beta, \delta \in \mathbb{R}, \tag{1.32}$$

where $(\delta)_k$ denotes the Pochhammer symbol

$$(\delta)_k = \delta(\delta+1)\ldots(\delta+k-1), \quad k \in \mathbb{N}, \quad (\delta)_0 = 1.$$

It is a generalization of the classical Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$:

$$E_\alpha(z) = E^1_{\alpha,1}(z), \quad E_{\alpha,\beta}(z) = E^1_{\alpha,\beta}(z).$$

The asymptotic behavior of the three-parameter Mittag-Leffler function of real negative argument can be obtained from the expansion

$$E^{\delta}_{\alpha,\beta}(-t) = \sum_{j=0}^{\infty} \frac{t^{-\delta-j}}{\Gamma(\beta - \alpha(\delta+j))} \frac{(\delta)_j}{j!}, \qquad t \to +\infty. \tag{1.33}$$

Recall also the Laplace transform pair

$$\mathcal{L}\{t^{\beta-1} E^{\delta}_{\alpha,\beta}(-\lambda t^\alpha)\}(s) = \frac{s^{\alpha\delta-\beta}}{(s^\alpha + \lambda)^\delta}. \tag{1.34}$$

The following relations can be established by the use of identity (1.34)

$$J^\gamma_t \left(t^{\beta-1} E^{\delta}_{\alpha,\beta}(at^\alpha)\right) = t^\beta E^{\delta}_{\alpha,\beta+\gamma}(at^\alpha), \tag{1.35}$$

$$\left(t^{\beta-1} E^{\delta}_{\alpha,\beta}(at^\alpha)\right) * \left(t^{\beta_0-1} E^{\delta_0}_{\alpha,\beta_0}(at^\alpha)\right) = t^{\beta+\beta_0-1} E^{\delta+\delta_0}_{\alpha,\beta+\beta_0}(at^\alpha). \tag{1.36}$$

The Prabhakar type function obeys the following complete monotonicity property:

$$\text{If } 0 < \alpha \le 1,\ 0 < \alpha\delta \le \beta \le 1, \ \text{ then } \ t^{\beta-1} E^{\delta}_{\alpha,\beta}(-t^\alpha) \in \mathcal{CMF}. \tag{1.37}$$

Further details on Fractional Calculus and Mittag-Leffler functions can be found in [58, 89, 45, 43, 86]. For generalizations and a survey of special functions related to Fractional Calculus we refer to [59, 60].

1.5 Functions of Wright type

The Wright function $W_{\lambda,\mu}(\cdot)$ was introduced and studied by E. Maitland Wright in a series of papers [106, 107]. It is an entire function defined by the series expansion

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1,\ \mu \in \mathbb{C},\ z \in \mathbb{C}.$$

In particular, the function with $\lambda = -\gamma$, $\mu = 1-\gamma$, where $0 < \gamma < 1$, plays a crucial role in the study of fractional evolution equations and is sometimes referred to as M-Wright or Mainardi function. The Mainardi function is an entire function of Wright type defined as [75, 45]

$$M_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \sigma^{\gamma-1} \exp\left(\sigma - z\sigma^\gamma\right) d\sigma, \qquad (1.38)$$

where $0 < \gamma < 1$, $z \in \mathbb{C}$, and Γ is a contour which starts and ends at $-\infty$ and encircles the origin counterclockwise. The Mainardi function is related to the Mittag-Leffler function $E_\gamma(\cdot)$ through the Laplace transform identity

$$\mathcal{L}\{M_\gamma(\cdot)\}(s) = \int_0^\infty e^{-sr} M_\gamma(r)\, dr = E_\gamma(-s), \quad 0 < \gamma < 1. \qquad (1.39)$$

It is proven in the original paper [107] that the function $M_\gamma(z)$ admits the following asymptotic expansion in the sector $|\arg z| < \min\{(1-\gamma)3\pi/2, \pi\}$

$$M_\gamma(z) \sim a(\gamma) z^{\frac{\gamma-1/2}{1-\gamma}} \exp\left(-b(\gamma) z^{\frac{1}{1-\gamma}}\right), \quad |z| \to \infty, \qquad (1.40)$$

where $a(\gamma)$ and $b(\gamma)$ are positive constants depending only on γ, $a(\gamma) = Ab(\gamma)^{\gamma-1/2}$, $b(\gamma) = (1-\gamma)\gamma^{\gamma/(1-\gamma)}$, $A > 0$.

Consider the function $L_\gamma(\cdot)$ defined by the Laplace transform pair

$$\mathcal{L}\{L_\gamma(\cdot)\}(s) = \int_0^\infty e^{-sr} L_\gamma(r)\, dr = \exp(-s^\gamma), \quad 0 < \gamma < 1. \qquad (1.41)$$

It is referred to as Lévy extremal stable density (Lévy one-sided stable distribution), see e.g. [38, 76, 82].

The function $L_\gamma(z)$ is related to $M_\gamma(z)$ via the identity (see e.g. [75, 90])

$$L_\gamma(z) = \gamma z^{-\gamma-1} M_\gamma\left(z^{-\gamma}\right), \quad 0 < \gamma < 1, \qquad (1.42)$$

for $z \in \mathbb{C}$ cut along the negative real axis. The Lévy extremal stable density admits the series representation

$$L_\gamma(z) = \frac{1}{\pi z}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\Gamma(\gamma n+1)}{n!}\frac{\sin(\gamma n\pi)}{z^{\gamma n}}, \quad 0<\gamma<1, \tag{1.43}$$

which can be deduced from (1.38), (1.42), and the property of the Gamma function

$$\frac{\pi}{\sin(\gamma\pi)} = \Gamma(\gamma)\Gamma(1-\gamma). \tag{1.44}$$

The first term of the series in (1.43) provides the following asymptotic expression of $L_\gamma(z)$ for large $|z|$ in the complex plane cut along the negative real axis

$$L_\gamma(z) \sim \frac{c(\gamma)}{z^{\gamma+1}}, \quad |z|\to\infty, \tag{1.45}$$

with $c(\gamma) = \gamma/\Gamma(1-\gamma)$, where the property of the Gamma function (1.44) is used.

Applying (1.42), the asymptotic behavior of $L_\gamma(z)$ for small $|z|$ can be derived from (1.40)

$$L_\gamma(z) \sim \gamma a(\gamma) z^{-\frac{2-\gamma}{2(1-\gamma)}} \exp\left(-b(\gamma)z^{-\frac{\gamma}{1-\gamma}}\right), \quad |z|\to 0, \tag{1.46}$$

for z belonging to the sector $|\arg z| < \min\{(1/\gamma-1)3\pi/2, \pi\}$. We notice that, by restricting z to the real positive half-line $z=t\in(0,\infty)$ in (1.46), resp. (1.45), we recover the asymptotic formulae established in [82].

The functions M_γ and L_γ, $0<\gamma<1$, are unilateral probability density functions (PDF), that is

$$M_\gamma(r)\geq 0, \quad r\geq 0; \qquad \int_0^\infty M_\gamma(r)\,dr = 1, \tag{1.47}$$

and

$$L_\gamma(r)\geq 0, \quad r\geq 0; \qquad \int_0^\infty L_\gamma(r)\,dr = 1. \tag{1.48}$$

The properties can be deduced from the Laplace transform pairs (1.39) and (1.41). The non-negativity of M_γ and L_γ follows from the complete monotonicity of $E_\gamma(-s)$ and $\exp(-s^\gamma)$ for $s>0$ and $0<\gamma\leq 1$ by the use of Bernstein's theorem.

In the particular case $\gamma = 1/2$ series (1.38) and (1.42) yield the representations

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp(-z^2/2) \tag{1.49}$$

and

$$L_{1/2}(z) = \frac{1}{2\sqrt{\pi} z^{3/2}} \exp(-1/(4z)). \tag{1.50}$$

For more details on the functions M_γ and L_γ we refer to [45, 75], see also [76], where these two functions appear in the context of the one-dimensional space-time fractional diffusion-wave equation.

Chapter 2

Introduction to subordination principle

Bernstein functions and related classes of functions play a prominent role in the theory of generalized fractional evolution equations and appear in applications quite naturally. First we list definitions and basic properties of these classes of functions. For a unified approach to the variety of evolution equations with fractional derivatives we use the framework of abstract Volterra equations, a short introduction to which is given next. The rest of the chapter is devoted to subordination principle for evolution equations with the main emphasis on two general theorems.

2.1 Bernstein functions

Four special classes of functions play an essential role in this dissertation: the classes of completely monotone functions ($\mathcal{CMF}$), Bernstein functions ($\mathcal{BF}$), Stieltjes functions ($\mathcal{SF}$), and complete Bernstein function ($\mathcal{CBF}$). Their definitions and basic properties, which are used in the dissertation, are summarized next. We use the terminology of the monograph [101].

The class $\mathcal{CMF}$ of completely monotone functions consists of all real-valued infinitely differentiable on $\mathbb{R}_+$ function $\phi(t)$, satisfying inequalities (1.1). Basic examples of completely monotone functions are e^{-at} for $a \geq 0$, $t^{\alpha-1}$ for $0 \leq \alpha \leq 1$, as well as some generalizations of these two functions in terms of Mittag-Leffler functions, see (1.26) and (1.37).

The class $\mathcal{BF}$ of Bernstein function consists of all non-negative functions $\phi(t)$ defined on $\mathbb{R}_+$, such that $\phi'(t) \in \mathcal{CMF}$.

The class of Stieltjes functions ($\mathcal{SF}$) consists of all functions defined on $\mathbb{R}_+$ which have the representation (see [62])

$$\phi(s) = \frac{a}{s} + b + \int_0^\infty e^{-s\tau}\psi(\tau)\,d\tau, \quad s > 0, \tag{2.1}$$

where $a, b \geq 0$, $\psi \in \mathcal{CMF} \cap L^1_{loc}(\mathbb{R}_+)$, and the Laplace transform of ψ exists for any $s > 0$.

A function ϕ defined on $\mathbb{R}_+$ is said to be a complete Bernstein function ($\phi \in \mathcal{CBF}$) if and only if $\phi(s)/s \in \mathcal{SF}$, $s > 0$.

Basic examples of Stieltjes and complete Bernstein functions are the following:

$$\text{if } \alpha \in [0,1] \text{ then } s^{-\alpha} \in \mathcal{SF},\ s^{\alpha} \in \mathcal{CBF}.$$

This follows by plugging $\psi(t) = \omega_\alpha(t)$ in (2.1), and taking into account (1.4). Note that $\omega_\alpha(t)$, defined in (1.3), is completely monotone for $\alpha \in (0,1)$.

A selection of properties of the above classes of functions is given in the next proposition. The sign $\circ$ denotes composition of functions from the corresponding classes. For the sake of brevity here and throughout the whole dissertation the abbreviation "iff" is used instead of "if and only if".

Proposition 2.1. *Let $s > 0$. The following properties are satisfied:*

(P1) *The class $\mathcal{CMF}$ is closed under pointwise addition, multiplication, and convergence.*

(P2) *The classes $\mathcal{BF}$, $\mathcal{CBF}$, and $\mathcal{SF}$ are closed under pointwise addition, multiplication with positive numbers, and convergence.*

(P3) $\mathcal{SF} \subset \mathcal{CMF}$, $\mathcal{CBF} \subset \mathcal{BF}$.

(P4) *If $\phi \in \mathcal{BF}$ then $\phi(s)/s \in \mathcal{CMF}$.*

(P5) *Let $\phi(s) > 0$. Then $\phi \in \mathcal{BF}$ iff $\psi \circ \phi \in \mathcal{CMF}$ for every $\psi \in \mathcal{CMF}$.*

(P6) *Let $\phi \in L^1_{loc}(\mathbb{R}_+)$. Then $\phi \in \mathcal{CMF}$ iff $\widehat{\phi}(s) \in \mathcal{SF}$ and $\lim_{s\to+\infty} \widehat{\phi}(s) = 0$.*

(P7) $\phi \in \mathcal{SF}$ *iff* $s\phi(s) \in \mathcal{CBF}$.

(P8) *Let $\phi \neq 0$. Then $\phi(s) \in \mathcal{CBF}$ iff $(\phi(s))^{-1} \in \mathcal{SF}$.*

(P9) *Let $\phi \neq 0$. Then $\phi(s) \in \mathcal{CBF}$ iff $s/\phi(s) \in \mathcal{CBF}$.*

(P10) *Let* $\phi \neq 0$. *Then* $\phi(s) \in \mathcal{SF}$ *iff* $(s\phi(s))^{-1} \in \mathcal{SF}$.

(P11) $\mathcal{CBF} \circ \mathcal{CBF} \subset \mathcal{CBF}$.

(P12) $\mathcal{CBF} \circ \mathcal{SF} \subset \mathcal{SF}$.

(P13) *Let* $\phi, \psi \in \mathcal{CBF}$. *Assume* $\alpha_1, \alpha_2 \in (0,1)$ *are such that* $\alpha_1 + \alpha_2 \leq 1$. *Then*

$$\phi^{\alpha_1}(s) \cdot \psi^{\alpha_2}(s) \in \mathcal{CBF}.$$

(P14) *Let* $\phi, \psi \in \mathcal{CBF}$ *and* $\alpha \in [-1,1]\backslash\{0\}$. *Then*

$$(\phi^{\alpha}(s) + \psi^{\alpha}(s))^{1/\alpha} \in \mathcal{CBF}.$$

(P15) *Every function* ϕ *from the classes* $\mathcal{CBF}$ *and* $\mathcal{SF}$ *admits an analytic extension to* $\mathbb{C}\backslash(-\infty, 0]$, *such that* $(\phi(z))^* = \phi(z^*)$, *where* $*$ *denotes the complex conjugate, and*

$$|\arg \phi(z)| \leq |\arg z|, \;\; z \in \mathbb{C}\backslash(-\infty, 0].$$

Moreover, $\Im z \cdot \Im\phi(z) \geq 0$ *for* $\phi \in \mathcal{CBF}$ *and* $\Im z \cdot \Im\phi(z) \leq 0$ *for* $\phi \in \mathcal{SF}$.

Proof. Properties **(P1)**, **(P2)**, the first part of **(P3)**, and **(P7)** follow directly from the definitions of the spaces. Properties **(P9)** and **(P10)** follow easily from **(P8)**, taking into account **(P7)**. For proofs of the rest of the statements we refer to [101], Chapters 6 and 7, and [49], Theorem 2.6, see also [93], Section 4.1. □

We close this section with some remarks.

According to **(P1)** the product of two completely monotone functions is again completely monotone. Such a nice property does not hold for the other three classes. However, according to **(P13)**

$$\text{if} \;\; \phi, \psi \in \mathcal{CBF} \;\; \text{then} \;\; \sqrt{\phi \cdot \psi} \in \mathcal{CBF}. \tag{2.2}$$

A useful result is the following: If $\phi \in \mathcal{BF}$ then for any $\tau > 0$

$$\frac{\phi(s)}{s} e^{-\tau\phi(s)} \in \mathcal{CMF}, \quad s > 0. \tag{2.3}$$

Indeed, according to **(P5)**, $\varphi(s) \in \mathcal{BF}$ is equivalent to $e^{-\tau\varphi(s)} \in \mathcal{CMF}$. Moreover, **(P4)** yields $\phi(s)/s \in \mathcal{CMF}$. Then **(P1)** implies that the product of these two completely monotone functions is again completely monotone, i.e. (2.3) is satisfied.

For convenience we define also the class of functions $\mathcal{CMF}_0$. The function $\phi \in \mathcal{CMF}_0$ if the Laplace transform $\widehat{\phi}(s)$ exists for all $s > 0$ and $\widehat{\phi}(s) \in \mathcal{SF}$. According to (2.1) the functions from the class $\mathcal{CMF}_0$ admit the representation

$$\phi(t) = \phi_0 \delta(t) + \phi_1(t), \tag{2.4}$$

where $\phi_0 \geq 0$ is a constant, $\delta(\cdot)$ denotes the Dirac delta function, and $\phi_1(t) \in L^1_{loc}(\mathbb{R}_+)$ is a completely monotone function. Property **(P6)** implies

$$\phi_0 = \lim_{s \to \infty} \widehat{\phi}(s).$$

2.2 Abstract Volterra integral equations

Evolution equations are equations that can be interpreted as the differential or integro-differential law of the development (evolution) in time of a system. An example is the classical one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{2.5}$$

where $u(x,t)$ is the state of a system at time $t > 0$ at point $x \in \mathbb{R}$. If the solution $u(x,t)$ of such an equation is regarded as an element of some space of functions in x that depend on a parameter t, then one arrives at abstract evolution equations.

Since many of the results in this dissertation will be formulated in the general setting of abstract evolution equations, we give some basic definitions.

Let X be a Banach space with norm $\|.\|$. Let A be a closed linear operator in X with dense domain $D(A) \subset X$, equipped with the graph norm $\|.\|_A$,

$$\|x\|_A := \|x\| + \|Ax\|, \ x \in D(A).$$

Denote by $\varrho(A)$ the resolvent set of A and by $R(s,A)$ the resolvent operator of A: $R(s,A) = (s - A)^{-1}$, $s \in \varrho(A)$.

If Y is another Banach space, by $\mathcal{B}(X,Y)$ we denote the space of all bounded linear operators from X to Y; $\mathcal{B}(X) = \mathcal{B}(X,X)$.

The most prominent abstract evolution equation is the classical first order Cauchy problem [2, 36, 108]

$$u'(t) = Au(t), \quad t > 0; \quad u(0) = v \in X, \tag{2.6}$$

with its equivalent formulation - the semigroup theory. Classical models of relaxation or diffusion processes, such as (2.5), are particular cases of (2.6).

The second order abstract Cauchy problem [2]

$$u''(t) = Au(t), \quad t > 0; \quad u(0) = v \in X, \ u'(0) = 0, \tag{2.7}$$

is another classical example of abstract evolution equation. Although the second initial condition can be an arbitrary element of X, for continuity reasons in this dissertation we consider only initial conditions of the above type. Equations modeling different oscillation or wave phenomena are particular cases of (2.7).

In this dissertation we consider evolution equations, which contain operators of Fractional Calculus. In contrast to classical differential operators, the fractional derivatives have a nonlocal character, which makes them relevant in modeling of materials and processes with memory.

Let ${}^C D_t^\alpha$ be the Caputo fractional derivative of order $\alpha \in (0, 2]$. The most extensively studied fractional evolution equation is

$${}^C D_t^\alpha u(t) = Au(t), \quad t > 0, \tag{2.8}$$

with the following initial conditions:

$$\begin{aligned} &u(0) = v \in X \text{ for } \ \alpha \in (0, 1]; \\ &u(0) = v \in X, \ u'(0) = 0 \text{ for } \ \alpha \in (1, 2]. \end{aligned}$$

The classical abstract Cauchy problems (2.6) and (2.7) are particular cases of (2.8) obtained for $\alpha = 1$ and $\alpha = 2$, respectively.

For a unified approach to the different evolution equations in this dissertation, we rewrite them as equivalent Volterra integral equations and apply for the study of the obtained weaker formulations the theory developed in the monograph [93]. The notions of solution, well-posedness, and solution operator, defined next for Volterra integral equations, are used also for the corresponding equivalent Cauchy problems.

Let A be a closed linear unbounded operator, densely defined in a Banach space X. Consider the Volterra integral equation

$$u(t) = \int_0^t k(t - \tau) Au(\tau) \, d\tau + f(t), \quad t > 0, \tag{2.9}$$

with a scalar kernel $k(t) \in L^1_{loc}(\mathbb{R}_+)$.

Definition 2.1. *A function $u \in C(\mathbb{R}_+; X)$ is called a strong solution of equation (2.9) if $u \in C(\mathbb{R}_+; D(A))$ and (2.9) holds on $\mathbb{R}_+$.*

Definition 2.2. *Equation (2.9) is said to be well posed if for each $v \in D(A)$, there is a unique strong solution $u(t; v)$ of*

$$u(t) = v + \int_0^t k(t-\tau) A u(\tau)\, d\tau, \qquad t > 0, \quad v \in D(A), \tag{2.10}$$

and $\{v_n\} \subset D(A)$, $v_n \to 0$ imply $u(t; v_n) \to 0$ in X, uniformly on compact intervals.

Suppose (2.9) is well posed. Then the solution operator $S(t)$ for (2.9) is defined by:

$$S(t)v = u(t; v), \quad v \in D(A),\ t \geq 0.$$

The solution operator $S(t)$ is linear for each $t \geq 0$. Since $S(t)$ is a bounded operator, it admits extension to all of X, $S(t)v$ is continuous for each $v \in X$.

Since (2.9) is a convolution equation, it is natural to employ the Laplace transform for its study. Suppose the Laplace transform $\widehat{k}(s)$ of the kernel $k(t)$ exists and $\widehat{k}(s) \neq 0$ for all $s > 0$ and set for the sake of brevity

$$g(s) = \left(\widehat{k}(s)\right)^{-1}, \quad s > 0. \tag{2.11}$$

Assume moreover that $g(s) \in \varrho(A)$ for any $s > 0$.

For instance, the Cauchy problem (2.8) corresponds to (2.10) with $k(t) = \omega_\alpha(t)$ and $g(s) = s^\alpha$. Let us denote by $S_\alpha(t)$ the related solution operator. In particular, the solution operator $S_1(t)$ of (2.6) is a C_0-semigroup and the solution operator $S_2(t)$ of (2.7) is a strongly continuous cosine family, see [2].

Definition 2.3. *A solution operator $S(t)$ is called bounded if there exists a constant $C \geq 1$ such that*

$$\|S(t)\| \leq C \quad \textit{for all } t \geq 0.$$

Suppose $S(t)$ is a bounded solution operator for (2.10). Then the Laplace transform

$$H(s) = \int_0^\infty e^{-st} S(t)\, dt \tag{2.12}$$

is well defined for $\Re s > 0$ and is given by

$$H(s) = \frac{g(s)}{s}(g(s) - A)^{-1}, \tag{2.13}$$

where the function $g(s)$ is defined in (2.11).

The Generation Theorem for abstract Volterra equations ([93], Theorem 1.3) is formulated next.

Theorem 2.1. [93] *Equation (2.9) is well posed and admits a bounded solution operator $S(t)$ satisfying $\|S(t)\| \leq C,\ t \geq 0$, iff the following conditions hold.*
(H1) $\widehat{k}(s) \neq 0$ *and* $(\widehat{k}(s))^{-1} \in \varrho(A)$ *for all* $s > 0$*;*
(H2) *the estimates*

$$\|H^{(n)}(s)\| \leq C\frac{n!}{s^{n+1}} \quad \text{for all } s > 0,\ n \in \mathbb{N}_0, \tag{2.14}$$

are satisfied, where $H(s)$ is defined in (2.13).

In the case of classical Cauchy problem (2.6) ($k(t) \equiv 1$, $g(s) = s$, $H(s) = (s-A)^{-1}$) Theorem 2.1 is known as the Hille-Yosida theorem for C_0 semigroups, see e.g. [2, 108].

A generalization of the definition of bounded analytic semigroup (see e.g. [2], Def. 3.7.3) is given next.

Definition 2.4. *A solution operator $S(t)$ is said to be a bounded analytic solution operator of angle $\theta_0 \in (0, \pi/2]$ if $S(\cdot) : \mathbb{R}_+ \to \mathcal{B}(X)$ admits an analytic extension $S(z)$ to the sector $|\arg z| < \theta_0$, which is bounded on each subsector $|\arg z| \leq \theta$, where $\theta < \theta_0$.*

Let us note that a solution operator $S(t)$, which is bounded (i.e. bounded for $t \in [0, \infty)$), and admits an analytic extension to some sector in the complex plane, is not necessarily a bounded analytic solution operator.

Next we formulate the generation theorem for bounded analytic solution operators ([93], Theorem 2.1).

Theorem 2.2. [93] *Equation (2.9) admits a bounded analytic solution operator $S(t)$ of angle $\theta_0 \in (0, \pi/2]$ iff the following conditions hold.*
(A1) $\widehat{k}(s)$ *admits meromorphic extension to* $\Sigma(\theta_0 + \pi/2)$*;*
(A2) $\widehat{k}(s) \neq 0$ *and* $(\widehat{k}(s))^{-1} \in \varrho(A)$ *for all* $s \in \Sigma(\theta_0 + \pi/2)$*;*
(A3) *For each $\theta < \theta_0$ there is a constant $C = C(\theta)$ such that $H(s)$, defined in (2.13), satisfies the estimate*

$$\|H(s)\| \leq \frac{C}{|s|} \quad \text{for all } s \in \Sigma(\theta + \pi/2). \tag{2.15}$$

2.3 Subordination for fractional evolution equations

Subordination principle for fractional evolution equations is studied in [9]. The main results are summarized in the next theorem.

Theorem 2.3. *Let A be a closed linear and densely defined operator in X. Assume problem (2.8) is well posed for some α, $0 < \alpha \leq 2$. Let β be such that $0 < \beta < \alpha$ and set $\gamma = \beta/\alpha$. Then problem (2.8) with α replaced by β is well posed and the corresponding solution operators S_α and S_β are related by the identity*

$$S_\beta(t) = \int_0^\infty \varphi_\gamma(t,\tau) S_\alpha(\tau)\, d\tau, \quad t > 0. \tag{2.16}$$

The subordination kernel $\varphi_\gamma(t,\tau)$ admits the representation

$$\varphi_\gamma(t,\tau) = t^{-\gamma} M_\gamma(\tau t^{-\gamma}),$$

where $M_\gamma(\cdot)$ is the Mainardi function (1.38). Moreover, if S_α is a bounded solution operator, then S_β is a bounded analytic solution operator of angle

$$\theta(\gamma) = \min\left\{ \left(\frac{1}{\gamma} - 1\right)\frac{\pi}{2}, \frac{\pi}{2} \right\}. \tag{2.17}$$

Let us note that due to the properties (1.47) of Mainardi function the subordination kernels $\varphi_\gamma(t,\tau)$ are unilateral probability density functions.

Two basic examples of applications of Theorem 2.3 are considered next.

Example 2.1. Consider the fractional relaxation-oscillation equation, which is the scalar version of equation (2.8), where $X = \mathbb{R}$, and the operator $A = -\lambda$, $\lambda > 0$, is multiplication with a constant. Set $v = 1$. Then the solution is given by the Mittag-Leffler function

$$S_\alpha(t) = E_\alpha(-\lambda t^\alpha), \quad t > 0.$$

Let $0 < \beta < \alpha$ and set $\gamma = \beta/\alpha$. Then subordination relation (2.16) yields the following relation between Mittag-Leffler functions

$$E_\beta(-\lambda t^\beta) = t^{-\gamma} \int_0^\infty M_\gamma\left(\tau t^{-\gamma}\right) E_\alpha(-\lambda \tau^\alpha)\, d\tau, \quad t > 0, \tag{2.18}$$

where $M_\gamma(\cdot)$ is the Mainardi function. Useful relations can be deduced from (2.18) by setting $\alpha = 1$, $\alpha = 2$, or $\beta = \alpha/2$. For example, (2.18) with $\alpha = 1$ implies (1.39), with $\alpha = 2$ gives

$$E_\beta(-\lambda t^\beta) = t^{-\beta/2} \int_0^\infty M_{\beta/2}\left(\tau t^{-\beta/2}\right) \cos(\sqrt{\lambda}\tau)\, d\tau, \quad t > 0,$$

and with $\beta = \alpha/2$ yields by the use of (1.49)

$$E_{\alpha/2}(-\lambda t^{\alpha/2}) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{\tau^2}{2t}} E_\alpha(-\lambda \tau^\alpha)\, d\tau, \quad t > 0.$$

Example 2.2. Consider the one-dimensional Cauchy problem for the fractional diffusion-wave equation:

$$^C D_t^\alpha u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R},\ t > 0, \quad 0 < \alpha \leq 2, \tag{2.19}$$

with conditions $u(\pm\infty, t) = 0,\ \ u(x,0) = f(x),\ u_t(x,0) = 0$ (the last one only for $1 < \alpha \leq 2$). Set

$$\begin{aligned} X &= L^p(\mathbb{R}),\ 1 \leq p < \infty, \quad \text{or} \quad X = C_0(\mathbb{R}); \qquad (2.20)\\ A &= \frac{\partial^2}{\partial x^2}, \quad D(A) = \{f \in X :\ f'' \in X\}, \end{aligned}$$

where $C_0(\mathbb{R})$ is the space of continuous functions vanishing at infinity.

For $\alpha = 2$ problem (2.19) is the second order Cauchy problem with solution operator $S_2(t)$ given by the d'Alembert formula

$$(S_2(t)f)(x) = \frac{1}{2}(f(x+t) + f(x-t)), \quad t \geq 0, x \in \mathbb{R}, \quad f \in X. \tag{2.21}$$

If we set $\alpha = 2$ in Theorem 2.3 and use (2.21) then for all $\alpha \in (0,2)$ we obtain

$$\begin{aligned} (S_\alpha(t)f)(x) &= t^{-\alpha/2} \int_0^\infty M_{\alpha/2}(\tau t^{-\alpha/2})(S_2(\tau)f)(x)\, d\tau \qquad (2.22)\\ &= \frac{1}{2} t^{-\alpha/2} \int_{-\infty}^\infty M_{\alpha/2}(|\tau| t^{-\alpha/2}) f(x-\tau)\, d\tau, \end{aligned}$$

where $t > 0$ and $x \in \mathbb{R}$. This is a well-known result, see e.g. [74]. Taking into account (1.49), we recover from (2.22) the solution formula for the first order Cauchy problem

$$(S_1(t)f)(x) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^\infty f(x-\tau) e^{-\tau^2/4t}\, d\tau, \quad t > 0,\ x \in \mathbb{R}. \tag{2.23}$$

For $\alpha \in (0,1)$ the fractional diffusion-wave equation (2.19) models anomalous diffusion, while for $\alpha \in (1,2)$ it governs the propagation of mechanical diffusive waves, i.e. interpolates between diffusion and wave propagation [75]. Indeed, the time-fractional diffusion-wave equation (2.19) with $\alpha \in (1,2)$ exhibits intermediate character regarding the response to a localized disturbance. In this case a disturbance spreads infinitely fast [41], which is typical for diffusion. On the other hand, the fundamental solution possesses a maximum that disperses with a finite speed [41, 72], which is typical for the classical wave equation. Moreover, the fundamental solution $\frac{1}{2}t^{-\alpha/2}M_{\alpha/2}(|\tau|t^{-\alpha/2})$ is a spatial probability density function evolving in time, which is unimodal in the diffusion regime and bimodal in the wave propagation regime [75].

2.4 General subordination theorems

The following generalization of Theorem 2.3 will play a central role in this dissertation. A more general theorem for completely positive kernels can be found in [93], Theorem 4.1. Our formulation is adapted to the framework of Bernstein functions, which is more convenient for application.

Theorem 2.4. *Let A be a closed linear and densely defined operator in X. Assume the Cauchy problem (2.8) is well posed for some α, $0 < \alpha \le 2$, and admits a bounded solution operator $S_\alpha(t)$. For the kernel $k(t)$ of the Volterra integral equation (2.9) assume $k(t) \in L^1_{loc}(\mathbb{R}_+)$, $\widehat{k}(s)$ exists for $s > 0$, $\widehat{k}(s) \neq 0$ and the function $g(s) = (\widehat{k}(s))^{-1}$ satisfies the condition*

$$g(s)^{1/\alpha} \in \mathcal{CBF}, \quad s > 0. \tag{2.24}$$

Then problem (2.9) admits a bounded solution operator $S(t)$, which is related to $S_\alpha(t)$ via the subordination identity

$$S(t) = \int_0^\infty \varphi(t,\tau) S_\alpha(\tau)\, d\tau, \quad t > 0, \tag{2.25}$$

where

$$\varphi(t,\tau) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{g(s)^{1/\alpha}}{s} \exp\left(st - \tau g(s)^{1/\alpha}\right) ds, \quad c > 0.$$

The subordination kernel $\varphi(t,\tau)$ is a unilateral probability density function (PDF) in τ when $t > 0$ is considered as a parameter, that is

$$\varphi(t,\tau) \geq 0, \quad \int_0^\infty \varphi(t,\tau)\, d\tau = 1, \qquad t,\tau > 0. \tag{2.26}$$

Proof. To prove that problem (2.9) is well posed and admits a bounded solution operator we will check **(H1)** and **(H2)** of the Generation theorem (Theorem 2.1). First, the existence of a bounded solution operator $S_\alpha(t)$ implies by the Generation theorem that $s^\alpha \in \varrho(A)$ for all $s > 0$, thus $\mathbb{R}_+ \subset \varrho(A)$. Assumption (2.24) implies that $g(s) : \mathbb{R}_+ \to \mathbb{R}_+$ and thus $g(s) \subset \varrho(A)$ for all $s > 0$. Therefore, **(H1)** is fulfilled and $H(s)$ in (2.13) is well defined.

Now we prove that the conditions **(H2)** of the Generation theorem are fulfilled. Set

$$h(s,\tau) = \frac{g(s)^{1/\alpha}}{s} \exp\left(-\tau g(s)^{1/\alpha}\right). \tag{2.27}$$

Assumption (2.24) implies that the function $h(s,\tau)$ is completely monotone as a function of s, $s > 0$, for any $\tau > 0$ considered as a parameter, see (2.3).

Let

$$H(s) = \int_0^\infty h(s,\tau) S_\alpha(\tau)\, d\tau, \tag{2.28}$$

where h is defined in (2.27). Since the Laplace transform of the solution operator $S_\alpha(t)$ is given by (2.13) with $g(s) = s^\alpha$, that is

$$\int_0^\infty e^{-st} S_\alpha(t)\, dt = s^{\alpha-1}(s^\alpha - A)^{-1}, \tag{2.29}$$

we derive

$$H(s) = \frac{g(s)^{1/\alpha}}{s} \int_0^\infty \exp\left(-\tau g(s)^{1/\alpha}\right) S_\alpha(\tau)\, d\tau = \frac{g(s)}{s}(g(s) - A)^{-1}.$$

Therefore, the function (2.13) admits the representation (2.28).

It is assumed that $S_\alpha(t)$ is a bounded solution operator, that is, $\|S_\alpha(t)\| \leq C,\ t \geq 0$, for some $C \geq 1$.

Define the operator

$$L_s^n = \frac{(-1)^n}{n!} \frac{\partial^n}{\partial s^n}.$$

To establish **(H2)** we will prove

$$\|L_s^n H(s)\| \leq \frac{C}{s^{n+1}}, \quad s > 0,\ n \in \mathbb{N}_0. \tag{2.30}$$

Since $L_s^n h(s,\tau) \geq 0$ due to the complete monotonicity of $h(s,\tau)$, we obtain from (2.28) for $s > 0,\ n \in \mathbb{N}_0$

$$\begin{aligned}
\|L_s^n H(s)\| &= \left\| \int_0^\infty L_s^n h(s,\tau) S_\alpha(\tau)\, d\tau \right\| \\
&\leq \int_0^\infty L_s^n h(s,\tau) \|S_\alpha(\tau)\|\, d\tau \\
&\leq C \int_0^\infty L_s^n h(s,\tau)\, d\tau \\
&= C L_s^n \int_0^\infty h(s,\tau)\, d\tau = C L_s^n \left(\frac{1}{s}\right) = \frac{C}{s^{n+1}},
\end{aligned}$$

where we have used

$$\begin{aligned}
\int_0^\infty h(s,\tau)\, d\tau = &= \int_0^\infty \frac{g(s)^{1/\alpha}}{s} \exp\left(-\tau g(s)^{1/\alpha}\right) d\tau \\
&= \frac{1}{s} \int_0^\infty \mathrm{e}^{-\xi}\, d\xi = \frac{1}{s}, \quad s > 0,
\end{aligned} \tag{2.31}$$

and $L_s^n(s^{-1}) = s^{-(n+1)}$, which can be easily established by induction. In this way we proved **(H1)** and **(H2)** for problem (2.9), which guarantees that it is well posed and admits a bounded solution operator.

Define the function $\varphi(t,\tau)$ such that its Laplace transform with respect to t satisfies

$$\widehat{\varphi}(s,\tau) = \int_0^\infty e^{-st} \varphi(t,\tau)\, d\tau = h(s,\tau). \tag{2.32}$$

Since $h(\cdot,\tau) \in \mathcal{CMF}$, according to Bernstein's theorem the function $\varphi(t,\tau)$ exists and is nonnegative, $\varphi(t,\tau) \geq 0$. Let us also check that the function $\varphi(t,\tau)$ is normalized. Applying (2.32) and (2.31) we deduce

$$\mathcal{L}\left\{\int_0^\infty \varphi(t,\tau)\, d\tau\right\}(s) = \int_0^\infty \widehat{\varphi}(s,\tau)\, d\tau = \int_0^\infty h(s,\tau)\, d\tau = \frac{1}{s}, \quad s > 0.$$

Then, taking the inverse Laplace transform, it follows

$$\int_0^\infty \varphi(t,\tau)\, d\tau = 1.$$

Let the operator $S(t)$ be defined by (2.25). Then (2.32) and identity (2.29) imply for the Laplace transform of $S(t)$

$$\begin{aligned}\int_0^\infty e^{-st} S(t)\,dt &= \int_0^\infty \widehat{\varphi}(s,\tau) S_\alpha(\tau)\,d\tau \\ &= \frac{g(s)^{1/\alpha}}{s}\int_0^\infty \exp\left(-\tau g(s)^{1/\alpha}\right) S_\alpha(\tau)\,d\tau \\ &= \frac{g(s)}{s}(g(s)-A)^{-1}.\end{aligned}$$

Therefore (2.13) is satisfied and $S(t)$ is the solution operator of problem (2.9) due to the uniqueness property of Laplace transform. □

Relation (2.25) suggests that the subordinated solution operator $S(t)$ inherits the main properties of $S_\alpha(t)$. Since the solution operators $S_1(t)$ and $S_2(t)$ of the classical problems (2.6) and (2.7) are well studied, Theorem 2.4 is most useful for $\alpha = 1$ and $\alpha = 2$. Let us note that according to condition (2.24) the general problem (2.9) is subordinated to the first order Cauchy problem (2.6) if $g(s) \in \mathcal{CBF}$ and to the second order Cauchy problem (2.7) provided $\sqrt{g(s)} \in \mathcal{CBF}$.

Property (2.24) implies $g(s)^{1/\alpha_1} \in \mathcal{CBF}$ for any $\alpha_1 > \alpha$. This follows from the representation

$$g(s)^{1/\alpha_1} = (g(s)^{1/\alpha})^{\alpha/\alpha_1},$$

which implies that $g(s)^{1/\alpha_1} = f_1(f_2(s))$ is a composition of two complete Bernstein functions $f_1(s) = s^{\alpha/\alpha_1}$ and $f_2(s) = g(s)^{1/\alpha}$. According to property **(P11)** in Proposition 2.1 it is again complete Bernstein function. For this reason it is useful to know the smallest $\alpha > 0$, for which (2.24) is satisfied.

Let us note that the property $h(s,\tau) \in \mathcal{CMF}$ holds under a weaker assumption, namely $\mathcal{CBF}$ in (2.24) can be replaced by $\mathcal{BF}$, see (2.3). However, there is an instructive example in [62] showing that the class $\mathcal{BF}$ can lead to discontinuous solutions. Therefore, in this dissertation we will work with condition (2.24).

The fact that the function $\varphi(t,\tau)$ in subordination identity (2.25) is a PDF has several important implications. One of them is that if $\|S_\alpha(t)\| \le C$ for $t \ge 0$, then the same holds for $S(t)$. Indeed, from (2.25) it follows

$$\|S(t)\| \le \int_0^\infty \varphi(t,\tau)\|S_\alpha(\tau)\|\,d\tau \le C\int_0^\infty \varphi(t,\tau)\,d\tau = C, \quad t \ge 0.$$

Another implication is that positivity of the solutions is preserved after subordination. To formulate this result in a general abstract setting we suppose that X is an ordered Banach space (for a simple introduction see e.g. [2], [36]). For example, such are the spaces of type $L^p(\Omega)$ or $C_0(\Omega)$ for some $\Omega \in \mathbb{R}^n$, $n \in \mathbb{N}$, where $C_0(\Omega)$ is the space of continuous functions vanishing at infinity. We consider the canonical ordering in these spaces: a function $v \in X$ is positive (in symbols: $v \geq 0$) if $v(x) \geq 0$ for (almost) all $x \in \Omega$.

A solution operator $S(t)$ in an ordered Banach space X is called positive if $v \geq 0$ implies $S(t)v \geq 0$ for any $t \geq 0$.

In other words, positivity of a solution operator means that positivity of the initial condition is preserved in time.

Corollary 2.1. *Let X be an ordered Banach space. Suppose that the conditions of Theorem 2.4 are satisfied and the solution operator $S_\alpha(t)$ is positive. Then the solution operator $S(t)$ is positive.*

In general, the integral representation (2.25) implies that $S(t)$ has at least the same regularity as $S_\alpha(t)$. More detailed results are given next.

According to property **(P15)** in Proposition 2.1 the assumption $g(s)^{1/\alpha} \in \mathcal{CBF}$ in (2.24) implies that $g(s)^{1/\alpha}$ admits an analytic extension to $\mathbb{C}\backslash(-\infty, 0]$ and

$$|\arg\{g(s)^{1/\alpha}\}| \leq |\arg s|, \; s \in \mathbb{C}\backslash(-\infty, 0].$$

Then, according to Theorem 2.2 if $S_\alpha(t)$ is a bounded analytic solution operator of angle $\phi_0 \in (0, \pi/2]$, the same holds true for the subordinated solution operator $S(t)$.

In the next theorem we make the following stronger assumption:

$$|\arg\{g(s)^{1/\beta}\}| \leq |\arg s|, \quad 0 < \beta < \alpha, \; s \in \mathbb{C}\backslash(-\infty, 0]. \tag{2.33}$$

Theorem 2.5. *Suppose the assumptions of Theorem 2.4 and condition (2.33) are satisfied. Then $S(t)$ is a bounded analytic solution operator of angle*

$$\theta_* = \min\left\{\left(\frac{\alpha}{\beta} - 1\right)\frac{\pi}{2}, \frac{\pi}{2}\right\}. \tag{2.34}$$

If, moreover, $S_\alpha(t)$ is a bounded analytic solution operator of angle $\phi_0 \in (0, \pi/2]$ then $S(t)$ is a bounded analytic solution operator of angle

$$\theta_0 = \min\left\{\frac{\alpha}{\beta}\phi_0 + \left(\frac{\alpha}{\beta} - 1\right)\frac{\pi}{2}, \frac{\pi}{2}\right\}. \tag{2.35}$$

Proof. Suppose first $S_\alpha(t)$ is a bounded solution operator, $\|S_\alpha(t)\| \le C$ for $t \ge 0$. Then the Laplace transform $H(s)$ is defined for $\Re s > 0$ and satisfies

$$\|H(s)\| = \left\| \int_0^\infty e^{-st} S_\alpha(t)\, dt \right\| \le \int_0^\infty e^{-(\Re s)t} \|S_\alpha(t)\|\, dt \le \frac{C}{\Re s}, \quad \Re s > 0.$$

Moreover, $s^\alpha \in \rho(A)$ for $\Re s > 0$ and $H(s) = s^{\alpha-1}\,(s^\alpha - A)^{-1}$. Therefore

$$\|s^{\alpha-1}\,(s^\alpha - A)^{-1}\,\| \le \frac{C}{\Re s}, \quad \Re s > 0,$$

which implies

$$\|s^{\alpha}\,(s^\alpha - A)^{-1}\,\| \le \frac{C}{\cos(\arg s)}, \quad \Re s > 0. \tag{2.36}$$

Let $s \in \Sigma(\theta + \pi/2)$ for some fixed $\theta < \theta_*$, where θ_* is defined in (2.34). Then

$$\left|\arg g(s)^{1/\alpha}\right| \le \frac{\beta}{\alpha}\left(\theta + \frac{\pi}{2}\right) < \frac{\beta}{\alpha}\left(\theta_* + \frac{\pi}{2}\right) \le \frac{\pi}{2}.$$

Thus, $\Re g(s)^{1/\alpha} > 0$ and we can replace s in (2.36) by $g(s)^{1/\alpha}$, which gives

$$\|g(s)\,(g(s) - A)^{-1}\,\| \le \frac{C}{\cos(\arg g(s)^{1/\alpha})} \le \frac{C}{\cos\left(\frac{\beta}{\alpha}\left(\theta + \frac{\pi}{2}\right)\right)} = C_1.$$

Therefore

$$\left\| \frac{g(s)}{s}\,(g(s) - A)^{-1} \right\| \le \frac{C_1}{|s|}, \;\; s \in \Sigma(\theta + \pi/2).$$

This implies (see Theorem 2.2) that $S(t)$ is a bounded analytic solution operator of angle θ_*.

Suppose now that $S_\alpha(t)$ is a bounded analytic solution operator of angle ϕ_0. Then, according to Theorem 2.2, $s^\alpha \in \rho(A)$ for $s \in \Sigma(\phi_0 + \pi/2)$ and for each $\phi < \phi_0$

$$\|s^{\alpha}\,(s^\alpha - A)^{-1}\,\| \le C, \quad s \in \Sigma(\phi + \pi/2),$$

which is equivalent to

$$\|z\,(z - A)^{-1}\,\| \le C, \quad z \in \Sigma\left(\alpha(\phi + \pi/2)\right).$$

Let $s \in \Sigma(\theta + \pi/2)$ for some fixed $\theta < \theta_0$, where θ_0 is defined in (2.35). Then $g(s) \in \Sigma\left(\alpha(\phi_0 + \pi/2)\right)$ and we can plug $z = g(s)$ in the above inequality, wnich yields

$$\left\| \frac{g(s)}{s}\,(g(s) - A)^{-1} \right\| \le \frac{C}{|s|}, \quad s \in \Sigma(\theta + \pi/2).$$

Therefore, according to Theorem 2.2, the solution operator $S(t)$ is bounded analytic of angle θ_0. $\square$

Chapter 3

Space-time fractional evolution equations

This chapter is devoted to subordination principle for the fractional evolution equation with the Caputo derivative of order $\beta \in (0,1)$ and operator $-(-A)^{\alpha}$, $\alpha \in (0,1)$, where A generates a strongly continuous one-parameter semigroup on a Banach space. Some properties of the subordination kernel are established and representations in terms of Mainardi function M_{β} and Lévy extremal stable densities L_{α} are derived. Analyticity of the solution operator is deduced by taking into account the asymptotic behavior of the subordination kernel. The subordination formulae are applied to the multi-dimensional space-time fractional diffusion equation to obtain some closed-form solutions. Integral representations in terms of Mittag-Leffler functions are derived for the fundamental solution and the subordination kernel.

3.1 Derivation of subordination formula

Assume the operator A is a closed densely defined operator in a Banach space X, which is the infinitesimal generator of a bounded C_0-semigroup. Therefore, the classical abstract Cauchy problem

$$u'(t) = Au(t), \quad t > 0; \qquad u(0) = v \in X, \tag{3.1}$$

is well posed with a bounded solution operator. The assumptions on the operator A in particular imply that $-A$ is a non-negative operator, i.e. $(-\infty, 0) \subset \varrho(-A)$ and

$$\|\lambda(\lambda - A)^{-1}\| \leq C < \infty, \quad \lambda > 0.$$

For $0 < \alpha < 1$ we define the fractional power $(-A)^\alpha$ of the non-negative operator $-A$ using the Balakrishnan definition [8, 108]

$$(-A)^\alpha v = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda - A)^{-1}(-Av)\, d\lambda, \quad v \in D(A). \tag{3.2}$$

Then $-(-A)^\alpha$ is a closed densely defined operator, which generates a bounded analytic C_0-semigroup [108]. This semigroup is the solution operator to the abstract Cauchy problem

$$u'(t) = -(-A)^\alpha u(t), \quad t > 0; \qquad u(0) = v \in X. \tag{3.3}$$

This chapter is devoted to the Cauchy problem for the fractional evolution equation

$${}^C D_t^\beta u(t) = -(-A)^\alpha u(t),\ t > 0; \quad u(0) = v \in X; \qquad 0 < \alpha, \beta \leq 1, \tag{3.4}$$

where ${}^C D_t^\beta$ is the Caputo time-fractional derivative. Applying Theorem 2.3, the well-posedness of problem (3.3) implies that problem (3.4) is well posed.

In this chapter we use the double-index notation $S_{\alpha,\beta}(t)$ for the solution operator of problem (3.4), where $0 < \alpha, \beta \leq 1$. In particular, the solution operator $S_{1,1}(t)$ to the classical problem (3.1) is the C_0 - semigroup of operators generated by the operator A. The solution operator corresponding to $\alpha = 1$ is denoted in this chapter by $S_{1,\beta}(t)$ (while in the rest of the dissertation the simpler notation $S_\beta(t)$ is used).

Our first aim is to obtain a subordination formula

$$S_{\alpha,\beta}(t) = \int_0^\infty \psi_{\alpha,\beta}(t,\tau) S_{1,1}(\tau)\, d\tau, \quad t > 0,$$

which relates the solution operator $S_{\alpha,\beta}(t)$ of problem (3.4) with the solution operator $S_{1,1}(t)$ of the classical abstract Cauchy problem (3.1), where $\psi_{\alpha,\beta}(t,\tau)$ is a unilateral probability density function in τ. To derive such a formula we apply successively two already known subordination results.

First, let us set $\beta = 1$ in (3.4) and apply a classical theorem (see [108], Chapter IX) according to which the operator $-(-A)^\alpha$ generates a bounded analytic semigroup $S_{\alpha,1}(t)$, related to the semigroup $S_{1,1}(t)$ via the identity

$$S_{\alpha,1}(t) = \int_0^\infty f_\alpha(t,\tau) S_{1,1}(\tau)\, d\tau, \quad t > 0, \tag{3.5}$$

where the subordination kernel $f_\alpha(t,\tau)$ is defined by the inverse Laplace integral

$$f_\alpha(t,\tau) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} e^{z\tau - tz^\alpha}\, dz, \quad \sigma > 0. \tag{3.6}$$

The semigroup $S_{\alpha,1}(t)$ is the solution operator to the Cauchy problem (3.3). It is worth noting that in the scalar case $-A = \lambda > 0$ relation (3.5) reads

$$e^{-\lambda^\alpha t} = \int_0^\infty f_\alpha(t,\tau) e^{-\lambda\tau}\, d\tau, \quad t > 0. \tag{3.7}$$

Second, according to the subordination principle for fractional evolution equations, see Theorem 2.3, the well-posedness of problem (3.3) implies well-posedness of problem (3.4) for all $\beta \in (0,1)$ and the corresponding solution operator $S_{\alpha,\beta}(t)$ is expressed by the formula

$$S_{\alpha,\beta}(t) = \int_0^\infty \varphi_\beta(t,\tau) S_{\alpha,1}(\tau)\, d\tau, \quad t > 0, \tag{3.8}$$

where

$$\varphi_\beta(t,\tau) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} z^{\beta-1} e^{zt - \tau z^\beta}\, dz, \quad \sigma > 0. \tag{3.9}$$

Since in the scalar case $-A = \lambda > 0$ the solution operator $S_{\alpha,\beta}(t)$ of problem (3.4) is given by the Mittag-Leffler function $E_\beta(-\lambda^\alpha t^\beta)$, the scalar version of relation (3.8) is

$$E_\beta(-\lambda^\alpha t^\beta) = \int_0^\infty \varphi_\beta(t,\tau) e^{-\lambda^\alpha \tau}\, d\tau, \quad t > 0. \tag{3.10}$$

This holds for any $0 < \alpha \le 1$, while the function $\varphi_\beta(t,\tau)$ is independent of α.

As a result of the successive application of the above two steps we deduce

$$\begin{aligned} S_{\alpha,\beta}(t) &= \int_0^\infty \varphi_\beta(t,\sigma) \int_0^\infty f_\alpha(\sigma,\tau) S_{1,1}(\tau)\, d\tau\, d\sigma \\ &= \int_0^\infty \left(\int_0^\infty \varphi_\beta(t,\sigma) f_\alpha(\sigma,\tau)\, d\sigma \right) S_{1,1}(\tau)\, d\tau. \end{aligned} \tag{3.11}$$

In this way we derived the following result.

Theorem 3.1. *Let A be a generator of a bounded C_0-semigroup $S_{1,1}(t)$. Then problem (3.4) admits a bounded solution operator $S_{\alpha,\beta}(t)$, which is related to $S_{1,1}(t)$ via the subordination identity*

$$S_{\alpha,\beta}(t) = \int_0^\infty \psi_{\alpha,\beta}(t,\tau) S_{1,1}(\tau)\, d\tau, \quad t > 0, \tag{3.12}$$

where the subordination kernel

$$\psi_{\alpha,\beta}(t,\tau) = \int_0^\infty \varphi_\beta(t,\sigma) f_\alpha(\sigma,\tau)\, d\sigma \tag{3.13}$$

is a unilateral probability density in τ, i.e. it satisfies

$$\psi_{\alpha,\beta}(t,\tau) \geq 0, \quad \int_0^\infty \psi_{\alpha,\beta}(t,\tau)\, d\tau = 1. \tag{3.14}$$

Proof. The subordination identity is derived in (3.11). It remains to prove (3.14). Since it is already well-known that f_α and φ_β are probability densities (which can be also directly checked using the defining identities (3.6) and (3.9)) the fact that $\psi_{\alpha,\beta}$ is a PDF can be derived from the composition rule (3.13).

Alternatively, let us note that in the scalar case the subordination identity (3.12) reduces to

$$E_\beta(-\lambda^\alpha t^\beta) = \int_0^\infty \psi_{\alpha,\beta}(t,\tau) e^{-\lambda\tau}\, d\tau, \quad t > 0. \tag{3.15}$$

Then the normalization identity can be derived by letting $\lambda \to 0$ in (3.15), which implies

$$\int_0^\infty \psi_{\alpha,\beta}(t,\tau)\, d\tau = E_\beta(0) = 1.$$

The nonnegativity of the function $\psi_{\alpha,\beta}(t,\tau)$ can be established from its Laplace transform (3.15) by applying Bernstein's theorem. Indeed, the Mittag-Leffler function $E_\beta(-\lambda^\alpha t^\beta)$ is completely monotone as a function of $\lambda > 0$ for any fixed $t > 0$ (as a composition of the completely monotone function $E_\beta(-ax)$ and the Bernstein function λ^α, see property **(P5)** in Proposition 2.1).

The subordination identity (3.12) and (3.14) imply that if $\|S_{1,1}(t)\| \leq C$, $t \geq 0$, then

$$\|S_{\alpha,\beta}(t)\| \leq \int_0^\infty \psi_{\alpha,\beta}(t,\tau) \|S_{1,1}(\tau)\|\, d\tau \leq C \int_0^\infty \psi_{\alpha,\beta}(t,\tau)\, d\tau = C$$

for any $t \geq 0$, $0 < \alpha \leq 1$ and $0 < \beta \leq 1$. □

Let us note that in the case of ordered Banach space X, subordination identity (3.12) shows that positivity of the C_0-semigroup $S_{1,1}(t)$ implies positivity of the solution operator $S_{\alpha,\beta}(t)$ for any $0 < \alpha \leq 1$ and $0 < \beta \leq 1$.

Proposition 3.1. *The subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ obeys the following Laplace transform pairs*

$$\int_0^\infty \psi_{\alpha,\beta}(t,\tau)e^{-\lambda\tau}\,d\tau = E_\beta(-\lambda^\alpha t^\beta), \tag{3.16}$$

and

$$\int_0^\infty \psi_{\alpha,\beta}(t,\tau)e^{-st}\,dt = s^{\beta-1}\tau^{\alpha-1}E_{\alpha,\alpha}(-s^\beta\tau^\alpha). \tag{3.17}$$

Proof. For convenience we use in this chapter the following notations for the Laplace transform

$$\mathcal{L}\{f(t,\tau); t \to s\} = \int_0^\infty e^{-st} f(t,\tau)\,dt$$

and the double Laplace transform

$$\mathcal{L}^2\{f(t,\tau); t \to s, \tau \to \lambda\} = \int_0^\infty \int_0^\infty e^{-(st+\lambda\tau)} f(t,\tau)\,dt\,d\tau,$$

where, due to Fubini's theorem the order of integration may be switched for sufficiently well behaved functions. The Laplace transform of the subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ with respect to the variable τ is obtained in (3.15).

In order to find the Laplace transform of $\psi_{\alpha,\beta}(t,\tau)$ with respect to t, we note first that (3.15) and (1.27) imply

$$\mathcal{L}^2\{\psi_{\alpha,\beta}(t,\tau); t \to s, \tau \to \lambda\} = \mathcal{L}\{E_\beta(-\lambda^\alpha t^\beta); t \to s\} = \frac{s^{\beta-1}}{s^\beta + \lambda^\alpha}. \tag{3.18}$$

Then, taking inverse Laplace transform $\mathcal{L}^{-1}\{\cdot; \lambda \to \tau\}$ in (3.18) we deduce by the use of (1.27):

$$\mathcal{L}\{\psi_{\alpha,\beta}(t,\tau); t \to s\} = s^{\beta-1}\tau^{\alpha-1}E_{\alpha,\alpha}(-s^\beta\tau^\alpha).$$

□

Let us note that in the limiting case $\alpha = 1$ and $\beta = 1$ the subordination kernels are Dirac delta functions

$$f_1(t,\tau) = \varphi_1(t,\tau) = \psi_{1,1}(t,\tau) = \delta(t-\tau). \tag{3.19}$$

Moreover, the kernels $f_\alpha(t,\tau)$ and $\varphi_\beta(t,\tau)$ are particular cases of the composite kernel $\psi_{\alpha,\beta}(t,\tau)$, namely

$$f_\alpha(t,\tau) = \psi_{\alpha,1}(t,\tau), \quad \varphi_\beta(t,\tau) = \psi_{1,\beta}(t,\tau). \tag{3.20}$$

Therefore, the Laplace transform pairs for $f_\alpha(t,\tau)$ and $\varphi_\beta(t,\tau)$ can be derived from the identities (3.15), (3.18), and (3.17), taking $\beta = 1$ or $\alpha = 1$, respectively.

Remark 3.1. *Let us emphasize that the integral expression in (3.13) is not commutative:* $\psi_{\alpha,\beta}(t,\tau) \not\equiv \int_0^\infty f_\alpha(t,\sigma)\varphi_\beta(\sigma,\tau)\,d\sigma$*. This is due to the fact that the order of the two steps in the derivation procedure of subordination identity (3.12) is essential.*

For example, let us consider the case $\alpha = \beta = 1/2$, in which the subordination kernels can be expressed in terms of elementary functions as follows (e.g. [108, 2]):

$$f_{1/2}(t,\tau) = \frac{te^{-t^2/4\tau}}{2\sqrt{\pi}\tau^{3/2}}, \quad \varphi_{1/2}(t,\tau) = \frac{1}{\sqrt{\pi t}}e^{-\tau^2/4t}. \tag{3.21}$$

Plugging expressions (3.21) in the composition rule (3.13) we get

$$\psi_{1/2,1/2}(t,\tau) = \int_0^\infty \varphi_{1/2}(t,\sigma)f_{1/2}(\sigma,\tau)\,d\sigma = \frac{\sqrt{t}}{\pi\sqrt{\tau}(t+\tau)}. \tag{3.22}$$

The last formula can also be directly derived from eq. (3.54). On the other hand,

$$\int_0^\infty f_{1/2}(t,\sigma)\varphi_{1/2}(\sigma,\tau)\,d\sigma = \frac{2t}{\pi(t^2+\tau^2)}, \tag{3.23}$$

which can be obtained by introducing a new integration variable $(t^2+\tau^2)/4\sigma$.

A comparison of identities (3.22) and (3.23) confirms the non-commutativity pointed out in the remark.

Next we establish some properties of the subordination kernels based on the Laplace transform pairs (3.15), (3.18), (3.17), and identities (3.20).

Proposition 3.2. *Assume* $0 < \alpha, \alpha', \beta, \beta' \le 1$*. Then*

$$f_{\alpha\alpha'}(t,\tau) = \int_0^\infty f_\alpha(t,\sigma)f_{\alpha'}(\sigma,\tau)\,d\sigma, \tag{3.24}$$

$$\varphi_{\beta\beta'}(t,\tau) = \int_0^\infty \varphi_\beta(t,\sigma)\varphi_{\beta'}(\sigma,\tau)\,d\sigma. \tag{3.25}$$

Proof. To prove (3.24) we apply Laplace transform with respect to τ and obtain by using (3.15) with $\beta = 1$ and Fubini's theorem $\mathcal{L}\{f_{\alpha\alpha'}(t,\tau); \tau \to \lambda\} = e^{-t\lambda^{\alpha\alpha'}}$ and

$$\mathcal{L}\left\{\int_0^\infty f_\alpha(t,\sigma) f'_\alpha(\sigma,\tau)\, d\sigma; \tau \to \lambda\right\} = \int_0^\infty f_\alpha(t,\sigma) e^{-\sigma\lambda^{\alpha'}}\, d\sigma = e^{-t\lambda^{\alpha\alpha'}}.$$

In order to prove (3.25) we apply double Laplace transform, which gives by using (3.15) and (3.18) with $\alpha = 1$

$$\begin{aligned}
&\mathcal{L}^2\left\{\int_0^\infty \varphi_\beta(t,\sigma)\varphi_{\beta'}(\sigma,\tau)\, d\sigma; t \to s, \tau \to \lambda\right\} \\
&= s^{\beta-1}\int_0^\infty e^{-\sigma s^\beta} E_\beta(-\lambda\sigma^{\beta'})\, d\sigma = s^{\beta-1}\frac{s^{\beta(\beta'-1)}}{s^{\beta\beta'}+\lambda} = \frac{s^{\beta\beta'-1}}{s^{\beta\beta'}+\lambda} \\
&= \mathcal{L}^2\{\varphi_{\beta\beta'}(t,\tau); t \to s, \tau \to \lambda\}.
\end{aligned}$$

To finish the proof it remains to apply the uniqueness property of Laplace transform. □

Let us note that (3.24) is equivalent to the following natural operator identity

$$((-A)^\alpha)^{\alpha'} = ((-A)^{\alpha'})^\alpha = (-A)^{\alpha\alpha'}, \quad 0 < \alpha, \alpha' \le 1, \tag{3.26}$$

for a generator $-A$ of a bounded C_0-semigroup. Indeed, (3.24) together with subordination formula (3.5) shows that any of the operators in (3.26) is infinitesimal generator of one and the same semigroup: $S_{\alpha\alpha',1}(t)$. For a different proof of (3.26) see e.g. [108], Chapter IX.

Identity (3.25) is related to successive application of the subordination principle for time-fractional evolution equations and is in agreement with Theorem 2.3. Let us note that the composite function $\psi_{\alpha,\beta}$ does not satisfy a property, analogous to those in Proposition 3.2. This is due to the non-commutativity of definition (3.13), see Remark 3.1.

Corollary 3.1. *Let $0 < \alpha \le \beta < 1$. Under the conditions of Theorem 3.1 the solution operator $S_{\alpha,\beta}(t)$ admits the representation*

$$S_{\alpha,\beta}(t) = \int_0^\infty \psi_{\beta,\beta}(t,\tau) S_{\alpha/\beta,1}(\tau)\, d\tau, \quad t > 0, \tag{3.27}$$

where $S_{\alpha/\beta,1}(t)$ is the C_0-semigroup generated by the operator $-A^{\alpha/\beta}$ and the function $\psi_{\beta,\beta}(t,\tau)$ is defined in (3.54).

Proof. This subordination identity is derived by plugging in (3.12) the identity

$$f_\alpha(t,\tau) = \int_0^\infty f_\beta(t,\sigma) f_{\alpha/\beta}(\sigma,\tau)\, d\sigma$$

following from (3.24), and applying (3.13), Fubini's theorem, and (3.5). □

3.2 The subordination kernel

In this section some representation formulae for the subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ are derived.

3.2.1 Relations to L_α and M_β

We start with scaling laws for the subordination kernels. From the definitions (3.6), (3.9) and (3.13) of the subordination kernels f_α, φ_β and $\psi_{\alpha,\beta}$ we derive the following self-similarity properties

$$f_\alpha(t,\tau) = t^{-1/\alpha} f_\alpha(1,\tau t^{-1/\alpha}), \quad \varphi_\beta(t,\tau) = t^{-\beta}\varphi_\beta(1,\tau t^{-\beta}), \tag{3.28}$$
$$\psi_{\alpha,\beta}(t,\tau) = t^{-\beta/\alpha}\psi_{\alpha,\beta}(1,\tau t^{-\beta/\alpha}). \tag{3.29}$$

Introducing the functions of one variable L_α, M_β and $K_{\alpha,\beta}$ as follows

$$L_\alpha(r) = f_\alpha(1,r),\ M_\beta(r) = \varphi_\beta(1,r),\ K_{\alpha,\beta}(r) = \psi_{\alpha,\beta}(1,r), \tag{3.30}$$

we deduce from (3.28) and (3.29) the following representations for the subordination kernels

$$f_\alpha(t,\tau) = t^{-1/\alpha} L_\alpha(\tau t^{-1/\alpha}), \quad \varphi_\beta(t,\tau) = t^{-\beta} M_\beta(\tau t^{-\beta}), \tag{3.31}$$
$$\psi_{\alpha,\beta}(t,\tau) = t^{-\beta/\alpha} K_{\alpha,\beta}(\tau t^{-\beta/\alpha}). \tag{3.32}$$

In addition, identities (3.19) and (3.20) imply for the new functions

$$L_\alpha(r) = K_{\alpha,1}(r),\ M_\beta(r) = K_{1,\beta}(r), \tag{3.33}$$

and $L_1(r) = M_1(r) = K_{1,1}(r) = \delta(r-1)$.

From (3.30) and (3.15) we deduce:

$$\mathcal{L}\{K_{\alpha,\beta}(r); r \to \lambda\} = E_\beta(-\lambda^\alpha), \tag{3.34}$$

i.e. $K_{\alpha,\beta}(r)$ can be defined as the inverse Laplace transform of the Mittag-Leffler function $E_\beta(-\lambda^\alpha)$.

Theorem 3.2. *The subordination kernel admits the representation*

$$\psi_{\alpha,\beta}(t,\tau) = t^{-\beta/\alpha} K_{\alpha,\beta}(\tau t^{-\beta/\alpha}),$$

where $K_{\alpha,\beta}(r)$ *is a unilateral probability density function, i.e.*

$$K_{\alpha,\beta}(r) \geq 0, \quad \int_0^\infty K_{\alpha,\beta}(r)\, dr = 1,$$

defined as the inverse Laplace transform of a Mittag-Leffler function:

$$E_\beta(-\lambda^\alpha) = \int_0^\infty e^{-\lambda r} K_{\alpha,\beta}(r)\, dr.$$

The Laplace transform pairs (1.41) and (1.39) for the functions $L_\alpha(r)$ and $M_\beta(r)$ can be derived from (3.34), taking $\beta = 1$ and $\alpha = 1$, respectively. Therefore, we recognize the Lévy extremal stable density $L_\alpha(r)$ and the Mainardi function $M_\beta(r)$, see Section 1.5.

In the next theorem we derive representations for the function $K_{\alpha,\beta}(r)$ in terms of $L_\alpha(r)$ and $M_\beta(r)$.

Theorem 3.3. *The function* $K_{\alpha,\beta}(r)$ *admits the following representations*

$$K_{\alpha,\beta}(r) = \int_0^\infty \sigma^{-1/\alpha} L_\alpha(r\sigma^{-1/\alpha}) M_\beta(\sigma)\, d\sigma, \tag{3.35}$$

$$K_{\alpha,\beta}(r) = \int_0^\infty \sigma^{\beta/\alpha} L_\alpha(r\sigma^{\beta/\alpha}) L_\beta(\sigma)\, d\sigma, \tag{3.36}$$

$$K_{\alpha,\beta}(r) = \alpha r^{\alpha-1} \int_0^\infty \sigma M_\alpha(\sigma) M_\beta(\sigma r^\alpha)\, d\sigma. \tag{3.37}$$

Moreover, in the particular case $\alpha = \beta$ *it holds*

$$K_{\alpha,\alpha}(r) = \frac{1}{\pi} \frac{r^{\alpha-1} \sin \alpha\pi}{r^{2\alpha} + 2r^\alpha \cos \alpha\pi + 1} \tag{3.38}$$

and if $0 < \alpha \leq \beta \leq 1$ *then*

$$K_{\alpha,\beta}(r) = \int_0^\infty \sigma^{-\beta/\alpha} L_{\alpha/\beta}(r\sigma^{-\beta/\alpha}) K_{\beta,\beta}(\sigma)\, d\sigma. \tag{3.39}$$

Here $L_\alpha(r)$ *is the Lévy extremal stable density and* $M_\beta(r)$ *is the Mainardi function.*

Proof. Expression (3.35) follows directly from (3.13), (3.31) and (3.32). Representations (3.36) and (3.37) can be deduced from (3.35) after applying the formula (1.42) Representation (3.38) follows directly from (1.28) and (3.34) for $\alpha = \beta$. To prove (3.39) we find the inverse Laplace transform of $E_\beta(-\lambda^\alpha)$ by using (1.28) and the following property (see e.g. [26]): If $I(\lambda) = \mathcal{L}\{H(r); r \to \lambda\}$ and $I(\lambda^\alpha) = \mathcal{L}\{H_\alpha(r); r \to \lambda\}$ then

$$H_\alpha(r) = \int_0^\infty \sigma^{-1/\alpha} L_\alpha(r\sigma^{-1/\alpha}) H(\sigma)\, d\sigma, \quad 0 < \alpha \le 1. \tag{3.40}$$

(In fact, formula (3.40) can be verified by proving that Laplace transforms of both sides are equal.) From (3.40) and (3.34) it follows for $0 < \alpha \le \beta \le 1$

$$K_{\alpha,\beta}(r) = \mathcal{L}^{-1}\{E_\beta(-(\lambda^{\alpha/\beta})^\beta; \lambda \to r\} = \int_0^\infty \sigma^{-\beta/\alpha} L_{\alpha/\beta}(r\sigma^{-\beta/\alpha}) K_{\beta,\beta}(\sigma)\, d\sigma$$

and the last identity is proved. □

3.2.2 Other representations and properties of $K_{\alpha,\beta}$

Next, representations of the function $K_{\alpha,\beta}(r)$ are deduced by direct inversion of the Laplace transform in (3.34).

Theorem 3.4. *Assume*

$$0 < \alpha < \max\{\beta, 1 - \beta/2\} < 1. \tag{3.41}$$

Then the function $K_{\alpha,\beta}(r)$ admits the representations

$$K_{\alpha,\beta}(r) = \frac{-1}{\pi} \int_0^\infty e^{-r\sigma} \Im\{E_\beta(-\sigma^\alpha e^{i\alpha\pi})\}\, d\sigma, \tag{3.42}$$

$$K_{\alpha,\beta}(r) = \frac{1}{\pi r} \sum_{n=0}^\infty (-1)^{n+1} \frac{\Gamma(\alpha n + 1)}{\Gamma(\beta n + 1)} \frac{\sin \alpha n \pi}{r^{\alpha n}}. \tag{3.43}$$

Proof. Applying the complex Laplace inversion formula to (3.34) yields:

$$K_{\alpha,\beta}(r) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{\lambda r} E_\beta(-\lambda^\alpha)\, d\lambda, \quad c > 0. \tag{3.44}$$

For the multivalued function $\lambda^\alpha = \exp(\alpha \ln \lambda)$ the principal branch is considered.

Let first $0 < \alpha, \beta < 1$ are arbitrary and fix some θ_0, which satisfies the inequalities

$$\frac{\pi}{2} < \theta_0 < \min\left\{\frac{1-\beta/2}{\alpha}\pi, \pi\right\}. \tag{3.45}$$

Since the Mittag-Leffler function is an entire function, $E_\beta(-\lambda^\alpha)$ is analytic for $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. Therefore, by the Cauchy's theorem, the integration in (3.44) can be replaced by integration on the composite contour $\Gamma_1^+ \cup \Gamma_2^+ \cup \Gamma_3^+ \cup \Gamma_4 \cup \Gamma_3^- \cup \Gamma_2^- \cup \Gamma_1^-$, where (with appropriate orientation)

$$\begin{aligned}
&\Gamma_1^\pm = \{\lambda = q \pm \mathrm{i}R,\quad q \in (0,c)\},\\
&\Gamma_2^\pm = \{\lambda = Re^{\pm\mathrm{i}\theta},\quad \theta \in (\pi/2, \theta_0)\},\quad R \to \infty,\\
&\Gamma_3^\pm = \{\lambda = \sigma e^{\pm\mathrm{i}\theta_0},\quad \sigma \in (\varepsilon, R)\},\\
&\Gamma_4 = \{\lambda = \varepsilon e^{\mathrm{i}\theta},\ \theta \in (-\theta_0, \theta_0)\},\quad \varepsilon \to 0.
\end{aligned}$$

For the integration on Γ_1^+ we obtain

$$\left|\int_{\Gamma_1^+} e^{\lambda r} E_\beta(-\lambda^\alpha)\, d\lambda\right| \le \int_0^c e^{qr} |E_\beta(-(q+\mathrm{i}R)^\alpha)|\, dq \to 0,\quad R\to\infty, \tag{3.46}$$

due to the asymptotic expansion (1.22) for the Mittag-Leffler function in the integrand, which is satisfied since $(q+\mathrm{i}R)^\alpha \sim R^\alpha e^{\mathrm{i}\alpha\pi/2}$ as $R \to \infty$ and $\pi - \alpha\pi/2 > \beta\pi/2$. The integral on Γ_1^- is treated in an analogous way. Further,

$$\left|\int_{\Gamma_2^+} e^{\lambda r} E_\beta(-\lambda^\alpha)\, d\lambda\right| \le \int_{\pi/2}^{\theta_0} e^{Rr\cos\theta} |E_\beta(-R^\alpha e^{\mathrm{i}\alpha\theta})| R\, d\theta \to 0,\quad R\to\infty, \tag{3.47}$$

where we have taken into account $\cos\theta < 0$ and the asymptotic expansion (1.22) for the Mittag-Leffler function under the integral sign, which is satisfied due to the inequality $\pi - \alpha\theta > \beta\pi/2$, implied by assumption (3.45). The integral on Γ_2^- is estimated analogously. Concerning the integral over Γ_4 we have

$$\left|\int_{\Gamma_4} e^{\lambda r} E_\beta(-\lambda^\alpha)\, d\lambda\right| \le \int_{-\theta_0}^{\theta_0} e^{\varepsilon r\cos\theta} |E_\beta(-\varepsilon^\alpha e^{\mathrm{i}\alpha\theta})| \varepsilon\, d\theta \to 0,\quad \varepsilon\to 0, \tag{3.48}$$

since the Mittag-Leffler function is bounded as $\varepsilon \to 0$. Therefore, (3.44), (3.46), (3.47), and (3.48) imply that $K_{\alpha,\beta}(r)$ is given by the integral over $\Gamma_3^+ \cup \Gamma_3^-$ with

$\varepsilon \to 0$ and $R \to \infty$, that is

$$
\begin{aligned}
K_{\alpha,\beta}(r) &= \lim_{\varepsilon\to 0, R\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_3^+ \cup \Gamma_3^-} e^{\lambda r} E_\beta(-\lambda^\alpha)\, d\lambda \\
&= \lim_{\varepsilon\to 0, R\to\infty} \frac{1}{2\pi \mathrm{i}} \Bigg(\int_\varepsilon^R e^{\sigma r(\cos\theta_0 + \mathrm{i}\sin\theta_0)} E_\beta(-\sigma^\alpha e^{\mathrm{i}\alpha\theta_0}) e^{\mathrm{i}\theta_0}\, d\sigma \\
&\qquad + \int_R^\varepsilon e^{\sigma r(\cos\theta_0 - \mathrm{i}\sin\theta_0)} E_\beta(-\sigma^\alpha e^{-\mathrm{i}\alpha\theta_0}) e^{-\mathrm{i}\theta_0}\, d\sigma \Bigg).
\end{aligned}
$$

Therefore, for $0 < \alpha, \beta < 1$ the function $K_{\alpha,\beta}(r)$ admits the integral representation

$$
K_{\alpha,\beta}(r) = \frac{1}{\pi} \int_0^\infty e^{r\sigma\cos\theta_0} \Im\{ e^{\mathrm{i}(\theta_0 + r\sigma\sin\theta_0)} E_\beta(-\sigma^\alpha e^{\mathrm{i}\alpha\theta_0})\}\, d\sigma, \tag{3.49}
$$

with θ_0 satisfying (3.45).

Suppose now that the parameters α and β obey the assumption (3.41). Assume first $\alpha < 1 - \beta/2$. Then $\min\left\{\frac{1-\beta/2}{\alpha}\pi, \pi\right\} = \pi$ in (3.45) and we can let $\theta_0 \to \pi$ in (3.49), which implies (3.42). Let now $1 - \beta/2 < \alpha < \beta$. Then we can repeat the above argument to prove (3.46) and (3.48) with $\theta_0 = \pi$. Let us prove that the integral in (3.47) with $\theta_0 = \pi$ vanishes as $R \to \infty$. In this case

$$
\left| \int_{\Gamma_2^+} e^{\lambda r} E_\beta(-\lambda^\alpha)\, d\lambda \right| \le \int_{\pi/2}^{\pi} e^{Rr\cos\theta} |E_\beta(-R^\alpha e^{\mathrm{i}\alpha\theta})| R\, d\theta. \tag{3.50}
$$

We split the integral in (3.50) into two integrals $\int_{\pi/2}^{\pi} = \int_{\pi/2}^{\theta_0} + \int_{\theta_0}^{\pi}$, where θ_0 satisfies (3.45). It is already proven in (3.47) that the first integral vanishes as $R \to \infty$. For the Mittag-Leffler function in the second integral the asymptotic expansion (1.22) holds, implying

$$
|E_\beta(-R^\alpha e^{\mathrm{i}\alpha\theta})| \le \frac{1}{\beta} \exp\left(R^{\alpha/\beta} \cos\frac{\pi - \alpha\theta}{\beta} \right) + \frac{C}{1 + R^\alpha}, \quad R \to \infty. \tag{3.51}
$$

Note that $\cos\frac{\pi-\alpha\theta}{\beta} \ge 0$ for $\theta \in (\theta_0, \pi)$, so that the exponent in (3.51) is not bounded as $R \to \infty$. However, due to the assumption $\alpha < \beta$, the dominant term of the integrand $e^{Rr\cos\theta}|E_\beta(-R^\alpha e^{\mathrm{i}\alpha\theta})|R$ as $R \to \infty$ is $e^{Rr\cos\theta}$ and therefore the integral vanishes. Therefore, again (3.49) holds with $\theta_0 = \pi$. This completes the proof of the integral representation (3.42).

The series representation (3.43) can be deduced from (3.42) by inserting the definition(1.21) of the Mittag-Leffler function under the integral sign. □

Next, the regularity of the function $K_{\alpha,\beta}(r)$ is discussed briefly. We start with the asymptotic identity implied by (1.22)

$$E_\beta(-\lambda^\alpha) \sim \frac{\lambda^{-\alpha}}{\Gamma(1-\beta)}, \quad \lambda \to +\infty, \tag{3.52}$$

which, by (3.34) and applying Karamata-Feller Tauberian theorem, is equivalent to

$$K_{\alpha,\beta}(r) \sim \frac{r^{\alpha-1}}{\Gamma(\alpha)\Gamma(1-\beta)}, \quad r \to 0+. \tag{3.53}$$

Therefore, if $\alpha < 1$ and $\beta < 1$, then $K_{\alpha,\beta}(r)$ has a singularity at the origin: $K_{\alpha,\beta}(r) \to +\infty$ as $r \to 0+$. This is in contrast with the regular behaviour of any of the functions $L_\alpha(r) = K_{\alpha,1}(r)$ and $M_\beta(r) = K_{1,\beta}(r)$, which can be seen again from (3.53), taking $\beta = 1$ or $\alpha = 1$ and noting that $\Gamma(0) = \infty$.

Moreover, the Laplace transforms (1.41) and (1.39) of $L_\alpha(r)$ and $M_\beta(r)$ satisfy (ii) of Proposition 1.3, which means that these functions admit bounded analytic extensions to appropriate sectors of the complex plane. In contrast, if $\alpha < 1$ and $\beta < 1$, then (3.52) implies that $\lambda E_\beta(-\lambda^\alpha) \to +\infty$ as $\lambda \to +\infty$, therefore (ii) is not satisfied, thus, $K_{\alpha,\beta}(r)$ does not admit a bounded analytic extension to any sector of the complex plane.

In contrast to the singular behaviour of $K_{\alpha,\beta}(r)$ when $\alpha < 1$ and $\beta < 1$, the related subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ exhibits a regular behaviour in t (see Proposition 3.3). As an illustration let us consider the particular case $\alpha = \beta$, in which formula (3.38) together with (3.32) yields

$$\psi_{\alpha,\alpha}(t,\tau) = t^{-1}K_{\alpha,\alpha}(\tau t^{-1}) = \frac{1}{\pi}\frac{t^\alpha \tau^{\alpha-1}\sin\alpha\pi}{t^{2\alpha} + 2t^\alpha\tau^\alpha\cos\alpha\pi + \tau^{2\alpha}}. \tag{3.54}$$

Indeed, $K_{\alpha,\alpha}(r) \to \infty$ as $r \to 0$, while $\psi_{\alpha,\alpha}(t,\tau) \to 0$ as $t \to 0$ and $t \to \infty$.

3.2.3 Integral representation for the subordination kernel

Representations of the subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ are useful in view of the integral expression (3.77) for the fundamental solution. In a limited number of particular cases the subordination kernel can be expressed in terms of elementary functions, e.g. the relation (3.54), (3.21), (3.22). However, for arbitrary values of the fractional parameters explicit expressions are not available and other types of representations are needed.

Next we deduce an integral representation of the subordination kernel $\psi_{\alpha,\beta}$ by inversion of the Laplace transform pair (3.17). We choose (3.17) instead of (3.16), because of the faster decay for large arguments of the correponding Mittag-Leffler function, see (1.24).

Assume $0 < \alpha, \beta \leq 1$ and $\alpha\beta \neq 1$. Applying the complex Laplace inversion formula to (3.17) yields:

$$\psi_{\alpha,\beta}(t,\tau) = \frac{\tau^{\alpha-1}}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st} s^{\beta-1} E_{\alpha,\alpha}(-\tau^{\alpha} s^{\beta})\, ds, \quad c > 0, \tag{3.55}$$

where $s^{\beta} = \exp(\beta \ln s)$ means the principal branch of the corresponding multi-valued function defined in the whole complex plane cut along the negative real semi-axis. Since the Mittag-Leffler function is an entire function, $E_{\alpha,\alpha}(-\tau^{\alpha} s^{\beta})$ is analytic for $s \in \mathbb{C}\backslash(-\infty, 0]$.

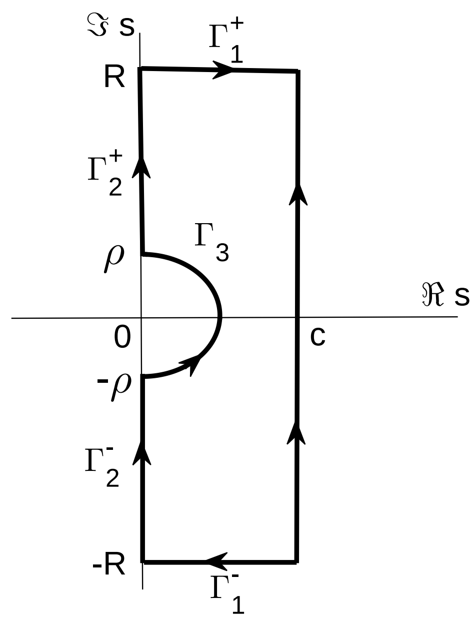


Figure 3.1: Contour Γ.

Therefore, by the Cauchy's theorem, the integral in (3.55) can be replaced by an integral over the composite contour $\Gamma = \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3 \cup \Gamma_2^+ \cup \Gamma_1^+$, where

$$\Gamma_1^{\pm} = \{s = q \pm \mathrm{i}R, \quad q \in (0, c)\}, \quad \Gamma_2^{\pm} = \{s = re^{\pm \mathrm{i}\pi/2}, \quad r \in (\rho, R)\},$$
$$\Gamma_3 = \{s = \rho e^{\mathrm{i}\theta}, \; \theta \in (-\pi/2, \pi/2)\},$$

with appropriate orientation (see Figure 3.1) and letting $\rho \to 0$, $R \to \infty$.

Since $(q+\mathrm{i}R)^\beta \sim R^\beta e^{\mathrm{i}\beta\pi/2}$ as $R \to \infty$, for the integration on Γ_1^+ as $R \to \infty$ we obtain

$$\left| \int_{\Gamma_1^+} e^{st} s^{\beta-1} E_{\alpha,\alpha}(-\tau^\alpha s^\beta)\, ds \right| \le C \int_0^c e^{qt} R^{\beta-1} |E_{\alpha,\alpha}(-\tau^\alpha R^\beta e^{\mathrm{i}\beta\pi/2})|\, dq \to 0 \tag{3.56}$$

as $R \to \infty$ due to the asymptotic expansion (1.24) for the Mittag-Leffler function, which is satisfied since $|\arg(\tau^\alpha R^\beta e^{\mathrm{i}\beta\pi/2})| = \beta\pi/2 < (1-\alpha/2)\pi$. The integral on Γ_1^- is treated in the same way.

Concerning the integral over Γ_3 we have for $\rho \to 0$

$$\left| \int_{\Gamma_3} e^{st} s^{\beta-1} E_{\alpha,\alpha}(-\tau^\alpha s^\beta)\, ds \right| \le \int_{-\pi/2}^{\pi/2} e^{\rho t \cos\theta} \varepsilon^\beta |E_{\alpha,\alpha}(-\tau^\alpha \rho^\beta e^{\mathrm{i}\beta\theta})|\, d\theta \to 0, \tag{3.57}$$

since the Mittag-Leffler function under the integral sign is bounded as $\rho \to 0$. Therefore, (3.55), (3.56), and (3.57) imply that $\psi_{\alpha,\beta}(t,\tau)$ is given by the integral over $\Gamma_2^+ \cup \Gamma_2^-$ along the imaginary axis with $\rho \to 0$ and $R \to \infty$. This implies

$$\begin{aligned}
\psi_{\alpha,\beta}(t,\tau) &= \frac{\tau^{\alpha-1}}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} e^{st} s^{\beta-1} E_{\alpha,\alpha}(-\tau^\alpha s^\beta)\, ds \\
&= \frac{\tau^{\alpha-1}}{2\pi\mathrm{i}} \left(\int_0^\infty \exp(rte^{\mathrm{i}\pi/2}) r^{\beta-1} e^{\mathrm{i}\beta\pi/2} E_{\alpha,\alpha}(-\tau^\alpha r^\beta e^{\mathrm{i}\beta\pi/2})\, dr \right. \\
&\qquad \left. + \int_0^\infty \exp(rte^{-\mathrm{i}\pi/2}) r^{\beta-1} e^{-\mathrm{i}\beta\pi/2} E_{\alpha,\alpha}(-\tau^\alpha r^\beta e^{-\mathrm{i}\beta\pi/2}\, dr \right).
\end{aligned}$$

Therefore

$$\psi_{\alpha,\beta}(t,\tau) = \frac{\tau^{\alpha-1}}{\pi} \int_0^\infty r^{\beta-1} \Im \left\{ e^{\mathrm{i}(rt+\beta\pi/2)} E_{\alpha,\alpha}(-\tau^\alpha r^\beta e^{\mathrm{i}\beta\pi/2}) \right\} dr. \tag{3.58}$$

We observe that the integral in (3.58) is convergent since the integrand behaves as $r^{\beta-1}$ for $r \to 0$ and as $r^{-\beta-1}$ for $r \to \infty$ due to the asymptotic expansion (1.24) for the Mittag-Leffler function. In this way, from (3.58) we obtain the following

Theorem 3.5. *Let $0 < \alpha \le 1$, $0 < \beta \le 1$, and $\alpha\beta \ne 1$. Then the subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ admits the integral representation*

$$\psi_{\alpha,\beta}(t,\tau) = \frac{\tau^{\alpha-1}}{\pi} \int_0^\infty r^{\beta-1} \left(C_\beta(r,t) I_{\alpha,\beta}(r,\tau) + S_\beta(r,t) R_{\alpha,\beta}(r,\tau) \right) dr, \tag{3.59}$$

where $C_\beta(r,t) = \cos(rt + \beta\pi/2)$, $S_\beta(r,t) = \sin(rt + \beta\pi/2)$, *and*

$$I_{\alpha,\beta}(r,\tau) = \Im\{\tau^{\alpha-1}E_{\alpha,\alpha}(-\tau^\alpha r^\beta e^{i\beta\pi/2})\} = \sum_{k=0}^{\infty} \frac{(-1)^k \tau^{\alpha k+\alpha-1} r^{\beta k} \sin k\beta\pi/2}{\Gamma(\alpha k + \alpha)},$$

$$R_{\alpha,\beta}(r,\tau) = \Re\{\tau^{\alpha-1}E_{\alpha,\alpha}(-\tau^\alpha r^\beta e^{i\beta\pi/2})\} = \sum_{k=0}^{\infty} \frac{(-1)^k \tau^{\alpha k+\alpha-1} r^{\beta k} \cos k\beta\pi/2}{\Gamma(\alpha k + \alpha)}.$$

For the numerical implementation of formula (3.59) the above real and imaginary parts can be numerically calculated employing a method of computation of the Mittag-Leffler function of complex argument.

In the particular case $\alpha = 1$ (time-fractional diffusion) representation (3.59) yields the following simpler formula for the subordination kernel

$$\varphi_\beta(t,\tau) = \frac{1}{\pi}\int_0^\infty r^{\beta-1} \sin\left(rt + \beta\pi/2 - \tau r^\beta \sin\beta\pi/2\right) \exp(-\tau r^\beta \cos\beta\pi/2)\, dr.$$

Let us recall the relation $\varphi_\beta(t,\tau) = t^{-\beta} M_\beta(\tau t^{-\beta})$, where $M_\beta(\cdot)$ denotes the Mainardi function. In this way, as a byproduct, we obtained also an integral representation of this function.

The technique used in this section for deriving of the integral representation for the subordination kernel does not rely on the scaling property and can be extended to equations with more general nonlocal operators in space and operators with a general memory kernel in time, which are considered in Chapter 5.

3.3 Analyticity of the solution operator

First we prove that the subordination kernel $\psi_{\alpha,\beta}(t,\tau)$, considered as a function of t, admits a bounded analytic extension to a sector in the complex plane.

Proposition 3.3. *Assume* $0 < \alpha, \beta \leq 1$, $\alpha\beta \neq 1$, *and let*

$$\theta_0 = \min\left\{\frac{(2-\alpha-\beta)\pi}{2\beta}, \frac{\pi}{2}\right\}. \qquad (3.60)$$

For any $\tau > 0$ *the function* $\psi_{\alpha,\beta}(t,\tau)$ *as a function of* t *admits analytic extension to the sector* $|\arg t| < \theta_0$, *which is bounded on each sector* $|\arg t| \leq \theta$, $0 < \theta < \theta_0$.

Proof. We apply Proposition 1.3 for the Laplace transform pair (3.17): $F(s) = s^{\beta-1}\tau^{\alpha-1}E_{\alpha,\alpha}(-s^\beta\tau^\alpha)$, and $f(t) = \psi_{\alpha,\beta}(t,\tau)$, where $\tau > 0$ is considered as a parameter. The function $F(s)$ admits analytic extension to $\mathbb{C}$ cut along the negative real axis. According to the estimate (1.23) for the Mittag-Leffler function

$$|sF(s)| \le C\tau^{-1}\frac{\tau^\alpha|s|^\beta}{1+\tau^\alpha|s|^\beta} < C\tau^{-1},$$

for all $s \in \mathbb{C}$ such that

$$|\arg(s)| \le \min\left\{\frac{(2-\alpha)\pi}{2\beta} - \varepsilon, \pi\right\}.$$

To obtain the desired statement it remains to apply implication (i) ⇒ (ii). □

Proposition 3.3 together with subordination formula (3.12) implies that $S_{\alpha,\beta}(t)$ is a bounded analytic solution operator according to the above definition. The proof is similar to the one in [9], Theorem 3.2, where, based on analogous property for the function $\varphi_\beta(t,\tau)$, analyticity of the subordinated solution operator for the time-fractional evolution equation is established.

Taking into account relations (3.20) we can derive the corresponding sectors of existence of bounded analytic extensions for the functions $f_\alpha(t,\tau)$ and $\varphi_\beta(t,\tau)$ (setting in (3.60) $\beta = 1$ or $\alpha = 1$). In this way known results for analyticity of the semigroup $S_{\alpha,1}(t)$ [108, 66] and of the solution operator $S_{1,\beta}(t)$ [9, 66] are recovered.

Proposition (3.3) implies that if $0 < \alpha, \beta \le 1$, $\alpha\beta \ne 1$, and if A is the generator of a bounded C_0-semigroup $S(t)$ on X, then problem (3.4) admits a bounded analytic solution operator $S_{\alpha,\beta}(t)$ of angle θ, where

$$\theta = \min\left\{\frac{(2-\alpha-\beta)\pi}{2\beta}, \frac{\pi}{2}\right\}. \tag{3.61}$$

Now we suppose that the C_0-semigroup generated by the operator A is a bounded analytic semigroup. In this case we expect that the solution operator $S_{\alpha,\beta}(t)$ will be bounded analytic in a larger sector of the complex plane. To formulate the precise result we prove some estimates for the functions $M_\gamma(z)$ and $L_\gamma(z)$ in appropriate sectors of the complex plane.

The sectors in the complex plane, in which the functions $M_\gamma(z)$ and $L_\gamma(z)$ are bounded analytic functions can be easily found from the behavior of their

Laplace transforms (1.39) and (1.41), applying Proposition 1.3. The asymptotic expansion for the Mittag-Leffler function (1.22) implies that

$$s\mathcal{L}\{M_\gamma(\cdot)\}(s) = sE_\gamma(-s)$$

is bounded for $s \in \Sigma((1-\gamma/2)\pi)$. On the other hand,

$$s\mathcal{L}\{L_\gamma(\cdot)\}(s) = s\exp(-s^\gamma)$$

is bounded when $|\arg(s^\gamma)| < \pi/2$, i.e. for $s \in \Sigma(\gamma^{-1}\pi/2)$. Based on the above statement, we deduce that the functions $M_\gamma(z)$ and $L_\gamma(z)$ are analytic in the sectors $\Sigma(\theta_M)$ and $\Sigma(\theta_L)$, respectively, and bounded on each proper subsector of these sectors, where

$$\theta_M(\gamma) = (1-\gamma)\pi/2 \;\text{ for }\; M_\gamma(z); \qquad \theta_L(\gamma) = (1/\gamma-1)\pi/2 \;\text{ for }\; L_\gamma(z). \tag{3.62}$$

In fact, the asymptotic expansions for the functions M_γ and L_γ imply a stronger property in the same sectors as we see next.

Concerning M_γ, expansion (1.40) implies that there exists r^*, such that for any $r > r^*$

$$|M_\gamma(re^{i\theta}))| \le a(\gamma) r^{\frac{\gamma-1/2}{1-\gamma}} \exp\left(-b(\gamma) r^{\frac{1}{1-\gamma}} \cos\left(\frac{\theta}{1-\gamma}\right)\right).$$

Therefore, this function is integrable at $r \to \infty$ provided

$$|\theta| < (1-\gamma)\pi/2 = \theta_M(\gamma),$$

recovering the same angle as in (3.62). In addition, (1.38) shows that $|M_\gamma(re^{i\theta})|$ is a bounded function as $r \to 0$. Therefore, the following integral is uniformly bounded

$$\int_0^\infty |M_\gamma(re^{i\theta})|\, dr \le C_M, \quad |\theta| < \theta_M(\gamma). \tag{3.63}$$

Concerning the function L_γ, expansion (1.45) shows that $|L_\gamma(re^{i\theta})|$ admits an integrable singularity for $r \to \infty$. For small r the asymptotic expression (1.46) implies the estimate

$$|L_\gamma(re^{i\theta})| \le \gamma a(\gamma) r^{-\frac{2-\gamma}{2(1-\gamma)}} \exp\left(-b(\gamma) r^{-\frac{\gamma}{1-\gamma}} \cos\left(\frac{\gamma\theta}{1-\gamma}\right)\right).$$

This shows that the function $|L_\gamma(re^{i\theta})|$ is integrable for $r \to 0$ provided

$$|\theta| < (1/\gamma - 1)\pi/2 = \theta_L(\gamma),$$

the same angle as defined in (3.62). Therefore, we established the uniform boundedness of the integral

$$\int_0^\infty |L_\gamma(re^{\mathrm{i}\theta})|\, dr \le C_L, \quad |\theta| < \theta_L(\gamma). \tag{3.64}$$

The constants C_M and C_L in (3.63) and (3.64) depend on γ, but do not depend on θ.

In this way we obtained the sectors $\Sigma(\theta_M(\gamma))$ and $\Sigma(\theta_L(\gamma))$ of "good behavior" of the functions $M_\gamma(z)$ and $L_\gamma(z)$, respectively.

Theorem 3.6. *If* $0 < \alpha, \beta \le 1$, $\alpha\beta \ne 1$, *and if A is the generator of a bounded analytic semigroup of angle* $\phi_0 \in (0, \pi/2]$, *then* $S_{\alpha,\beta}(t)$ *is a bounded analytic solution operator of angle* θ_0, *where*

$$\theta_0 = \min\left\{\frac{\alpha\phi_0}{\beta} + \frac{(2-\alpha-\beta)\pi}{2\beta}, \frac{\pi}{2}\right\}. \tag{3.65}$$

Proof. The proof of this statement is divided into two steps, based on subordination identities (3.5) and (3.8). We use the uniform boundedness of the integrals in (3.63) and (3.64), established in the previous section.

First step. Assume $S(t)$ is a bounded analytic semigroup of angle ϕ_0, i.e. $S(t)$ admits an analytic extension to the sector $\Sigma(\phi_0)$ and it is bounded on each proper subsector, i.e.

$$\|S(z)\| \le C, \quad z \in \overline{\Sigma(\phi)}, \quad \phi < \phi_0. \tag{3.66}$$

We start from the subordination identity (3.5). Let us consider the path in the complex plane

$$\Gamma_{R,\phi} = \{z = r, r \in [0, R]\} \cup \{z = Re^{\mathrm{i}\varphi}, \varphi \in [0, \phi]\} \cup \{z = re^{\mathrm{i}\phi}, r \in [0, R]\} \tag{3.67}$$

with counter-clockwise orientation, where $\phi \in (-\phi_0, \phi_0)$, $R > 0$. An application of Cauchy's theorem shows that $\int_{\Gamma_{R,\phi}} f_\alpha(t, z) S(z)\, dz = 0$ and, letting $R \to \infty$, we obtain from (3.5) and (3.31)

$$S_{\alpha,1}(t) = \int_0^\infty f_\alpha(t, re^{\mathrm{i}\phi}) S(re^{\mathrm{i}\phi}) e^{\mathrm{i}\phi}\, dr = t^{-1/\alpha} \int_0^\infty L_\alpha(re^{\mathrm{i}\phi} t^{-1/\alpha}) S(re^{\mathrm{i}\phi}) e^{\mathrm{i}\phi}\, dr.$$

Now let

$$S_{\alpha,1}(z) = z^{-1/\alpha} \int_0^\infty L_\alpha(re^{i\phi} z^{-1/\alpha}) S(re^{i\phi}) e^{i\phi}\, dr, \tag{3.68}$$

where $\alpha\phi - (1-\alpha)\pi/2 < \arg z < \alpha\phi + (1-\alpha)\pi/2$. Let us set $z = \rho e^{i\psi}$ with $\rho > 0$. Then (3.68) implies $|\phi - \psi/\alpha| < \theta_L(\alpha)$ with θ_L defined in (3.60) and

$$\begin{aligned} S_{\alpha,1}(z) &= \rho^{-1/\alpha} e^{-\mathrm{i}\psi/\alpha} \int_0^\infty L_\alpha(re^{\mathrm{i}\phi}\rho^{-1/\alpha}e^{-\mathrm{i}\psi/\alpha}) S(re^{\mathrm{i}\phi}) e^{\mathrm{i}\phi}\, dr \\ &= e^{\mathrm{i}(\phi-\psi/\alpha)} \int_0^\infty L_\alpha(\sigma e^{\mathrm{i}(\phi-\psi/\alpha)}) S(\sigma\rho^{1/\alpha}e^{\mathrm{i}\phi})\, d\sigma, \end{aligned}$$

where we have set $\sigma = \rho^{-1/\alpha} r$. Therefore, applying (3.66) and (3.64) we deduce

$$\begin{aligned} \|S_{\alpha,1}(z)\| &\le \int_0^\infty |L_\alpha(\sigma e^{\mathrm{i}(\phi-\psi/\alpha)})|\, \|S(\sigma\rho^{1/\alpha}e^{\mathrm{i}\phi})\|\, d\sigma \\ &\le C\int_0^\infty |L_\alpha(\sigma e^{\mathrm{i}(\phi-\psi/\alpha)})|\, d\sigma \le C_1. \end{aligned} \tag{3.69}$$

Varying $\phi \in (-\phi_0, \phi_0)$ in (3.68) provides an analytic extension of $S_{\alpha,1}$ to the sector $\Sigma(\phi_\alpha)$, which is bounded on each proper subsector, where

$$\phi_\alpha = \alpha\phi_0 + (1-\alpha)\pi/2. \tag{3.70}$$

Second step. Let us apply now the subordination identity (3.8), where $S_{\alpha,1}(t)$ is a bounded analytic solution operator of angle ϕ_α, defined in (3.70). We proceed in a way analogous to the first step. Take $\phi \in (-\phi_\alpha, \phi_\alpha)$ and consider the path (3.67). By applying the Cauchy's theorem it follows that

$$\int_{\Gamma_{R,\phi}} \varphi_\beta(t,z) S_{\alpha,1}(z)\, dz = 0$$

for $\phi \in (-\phi_\alpha, \phi_\alpha)$. Therefore, for $R \to \infty$ we obtain from (3.8), taking into account (3.31),

$$S_{\alpha,\beta}(t) = \int_0^\infty \varphi_\beta(t, re^{\mathrm{i}\phi}) S_{\alpha,1}(re^{\mathrm{i}\phi}) e^{\mathrm{i}\phi}\, dr = t^{-\beta}\int_0^\infty M_\beta(re^{\mathrm{i}\phi}t^{-\beta}) S_{\alpha,1}(re^{\mathrm{i}\phi}) e^{\mathrm{i}\phi}\, dr.$$

Consider the operator-valued function

$$S_{\alpha,\beta}(z) = z^{-\beta}\int_0^\infty M_\beta(re^{\mathrm{i}\phi}z^{-\beta}) S_{\alpha,1}(re^{\mathrm{i}\phi}) e^{\mathrm{i}\phi}\, dr, \tag{3.71}$$

where $\phi/\beta - (1/\beta - 1)\pi/2 < \arg z < \phi/\beta + (1/\beta - 1)\pi/2$. Let $z = \rho e^{\mathrm{i}\psi}$, $\rho > 0$. Then (3.71) implies $|\phi - \beta\psi| < \theta_M(\beta)$ with θ_M defined in (3.60) and

$$\begin{aligned} S_{\alpha,\beta}(z) &= \rho^{-\beta} e^{-\mathrm{i}\beta\psi} \int_0^\infty M_\beta(re^{\mathrm{i}\phi}\rho^{-\beta}e^{-\mathrm{i}\beta\psi}) S_{\alpha,1}(re^{\mathrm{i}\phi}) e^{\mathrm{i}\phi}\, dr \\ &= e^{\mathrm{i}(\phi-\beta\psi)} \int_0^\infty M_\beta(\sigma e^{\mathrm{i}(\phi-\beta\psi)}) S_{\alpha,1}(\sigma\rho^\beta e^{\mathrm{i}\phi})\, d\sigma, \end{aligned}$$

where we have set $\sigma = \rho^{-\beta} r$. Applying (3.69) and (3.63) it follows

$$\begin{aligned} \|S_{\alpha,\beta}(z)\| &\le \int_0^\infty |M_\beta(\sigma e^{\mathrm{i}(\phi-\beta\psi)})| \, \|S_{\alpha,1}(\sigma\rho^\beta e^{\mathrm{i}\phi})\| \, d\sigma, \\ &\le C_1 \int_0^\infty |M_\beta(\sigma e^{\mathrm{i}(\phi-\beta\psi)})| \, d\sigma \le C_2. \end{aligned}$$

Therefore, varying $\phi \in (-\phi_\alpha, \phi_\alpha)$ in (3.71) provides an analytic extension of $S_{\alpha,\beta}$ to the sector $\Sigma(\theta_0)$, which is bounded on each proper subsector, where

$$\theta_0 = \phi_\alpha/\beta + (1/\beta - 1)\pi/2. \tag{3.72}$$

Combining the results of the above two steps and inserting the value (3.70) of ϕ_α in (3.72), we derive the angle of analyticity (3.65). In this way the statement is proven. □

Particular cases of this result can be found in [2], see Theorem 3.8.3, as well as in [9, 66, 10].

3.4 Multi-dimensional fundamental solution

First, let us recall that the Fourier transform of a function $v(x)$, $x \in \mathbb{R}^n$, is defined by

$$\mathcal{F}\{v\}(\xi) = \widetilde{v}(\xi) = \int_{\mathbb{R}^n} e^{\mathrm{i}\xi\cdot x} v(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

The Fourier transform pair corresponding to the Laplace operator Δ of a function $v(x)$, $x \in \mathbb{R}^n$, such that $\lim_{|x|\to\infty} v(x) = 0$, is (see e.g. [3, Chapter 15])

$$\mathcal{F}\{\Delta v\}(\xi) = -|\xi|^2 \mathcal{F}\{v\}(\xi), \quad \xi \in \mathbb{R}^n. \tag{3.73}$$

The main example of the considered abstract problem (3.4) is the space-time fractional diffusion equation

$${}^C\!D_t^\beta u(\mathbf{x}, t) = -(-\Delta)^\alpha u(\mathbf{x}, t), \quad t > 0, \ \ \mathbf{x} \in \mathbb{R}^n; \quad u(\mathbf{x}, 0) = v(\mathbf{x}); \tag{3.74}$$

where $0 < \alpha, \beta \le 1$, ${}^C\!D_t^\beta$ is the Caputo time-fractional derivative, and Δ is the full-space fractional Laplace operator in $\mathbb{R}^n$. Then the fractional power $(-\Delta)^\alpha$ coincides with the pseudo-differential operator defined as follows

$$\mathcal{F}\{(-\Delta)^\alpha f; \xi\} = |\xi|^{2\alpha} \mathcal{F}\{f; \xi\}, \quad \xi \in \mathbb{R}^n, \tag{3.75}$$

where $\mathcal{F}\{f;\xi\}$ denotes the Fourier transform of a function f at the point ξ. In particular, in the one-dimensional case $-(-\Delta)^\alpha$ coincides with the Riesz space-fractional derivative of order 2α. We refer to the survey paper [64], where the equivalence of ten different definitions of full-space fractional Laplacian is proven.

Assume X is one of the spaces $L^p(\mathbb{R}^n)$, $1 < p < \infty$, or the space $C_0(\mathbb{R}^n)$ of continuous functions vanishing at infinity. Let $A = \Delta$ be the Laplace operator defined on X with maximal domain. For details on the definition of the full-space Laplace operator we refer to [79]. In this case the operator $(-A)^\alpha$ is defined by (3.75). The operator A is a generator of a bounded analytic C_0-semigroup $S_{1,1}(t)$ with corresponding Green function [2]

$$\mathcal{G}_{1,1,n}(\mathbf{x},t) = \frac{1}{(4\pi t)^{n/2}} e^{-|\mathbf{x}|^2/4t}, \quad \mathbf{x} \in \mathbb{R}^n,\ t > 0. \tag{3.76}$$

For any $0 < \alpha \le 1$ and $0 < \beta \le 1$ the solution operator $S_{\alpha,\beta}(t)$ of problem (3.74) is given by

$$(S_{\alpha,\beta}(t)v)(\mathbf{x}) = \int_{\mathbb{R}^n} \mathcal{G}_{\alpha,\beta,n}(\mathbf{y},t) v(\mathbf{x}-\mathbf{y})\, d\mathbf{y}, \quad v \in X, \quad t > 0,\ \mathbf{x} \in \mathbb{R}^n,$$

where $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$ is the corresponding Green function. Therefore, the subordination formula (3.12) can be written in terms of Green functions as follows

$$\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t) = \int_0^\infty \psi_{\alpha,\beta}(t,\tau) \mathcal{G}_{1,1,n}(\mathbf{x},\tau)\, d\tau, \quad \mathbf{x} \in \mathbb{R}^n. \tag{3.77}$$

It is worth noting that some known basic properties of $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$ follow in a straightforward way from the subordination relation (3.77), taking into account that the subordination kernel is a PDF. In this way we can prove that for any dimension $n \ge 1$ the fundamental solution $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$ is a spatial PDF evolving in time:

$$\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t) \ge 0, \quad \int_{\mathbb{R}^n} \mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)\, d\mathbf{x} = 1.$$

Therefore, $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$, $0 < \alpha, \beta \le 1$, inherits this property of the classical Gaussian kernel $\mathcal{G}_{1,1,n}(\mathbf{x},t)$. In a similar way, estimates for the fundamental solution $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$ can be derived from known estimates for the Gaussian kernel $\mathcal{G}_{1,1,n}(\mathbf{x},t)$. For example, since $\|\mathcal{G}_{1,1,n}(\cdot,t)\|_{L^1(\mathbb{R}^n)} = 1$ (see e.g.[2], Remark 3.7.10.), the subordination formula (3.77) together with properties (3.14) imply

$$\begin{aligned}\|\mathcal{G}_{\alpha,\beta,n}(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\le \int_0^\infty \psi_{\alpha,\beta}(t,\tau)\|\mathcal{G}_{1,1,n}(\cdot,\tau)\|_{L^1(\mathbb{R}^n)}\,d\tau \\ &\le \int_0^\infty \psi_{\alpha,\beta}(t,\tau)\,d\tau = 1.\end{aligned}$$

3.4.1 Some closed-form solutions

Based on (3.77) and (3.76), we find closed form expressions for $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$. Assume first $\alpha=\beta$. In this case Eq. (3.74) is the so-called α-fractional diffusion equation studied in [69]. Taking into account (3.54), the subordination formula (3.77) reads

$$\mathcal{G}_{\alpha,\alpha,n}(\mathbf{x},t)=\frac{1}{\pi}\int_0^\infty \frac{t^\alpha\tau^{\alpha-1}\sin\alpha\pi}{t^{2\alpha}+2t^\alpha\tau^\alpha\cos\alpha\pi+\tau^{2\alpha}}\mathcal{G}_{1,1,n}(\mathbf{x},\tau)\,d\tau,\quad \mathbf{x}\in\mathbb{R}^n. \quad (3.78)$$

For $\mathbf{x}=\mathbf{0}$ the integral in (3.78) is convergent only if $\alpha>n/2$. Therefore, $\mathcal{G}_{\alpha,\alpha,n}(\mathbf{0},t)$ is finite only for $n=1$ and $\alpha>1/2$. The same conclusion can be found in [69].

Next, applying the subordination formula (3.78), we derive a closed-form expression for the two-dimensional Green function. Plugging (3.76) with $n=2$ in (3.78) yields

$$\begin{aligned}\mathcal{G}_{\alpha,\alpha,2}(\mathbf{x},t) &= \frac{t^\alpha}{4\pi^2}\int_0^\infty \frac{\tau^{\alpha-2}\sin\alpha\pi}{t^{2\alpha}+2t^\alpha\tau^\alpha\cos\alpha\pi+\tau^{2\alpha}}e^{-|\mathbf{x}|^2/4\tau}\,d\tau \\ &= \frac{1}{4\pi t}\int_0^\infty \frac{\sigma^\alpha\sin\alpha\pi}{\sigma^{2\alpha}+2\sigma^\alpha\cos\alpha\pi+1}e^{-(|\mathbf{x}|^2/4t)\sigma}\,d\sigma,\end{aligned}$$

where we have made the change of variables $\sigma=t/\tau$. Formula (1.28) gives the following expression in terms of Mittag-Leffler functions

$$\mathcal{G}_{\alpha,\alpha,2}(\mathbf{x},t)=\frac{1}{4\pi t}(|\mathbf{x}|^2/4t)^{\alpha-1}E_{\alpha,\alpha}(-(|\mathbf{x}|^2/4t)^\alpha),\quad \mathbf{x}\in\mathbb{R}^2. \quad (3.79)$$

Expression (3.79) can be found also in [70].

Further, we restrict our attention to the special case $\alpha=\beta=1/2$. Plugging (3.22) in the subordination formula (3.77) yields

$$\mathcal{G}_{1/2,1/2,n}(\mathbf{x},t)=\frac{\sqrt{t}}{\pi}\int_0^\infty \frac{1}{\sqrt{\tau}(t+\tau)}\mathcal{G}_{1,1,n}(\mathbf{x},\tau)\,d\tau,\quad \mathbf{x}\in\mathbb{R}^n. \quad (3.80)$$

Inserting (3.76) in (3.80) and introducing a new integration variable $\sigma = t/\tau$ gives

$$\mathcal{G}_{1/2,1/2,n}(\mathbf{x},t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2^n \pi^{n/2+1} t^{n/2}} U\left(\frac{n+1}{2}, \frac{n+1}{2}, \frac{|\mathbf{x}|^2}{4t}\right), \quad \mathbf{x} \in \mathbb{R}^n, \qquad (3.81)$$

where U is the Tricomi's confluent hypergeometric function (3.100).

Let us first note that relation (3.101) confirms that formula (3.81) with $n = 2$ is the same as (3.79) with $\alpha = 1/2$.

Applying (3.103), expression (3.81) for the Green function can be rewritten in terms of the incomplete Gamma function (3.102) as follows

$$\mathcal{G}_{1/2,1/2,n}(\mathbf{x},t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2^n \pi^{n/2+1} t^{n/2}} e^{|\mathbf{x}|^2/4t} \Gamma\left(\frac{1-n}{2}, \frac{|\mathbf{x}|^2}{4t}\right), \quad \mathbf{x} \in \mathbb{R}^n. \qquad (3.82)$$

In the one-dimensional case representations (3.81) and (3.82) reduce to

$$\mathcal{G}_{1/2,1/2,1}(x,t) = \frac{1}{2\pi^{3/2}\sqrt{t}} e^{x^2/4t} \mathbb{E}_1\left(\frac{x^2}{4t}\right), \qquad (3.83)$$

where $\mathbb{E}_1(z)$ is the exponential integral (3.105). The asymptotic expansion (3.107) implies for $x^2/4t \to \infty$ (i.e. as $x \to \infty$ and $t > 0$ is fixed or $t \to 0$ and $x \neq 0$ is fixed)

$$\mathcal{G}_{1/2,1/2,1}(x,t) \sim \frac{2\sqrt{t}}{\pi^{3/2} x^2}, \quad x^2/4t \to \infty.$$

Similar asymptotic behaviour is observed for $\alpha > 1/2$ in [69], Eq. 25.

On the other hand, the expansion (3.106) of the exponential integral gives for $x \in \mathbb{R}, x \neq 0$ and $t > 0$

$$\mathcal{G}_{1/2,1/2,1}(x,t) = \frac{e^{x^2/4t}}{2\pi^{3/2}\sqrt{t}} \left(-\gamma - \ln\left(\frac{x^2}{4t}\right) - \sum_{k=1}^{\infty} \frac{\left(x^2/4t\right)^k}{k(k!)}\right).$$

This expansion implies the following asymptotic behaviour

$$\mathcal{G}_{1/2,1/2,1}(x,t) \sim \frac{\ln 4t - \ln x^2}{2\pi^{3/2}\sqrt{t}}, \quad x^2/4t \to 0.$$

Therefore, $\mathcal{G}_{1/2,1/2,1}(x,t) \to \infty$ with logarithmic growth as $x \to 0$ for any fixed $t > 0$ and $\mathcal{G}_{1/2,1/2,1}(x,t) \to 0$ as $t \to \infty$ and $x \neq 0$ is fixed. Let us note that when $\alpha > 1/2$ the asymptotic behaviour of $\mathcal{G}_{\alpha,\alpha,1}(x,t)$ as $x \to 0$ is of a power law type (see [69], Eq. 24), which is in contrast to the observed here logarithmic growth for $\alpha = 1/2$.

In addition, inequalities (3.108) imply that the Green function $\mathcal{G}_{1/2,1/2,1}(x,t)$ is bracketed for any $x \in \mathbb{R}, x \neq 0$, and $t > 0$ as follows

$$\frac{1}{4\pi^{3/2}\sqrt{t}} \ln\left(1 + \frac{8t}{x^2}\right) < \mathcal{G}_{1/2,1/2,1}(x,t) < \frac{1}{2\pi^{3/2}\sqrt{t}} \ln\left(1 + \frac{4t}{x^2}\right).$$

Expressions for the multi-dimensional Green function $\mathcal{G}_{1/2,1/2,n}(\mathbf{x},t)$ in terms of the exponential integral $\mathbb{E}_1$ (for odd dimensions) or the Mittag-Leffler function $E_{1/2,1/2}$ (for even dimensions) can be obtained from (3.83) and (3.79) by applying representation (3.82) and the recurrence relation (3.104) between the incomplete Gamma functions. For example, in this way we derive from (3.82), (3.83), and (3.104), the following expression for the three-dimensional Green function

$$\mathcal{G}_{1/2,1/2,3}(\mathbf{x},t) = \frac{1}{2\pi^{5/2}\sqrt{t}|\mathbf{x}|^2} - \frac{1}{8\pi^{5/2}t^{3/2}} e^{|\mathbf{x}|^2/4t}\mathbb{E}_1\left(\frac{|\mathbf{x}|^2}{4t}\right), \quad \mathbf{x} \in \mathbb{R}^3.$$

Finally, we present an application of the subordination formula (3.27), which for $\beta = 2\alpha$ gives in terms of Green functions

$$\mathcal{G}_{\alpha/2,\alpha,n}(\mathbf{x},t) = \frac{1}{\pi}\int_0^\infty \frac{t^\alpha \tau^{\alpha-1}\sin\alpha\pi}{t^{2\alpha} + 2t^\alpha\tau^\alpha\cos\alpha\pi + \tau^{2\alpha}} \mathcal{G}_{1/2,1,n}(\mathbf{x},\tau)\, d\tau, \quad \mathbf{x} \in \mathbb{R}^n, \tag{3.84}$$

where $\mathcal{G}_{1/2,1,n}(\mathbf{x},t)$ is the n-dimensional Poisson kernel [2]

$$\mathcal{G}_{1/2,1,n}(\mathbf{x},t) = \frac{\Gamma\left(\frac{n+1}{2}\right)t}{\pi^{(n+1)/2}(t^2 + |\mathbf{x}|^2)^{(n+1)/2}}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{3.85}$$

For example, applying (3.84) for $n = 1$, we can recover the following known closed-form expression (see e.g. [76], Eq. (4.38))

$$\mathcal{G}_{\alpha/2,\alpha,1}(x,t) = \frac{1}{\pi}\frac{t^\alpha x^{\alpha-1}\sin(\alpha\pi/2)}{t^{2\alpha} + 2t^\alpha x^\alpha\cos(\alpha\pi/2) + x^{2\alpha}}, \quad x > 0, t > 0. \tag{3.86}$$

Indeed, starting from the following integral obtained from (3.84) and (3.85)

$$\mathcal{G}_{\alpha/2,\alpha,1}(x,t) = \frac{1}{\pi^2}\int_0^\infty \frac{t^\alpha\tau^{\alpha-1}\sin\alpha\pi}{t^{2\alpha} + 2t^\alpha\tau^\alpha\cos\alpha\pi + \tau^{2\alpha}}\, \frac{\tau}{\tau^2 + x^2}\, d\tau, \tag{3.87}$$

inserting in (3.87) the identity

$$\frac{1}{\tau^2 + x^2} = \frac{1}{x} \int_0^\infty e^{-\tau\sigma} \sin x\sigma \, d\sigma$$

and changing the order of integration we obtain by the use of (1.28)

$$\mathcal{G}_{\alpha/2,\alpha,1}(x,t) = \frac{t^\alpha}{\pi x} \int_0^\infty \sigma^{\alpha-1} E_{\alpha,\alpha}(-t^\alpha \sigma^\alpha) \sin x\sigma \, d\sigma. \quad (3.88)$$

Note that the integral in (3.88) is convergent due to the asymptotic expansion (1.22) of the Mittag-Leffler function. Formal integration in (3.88) by using the identity $\sin a = \Im\{e^{ia}\}$ and the Laplace transform pair (1.27) yields (3.86).

3.4.2 Integral representations

According to the subordination relation (3.77) and the formula for the Gaussian kernel (3.76), the fundamental solution of problem (3.4) admits the representation

$$\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \psi_{\alpha,\beta}(t,\tau) \tau^{-n/2} e^{-|\mathbf{x}|^2/4\tau} \, d\tau, \quad \mathbf{x} \in \mathbb{R}^n, \ t > 0. \quad (3.89)$$

Subordination formula (3.89) yields after the change of variables $\sigma = 1/\tau$

$$\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty \psi_{\alpha,\beta}(t,\sigma^{-1}) \sigma^{n/2-2} e^{-a\sigma} \, d\sigma, \quad a = |\mathbf{x}|^2/4. \quad (3.90)$$

Applying the formula for the Laplace transform ([37], Section 4.1, Eq. (25))

$$\int_0^\infty \sigma^{\nu-1} f(\sigma^{-1}) e^{-a\sigma} \, d\sigma = a^{-\frac{1}{2}\nu} \int_0^\infty s^{\frac{1}{2}\nu} J_\nu(2\sqrt{as}) \widehat{f}(s) \, ds, \quad Re\, \nu > -1,$$

where $J_\nu(\cdot)$ denotes the Bessel function (3.109) and $\widehat{f}$ is the Laplace transform of f, we deduce from (3.90) and (3.16) the following representation

$$\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t) = \frac{|\mathbf{x}|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^\infty \sigma^{\frac{n}{2}} J_{\frac{n}{2}-1}(|\mathbf{x}|\sigma) E_\beta(-\sigma^{2\alpha} t^\beta) \, d\sigma. \quad (3.91)$$

The obtained integral representation (3.91) is not new, see e.g. [70, 30], where it is deduced applying a different argument.

Let us first note that for $\beta = 1$ the integral in (3.91) is always convergent and gives the following representation for the fundamental solution to the space-fractional diffusion equation:

$$\mathcal{G}_{\alpha,1,n}(\mathbf{x},t) = \frac{|\mathbf{x}|^{1-\frac{n}{2}}}{(2\pi)^{n/2}} \int_0^\infty \sigma^{n/2} J_{\frac{n}{2}-1}(|\mathbf{x}|\sigma) \exp(-\sigma^{2\alpha} t)\, d\sigma.$$

We observe, however, that if $\beta < 1$ the integral in (3.91) is convergent only for very limited ranges for the values of the other two parameters. Indeed, according to the asymptotic expansions of the Bessel and the Mittag-Leffler functions, (3.111) and (1.24), the integral in (3.91) is convergent only in the following cases: $n = 1$ and $\alpha > 1/2$ or $n = 2$ and $\alpha > 3/4$. If $n \geq 3$ the integral is divergent for any $\alpha \in (0,1)$. Our aim here is to derive from (3.91) convergent integral representations for $n = 1, 2, 3$, which hold for all $\alpha, \beta \in (0,1)$.

Let first $n = 1$. Plugging in (3.91) the representation for $J_{-\frac{1}{2}}(\cdot)$ from (3.110) yields

$$\mathcal{G}_{\alpha,\beta,1}(x,t) = \frac{1}{\pi} \int_0^\infty \cos(|x|\sigma) E_\beta(-\sigma^{2\alpha} t^\beta)\, d\sigma, \tag{3.92}$$

which, according to (1.24), is convergent at $+\infty$ only if $2\alpha > 1$, unless $\beta = 1$. However, we can improve the convergence by performing integration by parts in (3.92). We use the identity

$$\frac{d}{d\sigma} E_\beta(-\sigma^{2\alpha} t^\beta) = -\frac{2\alpha}{\beta} \sigma^{2\alpha-1} t^\beta E_{\beta,\beta}(-\sigma^{2\alpha} t^\beta), \tag{3.93}$$

which is derived from (1.25). In this way the following integral representation is established.

Theorem 3.7. *Let* $0 < \alpha, \beta \leq 1$ *and* $\alpha\beta \neq 1$. *Then*

$$\mathcal{G}_{\alpha,\beta,1}(x,t) = \frac{2\alpha}{\beta} \frac{t^\beta}{\pi |x|} \int_0^\infty \sin(|x|\sigma) \sigma^{2\alpha-1} E_{\beta,\beta}(-\sigma^{2\alpha} t^\beta)\, d\sigma. \tag{3.94}$$

The asymptotic expression (1.24) indicates that the integral in (3.94) is convergent for all $0 < \alpha, \beta \leq 1$. In the particular case $\alpha = \beta/2$ representation (3.94) coincides with eq. (3.88).

Let us consider now $n = 3$. Plugging in the general formula (3.91) the representation for $J_{\frac{1}{2}}(\cdot)$ from (3.110) yields

$$\mathcal{G}_{\alpha,\beta,3}(\mathbf{x},t) = \frac{1}{2\pi^2 |\mathbf{x}|} \int_0^\infty \sigma \sin(|\mathbf{x}|\sigma) E_\beta(-\sigma^{2\alpha} t^\alpha)\, d\sigma.$$

This integral is divergent for all $0 < \alpha, \beta < 1$. Integration by parts gives

$$\mathcal{G}_{\alpha,\beta,3}(\mathbf{x},t) = \frac{1}{2\pi^2|\mathbf{x}|^2}\int_0^\infty \cos(|\mathbf{x}|\sigma)\frac{d}{d\sigma}\left(\sigma E_\beta(-\sigma^{2\alpha}t^\alpha)\right)\, d\sigma$$

and, by applying formula (3.93), we obtain the following integral expression for the three-dimensional fundamental solution

$$\mathcal{G}_{\alpha,\beta,3}(\mathbf{x},t) = \frac{1}{2\pi^2|\mathbf{x}|^2}\int_0^\infty \cos(|\mathbf{x}|\sigma)F_{\alpha,\beta}(\sigma,t)\, d\sigma, \tag{3.95}$$

where

$$F_{\alpha,\beta}(\sigma,t) = E_\beta(-\sigma^{2\alpha}t^\beta) - \frac{2\alpha}{\beta}\sigma^{2\alpha}t^\beta E_{\beta,\beta}(-\sigma^{2\alpha}t^\beta). \tag{3.96}$$

The asymptotic expansions (1.24) of the Mittag-Leffler functions imply that the integral in (3.95) is convergent for $1/2 < \alpha < 1$ and $0 < \beta \le 1$. Applying again integration by parts in (3.95) yields

$$\mathcal{G}_{\alpha,\beta,3}(\mathbf{x},t) = \frac{1}{2\pi^2|\mathbf{x}|^3}\int_0^\infty \sin(|\mathbf{x}|\sigma)H_{\alpha,\beta}(\sigma,t)\, d\sigma, \tag{3.97}$$

where $H_{\alpha,\beta}(\sigma,t) = -\frac{d}{d\sigma}F_{\alpha,\beta}(\sigma,t)$. Therefore, from (3.96), (1.25), and the identity

$$\frac{d}{dz}\left(z^{\alpha-1}E_{\alpha,\alpha}(-z^\alpha)\right) = z^{\alpha-2}E_{\alpha,\alpha-1}(-z^\alpha) \tag{3.98}$$

it follows

$$H_{\alpha,\beta}(\sigma,t) = \mu\sigma^{2\alpha-1}t^\beta\left((1+\mu)\, E_{\beta,\beta}(-\sigma^{2\alpha}t^\beta) + \mu E_{\beta,\beta-1}(-\sigma^{2\alpha}t^\beta)\right), \tag{3.99}$$

where $\mu = 2\alpha/\beta$. The asymptotic behavior of the Mittag-Leffler functions (1.24) implies that the integral in (3.97) is convergent for all $0 < \alpha, \beta < 1$.

We summarize the result for the three-dimensional case as follows.

Theorem 3.8. *Let $0 < \alpha, \beta \le 1$, $\alpha\beta \ne 1$, and $n = 3$. Then the fundamental solution admits the integral representation (3.97), where the function $H_{\alpha,\beta}$ is defined in (3.99).*

In an analogous way, for $n = 2$ we deduce from (3.91) and (3.110) the following result.

Theorem 3.9. *Let* $0 < \alpha, \beta \le 1$ *and* $\alpha\beta \neq 1$. *Then*

$$\mathcal{G}_{\alpha,\beta,2}(\mathbf{x},t) = -\frac{1}{2\pi^2|\mathbf{x}|^2}\int_0^\pi \frac{1}{\cos^2\theta}\left(1+\int_0^\infty \cos(|\mathbf{x}|\sigma\cos\theta)H_{\alpha,\beta}(\sigma,t)\,d\sigma\right)d\theta,$$

where the function $H_{\alpha,\beta}$ *is defined in (3.99).*

Let us emphasize that identities for the derivatives of Mittag-Leffler functions (1.25) and (3.98) together with the asymptotic expansions (1.24) show a faster decay for large $|z|$ after repeated differentiation, which is essentially used in the above representations.

3.5 Appendix: some special functions

The Tricomi's confluent hypergeometric function can be defined by the Laplace integral ([1], Eq. 13.2.5)

$$U(a,c,z) = \frac{1}{\Gamma(a)}\int_0^\infty \xi^{a-1}(1+\xi)^{c-a-1}e^{-z\xi}\,d\xi,\ \ a>0, z>0. \tag{3.100}$$

The following representations of Mittag-Leffler functions in terms of Tricomi's confluent hypergeometric function can be obtained from (1.28) for $\alpha = \beta = 1/2$

$$\begin{aligned} E_{1/2}(-t^{1/2}) &= \frac{1}{\sqrt{\pi}}U(1/2,1/2,t);\\ t^{-1/2}E_{1/2,1/2}(-t^{1/2}) &= \frac{1}{2\sqrt{\pi}}U(3/2,3/2,t). \end{aligned} \tag{3.101}$$

For $a = c$ the Tricomi's confluent hypergeometric function (3.100) is related to the upper incomplete Gamma function

$$\Gamma(a,z) = \int_z^\infty \xi^{a-1}e^{-\xi}\,d\xi \tag{3.102}$$

as follows ([1], Eq. 13.6.28)

$$U(a,a,z) = e^z\Gamma(1-a,z). \tag{3.103}$$

Integration by parts in (3.102) yields the following recurrence relation

$$\Gamma(a+1,z) = z^a e^{-z} + a\Gamma(a,z). \tag{3.104}$$

The incomplete Gamma function (3.102) with $a = 0$ gives the exponential integral

$$\mathbb{E}_1(z) = \int_z^\infty \frac{e^{-\xi}}{\xi}\, d\xi, \tag{3.105}$$

which satisfies $\mathbb{E}_1(z) = \Gamma(0, z) = e^{-z} U(1, 1, z)$. For real or complex arguments off the negative real axis, it can be expressed as ([1], Eq. 5.1.11)

$$\mathbb{E}_1(z) = -\gamma - \ln z - \sum_{k=1}^{\infty} \frac{(-z)^k}{k(k!)}, \quad |\arg z| < \pi, \tag{3.106}$$

where γ is the Euler-Mascheroni constant. For large values of $\Re z$ the following approximation is valid [27]

$$\mathbb{E}_1(z) \sim \frac{e^{-z}}{z} \sum_{k=0}^{N-1} \frac{k!}{(-z)^k}. \tag{3.107}$$

For real positive values of the argument the exponential integral can be bracketed by elementary functions as follows ([1], Eq. 5.1.20)

$$0.5e^{-x} \ln (1 + 2/x) < \mathbb{E}_1(x) < e^{-x} \ln (1 + 1/x), \quad x > 0. \tag{3.108}$$

The Bessel function of the first kind $J_\nu(z)$ is defined by the series [1]

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(\nu + k + 1)}. \tag{3.109}$$

The following particular expressions are of interest in this dissertation

$$\begin{aligned} J_{-1/2}(z) &= \sqrt{\frac{2}{\pi z}} \cos z, \\ J_{1/2}(z) &= \sqrt{\frac{2}{\pi z}} \sin z, \\ J_0(z) &= \frac{1}{\pi} \int_0^\pi \cos(z \cos\theta)\, d\theta. \end{aligned} \tag{3.110}$$

The asymptotic expansions of the Bessel function $J_\nu(r)$ for small and large real arguments are as follows

$$J_\nu(r) \sim \begin{cases} \dfrac{1}{\Gamma(\nu+1)} \left(\dfrac{r}{2}\right)^\nu, & r \to 0; \\ \sqrt{\dfrac{2}{\pi r}} \cos(r - \nu\pi/2 - \pi/4), & r \to +\infty. \end{cases} \tag{3.111}$$

Chapter 4

Transition from diffusion to wave propagation

In this chapter we study a fractional Jeffreys-type heat conduction equation as a model problem to demonstrate the application of the general subordination theorem (Theorem 2.4). This is an evolution equation containing time-fractional differential operators, which, depending on the model parameters, obeys two different subordination properties, and, respectively, two fundamentally different types of behaviour: diffusion and wave propagation. The one-dimensional Cauchy problem is studied and explicit representations for the fundamental solution and the mean squared displacement are derived. The fundamental solution is shown to be a spatial probability density function evolving in time, which is unimodal in the diffusion regime and bimodal in the propagation regime.

4.1 Problem formulation

The heat conduction equation with a fractional Jeffreys-type constitutive law in abstract form reads as follows [6], Chapter 7,

$$(1 + aD_t^{\alpha})\, u'(t) = (1 + bD_t^{\alpha})\, Au(t), \quad t > 0, \tag{4.1}$$

where D_t^{α} is the fractional Riemann-Liouville derivative of order $\alpha \in (0, 1]$, $a \geq 0, b \geq 0$ are given parameters, and A is a closed linear densely defined operator in a Banach space X (usually A is some realization of the Laplace operator, one example is the operator defined in (2.20)). The equation (4.1) is supplemented with the usual initial conditions $u(0) = v \in X$ and $u'(0) = 0$. For derivation details and relevant references we refer to [22].

In this chapter the main emphasis will be on the differences in the model governed by equation (4.1) in the two cases: $a < b$ and $a > b$, where $\alpha \in (0, 1]$ is arbitrarily chosen. Based on the theory of Bernstein functions, we establish two different types of subordination principles in the two cases: for $a < b$ the equation is subordinated to the first order Cauchy problem (2.6), while for $a < b$ it is subordinated to the second order Cauchy problem (2.7). Accordingly, two fundamentally different types of behavior are established in the two cases: diffusion regime for $a < b$ and wave propagation regime for $a > b$. This is a demonstration how the subordination principle can be applied for the proper classification and understanding of the variety of mathematical models in the form of generalized fractional evolution equations.

4.2 Subordination results

We first recast the fractional Jeffreys-type heat conduction equation (4.1) as a Volterra integral equation. By the use of (1.17) equation (4.1) with initial conditions $u(0) = v$ and $u'(0) = 0$ in Laplace domain reads

$$\widehat{u}(s) = v\frac{1}{s} + \frac{1 + bs^{\alpha}}{s(1 + as^{\alpha})} A\widehat{u}(s). \tag{4.2}$$

Taking the inverse Laplace transform in (4.2) yields the Volterra integral equation

$$u(t) = v + \int_0^t k(t - \tau)Au(\tau)\, d\tau, \tag{4.3}$$

with kernel $k(t)$ satisfying $\widehat{k}(s) = 1/g(s)$, where

$$g(s) = \frac{s(1 + as^{\alpha})}{1 + bs^{\alpha}}, \quad s > 0. \tag{4.4}$$

Applying inverse Laplace transform we derive by the use of (1.27) the explicit representation for the kernel

$$k(t) = 1 - \left(1 - \frac{b}{a}\right) E_{\alpha}\left(-\frac{1}{a}t^{\alpha}\right). \tag{4.5}$$

Let us note that $k(t) > 0$. Indeed, according to (1.25), the first derivative of $k(t)$ is

$$k'(t) = \frac{1}{a}\left(1 - \frac{b}{a}\right) t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{1}{a}t^{\alpha}\right).$$

Therefore, for $a < b$ the function $k(t)$ is decreasing from $k(0) = b/a > 1$ to $k(+\infty) = 1$ and for $a > b$ the function $k(t)$ is increasing from $k(0) = b/a < 1$ to $k(+\infty) = 1$. Moreover, the complete monotonicity of the Mittag-Leffler type function $t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}/a)$ implies $k(t) \in \mathcal{CMF}$ for $a < b$ and $k(t) \in \mathcal{BF}$ for $a > b$.

The following properties of the function $g(s)$ play a crucial role in the study of equations (4.3) and (4.1).

Proposition 4.1. *Let $0 < \alpha \leq 1$. For any $a, b \geq 0$ the function $\sqrt{g(s)}$ is a complete Bernstein function. If moreover $0 \leq a < b$ then $g(s)$ is a complete Bernstein function.*

Proof. We use the relation $g(s) = sf(s)$, where

$$f(s) = \frac{1 + as^{\alpha}}{1 + bs^{\alpha}}.$$

The function $f(s)$ satisfies the properties:

- $f(s) \in \mathcal{SF}$ for $a < b$;
- $f(s) \in \mathcal{CBF}$ for $a > b$.

Taking into account **(P8)** in Proposition 2.1 it is sufficient to prove only one of these two properties. Let $a < b$ and use the representation

$$f(s) = \frac{a}{b}\left(\frac{1}{a} - \frac{1}{b}\right)\frac{1}{s^{\alpha} + 1/b} + \frac{a}{b}. \tag{4.6}$$

Since $s^{\alpha} \in \mathcal{CBF}$, then $s^{\alpha} + 1/b \in \mathcal{CBF}$ and therefore, **(P8)** in Proposition 2.1 implies $(s^{\alpha} + 1/b)^{-1} \in \mathcal{SF}$. Therefore, $f(s)$ is a Stieltjes function for $a < b$ as a linear combination with positive coefficients of two Stieltjes functions. In this way the properties of $f(s)$ are proved.

Let $a < b$. Then $f(s) \in \mathcal{SF}$, which according to **(P8)** in Proposition 2.1 implies $g(s) \in \mathcal{CBF}$. Since square root of a complete Bernstein function is again a complete Bernstein function (by **(P11)** in Proposition 2.1), this also gives $\sqrt{g(s)} \in \mathcal{CBF}$.

If $a > b$ then $f(s) \in \mathcal{CBF}$. Therefore, $g(s)$ is a product of two complete Bernstein functions (s and $f(s)$) and (2.2) implies $\sqrt{g(s)} \in \mathcal{CBF}$.

An alternative way to check that $\sqrt{g(s)} \in \mathcal{CBF}$ for $a > b$ is as follows. Consider the function $h(s) = \sqrt{g(s)}/s$. Then

$$h^2(s) = \frac{1 + as^\alpha}{s(1 + bs^\alpha)}$$

is a Stieltjes function since, according to (1.27), it is Laplace transform of the completely monotone function $1 + (a/b - 1)E_\alpha(-t^\alpha/b)$. On the other hand, $s^{1/2}$ is a complete Bernstein function. This together with property **(P12)** in Proposition 2.1 implies that $h(s)$ is a Stieltjes function. Then property **(P7)** implies $\sqrt{g(s)} = sh(s)$ is a complete Bernstein function. □

Let us point out that $g(s) \notin \mathcal{CBF}$ for $a > b$. In fact, $g(s)$ is not even a Bernstein function in this case. Indeed, for $s \to 0$ we have the asymptotic expansion $g(s) \sim s + (a - b)s^{\alpha+1}$ and thus $g''(s) \sim (a - b)(\alpha + 1)\alpha s^{\alpha-1} > 0$ for $a > b$, hence $g(s) \notin \mathcal{BF}$.

Now we can apply the general subordination theorem (Theorem 2.4) and formulate the following subordination principles.

Theorem 4.1. *Let $a, b \geq 0$ and $0 < \alpha \leq 1$. Assume the operator A is a generator of a bounded strongly continuous cosine function $S_2(t)$. Then problem (4.1) is well posed and the corresponding solution operator $S(t)$ satisfies the following subordination relation*

$$S(t) = \int_0^\infty \varphi_1(t, \tau) S_2(\tau)\, d\tau, \quad t > 0, \tag{4.7}$$

where the kernel $\varphi_1(t, \tau)$ is a unilateral probability density (i.e. satisfies (2.26)), which is defined via the Laplace transform

$$\widehat{\varphi}_1(s, \tau) = \frac{\sqrt{g(s)}}{s} \exp\left(-\tau\sqrt{g(s)}\right), \quad s, \tau > 0. \tag{4.8}$$

Theorem 4.2. *Let $0 \leq a < b$ and $0 < \alpha \leq 1$. Suppose the operator A is a generator of a bounded C_0-semigroup of operators $S_1(t)$. Then problem (4.1) is well posed with solution operator $S(t)$ satisfying the subordination relation*

$$S(t) = \int_0^\infty \varphi_2(t, \tau) S_1(\tau)\, d\tau, \quad t > 0.$$

The kernel $\varphi_2(t,\tau)$ is a unilateral probability density (i.e. satisfies (2.26)), which is defined via the Laplace transform

$$\widehat{\varphi}_2(s,\tau) = \frac{g(s)}{s} \exp\left(-\tau g(s)\right), \quad s,\tau > 0.$$

A stronger subordination result in the case $a > b \geq 0$ is established next.

Theorem 4.3. *Let $a > b \geq 0$ and $0 < \alpha \leq 1$. Suppose the fractional evolution equation (2.8) of order $\alpha+1$ is well posed and admits a bounded solution operator $S_{\alpha+1}(t)$. Then problem (4.1) is well posed and the corresponding solution operator $S(t)$ satisfies the subordination relation*

$$S(t) = \int_0^\infty \varphi(t,\tau) S_{\alpha+1}(\tau)\, d\tau, \quad t > 0.$$

The kernel $\varphi(t,\tau)$ is a unilateral probability density (i.e. satisfies (2.26)), which is defined via the Laplace transform

$$\widehat{\varphi}(s,\tau) = \frac{g(s)^{1/(\alpha+1)}}{s} \exp\left(-\tau g(s)^{1/(\alpha+1)}\right), \quad s,\tau > 0.$$

Proof. According to Theorem 2.4 we need to prove

$$g(s)^{1/(\alpha+1)} \in \mathcal{CBF} \tag{4.9}$$

for the function $g(s)$, defined in (4.4). According to property **(P7)** in Proposition 2.1) (4.9) holds if and only if

$$h(s) = \frac{g(s)^{1/(\alpha+1)}}{s} \in \mathcal{SF}.$$

We observe that

$$(h(s))^{\alpha+1} = \frac{g(s)}{s^{\alpha+1}} = \frac{1+as^\alpha}{s^\alpha(1+bs^\alpha)} = \frac{1}{s^\alpha} + \frac{a/b-1}{s^\alpha+1/b} \in \mathcal{SF},$$

since $s^\alpha \in \mathcal{CBF}$, $a > b > 0$, and using properties **(P2)** and **(P8)** from Proposition 2.1. Let $F(s) = s^{1/(\alpha+1)}$. Then $F(s) \in \mathcal{CBF}$ for $0 < \alpha < 1$. Then $h(s) = F(h(s))^{\alpha+1})$ composition property **(P12)** yields $h(s) \in \mathcal{SF}$. □

Let us note that property (4.9) can be used to give an example of application of Theorem 2.5. Indeed, it implies for the analytic extension of $\sqrt{g(s)}$ to $\mathbb{C}\backslash(-\infty,0]$

$$|\arg\sqrt{g(z)}| \le \frac{\alpha+1}{2}|\arg z|, \quad z\in\mathbb{C}\backslash(-\infty,0]. \tag{4.10}$$

If $\alpha < 1$ and the operator A is a generator of a bounded cosine function $S_2(t)$ then since $\frac{\alpha+1}{2} < 1$ the subordinated solution operator $S(t)$ is a bounded analytic solution operator in some sector of the complex plane.

4.3 One-dimensional Cauchy problem

Consider the one-dimensional Cauchy problem for the fractional Jeffreys' heat conduction equation

$$(1+aD_t^\alpha)\frac{\partial}{\partial t}u(x,t) = (1+bD_t^\alpha)\frac{\partial^2}{\partial x^2}u(x,t), \quad x\in\mathbb{R},\ t>0, \tag{4.11}$$

$$u(x,0)=u_0(x); \quad \lim_{t\to 0^+}\frac{\partial}{\partial t}u(x,t)=0, \quad x\in\mathbb{R}, \tag{4.12}$$

$$\lim_{|x|\to\infty} u(x,t)=0, \quad t>0. \tag{4.13}$$

Problem (4.11)-(4.12)-(4.13) is conveniently treated using Laplace transform with respect to the temporal variable and Fourier transform with respect to the spatial variable.

By applying Laplace and Fourier transforms to equation (4.11) and taking into account initial conditions (4.12), the boundary condition (4.13), and identities (1.17) and (3.73) we derive the solution in Fourier-Laplace domain

$$\widehat{\widetilde{u}}(\xi,s) = \frac{g(s)/s}{g(s)+|\xi|^2}\widetilde{u_0}(\xi), \quad \xi\in\mathbb{R}, \quad s>0, \tag{4.14}$$

where $g(s)$ denotes the characteristic function, defined in (4.4).

Therefore, the solution of the Cauchy problem (4.11)-(4.12)-(4.13) is given by the integral

$$u(x,t) = \int_{-\infty}^{\infty}\mathcal{G}(x-y,t)u_0(y)\,dy, \quad x\in\mathbb{R}, \quad t>0.$$

where $\mathcal{G}(x,t)$ is the fundamental solution (Green function), defined in Fourier-Laplace domain as

$$\widehat{\widetilde{\mathcal{G}}}(\xi,s) = \frac{g(s)/s}{g(s)+|\xi|^2}, \quad \xi \in \mathbb{R}, \quad s>0. \tag{4.15}$$

By inversion of the Fourier transform and using the well-known formula

$$\mathcal{F}\left\{\exp(-c|x|)\right\}(\xi) = \frac{2c}{c^2+\xi^2}, \quad c>0; \quad x,\xi \in \mathbb{R},$$

we derive the Laplace transform of the fundamental solution

$$\widehat{\mathcal{G}}(x,s) = \frac{\sqrt{g(s)}}{2s} \exp\left(-|x|\sqrt{g(s)}\right), \quad x \in \mathbb{R}. \tag{4.16}$$

Let us note the relation with the subordination kernel, which Laplace transform is given in (4.8).

Let us note that in the special case $a=b$ equation (4.11) reduces to the classical diffusion equation. The fundamental solution in this special case is the Gaussian function

$$\mathcal{G}_1(x,t) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-|x|^2/4t\right), \quad x \in \mathbb{R}, \ t>0.$$

4.3.1 Fundamental solution

To study the behavior of the fundamental solution $\mathcal{G}(x,t)$, the properties of the characteristic function $g(s)$ from Proposition 4.1 are used.

Theorem 4.4. *The fundamental solution $\mathcal{G}(x,t)$ is a spatial probability density function evolving in time.*

Proof. For the proof we use representation (4.16). First, cccording to Proposition 4.1 $\sqrt{g(s)} \in \mathcal{CBF} \subset \mathcal{BF}$ for all values of the parameters a,b. Then property (2.3) yields $\widehat{\mathcal{G}}(x,s) \in \mathcal{CMF}$. Therefore, by Bernstein's theorem, $\mathcal{G}(x,t) \geq 0$. Further, (4.16) yields

$$\mathcal{L}\left\{\int_{-\infty}^{\infty} \mathcal{G}(x,t)\mathrm{d}x\right\} = \int_{-\infty}^{\infty} \widehat{\mathcal{G}}(x,s)dx = \frac{\sqrt{g(s)}}{s}\int_0^{\infty} \exp\left(-x\sqrt{g(s)}\right)dx = \frac{1}{s}$$

and, applying inverse Laplace transform we obtain

$$\int_{-\infty}^{\infty} \mathcal{G}(x,t)dx = 1.$$

The theorem is proved. □

To find explicit integral expression for the fundamental solution $\mathcal{G}(x,t)$ we apply integral inversion formula for Laplace transform to (4.16), which yields for $x \in \mathbb{R}\backslash\{0\}$ and $t > 0$

$$\mathcal{G}(x,t) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{st} \widehat{\mathcal{G}}(x,s)\, ds = \frac{1}{4\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\sqrt{g(s)}}{s} \exp\left(st - |x|\sqrt{g(s)}\right)\, ds,$$

where $c > 0$.

The function under the integral sign is holomorphic with $s = 0$ and $s = \infty$ as the only branch points. It has no singularities in $\mathbb{C}$ cut along the negative real axis. Hence, the integral can be evaluated using the Cauchy's theorem and the integration on the Bromwich path $\{s \in \mathbb{C} | \Re s = c, \Im s \in (-\infty, +\infty)\}$ can be replaced by an integral over the composite contour

$$\Gamma = \Gamma_1^- \cup \Gamma_2^- \cup \Gamma_3 \cup \Gamma_2^+ \cup \Gamma_1^+,$$

where

$$\begin{aligned} &\Gamma_1^\pm = \{s = q \pm \mathrm{i}R, \quad q \in (0,c)\}, \quad \Gamma_2^\pm = \{s = re^{\pm \mathrm{i}\pi/2}, \quad r \in (\rho, R)\}, \\ &\Gamma_3 = \{s = \rho e^{\mathrm{i}\theta}, \ \theta \in (-\pi/2, \pi/2)\}, \end{aligned}$$

with appropriate orientation (see Figure 3.1) and letting $\rho \to 0$, $R \to \infty$.

The integrals on the contours Γ_1^+ and Γ_1^- vanish for $R \to \infty$ due to the following asymptotic expression

$$\left|\sqrt{g(s)}\right| \sim \sqrt{\frac{a}{b}|s|} = \left(\frac{a}{b}(q^2 + R^2)^{1/2}\right)^{1/2}, \quad R \to \infty,$$

and

$$\Re\sqrt{g(s)} \sim \sqrt{\frac{a}{b}|s|} \cos\frac{\arg s}{2} \sim \left(\frac{a}{b}(q^2 + R^2)^{1/2}\right)^{1/2} \cos(\pm\pi/4), \quad R \to \infty.$$

Moreover, since

$$\lim_{|s| \to 0} s \left(\frac{\sqrt{g(s)}}{s} \exp\left(st - |x|\sqrt{g(s)}\right)\right) = 0,$$

it follows that the integral on the semi-circular contour Γ_3 also vanishes. Therefore, the original integral yields

$$\begin{aligned}\mathcal{G}(x,t) = &= \frac{1}{4\pi \mathrm{i}} \lim_{\rho\to 0, R\to\infty} \int_{\Gamma_2^- \cup \Gamma_2^+} \frac{\sqrt{g(s)}}{s} \exp\left(st - |x|\sqrt{g(s)}\right) ds \\ &= \frac{1}{2\pi} \int_0^\infty \Im\left\{\sqrt{g(\mathrm{i}r)} \exp\left(\mathrm{i}rt - |x|\sqrt{g(\mathrm{i}r)}\right)\right\} \frac{dr}{r}.\end{aligned}$$

To express the imaginary part under the integral sign in terms of elementary real functions we apply the formula for real and imaginary parts of the square root of a complex number z:

$$\begin{aligned}\Re\left\{z^{1/2}\right\} &= \frac{1}{\sqrt{2}}\left(a + \left(a^2+b^2\right)^{1/2}\right)^{1/2}, \\ \Im\left\{z^{1/2}\right\} &= \frac{b}{|b|\sqrt{2}}\left(-a + \left(a^2+b^2\right)^{1/2}\right)^{1/2},\end{aligned} \tag{4.17}$$

where $a = \Re z$, $b = \Im z$. In this way we obtain after some standard manipulations the following result.

Theorem 4.5. *The fundamental solution $\mathcal{G}(x,t)$ of the Cauchy problem (4.11)-(4.12)-(4.13) admits the following integral representation for $x \in \mathbb{R}\backslash\{0\}$ and $t > 0$:*

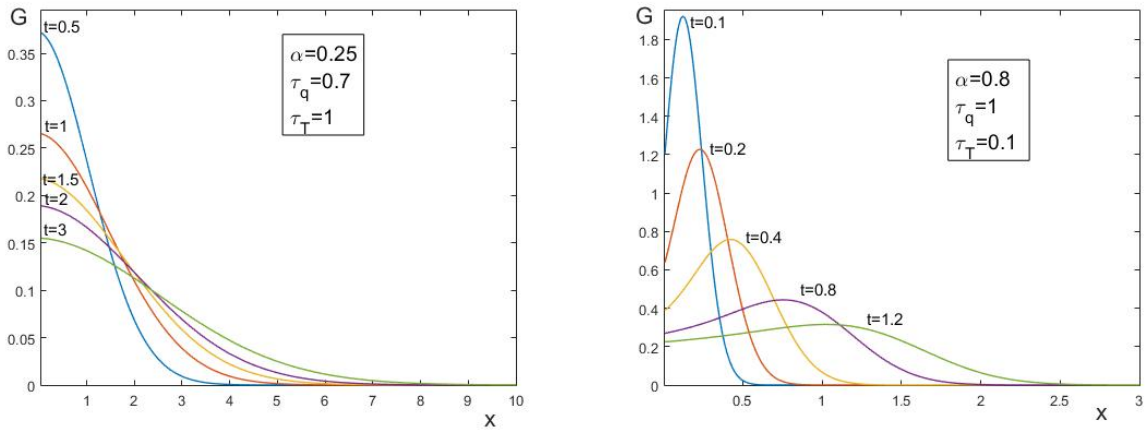
$$\begin{aligned}\mathcal{G}(x,t) &= \frac{1}{2\pi}\int_0^\infty \exp\left(-|x|K^-(r)\right)\left(K^-(r)\sin\left(rt - |x|K^+(r)\right)\right. \\ &+ \left. K^+(r)\cos\left(rt - |x|K^+(r)\right)\right) \frac{dr}{r},\end{aligned} \tag{4.18}$$

where the functions $K^\pm(r)$ are defined by

$$K^\pm(r) = \left(\frac{r}{2}\right)^{1/2}\left(\left(A^2(r)+B^2(r)\right)^{1/2} \pm A(r)\right)^{1/2} \tag{4.19}$$

with

$$\begin{aligned}A(r) &= \frac{(a-b)r^\alpha \sin(\alpha\pi/2)}{1 + 2br^\alpha\cos(\alpha\pi/2) + b^2r^{2\alpha}}, \\ B(r) &= \frac{1 + (a+b)r^\alpha\cos(\alpha\pi/2) + abr^{2\alpha}}{1 + 2br^\alpha\cos(\alpha\pi/2) + b^2r^{2\alpha}}.\end{aligned}$$



(**a**) (**b**)

Figure 4.1: Plots of the the fundamental solution $\mathcal{G}(x,t)$ versus x ($x > 0$) for different values of t; $a = \tau_q$, $b = \tau_T$; (**a**) diffusion regime ($a < b$); (**b**) propagation regime ($a > b$).

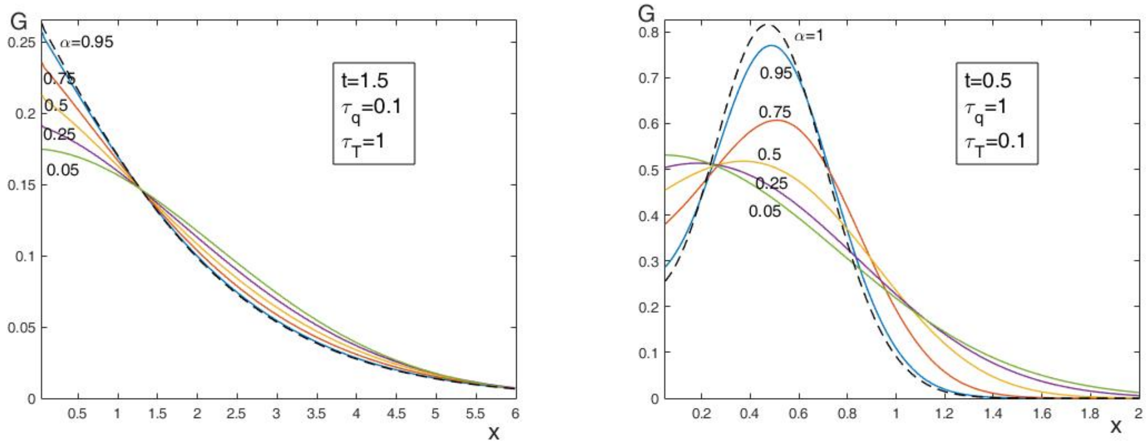
We note that the convergence of the integral in (4.18) is guaranteed by the following properties of the functions $K^{\pm}(r)$: $K^{\pm}(r) > 0$, $K^{\pm}(r) \sim r^{(1-\alpha)/2}$ as $r \to +\infty$ and $K^{\pm}(r) \sim r^{(1+\alpha)/2}$as $r \to 0$.

In [22] the integral representation (4.18) is used for numerical computation and visualization of the fundamental solution. Figures 4.1 and 4.2 are from this publication.

Fig. 4.1 shows the evolution in time of the fundamental solution $\mathcal{G}(x,t)$, starting from a delta function $\delta(x)$ at $t = 0$. The solution is plotted for five different time instances. In the diffusion regime (a) the maximum remains at $t = 0$, i.e. the probability density function is unimodal. In the propagation regime (b) the maximum moves away from the origin, the PDF is bimodal.

In Fig. 4.2 the fundamental solution is plotted for different values of the fractional parameter $\alpha \in (0,1]$. For $\alpha \to 1$ the solution approaches that of the classical Jeffreys' heat conduction equation. For $\alpha \to 0$ the fractional derivatives become identity operators and (4.11) approaches the classical diffusion equation with the one-dimensional Gaussian as fundamental solution (see the plots for $\alpha = 0.05$, which are qualitatively close to a Gaussian function).

In all figures we observe behavior, typical for a diffusion process for $a < b$: the fundamental solution is monotonically decreasing in x for $x > 0$. For $a > b$ the behavior is typical for a wave propagation process, with a maximum moving away from the origin. In this respect there is a strong analogy with the fractional diffusion-wave equation with Caputo time-derivative of order $\alpha \in (0,2)$ with



(**a**) (**b**)

Figure 4.2: Plots of the the fundamental solution $\mathcal{G}(x,t)$ versus x ($x > 0$) for fixed t and different values of α, $\alpha = 0.05, 0.25, 0.5, 0.75, 0.95$, compared to $\alpha = 1$ (dashed line); $a = \tau_q$, $b = \tau_T$; (**a**) diffusion regime ($a < b$); (**b**) propagation regime ($a > b$).

the two corresponding regimes: subdiffusion ($0 < \alpha < 1$) and diffusion-wave propagation ($1 < \beta < 2$), c.f. [75, Fig. 6.1], [51].

Let us briefly give analytical arguments for this behavior of $\mathcal{G}(x,t)$ on the basis of the properties of the characteristic function $g(s)$. Since the function $\mathcal{G}(x,t)$ is a spatial PDF, it is positive and $\mathcal{G}(x,t) \to 0$ as $x \to \infty$ for any fixed t. Since $\mathcal{G}(x,t) = \mathcal{G}(-x,t)$, we further consider only $x > 0$.

First, let $a < b$. In this case $g(s) \in \mathcal{BF}$ and $\sqrt{g(s)} \in \mathcal{BF}$, see Proposition 4.1. Therefore, (2.3) implies

$$\frac{g(s)}{s} \exp(-|x|\sqrt{g(s)}) \in \mathcal{CMF}, \quad a < b. \tag{4.20}$$

Differentiation of (4.16) yields (the differentiation under the improper integral can be justified in a standard way)

$$\mathcal{L}\left\{\frac{\partial \mathcal{G}}{\partial x}\right\}(x,s) = \frac{\partial}{\partial x}\widehat{\mathcal{G}}(x,s) = -\frac{g(s)}{2s}\exp\left(-x\sqrt{g(s)}\right), \quad x \geq 0, s > 0. \tag{4.21}$$

For $a < b$ this function is completely monotone, see (4.20), and Bernstein's theorem implies that $\frac{\partial \mathcal{G}}{\partial x} \geq 0$ and, thus, $\mathcal{G}(x,t)$ is monotonically decreasing in x for $x > 0$.

Let now $a > b$. We show that in this case the solution $\mathcal{G}(x,t)$ is an increasing function of x near the origin, for all $t > 0$. Indeed, (4.21) yields

$$\lim_{x\to 0+} \frac{\partial}{\partial x}\widehat{\mathcal{G}}(x,s) = -\frac{g(s)}{2s} = -\frac{1+as^\alpha}{2(1+bs^\alpha)} = -\frac{a}{2b}\left(\frac{b-a}{ab}\frac{1}{s^\alpha+1/b}+1\right),$$

which, after inverting the Laplace transform by the use of (1.27) implies

$$\lim_{x\to 0+} \frac{\partial}{\partial x}\mathcal{G}(x,t) = -\frac{a}{2b}\left(\frac{b-a}{ab}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{t^\alpha}{b}\right)+\delta(t)\right).$$

Therefore, due to the fact that the Mittag-Leffler function is positive, it follows for all $t > 0$

$$\lim_{x\to 0+} \frac{\partial}{\partial x}\mathcal{G}(x,t) > 0, \quad a > b,$$

and $\mathcal{G}(x,t)$ is an increasing function of x near the origin.

4.3.2 Mean squared displacement

Next, we study the temporal behavior of the mean squared displacement (MSD)

$$\left\langle |x|^2(t)\right\rangle = \int_{\mathbb{R}} x^2\mathcal{G}(x,t)\,dx,$$

which is determining for the character of the solution.

Representation (4.16) implies for the MSD in Laplace domain

$$\left\langle |x|^2(s)\right\rangle = \int_{\mathbb{R}} x^2\widehat{\mathcal{G}}(x,s)\,dx = \frac{\sqrt{g(s)}}{s}\int_0^\infty x^2\exp\left(-x\sqrt{g(s)}\right)\,dx.$$

Calculation of the integral yields

$$\left\langle |x|^2(s)\right\rangle = \frac{2}{sg(s)} = \frac{2(1+bs^\alpha)}{s^2(1+as^\alpha)}, \tag{4.22}$$

where $g(s)$ is the characteristic function (4.4). By the use of Laplace transform pair (1.27) we invert (4.22) and get two equivalent expressions in terms of Mittag-Leffler functions

$$\left\langle |x|^2(t)\right\rangle = 2t + 2\left(\frac{b}{a}-1\right)tE_{\alpha,2}\left(-\frac{t^\alpha}{a}\right) \tag{4.23}$$

$$= 2\frac{b}{a}t + \frac{2}{a}\left(1-\frac{b}{a}\right)t^{\alpha+1}E_{\alpha,\alpha+2}\left(-\frac{t^\alpha}{a}\right). \tag{4.24}$$

Both expressions are valid for all $a, b \geq 0$. However we give the two different forms, since (4.23) seems more natural for the diffusion regime ($a < b$) and (4.24) for the wave propagation regime ($a > b$). Let us note that the Mittag-Leffler functions in the MSD representations are positive functions, due to the relation

$$t^\beta E_{\alpha,\beta+1}(-at^\alpha) = \int_0^t \tau^{\beta-1} E_{\alpha,\beta}(-a\tau^\alpha)\, d\tau.$$

From the definition (1.21) of the Mittag-Leffler functions and their asymptotics (1.33) we derive the following asymptotic behavior for the MSD for short and long times

$$\langle |x|^2(t) \rangle \sim \begin{cases} 2\dfrac{b}{a}t\left(1 + \dfrac{a-b}{ab}\dfrac{t^\alpha}{\Gamma(\alpha+2)}\right), & t \to 0, \\ 2t\left(1 + (b-a)\dfrac{t^{-\alpha}}{\Gamma(2-\alpha)}\right), & t \to \infty. \end{cases}$$

The established asymptotic expansions show linear asymptotic behavior for short and long times. We also observe that the dominating term in the gradient of the MSD is $2b/a$ for $t \to 0$ versus 2 for $t \to \infty$. Therefore, in the diffusion regime ($a < b$) the MSD increases faster near the origin than for large times. The opposite behavior is observed in the wave propagation regime ($a > b$): the MSD increases slower near the origin than for large times. Let us note that qualitatively comparable asymptotic behavior of the MSD is observed in the fractional diffusion-wave equation (2.19), where MSD is proportional to t^α with $\alpha \in (0, 1)$ in the diffusion regime and $\alpha \in (1, 2)$ in the wave propagation regime.

4.4 Generalized diffusion-wave equations

4.4.1 Diffusion regime

A generalized subdiffusion equation has the form [62, 98]

$$\int_0^t \kappa(t-\tau)\frac{\partial}{\partial\tau}u(x,\tau)\, d\tau = \frac{\partial^2}{\partial x^2}u(x,t), \tag{4.25}$$

where $\kappa(t) \in L^1_{loc}(\mathbb{R}_+)$ is a non-negative function, such that its Laplace transform $\widehat{\kappa}(s) \in \mathcal{SF}$.

By the use of Laplace transform we recast problem (4.11)-(4.12)-(4.13) into the generalized diffusion equation (4.25) with kernel $\kappa(t)$, such that

$$\widehat{\kappa}(s) = \frac{1 + as^{\alpha}}{1 + bs^{\alpha}}. \tag{4.26}$$

In the proof of Proposition 4.1 we established that $\widehat{\kappa}(s)$ is a Stieltjes function for $a < b$. Taking into account (4.6) and (1.27) we get from (4.26) the explicit form of the kernel $\kappa(t)$

$$\kappa(t) = \frac{a}{b}\delta(t) + \left(1 - \frac{a}{b}\right)\frac{1}{b}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{b}t^{\alpha}\right), \tag{4.27}$$

where $\delta(t)$ is the Dirac delta function. Therefore, in the considered in this subsection case $0 < a/b < 1$ the function $\kappa(t)$ is non-negative. In this way we proved that the required conditions on the kernel $\kappa(t)$ in equation (4.25) are satisfied.

Plugging the expression (4.27) for the kernel into the diffusion equation (4.25) gives the following representation of the fractional Jeffreys' equation in the diffusion case

$$\frac{\partial u}{\partial t} + \frac{b-a}{ab}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{(t-\tau)^{\alpha}}{b}\right)\frac{\partial u}{\partial \tau}\,d\tau = \frac{b}{a}\frac{\partial^2 u}{\partial x^2}. \tag{4.28}$$

4.4.2 Wave propagation regime

In the case $a > b$ we are looking for a representation of problem (4.11)-(4.12)-(4.13) as a generalized diffusion-wave equation of the form [99]

$$\int_0^t \eta(t-\tau)\frac{\partial^2}{\partial \tau^2}u(x,\tau)\,d\tau = \frac{\partial^2}{\partial x^2}u(x,t) \tag{4.29}$$

where $\eta(t) \in L^1_{loc}(\mathbb{R}_+)$ is a non-negative function, such that $\widehat{\eta}(s) \in \mathcal{SF}$.

The initial condition

$$\lim_{t\to 0^+}\frac{\partial}{\partial t}u(x,t) = 0, \tag{4.30}$$

implies

$$\mathcal{L}\left\{\frac{\partial^2}{\partial t^2}u\right\}(x,s) = s\mathcal{L}\left\{\frac{\partial}{\partial t}u\right\}(x,s).$$

With the help of this identity we obtain in Laplace domain

$$\frac{1+as^\alpha}{s(1+bs^\alpha)}\mathcal{L}\left\{\frac{\partial^2}{\partial t^2}u\right\}(x,s) = \frac{\partial^2}{\partial x^2}\widehat{u}(x,s). \tag{4.31}$$

Therefore, the fractional Jeffreys' equation is equivalent to the generalized wave equation (4.29) with kernel $\eta(t)$, such that

$$\widehat{\eta}(s) = \frac{1+as^\alpha}{s(1+bs^\alpha)} = \frac{1}{s} + \left(\frac{a}{b}-1\right)\frac{s^{\alpha-1}}{s^\alpha + 1/b}. \tag{4.32}$$

Applying inverse Laplace transform in (4.32) gives by the use of (1.27)

$$\eta(t) = 1 + \left(\frac{a}{b}-1\right)E_\alpha\left(-\frac{1}{b}t^\alpha\right). \tag{4.33}$$

Since $a/b > 1$ the kernel $\eta(t)$ is a non-negative function. Moreover, $\eta(t) \in \mathcal{CMF}$ and therefore $\widehat{\eta}(s) \in \mathcal{SF}$. Hence, the requirements on the kernel $\eta(t)$ are satisfied.

We point out some general relations of the kernels κ and η to the kernel k of the equivalent Volterra integral equation (4.3). First, in the diffusion regime, $g(s) = s\widehat{\kappa}(s) \in \mathcal{CBF}$ if and only if $\widehat{\kappa}(s) \in \mathcal{SF}$, see property **(P7)** in Proposition 2.1. Therefore, a generalized subdiffusion equation is subordinated to the classical diffusion equation. For the wave propagation regime we note that $g(s) = s^2\widehat{\eta}(s)$. Then the property $\widehat{\eta}(s) \in \mathcal{SF}$ implies that $g(s)$ is a product of two complete Bernstein functions (s and $s\widehat{\eta}(s)$), and thus $\sqrt{g(s)} \in \mathcal{CBF}$, see (2.2). Therefore, a generalized diffusion-wave equation is subordinated to the classical wave equation.

In this section we studied the fractional Jeffreys' type heat conduction equation as a model problem. Depending on the model parameters it governs the two fundamental types of behavior, considered in the dissertation: subdiffusion and propagation of diffusive waves. By employing the Bernstein functions technique we establish diffusion regime for $a/b < 1$ and wave propagation regime for $a/b > 1$. The two regimes are related to two different subordination principles. In the diffusion regime the abstract Cauchy problem for the Jeffreys' equation is subordinated to the first order Cauchy problem (2.6), while in the wave propagation regime it is subordinated to the second order Cauchy problem (2.7). The fractional Jeffreys-type heat conduction equation in the two different regimes is represented as a generalized subdiffusion equation, and generalized diffusion-wave equation, respectively.

The established properties indicate a strong analogy between the fractional Jeffreys-type equation (4.1) and the fractional diffusion-wave equation with the Caputo fractional time-derivative (2.8) with its two different regimes: it is a subdiffusion equation for $0 < \alpha \leq 1$ and a diffusion-wave equation for $1 < \alpha \leq 2$.

We close the chapter with a remark concerning the terminology in this dissertation. For the sake of brevity, equations subordinated to the first order Cauchy problem (2.6) are called generalized subdiffusion equations; equations subordinated to the second order Cauchy problem (2.7), which are not generalized subdiffusion equations, are called generalized diffusion-wave equations. In other words, if we set

$$\alpha_* = \min\{\alpha > 0 |\ g(s)^{1/\alpha} \in \mathcal{CBF}, \quad s > 0\},$$

where $g(s)$ is the function (2.11), then the abstract Volterra equation (2.10) (respectively, any fractional order integro-differential equation, which is equivalent to (2.10)), is called generalized subdiffusion equation if A is a generator of C_0 semigroup and $\alpha_* \in (0, 1]$ and generalized diffusion-wave equation if A is a generator of a cosine family and $\alpha_* \in (1, 2]$.

Chapter 5

Generalized subdiffusion equations

First, the abstract Cauchy problem for the distributed order fractional evolution equation in the Caputo and in the Riemann-Liouville sense is studied for operators generating a strongly continuous one-parameter semigroup on a Banach space. Continuous as well as discrete distribution of fractional time-derivatives of orders in the interval $[0,1]$ are considered. The problem with a general convolutional derivative is studied next and two types of subordination results are established. The subordination principle in the particular case of general relaxation equation is applied to derive estimates for the relaxation functions, which are applied to prove uniqueness and stability for an inverse source problem.

5.1 Distributed order diffusion equations

In this section, we consider the fractional evolution equation of distributed order in the following two alternative forms:

$$\int_0^1 \mu(\beta)\,{}^C D_t^\beta u(t)\,d\beta = Au(t), \quad t > 0, \tag{5.1}$$

and

$$u'(t) = \int_0^1 \mu(\beta) D_t^\beta A u(t)\,d\beta, \quad t > 0, \tag{5.2}$$

with the initial condition $u(0) = a \in X$. Here ${}^C D_t^\beta$ and D_t^β are fractional time-derivatives in the Caputo and in the Riemann-Liouville sense, respectively, and

A is a closed linear operator densely defined in a Banach space X. The weight function μ may be a generalized function in the sense of Gelfand and Shilov [42] that represents a nonnegative measure.

For the weight function μ two cases are considered:

- discrete distribution

$$\mu(\beta) = \delta(\beta - \alpha) + \sum_{j=1}^{m} b_j \delta(\beta - \alpha_j), \tag{5.3}$$

where $1 > \alpha > \alpha_1 ... > \alpha_m > 0,\ b_j > 0,\ j = 1, ..., m,\ m \geq 0$, and δ is the Dirac delta function;

- continuous distribution

$$\mu \in C[0,1],\ \mu(\beta) \geq 0,\ \beta \in [0,1], \tag{5.4}$$

and $\mu(\beta) \neq 0$ on a set of a positive measure.

In the case of discrete distribution, equations (5.1) and (5.2) are reduced to the multi-term time-fractional equations

$${}^C D_t^{\alpha} u(t) + \sum_{j=1}^{m} b_j \, {}^C D_t^{\alpha_j} u(t) = Au(t), \quad t > 0, \tag{5.5}$$

and

$$u'(t) = D_t^{\alpha} Au(t) + \sum_{j=1}^{m} b_j \, D_t^{\alpha_j} Au(t), \quad t > 0, \tag{5.6}$$

respectively. Note that if $m = 0$ (single-term fractional evolution equation) problem (5.5) is equivalent to (5.6) with α replaced by $1 - \alpha$. However, in general, similar equivalence does not hold for equations (5.1) and (5.2).

In this chapter, it is assumed that the operator A is a generator of a C_0-semigroup, i.e. that the classical abstract Cauchy problem

$$u'(t) = Au(t), \quad t > 0; \qquad u(0) = a \in X, \tag{5.7}$$

is well-posed. Reformulating problems (5.1) and (5.2) as abstract Volterra integral equations, we propose a unified approach to their study. We prove that the scalar kernels of the corresponding integral equations have certain complete monotonicity properties and derive useful consequences for the original equations (5.1) and (5.2) based mainly on these properties.

5.1.1 Integral reformulation and properties of the kernels

We reformulate problems (5.1) and (5.2) as Volterra integral equations of the form (2.10) with appropriate kernels $k(t)$. By applying (formally) the Laplace transform and, by the use of properties (1.17) and (1.18), it follows for the solution of (2.10)

$$\widehat{u}(s) = \frac{1}{s}(1 - \widehat{k}(s)A)^{-1}a \tag{5.8}$$

and for the solutions of problems (5.1) and (5.2), respectively

$$\widehat{u}(s) = \frac{h(s)}{s}\left(h(s) - A\right)^{-1} a, \quad \widehat{u}(s) = \frac{1}{s}\left(1 - \frac{h(s)}{s}A\right)^{-1} a, \tag{5.9}$$

where

$$h(s) = \int_0^1 \mu(\beta)s^\beta \, d\beta. \tag{5.10}$$

Note that in the discrete distribution case (5.3) $h(s)$ admits the representation

$$h(s) = s^\alpha + \sum_{j=1}^{m} b_j s^{\alpha_j}. \tag{5.11}$$

Comparing (5.9) to (5.8), it follows for the kernels $k_1(t)$ and $k_2(t)$, corresponding to problems (5.1) and (5.2), respectively:

$$\widehat{k_1}(s) = \left(h(s)\right)^{-1}, \; \widehat{k_2}(s) = h(s)/s. \tag{5.12}$$

Define the functions

$$g_i(s) = 1/\widehat{k_i}(s), \quad i = 1, 2,$$

that is

$$g_1(s) = h(s), \; g_2(s) = s/h(s), \tag{5.13}$$

where $h(s)$ is defined in (5.10).

Some useful properties of the functions $k_i(t)$ and $g_i(s)$, $i = 1, 2$, are established in the next theorem.

Theorem 5.1. *Let $\mu(\beta)$ be either of the form (5.3) or of the form (5.4) with the additional assumptions $\mu \in C^3[0, 1]$, $\mu(1) \neq 0$, and $\mu(0) \neq 0$ or $\mu(\beta) = c\beta^\nu$ as $\beta \to 0$, where $c, \nu > 0$. Then the functions $k_i(t)$ and $g_i(s)$, $i = 1, 2$, have the following properties:*

(a) $k_i \in L^1_{loc}(\mathbb{R}_+)$, $\lim_{t\to 0} k_i(t) = +\infty$, *and* $\lim_{t\to+\infty} k_i(t) = 0$;
(b) $k_i(t) \in \mathcal{CMF}$ *for* $t > 0$;
(c) $k_1 * k_2 \equiv 1$;
(d) $g_i(s) \in \mathcal{CBF}$ *for* $s > 0$, $\lim_{s\to 0} g_i(s) = 0$ *and* $\lim_{s\to+\infty} g_i(s) = +\infty$;
(e) $\lim_{s\to 0} g_i(s)/s = +\infty$ *and* $\lim_{s\to+\infty} g_i(s)/s = 0$.

Let us first consider some particular cases. Applying (1.4), (1.27), (5.11), (5.12) and (5.13), it follows in the single-term case ((5.11) with $m = 0$):

$$k_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ k_2(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad g_1(s) = s^\alpha,\ g_2(s) = s^{1-\alpha},$$

and in the double-term case ((5.11) with $m = 1$):

$$k_1(t) = t^{\alpha-1} E_{\alpha-\alpha_1,\alpha}(-b_1 t^{\alpha-\alpha_1}),\ k_2(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + b_1 \frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)},$$
$$g_1(s) = s^\alpha + b_1 s^{\alpha_1},\ g_2(s) = \frac{s}{s^\alpha + b_1 s^{\alpha_1}}.$$

Thus, in the single-term case Theorem 5.1 is straightforward. In the double term case, statements (a) and (b) are trivial for k_2; for k_1 they follow from the asymptotic behavior of Mittag-Leffler function and the fact that the function $E_{\alpha,\beta}(-x) \in \mathcal{CMF}$ for $x > 0$, $0 \le \alpha \le 1, \beta \ge \alpha$. On the other hand, properties (d) and (e) are trivial for g_1. Since $g_2(s) = s/g_1(s)$, according to **(P9)** in Proposition 2.1 $g_1(s)$ and $g_2(s)$ are simultaneously complete Bernstein functions.

In the case of continuous distribution in its simplest form: constant weight function $\mu(\beta) \equiv 1$. Then (5.10) implies (taking $s^\beta = e^{\beta \log s}$)

$$g_1(s) = \frac{s-1}{\log s}, \quad g_2(s) = \frac{s \log s}{s-1}.$$

Based on these explicit representations, the positivity of the functions $g_i(s)$ for $s > 0$ and their limiting behavior as $s \to 0$ and $s \to +\infty$ can be straightforwardly established. However, the fact that $g_i(s) \in \mathcal{CBF}$ is not easily recognized.

Now, we proceed with the proof of Theorem 5.1.

Proof. Let us start with the kernel $k_2(t)$. Application of the inverse Laplace transform to $\widehat{k_2}(s) = h(s)/s$, see (5.12), implies by the use of (1.4):

$$k_2(t) = \int_0^1 \mu(\beta) \frac{t^{-\beta}}{\Gamma(1-\beta)}\, d\beta. \tag{5.14}$$

In the particular case of discrete distribution

$$k_2(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{j=1}^{m} b_j \frac{t^{-\alpha_j}}{\Gamma(1-\alpha_j)}.$$

Therefore

$$k_2(t) \sim \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t \to 0; \qquad k_2(t) \sim b_m \frac{t^{-\alpha_m}}{\Gamma(1-\alpha_m)}, \quad t \to \infty,$$

and thus (a) is satisfied for this kernel.

In the case of continuous distribution it is proven in [61], Proposition 2.1, that

$$k_2(t) \sim \frac{1}{t(\log t)^2}, \ t \to 0.$$

Therefore, it is integrable at $t = 0$ (note that the singularity at $t = 0$ is quite strong). Moreover, since $\Gamma(1-\beta) \geq 1$ for $\beta \in [0, 1]$, (5.14) implies for $t > 1$

$$0 \leq k_2(t) \leq \sup_{\beta \in [0,1]} |\mu(\beta)| \int_0^1 t^{-\beta} \, d\beta \leq C \frac{t-1}{t \log t}$$

and thus $k_2(t) \to 0$ as $t \to \infty$.

Complete monotonicity of $k_2(t)$ follows directly from (5.14) by noticing that $t^{-\beta} \in \mathcal{CMF}$, $\Gamma(1-\beta) > 0$ for $\beta \in (0, 1)$ and applying properties **(P1)** in Proposition 2.1. In this way, (a) and (b) are proven for the kernel $k_2(t)$ in both discrete and continuous case.

Consider now the kernel $k_1(t)$. The identity $\widehat{k_1}(s) = 1/h(s)$, see (5.12), implies the following representation for this kernel as an inverse Laplace integral:

$$k_1(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{st} \frac{1}{h(s)} \, ds, \quad \gamma > 0. \tag{5.15}$$

Consider first the discrete case in which $h(s)$ is defined by (5.11). The function $h(s)$ has no zeros in $\mathbb{C}$ cut along the negative real axis. Indeed, for $s = re^{i\phi}$, with $r > 0$, $\phi \in (-\pi, \pi)$,

$$\Im\{s^{\alpha} + \sum_{j=1}^{m} b_j s^{\alpha_j}\} = r^{\alpha} \sin \alpha\phi + \sum_{j=1}^{m} b_j r^{\alpha_j} \sin \alpha_j \phi \neq 0,$$

since $\sin \alpha\phi$ and $\sin \alpha_j \phi$ have the same sign and $b_j > 0$. Then the function under the integral sign in (5.15) is analytic in Σ_π and we can bend the integration contour to the contour $\Gamma_{\rho,\theta}$ defined by

$$\Gamma_{\rho,\theta} = \Gamma^-_{\rho,\theta} \cup \Gamma^0_{\rho,\theta} \cup \Gamma^+_{\rho,\theta}, \quad \rho > 0,\ \pi/2 < \theta < \pi,$$

where

$$\Gamma^\pm_{\rho,\theta} = \left\{ re^{\pm \mathrm{i}\theta} :\ r \geq \rho \right\},\ \Gamma^0_{\rho,\theta} = \left\{ \rho e^{\mathrm{i}\psi} :\ |\psi| \leq \theta \right\},$$

and $\Gamma_{\rho,\theta}$ is oriented in the direction of growth of $\arg s$. Hence

$$k_1(t) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\rho,\theta}} e^{st} \frac{1}{s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}}\, ds. \tag{5.16}$$

The integral over $\Gamma^0_{\rho,\theta}$ is a function from $C^\infty[0,\infty)$. Take $\rho = R$ so large that

$$|s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}| \geq |s|^\alpha - \sum_{j=1}^m b_j |s|^{\alpha_j} \geq |s|^\alpha/2, \quad |s| \geq R.$$

Then, noting that $\cos\theta < 0$ for $\pi/2 < \theta < \pi$, it follows

$$\left| \int_{\Gamma^-_{R,\theta} \cup \Gamma^+_{R,\theta}} e^{st} \frac{1}{s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}}\, ds \right| \leq C \int_R^\infty e^{rt\cos\theta} r^{-\alpha}\, dr \leq C t^{\alpha - 1}. \tag{5.17}$$

Therefore, $k_1(t) \sim t^{\alpha-1}$ for $t \to 0$ and thus it has an integrable singularity at $t = 0$. Since in the discrete case $\widehat{k_1}(s) \sim s^{-\alpha_m}$ as $s \to 0$, Karamata-Feller Tauberian theorem implies $k_1(t) \sim t^{\alpha_m - 1}$, $t \to \infty$. Thus (a) is proven for the discrete variant of $k_1(t)$. To prove its complete monotonicity we take $\rho \to 0$ and $\theta \to \pi$ in (5.16). Since

$$\left| \int_{\Gamma^0_{\rho,\theta}} e^{st} \frac{1}{s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}}\, ds \right| \leq C \int_0^\pi e^{\rho t \cos\psi} \rho^{1-\alpha_m}\, d\psi, \tag{5.18}$$

the integral over $\Gamma^0_{\rho,\theta}$ vanishes when $\rho \to 0$. Therefore, only the contributions of the integrals over $\Gamma^\pm_{\rho,\theta}$ remain in (5.16), implying

$$k_1(t) = \int_0^\infty e^{-rt} K(r)\, dr, \tag{5.19}$$

where

$$K(r) = -\frac{1}{\pi}\Im\left\{ \frac{1}{s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}}\bigg|_{s=re^{i\pi}} \right\}.$$

Simplifying this expression, we get

$$K(r) = \frac{1}{\pi}\frac{B(r)}{(A(r))^2 + (B(r))^2}$$

where

$$A(r) = r^\alpha \cos\alpha\pi + \sum_{j=1}^m b_j r^{\alpha_j}\cos\alpha_j\pi,\ \ B(r) = r^\alpha \sin\alpha\pi + \sum_{j=1}^m b_j r^{\alpha_j}\sin\alpha_j\pi,$$

and thus $K(r) > 0$ for $r > 0$. This together with representation (5.19) implies that $k_1(t) \in \mathcal{CMF}$.

In the case of continuous distribution it is proven in [61], Proposition 3.1, that for small values of t

$$k_1(t) \le C \log\frac{1}{t},$$

therefore this kernel has integrable singularity at $t \to 0$. Further, by [61], Proposition 2.2, (ii) and (iii),

$$\widehat{k_1}(s) \sim \left(\log\frac{1}{s}\right)^{\lambda+1}, \quad s \to 0, \tag{5.20}$$

where

$$\lambda = \begin{cases} 0 & \text{if } \mu(0) \neq 0, \\ \nu > 0 & \text{if } \mu(\beta) = c\beta^\nu \text{ as } \beta \to 0. \end{cases}$$

Applying again Karamata-Feller Tauberian theorem (Theorem 1.2) it follows

$$k_1(t) \sim \frac{(\log t)^\lambda}{t}, \quad t \to \infty,$$

and thus $k_1(t) \to 0$ as $t \to \infty$. Complete monotonicity of $k_1(t)$ in the case of continuous distribution is proven in [61], Propositions 3.1. In this way the proof of properties (a) and (b) is completed for all cases.

According to (5.12)

$$\widehat{k_1}(s)\widehat{k_2}(s) = 1/s$$

and taking the inverse Laplace transform of this identity we derive (c).

The fact that $g_i(s) \in \mathcal{CBF}$ can be established directly. Indeed, since $s^\beta \in \mathcal{CBF}$ for $\beta \in [0,1]$, it follows by applying **(P2)** in Proposition 2.1 that $h(s) \in \mathcal{CBF}$. Therefore, $g_1(s) \in \mathcal{CBF}$ and, taking into account **(P9)** in Proposition 2.1, it also follows $g_2(s) \in \mathcal{CBF}$.

The limiting behaviors of $g_i(s)$ as $s \to 0$ and $s \to +\infty$ are easily established for the cases of discrete distribution. For continuous distribution, inserting the limit (5.20) in the identities $g_1(s) = 1/\widehat{k}_1(s)$ and $g_2(s) = s\widehat{k}_1(s)$, it follows $\lim_{s\to 0} g_i(s) = 0$ and $\lim_{s\to 0} g_i(s)/s = +\infty$.

According to [61], Proposition 2.2, (i),

$$g_1(s) \sim \mu(1)\frac{s}{\log s}, \quad s \to +\infty.$$

This together with the identity $g_2(s) = s/g_1(s)$ implies $\lim_{s\to+\infty} g_i(s) = +\infty$ and $\lim_{s\to+\infty} g_i(s)/s = 0$. □

Let us note that kernels $k_1(t)$ and $k_2(t)$, satisfying property (c), are called a pair of Sonine kernels.

The limiting cases of the multi-term equations, (5.5) with $\alpha = 1$ and (5.6) with $\alpha_m = 0$ also deserve attention, since they appear in the modeling of some physical processes. The simplest two-term particular case of the first one was introduced in the modeling of fractal mobile-immobile solute transport [103], while the two-term case of the second is related to the Rayleigh-Stokes problem for a generalized second grade fluid [11, 109].

The abstract form of the fractal mobile-immobile solute transport equation is

$$u'(t) + b\, {}^C D_t^\alpha u(t) = Au(t), \quad t > 0, \quad u(0) = a \in X, \tag{5.21}$$

where ${}^C D_t^\alpha$ is the Caputo fractional derivative of order α, $0 < \alpha < 1$, $b > 0$, and A is an unbounded closed linear operator defined on a Banach space X.

The kernel of the equivalent Volterra equation obeys the relations

$$k(t) = E_{1-\alpha}(-bt^{1-\alpha}), \quad g(s) = (\widehat{k}(s))^{-1} = s + bs^\alpha, \quad \alpha \in (0,1).$$

The following properties of the functions $k(t)$ and $g(s)$ are easily derived:
(a) $k \in L^1_{loc}(\mathbb{R}_+) \cap \mathcal{CMF}$;
(b) $\lim_{t\to 0} k(t) = 1$, $\lim_{t\to+\infty} k(t) = 0$;
(c) $g(s) \in \mathcal{CBF}$ for $s > 0$;
(d) $\lim_{s\to 0} g(s) = 0$, $\lim_{s\to+\infty} g(s) = +\infty$;
(e) $\lim_{s\to 0} g(s)/s = +\infty$ and $\lim_{s\to+\infty} g(s)/s = 1$.

We see that, compared to the properties in Theorem 5.1, the only differences are in the limiting behaviour of the functions $k(t)$ and $g(s)$.

5.1.2 A limiting case

In this subsection we study in detail the second limiting case, which is given by the fractional differential equation:

$$u'(t) = Au(t) + \gamma D_t^\alpha Au(t),\ t > 0, \quad u(0) = a \in X, \tag{5.22}$$

where D_t^α is the Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$, $\gamma > 0$, A is an unbounded closed linear operator defined on a Banach space X.

Our motivation for the study of this equation comes from recent works where related problems appear in the modeling of unidirectional viscoelastic flows. For example, if the operator A is some realization of the Laplace operator, then the inhomogeneous version of (5.22) is the Rayleigh-Stokes problem for a generalized second-grade fluid, see e.g. [109].

Integral reformulation of the problem

Assume $u, Au \in C(\mathbb{R}_+, X)$. Integrating both sides of the governing equation in (5.22), by the use of the identity $(J_t^{1-\alpha}Au)(0) = 0$, we obtain:

$$u(t) = a + \int_0^t \left(1 + \gamma\omega_{1-\alpha}(t-\tau)\right) Au(\tau)\, d\tau \tag{5.23}$$

that is the Volterra integral equation (2.10) with kernel $k(t)$ given by:

$$k(t) = 1 + \gamma\omega_{1-\alpha}(t) \tag{5.24}$$

where the function ω_α is defined in (1.3). Conversely, differentiating both sides of (5.92) and using that:

$$\frac{d}{dt}\left(\omega_{1-\alpha} * Au\right) = \frac{d}{dt}\left(J_t^{1-\alpha}Au\right) = D_t^\alpha Au$$

we get back the governing equation in (5.22). Since $(k * Au)(0) = 0$, the initial condition is also satisfied.

We begin with summarizing some properties of the kernel $k(t)$ defined in (5.24) and the related function:

$$g(s) = (\widehat{k}(s))^{-1} = \frac{s}{1 + \gamma s^\alpha}. \tag{5.25}$$

Here, $\widehat{k}$ is the Laplace transform of k, and the Laplace transform pair (1.4) is used.

Theorem 5.2. *The functions $k(t)$ and $g(s)$, defined in (5.24) and (5.25), respectively, have the following properties:*
(a) $k \in L^1_{loc}(\mathbb{R}_+) \cap \mathcal{CMF}$;
(b) $\lim_{t\to 0} k(t) = +\infty$, $\lim_{t\to+\infty} k(t) = 1$;
(c) $g(s) \in \mathcal{CBF}$ *for* $s > 0$;
(d) $\lim_{s\to 0} g(s) = 0$, $\lim_{s\to+\infty} g(s) = +\infty$;
(e) $\lim_{s\to 0} g(s)/s = 1$ *and* $\lim_{s\to+\infty} g(s)/s = 0$.
(f) *the estimate holds true:*

$$|g(s)| \le C \min(|s|, |s|^{1-\alpha}), \;\; s \in \Sigma(\pi - \theta), \;\; \theta \in (0, \pi).$$

Proof. The function $k(t)$ is infinitely continuously differentiable for $t > 0$ with integrable singularity at $t = 0$, and its derivatives satisfy (5.72); thus, (a) is fulfilled. Limits (b) and (d) follow by direct check. Since $1 + \gamma s^\alpha \in \mathcal{CBF}$ for $\alpha \in (0, 1)$, then **(P9)** in Proposition 2.1 implies $g(s) \in \mathcal{CBF}$. Alternatively, this can be seen from the representation

$$g(s) = s\frac{1}{1 + \gamma s^\alpha} = s\mathcal{L}\{\gamma^{-1} t^{\alpha-1} E_{\alpha,\alpha}(-\gamma^{-1} t^\alpha)\}(s).$$

Since $\alpha \in (0, 1)$, the Mittag-Leffler function $E_{\alpha,\alpha}(-x) \in \mathcal{CMF}$ for $x > 0$. Then, the function $E(t) = \gamma^{-1} t^{\alpha-1} E_{\alpha,\alpha}(-\gamma^{-1} t^\alpha) \in \mathcal{CMF}$ for $t > 0$. Therefore, $\mathcal{L}\{E(t)\}(s) \in \mathcal{SF}$, see (2.1) and by **(P7)** in Proposition 2.1 $g(s) = s\mathcal{L}\{E(t)\}(s) \in \mathcal{BF}$. To prove property (f) we let $s \in \Sigma(\pi - \theta)$, i.e. $s = re^{i\psi}$, $|\psi| < \pi - \theta, r > 0$. Then

$$|1 + \gamma s^\alpha|^2 = 1 + 2\gamma r^\alpha \cos \alpha\psi + \gamma^2 r^{2\alpha} > 1 + 2\gamma r^\alpha \cos \alpha\pi + \gamma^2 r^{2\alpha}. \qquad (5.26)$$

Let $b = \cos \alpha\pi$. Since $f(x) = 1 + 2bx + x^2 \ge 1 - b^2$, it follows from (5.26) that

$$|1 + \gamma s^\alpha|^2 > 1 - \cos^2 \alpha\pi = \sin^2 \alpha\pi.$$

Since $\sin \alpha\pi > 0$ we obtain

$$|g(s)| = \left|\frac{s}{1 + \gamma s^\alpha}\right| < \frac{|s|}{\sin \alpha\pi}.$$

From (5.26) it also follows that

$$|1 + \gamma s^\alpha|^2 = (1 + \gamma r^\alpha \cos \alpha\pi)^2 + (\gamma r^\alpha \sin \alpha\pi)^2 \ge \gamma^2 \sin^2 \alpha\pi r^{2\alpha},$$

and consequently we get

$$|g(s)| \le \frac{r}{\gamma r^\alpha \sin \alpha\pi} = \frac{|s|^{1-\alpha}}{\gamma \sin \alpha\pi}.$$

This completes the proof of the theorem. □

Let us note that the limiting behaviour of the functions $k(t)$ and $g(s)$ in the above theorem is different from those in Theorem 5.1.

The scalar case

It is instructive to study first the scalar version of equation (5.22), where $A = -\lambda$ is a given negative constant. Consider the problem:

$$u'(t) + \lambda u(t) + \lambda\gamma D_t^\alpha u(t) = 0, \quad u(0) = 1, \tag{5.27}$$

where $\lambda > 0$. Denote its solution by $u(t;\lambda)$. To solve (5.27), we apply the Laplace transform and use the identities (1.17) and $\mathcal{L}\{u'\}(s) = s\mathcal{L}\{u\}(s) - u(0)$. In this way, for the Laplace transform of $u(t;\lambda)$, one gets:

$$\int_0^\infty e^{-st} u(t;\lambda)\, dt = \frac{1}{s + \gamma\lambda s^\alpha + \lambda}. \tag{5.28}$$

Theorem 5.3. *For any $\lambda > 0$, the solution $u(t;\lambda)$ of Problem (5.27) has the following properties:*
(a) *$u(t;\lambda)$ is a positive nonincreasing function for $t > 0$ and $u(t;\lambda) \to 0$ as $t \to +\infty$ with:*

$$u(t;\lambda) \sim -\frac{\gamma t^{-\alpha-1}}{\lambda\Gamma(-\alpha)}, \quad t \to +\infty, \tag{5.29}$$

(b) $u(t;\lambda) \in \mathcal{CMF},\ t > 0,$
(c) *The identity is satisfied:*

$$\lambda \int_0^\infty u(t;\lambda)\, dt = 1, \tag{5.30}$$

(d) *The solution admits the following explicit representation:*

$$u(t;\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\gamma^{k+1}\lambda^{k+1}} t^{(\alpha-1)(k+1)} E_{\alpha,\alpha(k+1)-k}^{k+1}(-\gamma^{-1} t^\alpha). \tag{5.31}$$

Proof. Properties (a) and (b), with the exception of the asymptotic estimate (5.29), are proven in [24], Theorem 2.2. To prove (5.29), we apply the Karamata–Feller–Tauberian theorem (see Theorem 1.2). Since for small $|s|$, the Laplace transform (5.28) of $u(t;\lambda)$ is dominated by the function

$$\frac{1}{\lambda\gamma s^\alpha + \lambda}$$

applying the asymptotic estimate (1.33) (note that $\Gamma(0)^{-1} = 0$), we obtain for large t:

$$u(t;\lambda) \sim \mathcal{L}^{-1}\left\{\frac{1}{\lambda\gamma s^\alpha + \lambda}\right\} = \frac{1}{\lambda\gamma} t^{\alpha-1} E_{\alpha,\alpha}(-\frac{1}{\gamma}t^\alpha) \sim -\frac{\gamma t^{-\alpha-1}}{\lambda\Gamma(-\alpha)}, \quad t \to +\infty.$$

Identity (5.30) is obtained taking $s \to 0$ in (5.28). Representation (5.31) in terms of three-parameter Mittag–Leffler functions is obtained by taking the inverse Laplace transform of function (5.28). If $\left|s\lambda^{-1}(\gamma s^\alpha + 1)^{-1}\right| < 1$, then:

$$\begin{aligned}\frac{1}{s + \gamma\lambda s^\alpha + \lambda} &= \frac{1}{\lambda(\gamma s^\alpha + 1)}\left(\frac{s}{\lambda(\gamma s^\alpha + 1)} + 1\right)^{-1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(\gamma\lambda)^{k+1}} \frac{s^k}{(s^\alpha + \gamma^{-1})^{k+1}}\end{aligned}$$

and, applying term-wise the inverse Laplace transform, we get (5.31) by the use of (1.34). □

Remark 5.1. *From the Laplace transform pair (5.28) it follows also the following representation as a multinomial Mittag-Leffler function (see Chapter 6)*

$$u(t;\lambda) = \mathcal{E}_{(1,1-\alpha),1}(t;\lambda,\gamma\lambda).$$

Subordination principle

Assume the operator A generates a bounded C_0 semigroup $S_1(t)$. The main goal now is to prove that in this case, problem (5.22) is well-posed, and its solution operator $S(t)$ satisfies the relationship:

$$S(t) = \int_0^\infty \varphi(t,\tau) S_1(\tau)\, d\tau, \quad t > 0, \tag{5.32}$$

with an appropriate function $\varphi(t,\tau)$.

We give next a complete proof of the subordination principle for problem (5.22), without the use of the general subordination theorem. Let the function $\varphi(t,\tau)$ be such that its Laplace transform with respect to t satisfies:

$$\int_0^\infty e^{-st}\varphi(t,\tau)\,dt = \frac{g(s)}{s}e^{-\tau g(s)}, \quad s,\tau>0, \tag{5.33}$$

where $g(s)$ is defined in (5.25). The reason is that in this case the operator $S(t)$, defined by (5.32), will satisfy (2.13). Indeed, by (5.32) and (5.33) and the identity for the Laplace transform of a C_0-semigroup

$$\int_0^\infty e^{-\mu\tau}S_1(\tau)\,d\tau = (\mu - A)^{-1},$$

it follows:

$$\begin{aligned}\int_0^\infty e^{-st}S(t)\,dt &= \int_0^\infty \left(\int_0^\infty e^{-st}\varphi(t,\tau)\,dt\right)S_1(\tau)\,d\tau \\ &= \frac{g(s)}{s}\int_0^\infty e^{-\tau g(s)}S_1(\tau)\,d\tau = \frac{g(s)}{s}(g(s)-A)^{-1}.\end{aligned} \tag{5.34}$$

Then, by the uniqueness of the Laplace transform, $S(t)$ will be the solution operator of problem (5.22). For the strict proof we refer to Theorem 2.4 with $\alpha = 1$.

Identity (5.33) implies that the function $\varphi(t,\tau)$ can be found by the inverse Laplace integral:

$$\varphi(t,\tau) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st-\tau g(s)}\,\frac{g(s)}{s}\,ds, \quad c>0,\ t,\tau>0. \tag{5.35}$$

Let us check that the function $\varphi(t,\tau)$ is well defined in this way. Since $g(s)\in\mathcal{CBF}$, then $\Re\{s\}>0$ implies $\Re\{g(s)\}>0$. More precisely, if $s = re^{\mathrm{i}\theta}$, then:

$$\Re\{g(s)\} = \frac{r\cos\theta + \gamma r^{\alpha+1}\cos(1-\alpha)\theta}{1+2\gamma r^\alpha\cos\alpha\theta + \gamma^2 r^{2\alpha}}.$$

Hence, when $r\to\infty$, $|\theta|\to\pi/2$, the dominant term of $\Re\{g(s)\}$ is

$$r^{1-\alpha}\sin\alpha\pi/2 > 0.$$

This together with the estimate (f) of Theorem 5.2 shows that the integral in (5.35) is absolutely convergent.

We are ready to formulate the main subordination result for problem (5.22).

Theorem 5.4. *Let A be a generator of a bounded C_0 semigroup $S_1(t)$, such that $\|S(t)\| \leq C$, $t \geq 0$. Then, problem (5.22) is well-posed, with bounded solution operator $S(t)$ satisfying the same bound. Moreover, the subordination identity (5.32) holds, where the function $\varphi(t,\tau)$ admits the representation for $t, \tau > 0$:*

$$\varphi(t,\tau) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{(\alpha-1)(k+1)}}{\gamma^{k+1}} \frac{\tau^k}{k!} E^{k+1}_{\alpha,\alpha(k+1)-k}(-\gamma^{-1}t^{\alpha}). \tag{5.36}$$

The function $\varphi(t,\tau)$ is a probability density function with respect to both variables t and τ, i.e., it satisfies the following properties for $t, \tau > 0$:

$$\begin{aligned} \varphi(t,\tau) &\geq 0 && (5.37)\\ \int_0^\infty \varphi(t,\tau)\, d\tau &= 1 && (5.38)\\ \int_0^\infty \varphi(t,\tau)\, dt &= 1. && (5.39) \end{aligned}$$

Proof. Let us find the Laplace transform of $\varphi(t,\tau)$ with respect to τ. Applying (5.35) and interchanging the order of integration, it follows:

$$\begin{aligned} \int_0^\infty e^{-\lambda\tau}\varphi(t,\tau)\, d\tau &= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st}\frac{g(s)}{s}\left(\int_0^\infty e^{-(\lambda+g(s))\tau}\, d\tau\right) ds \\ &= \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{st}\frac{g(s)}{s(g(s)+\lambda)}\, ds \end{aligned}$$

From the definition of $g(s)$ in (5.25):

$$\frac{g(s)}{s(g(s)+\lambda)} = \frac{1}{s+\gamma\lambda s^{\alpha}+\lambda}$$

Therefore, (5.28) implies that the last integral gives exactly the solution $u(t;\lambda)$ of the scalar equation (5.27), *i.e.*,

$$\int_0^\infty e^{-\lambda\tau}\varphi(t,\tau)\, d\tau = u(t;\lambda), \quad \lambda, t > 0. \tag{5.40}$$

Inserting representation (5.31) of $u(t;\lambda)$ in (5.40) and using (1.4), we deduce the series expansion of the function $\varphi(t,\tau)$ in (5.36). Alternatively, this expansion can be deduced, inserting the series expansion of the function $\frac{g(s)}{s}e^{-\tau g(s)}$ in (5.35) and using the Laplace transform pair (1.34).

The complete monotonicity of $u(t;\lambda)$ for $t>0$ and (5.40) imply the positivity of $\varphi(t,\tau)$ by applying Bernstein's theorem. Alternatively, the positivity of $\varphi(t,\tau)$ can be also deduced from (5.33), since $g(s)\in\mathcal{CBF}$ yields using (2.3)

$$\frac{g(s)}{s}e^{-\tau g(s)} \in \mathcal{CMF}.$$

Further, letting $s\to 0$ in (5.33) and $\lambda\to 0$ in (5.40) and noting that $u(t;0)=1$, we deduce by applying the dominated convergence theorem the integral identities, (5.38) and (5.39).

The definition (5.32), the estimate for $S_1(t)$ and the properties (5.37) and (5.38) imply:

$$\|S(t)\| = \int_0^\infty \varphi(t,\tau)\|S_1(\tau)\|\,d\tau \le C\int_0^\infty \varphi(t,\tau)\,d\tau = C, \quad t>0.$$

Next, we deduce the strong continuity of $S(t)$ at the origin from the strong continuity of $S_1(t)$ at the origin:

$$\lim_{t\to 0} S_1(t)a = a. \tag{5.41}$$

On the basis of the dominated convergence theorem and by the change of variables $\sigma = t^{\alpha-1}\tau$ in (5.32), we obtain:

$$\lim_{t\to 0} S(t)a = \lim_{t\to 0}\int_0^\infty t^{1-\alpha}\varphi(t,\sigma t^{1-\alpha})S_1(\sigma t^{1-\alpha})a\,d\sigma. \tag{5.42}$$

For the function under the integral sign, we get from (5.36):

$$t^{1-\alpha}\varphi(t,\sigma t^{1-\alpha}) = \sum_{k=0}^\infty \frac{(-1)^k}{k!}\frac{\sigma^k}{\gamma^{k+1}}E^{k+1}_{\alpha,\alpha(k+1)-k}(-\gamma^{-1}t^\alpha)$$

and thus:

$$\lim_{t\to 0}\left(t^{1-\alpha}\varphi(t,\sigma t^{1-\alpha})\right) = \sum_{k=0}^\infty \frac{(-1)^k\sigma^k}{\gamma^{k+1}k!\Gamma(\alpha(k+1)-k)} = \frac{1}{\gamma}M_{1-\alpha}\left(\frac{\sigma}{\gamma}\right)$$

where $M_\beta(z)$, $\beta\in(0,1)$, is the Mainardi function (1.38). Therefore, (5.42) together with (5.41) and the integral identity in (1.47) for the Mainardi function imply:

$$\lim_{t\to 0} S(t)a = \int_0^\infty \frac{1}{\gamma} M_{1-\alpha}\left(\frac{\sigma}{\gamma}\right)\, d\sigma a = \int_0^\infty M_{1-\alpha}(r)\, dra = a.$$

In this way, we proved that $S(t)$, defined by (5.32), is a strongly continuous bounded operator-valued function. Moreover, in (5.34), we proved that the Laplace transform of $S(t)$ satisfies:

$$\int_0^\infty e^{-st} S(t)\, dt = H(s) \tag{5.43}$$

where

$$H(s) = \frac{g(s)}{s}(g(s) - A)^{-1}.$$

After easily justified differentiation under the integral sign in (5.43), we obtain the estimates (2.14) and, thus, the well-posedness of problem (5.22). Then, identity (5.43) implies by the uniqueness of the Laplace transform that $S(t)$ is exactly the solution operator of (5.22). The proof of the theorem is completed. □

Let us note that in the case of single-term fractional evolution equation the subordinated solution operator $S(t)$ is always analytic in some sector without assuming the analyticity of the C_0-semigroup $S_1(t)$, see Theorem 2.3. However, this is not true for the considered here equation (5.22). The question of analyticity will be discussed later in this chapter.

5.2 General convolutional derivative

Generalized fractional derivative of Caputo type is introduced in [62] in the form

$$({}^C\mathbb{D}_t^{(\kappa)} f)(t) = \frac{d}{dt}\int_0^t \kappa(t-\tau) f(\tau)\, d\tau - \kappa(t) f(0), \quad t > 0, \tag{5.44}$$

where $\kappa(t)$ is a nonnegative locally integrable kernel. For the kernel $\kappa(t)$ we assume that its Laplace transform $\widehat{\kappa}(s)$ exists for all $s > 0$ and obeys

$$\widehat{\kappa}(s) \in \mathcal{SF} \quad \text{and} \quad \lim_{s\to+\infty} s\widehat{\kappa}(s) = +\infty, \tag{5.45}$$

where $\mathcal{SF}$ denotes the class of Stieltjes functions.

If f' is integrable function ($f \in W^{1,1}$) then by applying the identity

$$(\kappa * f)'(t) = (\kappa * f')(t) + \kappa(t)f(0) \tag{5.46}$$

we obtain the representation $({}^C\mathbb{D}_t^{(\kappa)} f)(t) = (\kappa * f')(t)$.

The operator ${}^C\mathbb{D}_t^{(\kappa)}$ reduces to the first-order derivative $\frac{d}{dt}$ when $\widehat{\kappa}(s) = 1$. It is the Caputo time-fractional derivative of order $\alpha \in (0, 1)$ when $\widehat{\kappa}(s) = s^{\alpha-1}$.

Let us note that assumptions (5.45) are weaker than those required in the original definition of the so-called general fractional derivative, introduced in [62]. More precisely, the operator ${}^C\mathbb{D}_t^{(\kappa)}$ is a general fractional derivative, if, along with (5.45), the following additional limiting behavior conditions are imposed: $\widehat{\kappa}(s) \to 0$ as $s \to \infty$; $\widehat{\kappa}(s) \to \infty$ and $s\widehat{\kappa}(s) \to 0$ as $s \to 0$. In order to cover some examples of physically meaningful models with corresponding memory kernels, which do not satisfy some of the additional conditions, they are not required here. Such examples are the subdiffusion equation with the truncated power-law memory kernel $\kappa(t) = e^{-\gamma t}\omega_{1-\alpha}(t)$, $\gamma > 0$, $\alpha \in (0, 1)$, considered in [98, 99], the Jeffreys' type heat conduction model in the diffusion regime, see Chapter 4, and the two examples of Section 5.1.2. On the other hand, the assumption $\widehat{\kappa}(s) \in \mathcal{SF}$ is typical for a subdiffusion model (see e.g., [98, 99]) and allows the use of the convenient Bernstein functions technique. It implies that the kernel $\kappa(t)$ admits the representation

$$\kappa(t) = \kappa_0\delta(t) + \kappa_1(t), \tag{5.47}$$

where $\kappa_0 \geq 0$, $\delta(\cdot)$ denotes the Dirac delta function, and $\kappa_1(t) \in L^1_{loc}(\mathbb{R}_+)$ is a completely monotone function. The space of functions, which admit representation (5.47) was denoted by $\mathcal{CMF}_0$, see (2.4).

For example, in the case of the first-order derivative $\kappa_0 = 1$ and $\kappa_1 \equiv 0$, while $\kappa_0 = 0$ and $\kappa_1 = \omega_{1-\alpha}(t)$ for the Caputo time-fractional derivative of order $\alpha \in (0, 1)$.

Along with the kernel $\kappa(t)$ we are also interested in the corresponding Sonine kernel $k(t) \in L^1_{loc}(\mathbb{R}_+)$, which is related to $\kappa(t)$ by the following identity

$$(\kappa * k)(t) \equiv 1. \tag{5.48}$$

In Laplace domain (5.48) reads $\widehat{\kappa}(s)\widehat{k}(s) = 1/s$. Therefore assumptions (5.45) imply $\widehat{k}(s) \in \mathcal{SF}$ (see property **(P10)** in Proposition 2.1) and $\lim_{s\to\infty} \widehat{k}(s) = 0$. Hence $\widehat{k}(s)$ obeys representation (2.1) with $b = 0$. Therefore, under the assumptions (5.45) a resolvent kernel $k(t)$ exists and $k(t) \in \mathcal{CMF}$.

Basic examples of kernels $\kappa(t)$ are considered next, together with their Sonine kernels $k(t)$. For the sake of brevity the notation (1.3) is used, as well as the Laplace transform pair (1.4).

Example 5.1. The power-law memory kernel

$$\begin{aligned} \kappa(t) &= \omega_{1-\alpha}(t), \quad \widehat{\kappa}(s) = s^{\alpha-1}, \quad 0<\alpha<1; \\ k(t) &= \omega_{\alpha}(t), \quad \widehat{k}(s) = s^{-\alpha}. \end{aligned} \tag{5.49}$$

Example 5.2. The multi-term power-law memory kernel:

$$\kappa(t) = \sum_{j=1}^{m} q_j \omega_{1-\alpha_j}(t), \quad \widehat{\kappa}(s) = \sum_{j=1}^{m} q_j s^{\alpha_j-1},$$

where $1 \geq \alpha_1 > \alpha_2 > \ldots > \alpha_m > 0$, $q_j > 0$, $j = 1, \ldots, m$, $m > 1$. Without loss of generality we assume $q_1 = 1$. In this case

$$\widehat{k}(s) = \frac{1}{\sum_{j=1}^{m} q_j s^{\alpha_j}} = \frac{s^{-\alpha_1}}{1 + \sum_{j=2}^{m} q_j s^{-(\alpha_1-\alpha_j)}}. \tag{5.50}$$

Therefore, $k(t)$ admits a representation as a multinomial Mittag-Leffler function (for the definition see Chapter 6)

$$k(t) = t^{\alpha_1-1} E_{(\alpha_1-\alpha_2,\ldots,\alpha_1-\alpha_m),\alpha_1}\left(-q_2 t^{\alpha_1-\alpha_2}, \ldots, -q_m t^{\alpha_1-\alpha_m}\right). \tag{5.51}$$

In particular, in the two-term case ($m = 2$)

$$\kappa(t) = \omega_{1-\alpha_1}(t) + q\omega_{1-\alpha_2}(t), \quad \widehat{\kappa}(s) = s^{\alpha_1-1} + q s^{\alpha_2-1}, \quad . \tag{5.52}$$

where $1 \geq \alpha_1 > \alpha_2 > 0, \quad q > 0$. Therefore,

$$\widehat{k}(s) = s^{-\alpha_2}/(s^{\alpha_1-\alpha_2} + q), \quad k(t) = t^{\alpha_1-1} E_{\alpha_1-\alpha_2,\alpha_1}(-q t^{\alpha_1-\alpha_2}), \tag{5.53}$$

where we have used the Laplace transform pair (1.34).

Example 5.3. The distributed-order memory kernel:

$$\kappa(t) = \int_0^1 \omega_{1-\alpha}(t)\mu(\alpha)\, d\alpha, \quad \widehat{\kappa}(s) = \int_0^1 s^{\alpha-1}\mu(\alpha)\, d\alpha, \tag{5.54}$$

where $\mu(\cdot)$ is a nonnegative weight function.

In the particular case of uniform distribution, $\mu \equiv 1$, the memory kernel becomes

$$\kappa(t) = \int_0^1 \omega_{1-\alpha}(t)\, d\alpha, \qquad \widehat{\kappa}(s) = \int_0^1 s^{\alpha-1}\, d\alpha = \frac{s-1}{s \log s}. \tag{5.55}$$

Therefore, the Sonine kernel $k(t)$ satisfies

$$\widehat{k}(s) = \frac{\log s}{s-1}$$

and, by applying the Titchmarsh theorem for the inverse Laplace transform we get

$$k(t) = \int_0^\infty e^{-rt} K(r)\, dr,$$

where

$$K(r) = -\frac{1}{\pi} \Im \left\{ \left. \frac{\log s}{s-1} \right|_{s=re^{i\pi}} \right\} = \frac{1}{r+1}.$$

This implies the representation

$$k(t) = \int_0^\infty \frac{e^{-rt}}{r+1}\, dr = e^t \mathbb{E}_1(t), \tag{5.56}$$

where $\mathbb{E}_1(t)$ denotes the exponential integral (see Section 3.5)

$$\mathbb{E}_1(t) = \int_t^\infty \frac{e^{-\xi}}{\xi}\, d\xi.$$

Any of the kernels $\kappa(t)$ in the above examples can be considered in a weighted form, $e^{-\gamma t}\kappa(t)$, where $\gamma > 0$. Indeed, if the kernel $\kappa(t)$ satisfies (5.45), then the Laplace transform relation

$$\mathcal{L}\{e^{-\gamma t}\kappa(t)\}(s) = \widehat{\kappa}(s+\gamma) \tag{5.57}$$

imply that requirements (5.45) are satisfied for the kernel $e^{-\gamma t}\kappa(t)$ as well. The next example is of this type.

Example 5.4. The truncated power-law memory kernel

$$\kappa(t) = e^{-\gamma t}\omega_{1-\alpha}(t), \quad \widehat{\kappa}(s) = (s+\gamma)^{\alpha-1}, \qquad 0 < \alpha < 1, \quad \gamma > 0. \tag{5.58}$$

In this case the Sonine kernel $\widehat{k}(s) = (s+\gamma)^{1-\alpha}s^{-1}$ and, therefore, identities (5.57) and (1.34) imply the representation

$$k(t) = e^{-\gamma t} t^{\alpha-1} E_{1,\alpha}(\gamma t)\,. \tag{5.59}$$

In any of the above examples both kernels κ and k satisfy (5.45) and therefore can be switched, that is the kernel k can be taken as kernel κ in the definition (5.44) for the operator ${}^C\mathbb{D}_t^{(\kappa)}$.

Example 5.5. Jeffreys kernel (4.27):

$$\kappa(t) = \frac{a}{b}\delta(t) + \left(1 - \frac{a}{b}\right)\frac{1}{b}t^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{1}{b}t^{\alpha}\right),$$

where $0 < a < b$, with corresponding Laplace transform

$$\widehat{\kappa}(s) = \frac{1 + as^{\alpha}}{1 + bs^{\alpha}}. \tag{5.60}$$

In this case the Sonine kernel $k(t)$ satisfies

$$\widehat{k}(s) = \frac{1 + bs^{\alpha}}{s(1 + as^{\alpha})}$$

and therefore

$$k(t) = 1 - \left(1 - \frac{b}{a}\right)E_{\alpha}\left(-\frac{1}{a}t^{\alpha}\right).$$

5.3 Subordination theorems

Consider the generalized subdiffusion equation

$${}^C\mathbb{D}_t^{(\kappa)}u(t) = Au(t), \quad t > 0; \quad u(0) = a \in X, \tag{5.61}$$

where ${}^C\mathbb{D}_t^{(\kappa)}$ is the general convolutional derivative (5.44) and A is a closed densely defined operator in the Banach space X, which generates a bounded C_0-semigroup.

Let us emphasize that the distributed-order equations (5.1) and (5.2), studied in Section 5.1, are special cases of equation (5.61) (this concerns equations in both Caputo and Riemann-Liouville sense, with continuous as well as discrete distribution, including the limiting cases (5.21) and (5.22)).

The equivalent to problem (5.61) Volterra integral equation is

$$u(t) = a + \int_0^t k(t-\tau)Au(\tau)\,d\tau, \quad t > 0, \quad a \in X, \tag{5.62}$$

where the kernel $k(t)$ is the Sonine kernel of $\kappa(t)$, i.e. (5.48) is satisfied. This can be proved by applying the convolution operator $k*$ to both sides of (5.61), which yields

$$\begin{aligned}\left(k * {}^C\mathbb{D}_t^{(\kappa)}u\right)(t) &= k*((\kappa*u)'(t)-\kappa(t)u(0))\\ &= k*(\kappa*u') = (k*\kappa)*u' = (1*u')(t)\\ &= u(t)-u(0)\end{aligned} \tag{5.63}$$

where we used the identity (5.46).

We continue with the study of the integral equation (5.62). Let us set

$$g(s) = \frac{1}{\widehat{k}(s)} = s\widehat{\kappa}(s).$$

According to **(P7)** in Proposition 2.1 the assumption $\widehat{\kappa}(s) \in \mathcal{SF}$ is equivalent to $g(s) \in \mathcal{CBF}$. Then the general subordination theorem (Theorem 2.4) with $\alpha = 1$ implies the following

Theorem 5.5. *Let A be a generator of a bounded C_0-semigroup $S_1(t)$ and assume the conditions (5.45) hold. Then problem (5.61) is well posed with bounded solution operator $S(t)$, which satisfies the subordination identity*

$$S(t) = \int_0^\infty \varphi(t,\tau)S_1(\tau)\,d\tau, \quad t>0, \tag{5.64}$$

with subordination kernel $\varphi(t,\tau)$ defined by

$$\varphi(t,\tau) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{st-\tau g(s)}\,\frac{g(s)}{s}\,ds, \quad \gamma,t,\tau>0, \tag{5.65}$$

where $g(s) = s\widehat{\kappa}(s)$. The function $\varphi(t,\tau)$ is a probability density function, i.e. it satisfies the properties (2.26).

An alternative subordination result is formulated next, which provides a generalization of the exponential representation for the solution of the classical Cauchy problem (5.7)

$$u(t) = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n} a.$$

Theorem 5.6. *Let A be a generator of a bounded C_0-semigroup. Assume the kernel κ satisfies (5.45). Then problem (5.61) is well-posed and its solution $u(t)$ admits the representation*

$$u(t) = \lim_{n\to\infty} \frac{1}{n!} (n/t)^{n+1} \sum_{k=0}^{n} \sum_{p=1}^{k} b_{n,k,p} (n/t) (g(n/t) - A)^{-(p+1)} a, \tag{5.66}$$

where the convergence is uniform on bounded intervals of $t > 0$. The functions $b_{n,k,p}(s)$ are nonnegative for $s > 0$ and are defined by

$$b_{n,k,p}(s) = (-1)^{n+p} \binom{n}{k} \left(\frac{g(s)}{s}\right)^{(n-k)} a_{k,p}(s)p!, \quad s > 0, \tag{5.67}$$

where $a_{k,p}(s)$ are given by the recurrence relation

$$\begin{aligned} &a_{k+1,p}(s) = a_{k,p-1}(s)g'(s) + a'_{k,p}(s), \quad 1 \le p \le k+1, \ k \ge 1, \\ &a_{k,0} = a_{k,k+1} \equiv 0, \quad a_{1,1}(s) = g'(s). \end{aligned} \tag{5.68}$$

Proof. As usual we denote by $R(s, A)$ the resolvent operator of A: $R(s, A) = (s - A)^{-1}$, $s \in \varrho(A)$. The assumptions on the operator A imply by the Hille-Yosida theorem that $(0, \infty) \subset \varrho(A)$ and

$$\|R(s, A)^n\| \le M/s^n, \quad s > 0, \ n \in \mathbb{N}. \tag{5.69}$$

To establish well posedness we prove estimates (2.14), where

$$H(s) = \frac{g(s)}{s} R(g(s), A), \quad s > 0.$$

Let us express $H^{(n)}(s)$ in terms of powers of

$$w(s) = R(g(s), A).$$

Note that $g(s) > 0$ for $s > 0$ (since $g(s) \in \mathcal{CBF}$) and thus $g(s) \in \varrho(A)$, i.e. the resolvent operator $R(g(s), A)$ is well defined. By the Leibniz rule it follows

$$H^{(n)}(s) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{g(s)}{s}\right)^{(n-k)} w^{(k)}(s). \tag{5.70}$$

The formula for the k-th derivative of a composite function (see [105]) gives

$$w^{(k)}(s) = \sum_{p=1}^{k} a_{k,p}(s)(-1)^p p!(R(g(s),A))^{p+1}, \qquad (5.71)$$

where the functions $a_{k,p}(s)$ are defined by (5.68).

We will prove inductively that for any $k \geq 1$ and $1 \leq p \leq k$

$$(-1)^{k+p} a_{k,p}(s) \in \mathcal{CMF}. \qquad (5.72)$$

For $k = p = 1$ this is fulfilled since $a_{1,1}(s) = g'(s)$ and $g'(s) \in \mathcal{CMF}$ by Theorem 3.1(d). Further, $a_{2,1} = g''$, $a_{2,2} = (g')^2$ and the assertion (5.72) holds for these functions applying Theorem 3.1.(d) and Proposition 2.1.(a). Now fix some $k_0 \geq 2$ and suppose that (5.72) holds for all $k \leq k_0$, $1 \leq p \leq k$. Then, (5.68) implies that (5.72) is satisfied for $k = k_0 + 1$, $1 \leq p \leq k_0$, since $(-1)^{k_0+p+1} a_{k_0,p-1}(s) g'(s) \in \mathcal{CMF}$ as a product of two completely monotone functions and $(-1)^{k_0+p+1} a'_{k_0,p}(s) \in \mathcal{CMF}$ by (5.72). In addition, by (5.68), $a_{k_0+1,k_0+1} = a_{k_0,k_0} g'$ and it is completely monotone since $a_{k_0,k_0} \in \mathcal{CMF}$ and $g' \in \mathcal{CMF}$. In this way the proof of (5.72) is completed.

In particular, (5.72) implies

$$(-1)^{k+p} a_{k,p}(s) \geq 0, \quad s > 0. \qquad (5.73)$$

On the other hand, by Theorem 3.1(e) $g(s)/s \in \mathcal{CMF}$, i.e.

$$(-1)^{n-k} \left(g(s)/s\right)^{(n-k)} \geq 0, \quad s > 0. \qquad (5.74)$$

Inserting (5.71) in (5.70) we get

$$(-1)^n H^{(n)}(s) = \sum_{k=0}^{n} \sum_{p=1}^{k} b_{n,k,p}(s)(R(g(s),A))^{p+1}, \qquad (5.75)$$

where the functions $b_{n,k,p}(s)$ are defined in (5.67). Moreover, inserting (5.73) and (5.74) in (5.67), it follows

$$b_{n,k,p}(s) \geq 0, \ s > 0. \qquad (5.76)$$

In addition, let us note that in the trivial case $A \equiv 0$ (5.75) implies the identity

$$(-1)^n (s^{-1})^{(n)} = \sum_{k=0}^{n} \sum_{p=1}^{k} b_{n,k,p}(s)(g(s))^{-(p+1)}. \qquad (5.77)$$

Now, applying successively (5.76), (5.69) and (5.77) we obtain from (5.75)

$$\begin{aligned}\|H^{(n)}(s)\| &\le \sum_{k=0}^{n}\sum_{p=1}^{k} b_{n,k,p}(s)\|(R(g(s),A))^{p+1}\| \\ &\le M\sum_{k=0}^{n}\sum_{p=1}^{k} b_{n,k,p}(s)((g(s))^{-(p+1)} \\ &= M(-1)^n(s^{-1})^{(n)} = Mn!s^{-(n+1)}, \quad s>0.\end{aligned}$$

Therefore, conditions (2.14) are satisfied and Theorem 2.1 implies that problem (5.62) is well-posed with bounded solution operator $S(t)$. Finally, since $u(t) = S(t)a$ is a continuous and bounded function for $t \ge 0$, the Post-Widder inversion theorem (Theorem 1.1) can be applied and gives the representation (5.66). □

The positivity of the coefficients $b_{n,k,p}$ in representation (5.66) has a useful direct consequence: it implies the positivity of the solution operator.

Corollary 5.1. *Let X be an ordered Banach space. Assume the conditions of Theorem 5.6 are satisfied and the solution operator $S_1(t)$ of the classical Cauchy problem (5.7) is positive. Then the solution operators $S(t)$ of problem (5.61) is positive.*

Proof. Since

$$R(s,A) = \int_0^\infty e^{-st}S_1(t)\,dt, \quad s>0,$$

the positivity of the C_0-semigroup $S_1(t)$ implies that the resolvent operator $R(s,A)$ is positive: if $a \in X$ and $a \ge 0$, then $R(s,A)a \ge 0$, $s > 0$. Therefore $R(g(s),A)a \ge 0$ for all $s > 0$. This together with the positivity of the coefficients (5.67) in the representation formula (5.66) implies the positivity of $S(t)$. □

5.4 Generalized relaxation equation

In this section we apply the subordination results to study the behavior of solution to the equation with generalized convolutional time-derivative in the scalar case. Consider the relaxation equation ($\lambda > 0$)

$$^{C}\mathbb{D}_t^{(\kappa)}u(t) + \lambda u(t) = f(t), \qquad t>0; \quad u(0) = a \in \mathbb{R}. \tag{5.78}$$

Denote by $u(t;\lambda)$ the fundamental solution and by $v(t;\lambda)$ the impulse-response solution, corresponding respectively to $a = 1$, $f \equiv 0$, and $a = 0$, $f(t) = \delta(t)$. The functions $u(t;\lambda)$ and $v(t;\lambda)$ are also referred to as relaxation functions. The unique solution of the relaxation equation (5.78) is given by

$$u(t) = au(t;\lambda) + \int_0^t v(\tau;\lambda) f(t-\tau)\,d\tau. \tag{5.79}$$

In the particular case when ${}^C\mathbb{D}_t^{(\kappa)}$ is the Caputo fractional derivative ${}^CD_t^{\alpha}$ it is known that the relaxation functions are expressed in terms of Mittag-Leffler functions: $u(t;\lambda) = E_\alpha(-\lambda t^\alpha)$ and $v(t;\lambda) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$, see (1.30). The main aim now is to generalize estimates (1.31) to the case of the generalized relaxation equation (5.78). The proof is based on two properties: subordination identity and analyticity of the relaxation functions for $t > 0$.

By applying Laplace transform to equation (5.78), we obtain the following representations of the fundamental and impulse-response solutions in Laplace domain

$$\widehat{u}(s;\lambda) = \frac{g(s)}{s(g(s)+\lambda)}, \quad \widehat{v}(s;\lambda) = \frac{1}{g(s)+\lambda}, \quad g(s) = s\widehat{\kappa}(s). \tag{5.80}$$

The assumptions (5.45) on the kernel $\kappa(t)$ are equivalent to the following assumptions on the function $g(s)$

$$g(s) \in \mathcal{CBF}; \quad g(s) \to \infty \text{ as } s \to \infty. \tag{5.81}$$

5.4.1 Subordination relations

Subordination relations for the relaxation functions $u(t;\lambda)$ and $v(t;\lambda)$ are formulated next.

Theorem 5.7. *The relaxation functions $u(t;\lambda)$ and $v(t;\lambda)$ of problem (5.78) satisfy the subordination identities*

$$u(t;\lambda) = \int_0^\infty \varphi(t,\tau)e^{-\lambda\tau}\,d\tau, \quad t > 0, \tag{5.82}$$

$$v(t;\lambda) = \int_0^\infty \psi(t,\tau)e^{-\lambda\tau}\,d\tau, \quad t > 0, \tag{5.83}$$

where the functions $\varphi(t,\tau)$ *and* $\psi(t,\tau)$ *obey the properties*

$$\varphi(t,\tau) \geq 0,\ \ \psi(t,\tau) \geq 0; \quad \int_0^\infty \varphi(t,\tau)\,d\tau = 1,\ \int_0^\infty \psi(t,\tau)\,d\tau = k(t), \quad (5.84)$$

where $k(t)$ *is the Sonine kernel of* $\kappa(t)$, *i.e.* $(k * \kappa)(t) = 1$.

Proof. Relation (5.82) is a particular scalar version of Theorem 5.5 with $A = -\lambda$, $S_1(t) = e^{-\lambda t}$.

To prove relation (5.83) let us define a function $\psi(t,\tau)$ via the Laplace transform pair

$$\widehat{\psi}(s,\tau) = \int_0^\infty e^{-st}\varphi(t,\tau)\,dt = e^{-\tau g(s)}, \quad s,\tau > 0,$$

Since $g(s) \in \mathcal{CBF}$ then $e^{-\tau g(s)} \in \mathcal{CMF}$, see (2.3). Then Bernstein's theorem imply that $\psi(t,\tau)$ exists and $\psi(t,\tau) \geq 0$.

If we define a function $v(t;\lambda)$ by (5.83), then for its Laplace transform we obtain

$$\begin{aligned} \int_0^\infty e^{-st} v(t;\lambda)\,dt &= \int_0^\infty e^{-st}\left(\int_0^\infty \psi(t,\tau)e^{-\lambda\tau}\,d\tau\right)dt \\ &= \int_0^\infty \widehat{\psi}(s,\tau)e^{-\lambda\tau}\,d\tau \\ &= \int_0^\infty e^{-\tau g(s)}e^{-\lambda\tau}\,d\tau = \frac{1}{g(s)+\lambda}. \end{aligned}$$

Comparing this result to (5.80), it follows by the uniqueness of the Laplace transform that $v(t;\lambda)$ defined by (5.83) and the impulse-response solution of (5.78) coincide. In this way (5.83) is established.

The integral identity in (5.84) for $\psi(t,\tau)$ follows as a particular case of (5.83) by letting $\lambda \to 0$ and taking into account that $v(t;0) = k(t)$ since $\widehat{v}(s;0) = (s\widehat{\kappa}(s))^{-1} = g(s)^{-1}$. □

5.4.2 Properties of the relaxation functions

In the next theorem, further useful properties of the relaxation functions are established (in the statements the functions $u(t;\lambda)$ and $v(t;\lambda)$ are considered as functions of one variable $t \geq 0$, while λ is a parameter).

Theorem 5.8. *For any $\lambda > 0$ the functions $u(t;\lambda)$ and $v(t;\lambda)$ admit holomorphic extensions to the half-plane $\mathbb{C}_+$ and*

$$u(t;\lambda), v(t;\lambda) \in \mathcal{CMF} \text{ in } t > 0; \tag{5.85}$$

$$u(0;\lambda) = 1; \quad 0 < u(t;\lambda) < 1, \quad v(t;\lambda) > 0, \quad t > 0; \tag{5.86}$$

$$\frac{d}{dt}u(t;\lambda) = -\lambda v(t;\lambda). \tag{5.87}$$

Moreover

$$u(t;\lambda) \leq \frac{1}{1 + \lambda(1 * k)(t)}, \tag{5.88}$$

where $k(t)$ is the resolvent kernel of $\kappa(t)$, i.e. $(k * \kappa)(t) = 1$.

For any $\lambda \geq \lambda_0 > 0$ and $t > 0$

$$u(t;\lambda) \leq u(t;\lambda_0), \quad v(t;\lambda) \leq v(t;\lambda_0), \tag{5.89}$$

and

$$C \leq \lambda \int_0^T v(t;\lambda)\, dt < 1, \quad T > 0, \tag{5.90}$$

where the constant $C = 1 - u(T;\lambda_0) > 0$ is independent of λ.

Proof. First, applying Proposition 1.3, we prove that the function $u(t;\lambda)$ admits holomorphic extensions to the half-plane $\mathbb{C}_+$. Since the function $f(t) = e^{-\lambda t}$ is holomorphic and bounded for $\Re t > 0$, then, using that **(ii)** implies **(i)**, it follows that the Laplace transform $\widehat{f}(s) = \frac{1}{s+\lambda}$ admits holomorphic extension to the sector $|\arg s| < \pi$ and

$$\left| s\widehat{f}(s) \right| = \left| \frac{s}{s+\lambda} \right| \leq M, \ |\arg s| \leq \theta, \ \forall\, \theta < \pi. \tag{5.91}$$

Since $g(s) \in \mathcal{CBF}$ this function admits holomorphic extension to $\mathbb{C}\backslash(-\infty, 0]$ and therefore, in view of (5.80), this will hold also for

$$\widehat{u}(s;\lambda) = \frac{g(s)}{s(g(s) + \lambda)}.$$

Moreover,

$$|\arg g(s)| \leq |\arg s|, \quad s \in \mathbb{C}\backslash(-\infty, 0],$$

which together with (5.91) gives

$$|s\widehat{u}(s;\lambda)| = \left| \frac{g(s)}{g(s) + \lambda} \right| \leq M, \ |\arg s| \leq \theta, \theta < \pi.$$

From implication **(i)⇒(ii)** in Proposition 1.3, it follows that function $u(t;\lambda)$ admits holomorphic extension to $\mathbb{C}_+$. The analyticity of $v(t;\lambda)$ is then inferred taking into account relation (5.87), which is proven below.

From (5.80) and $u(0;\lambda)=1$ we deduce

$$\mathcal{L}\left\{\frac{du}{dt}\right\}(s;\lambda)=\frac{g(s)}{g(s)+\lambda}-1=-\frac{\lambda}{g(s)+\lambda}=-\lambda\widehat{v}(s;\lambda).$$

Identity (5.87) then follows from the uniqueness property of the Laplace transform.

To prove that $u(t;\lambda)\in\mathcal{CMF}$ we first, note that $\frac{s}{s+\lambda}\in\mathcal{CBF}$, since $\left(\frac{s}{s+\lambda}\right)^{-1}=1+\lambda s^{-1}\in\mathcal{SF}$, see **(P9)** in Proposition 2.1. Therefore the function $\frac{g(s)}{g(s)+\lambda}\in\mathcal{CBF}$ as a composition of two complete Bernstein functions, see **(P11)** in Proposition 2.1. Therefore, the function $\frac{g(s)}{s(g(s)+\lambda)}\in\mathcal{SF}$ by **(P7)** and vanishes as $s\to+\infty$. Then property **(P6)** gives for the inverse Laplace transform $u(t;\lambda)\in\mathcal{CMF}$. Applying (5.87) it follows $v(t;\lambda)\in\mathcal{CMF}$.

Since $u(t;\lambda),v(t;\lambda)\in\mathcal{CMF}$, they are nonnegative and nonincreasing functions for $t>0$. This fact, together with their analyticity, implies that these functions are positive and strictly decreasing.

The relaxation functions $u(t;\lambda)$ satisfies the integral equation

$$u(t;\lambda)=1-\lambda\int_0^t k(t-\tau)u(\tau;\lambda)\,d\tau,\quad t>0, \tag{5.92}$$

where k is the resolvent kernel of κ. Taking into account the fact that $u(t;\lambda)$ are positive and decreasing functions, the integral equation (5.92) yields

$$1=u(t;\lambda)+\lambda\int_0^t k(t-\tau)u(\tau;\lambda)\,d\tau\geq u(t;\lambda)+\lambda u(t;\lambda)\int_0^t k(\tau)\,d\tau,$$

which implies estimates (5.88).

The inequalities (5.89) follow directly from the subordination identities (5.82) and (5.83). Indeed, for $\lambda\geq\lambda_0$

$$u(t;\lambda)=\int_0^\infty\varphi(t,\tau)e^{-\lambda\tau}\,d\tau\leq\int_0^\infty\varphi(t,\tau)e^{-\lambda_0\tau}\,d\tau=u(t;\lambda_0),$$

and analogously for $v(t;\lambda)$. Here the nonnegativity of the functions $\varphi(t,\tau)$ and $\psi(t,\tau)$ is essential.

Applying (5.87) we deduce

$$\lambda \int_0^T v(t;\lambda)\,dt = 1 - u(T;\lambda).$$

This together with the first inequality in (5.89) and $0 < u(T;\lambda) < 1$ implies (5.90). □

5.4.3 Application to an inverse source problem

As an application of the obtained estimates (5.90), uniqueness and a conditional stability result are established for an inverse source problem for the general time-fractional diffusion equation on a bounded domain.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and $T > 0$. Consider the initial-boundary-value problem

$$\begin{aligned} {}^C\mathbb{D}_t^{(\kappa)} u(x,t) &= \Delta u(x,t) + F(x,t), \quad x \in \Omega, \quad t \in (0,T), \\ u(x,t) &= 0, \quad x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) &= a(x), \quad x \in \Omega, \end{aligned} \tag{5.93}$$

where the operator ${}^C\mathbb{D}_t^{(\kappa)}$ is the general convolutional derivative that acts with respect to time variable and Δ is the Laplace operator acting on space variables.

Define the Laplace operator Δ in the Hilbert space $L^2(\Omega)$ with domain $D(\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$, where $H_0^1(\Omega)$ and $H^2(\Omega)$ are standard notations for Sobolev spaces, for more details we refer to [56, 96]. Denote by $\{-\lambda_n, \varphi_n\}_{n=1}^\infty$ the corresponding eigensystem. Then $0 < \lambda_1 \le \lambda_2 \le \dots$, $\lambda_n \to \infty$ as $n \to \infty$, and the functions $\{\varphi_n\}_{n=1}^\infty$ form an orthonormal basis of $L^2(\Omega)$.

Denote by $(.,.)$ the inner product in $L^2(\Omega)$.

An equivalent norm in the Hilbert space $H_0^1(\Omega) \cap H^2(\Omega)$ is given by (see e.g. [56])

$$\|v\|_{H_0^1(\Omega)\cap H^2(\Omega)} = \|\Delta v\|_{L^2(\Omega)},$$

where $\|\Delta v\|^2_{L^2(\Omega)} = \sum_{n=1}^\infty \lambda_n^2 (v,\varphi_n)^2$.

Applying eigenfunction decomposition, we obtain the following formal representation of the solution of problem (5.93)

$$u(x,t) = \sum_{n=1}^\infty a_n u_n(t) \varphi_n(x) + \sum_{n=1}^\infty \left(\int_0^t v_n(t-\tau) F_n(\tau)\,d\tau \right) \varphi_n(x) \tag{5.94}$$

where $u_n(t) = u(t;\lambda_n)$, $v_n(t) = v(t;\lambda_n)$ are the fundamental and impulse-response solution of the relaxation equation (5.78) with $\lambda = \lambda_n$, $n \in \mathbb{N}$, and

$$a_n = (a, \varphi_n), \quad F_n(t) = (F(.,t), \varphi_n), \quad n \in \mathbb{N}.$$

Assume now $a = 0$ and $F(x,t) = f(x)q(t)$, where the function $q \in C[0,T]$ is known and satisfies $q(t) \geq q_0 > 0$ for all $t \in [0,T]$. Consider the inverse problem to determine the solution $u(x,t)$ and source term $f(x)$, $(x \in \Omega, t \in (0,T))$, such that (5.93) is satisfied together with the additional overdetermination condition

$$u(x,T) = h(x), \; x \in \overline{\Omega}. \tag{5.95}$$

Theorem 5.9. *Let $T > 0$ be arbitrarily fixed. For any given $h \in H_0^1(\Omega) \cap H^2(\Omega)$, there exists a unique solution $(f(x), u(x,t))$ to problem (5.93), satisfying $f \in L^2(\Omega)$ and*

$$u \in C([0,T]; L^2(0,1)) \cap C((0,T]; H_0^1(\Omega) \cap H^2(\Omega)).$$

Moreover, there exist constants $\underline{C} > 0$ and $\overline{C} > 0$, such that

$$\underline{C}\|f\|_{L^2(\Omega)} \leq \|h\|_{H_0^1(\Omega)\cap H^2(\Omega)} \leq \overline{C}\|f\|_{L^2(\Omega)}. \tag{5.96}$$

If f satisfies the apriori bound condition $\|f\|_{H_0^1(\Omega)\cap H^2(\Omega)} \leq E$ then

$$\|f\|_{L^2(\Omega)} \leq \underline{C}^{-1/2} E^{1/2} \|h\|_{L^2(\Omega)}^{1/2}. \tag{5.97}$$

Proof. Taking $t = T$ in the formal expansion (5.94) of the solution of (5.93) we obtain

$$h(x) = \sum_{n=1}^{\infty} f_n \left(\int_0^T v_n(T-\tau) q(\tau)\, d\tau \right) \varphi_n(x), \tag{5.98}$$

where $f_n = (f, \varphi_n)$.

Introducing the notations $h_n = (h, \varphi_n)$ and $Q_n(t) = \int_0^t v_n(t-\tau)q(\tau)\, d\tau$, (5.98) gives

$$h_n = f_n Q_n(T). \tag{5.99}$$

Since $Q_n(T) \geq q_0 \int_0^T v_n(\tau)\, d\tau$ and $Q_n(T) \leq \|q\|_{C[0,T]} \int_0^T v_n(\tau)\, d\tau$, the bounds in (5.90) imply

$$0 < \underline{C}/\lambda_n \leq Q_n(T) \leq \overline{C}/\lambda_n, \tag{5.100}$$

where the constants $\underline{C}$ and $\overline{C}$ do not depend on n.

In particular, $Q_n(T) > 0$. This implies that the solution $\{f(x), u(x,t)\}$ of problem (5.93), (5.95) is unique. Indeed, if $h(x) = 0$ then $f(x) = 0$ by (5.99) and from the uniqueness of the direct problem, also $u(x,t) = 0$.

Estimates (5.100) for $Q_n(T)$ and (5.99) imply (5.96).

The lower bound in (5.100) can be used to prove the conditional stability result (5.97). Indeed, by (5.99)

$$\|f\|^2_{L^2(\Omega)} \le \sum_{n=1}^{\infty} f_n^2 = \sum_{n=1}^{\infty} \frac{h_n}{Q_n^2(T)} h_n \le \left(\sum_{n=1}^{\infty} \frac{h_n^2}{Q_n^4(T)}\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} h_n^2\right)^{\frac{1}{2}}. \quad (5.101)$$

Applying (5.100), the first term is estimated as follows

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{h_n^2}{Q_n^4(T)} &= \sum_{n=1}^{\infty} \frac{f_n^2}{Q_n^2(T)} \\ &\le \underline{C}^{-2} \sum_{n=1}^{\infty} \lambda_n^2 f_n^2 \\ &= \underline{C}^{-2} \|f\|^2_{H_0^1(\Omega)\cap H^2(\Omega)} \\ &\le \underline{C}^{-2} E^2. \end{aligned}$$

Plugging this bound in (5.101) completes the proof of (5.97). □

Chapter 6

Multinomial Mittag-Leffler type functions

We continue the study of the evolution equations with multiple derivatives in time with the main emphasis on the multinomial Mittag-Leffler function, which appear in the representation of their solutions. Basic properties of this function and its Prabhakar type generalization are studied, including complete monotonicity. Some subordination relations are established. The obtained results extend known properties of the classical Mittag-Leffler function.

6.1 Definition and basic relations

Various types of multi-index generalizations of the classical Mittag-Leffler function (1.21) are studied, see e.g. [86], the recent surveys [59, 60, 87] and the monographs [45, 85]. One of them is the multinomial Mittag-Leffler function

$$E_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m)=\sum_{k=0}^{\infty}\sum_{\substack{k_1+\ldots+k_m=k\\ k_1\geq 0,\ldots,k_m\geq 0}}\frac{k!}{k_1!\ldots k_m!}\frac{\prod_{j=1}^{m}z_j^{k_j}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_j k_j\right)},$$

where $z_j\in\mathbb{C}$, $\mu_j>0$, $\beta\in\mathbb{R}$, $j=1,\ldots,m$. It is proposed in [50] and used for solving multi-term fractional differential equations with constant coefficients by operational method in [71]. The multinomial Mittag-Leffler function plays a crucial role in the study of multi-term time-fractional diffusion equations. This is due to the fact that the time-dependent components in the eigenfunction expansion of the solution to initial-boundary value problems for multi-term equations are expressed in terms of multinomial Mittag-Leffler functions, see e.g. [56, 67, 68].

Initial-boundary-value problems for diffusion equations with multiple time derivatives and nonlocal boundary conditions are considered in [23]. The nonlocal character of the boundary conditions leads to a non-selfadjoint problem and multidimensional eigenspaces. This, in turn, implies that the time-dependent components in the generalized eigenfunction expansions of the solutions are expressed in terms of multinomial Mittag-Leffler functions and convolutions of them. It is known that convolution of two classical Mittag-Leffler functions is a Prabhakar function, see (1.36). Therefore, in the context of nonlocal boundary value problems for multi-term time-fractional differential equations the need of Prabhakar type generalization of the multinomial Mittag-Leffler function naturally emerge. Such a generalization, which is at the same time a multinomial generalization of the Prabhakar function (1.32) is defined next.

For the sake of brevity we use the vector notation $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$.

The multinomial Prabhakar function is defined as follows [15]

$$E^{\delta}_{\vec{\mu},\,\beta}(z_1, \ldots, z_m) = \sum_{k=0}^{\infty} \sum_{\substack{k_1+\ldots+k_m=k\\ k_1\geq 0,\ldots,k_m\geq 0}} \frac{(\delta)_k}{k_1! \cdots k_m!} \frac{\prod_{j=1}^{m} z_j^{k_j}}{\Gamma\left(\beta + \sum_{j=1}^{m} \mu_j k_j\right)}, \tag{6.1}$$

where $z_j \in \mathbb{C}$, $\mu_j, \beta, \delta \in \mathbb{R}$, $\mu_j > 0$, $j = 1, \ldots, m$. Here $(\delta)_k$ denotes the Pochhammer symbol

$$(\delta)_k = \frac{\Gamma(\delta + k)}{\Gamma(\delta)} = \delta(\delta + 1) \ldots (\delta + k - 1), \quad k \in \mathbb{N}, \quad (\delta)_0 = 1.$$

In general, the parameters μ_j, β, δ, are allowed to assume complex values with $\Re\mu_j > 0$. In this work, however, we restrict our attention to real parameters, which are of particular interest for the considered applications.

The classical Prabhakar function (1.32) is recovered from (6.1) for $m = 1$. The binomial variant ($m = 2$) of function (6.1) was recently introduced and studied in [39]. In the special case $\delta = 1$ the Pochhammer symbol yields $(1)_k = k!$ and the function (6.1) is the multinomial Mittag-Leffler function

$$E_{(\mu_1,\ldots,\mu_m),\beta}(z_1, \ldots, z_m) = E^1_{(\mu_1,\ldots,\mu_m),\beta}(z_1, \ldots, z_m). \tag{6.2}$$

If $\delta = -n$ is a negative integer then the Prabhakar function (6.1) is defined by a finite sum (since $(-n)_k = 0$ for $k \geq n + 1$), and $E^0_{\vec{\mu},\,\beta}(\cdot) = 1/\Gamma(\beta)$.

Let us note that the double summation in (6.1) can be formally replaced by the multiple summation, that is

$$E^{\delta}_{\vec{\mu},\beta}(z_1,\ldots,z_m)=\sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{(\delta)_{k_1+\ldots+k_m}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_j k_j\right)}\prod_{j=1}^{m}\frac{z_j^{k_j}}{k_j!}.$$

This yields a multiple power series, which converges absolutely and locally uniformly, and thus defines an entire function in each z_j, $j=1,\ldots,m$. Therefore, both representations are equivalent.

Applying successive term by term differentiation in (6.1) and using the identity $(\delta)_{k+1}=\delta(\delta+1)_k$ we deduce the relation

$$\left(\frac{\partial}{\partial z_j}\right)^n E^{\delta}_{\vec{\mu},\beta}(z_1,\ldots,z_m)=(\delta)_n E^{\delta+n}_{\vec{\mu},\,n\mu_j+\beta}(z_1,\ldots,z_m),$$

which generalizes a well-known identity for $m=1$, see e.g. [91], Eq.(2.1).

In the rest of this work we are concerned only with the following multinomial Prabhakar type function of a single variable $t>0$, which is of particular importance for the study of multi-term time-fractional equations

$$\mathcal{E}^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(t;a_1,\ldots,a_m):=t^{\beta-1}E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(-a_1t^{\mu_1},\ldots,-a_mt^{\mu_m}),\qquad(6.3)$$

where $\mu_j>0$, $\beta>0$, $\delta\in\mathbb{R}$, $a_j>0$, $j=1,\ldots,m$. For the sake of brevity the short notation $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})$ is used for the function (6.3). Definition (6.1) yields the series representation

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})=\sum_{k=0}^{\infty}\sum_{\substack{k_1+\ldots+k_m=k\\ k_1\geq 0,\ldots,k_m\geq 0}}\frac{(-1)^k(\delta)_k}{k_1!\ldots k_m!}\frac{\left(\prod_{j=1}^{m}a_j^{k_j}\right)t^{\beta-1+\sum_{j=1}^{m}\mu_j k_j}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_j k_j\right)}.\qquad(6.4)$$

The first terms in the power series (6.4) give the following asymptotic expansion for $t\to 0$:

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})\sim\frac{t^{\beta-1}}{\Gamma(\beta)}-\delta\sum_{j=1}^{m}a_j\frac{t^{\beta-1+\mu_j}}{\Gamma(\beta+\mu_j)},\quad t\to 0.\qquad(6.5)$$

We study the multinomial Prabhakar type function (6.3) applying Laplace transform technique. For this reason we are concerned only with locally integrable functions $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})$. Taking into account (6.5), this is guaranteed by the assumptions on the parameters of function (6.3).

Theorem 6.1. *The Laplace transform $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a})$ of the multinomial Prabhakar type function $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})$ is given by the identity*

$$\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a}) := \mathcal{L}\left\{\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})\right\}(s) = \frac{s^{-\beta}}{\left(1+\sum_{j=1}^{m} a_j s^{-\mu_j}\right)^{\delta}} \tag{6.6}$$

for $s \in \mathbb{C}$, such that $\Re s > 0$.

Proof. By applying term-wise Laplace transform to the series (6.4) and using the Laplace transform pair (1.4) for $\alpha > 0$, $\Re s > 0$, we obtain

$$\mathcal{L}\left\{\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})\right\}(s) = s^{-\beta}\sum_{k=0}^{\infty}\sum_{\substack{k_1+\ldots+k_m=k\\ k_1\ge 0,\ldots,k_m\ge 0}} \frac{(-1)^k(\delta)_k}{k_1!\ldots k_m!}\prod_{j=1}^{m}\left(a_j s^{-\mu_j}\right)^{k_j}. \tag{6.7}$$

By the use of the binomial series

$$(1+Z)^{-\delta} = \sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}(-Z)^k, \qquad |Z|<1, \tag{6.8}$$

and the multinomial theorem

$$(Z_1+\ldots+Z_m)^k = \sum_{\substack{k_1+\ldots+k_m=k\\ k_1\ge 0,\ldots,k_m\ge 0}} \frac{k!}{k_1!\ldots k_m!}\prod_{j=1}^{m} Z_j^{k_j},$$

(6.7) implies (6.6) for $\Re s > 0$, provided $\left|\sum_{j=1}^{m} a_j s^{-\mu_j}\right| < 1$. The last condition can be avoided by using analytic continuation. In this way the statement is established for any $s \in \mathbb{C}$, such that $\Re s > 0$. □

The Laplace transform pair (6.6) shows that, in general, the representation as a multinomial Prabhakar type function is not unique. For example, the identity $\mathcal{E}^{2\delta}_{\mu,\beta}(t;a) = \mathcal{E}^{\delta}_{(\mu,2\mu),\beta}(t;2a,a^2)$ can be proven by the use of (6.6). Moreover, the order of parameters μ_j in (6.3) can be changed (together with the corresponding a_j). For clarity, in what follows we choose the representation with minimal m and when a special arrangement of the parameters μ_j (resp. a_j) is assumed, this is explicitly stated.

A reduction of parameters result is established next.

Theorem 6.2. *For any $j = 1, \ldots, m$, there holds*

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{k!}(-a_j)^k \mathcal{E}^{k+\delta}_{(\mu_1,\ldots,\mu_{j-1},\mu_{j+1},\ldots,\mu_m),\mu_j k+\beta}(t;\vec{a}'), \tag{6.9}$$

where $\vec{a} = (a_1, \ldots, a_m)$ and $\vec{a}' = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m)$.

Proof. Representation (6.9) is deduced from the following identity obtained by the use of the binomial series (6.8)

$$\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a}) = s^{-\beta}\left(1+\Sigma'\right)^{-\delta}\left(1+\frac{a_j s^{-\mu_j}}{1+\Sigma'}\right)^{-\delta} = \sum_{k=0}^{\infty}\frac{(\delta)_k}{k!}(-a_j)^k \frac{s^{-\mu_j k-\beta}}{\left(1+\Sigma'\right)^{k+\delta}},$$

where $\Sigma' = \sum_l a_l s^{-\mu_l}$, $l = 1, 2, \ldots, j-1, j+1, \ldots, m$, by applying (6.6) and the uniqueness of Laplace transform. □

The integration, differentiation and convolution properties for the multinomial Prabhakar type functions, given next, extend those for the classical Prabhakar function (see e.g. [43]).

Theorem 6.3. *The following identities hold true*

$$J_t^{\alpha}\left(\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})\right) = \mathcal{E}^{\delta}_{\vec{\mu},\beta+\alpha}(t;\vec{a}), \quad \alpha > 0, \tag{6.10}$$

$$\left(\frac{d}{dt}\right)^n \mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) = \mathcal{E}^{\delta}_{\vec{\mu},\beta-n}(t;\vec{a}), \quad \beta > n, \tag{6.11}$$

$$\left(\mathcal{E}^{\delta}_{\vec{\mu},\beta}(\cdot;\vec{a})\right) * \left(\mathcal{E}^{\delta_0}_{\vec{\mu},\beta_0}(\cdot;\vec{a})\right)(t) = \mathcal{E}^{\delta+\delta_0}_{\vec{\mu},\beta+\beta_0}(t;\vec{a}), \tag{6.12}$$

where J_t^{α} is the Riemann-Liouville fractional integral and $*$ denotes the Laplace convolution.

The above identities can be verified directly from the series definition (6.4), or, proving by the use of (6.6) that the Laplace transforms of both sides coincide. Technically, the second method is shorter. Since the proofs are straightforward, they are omitted here. In the binomial case $m = 2$ the identities in Theorem 6.3 are proved in detail in [39].

6.2 Complete monotonicity

This section is devoted to the study of complete monotonicity property of the multinomial Prabhakar type function (6.3) for $t > 0$. Concerning the classical Prabhakar type function the current most general result states that the function $t^{\beta-1}E^{\delta}_{\mu,\beta}(-t^{\mu})$, $t>0$, is completely monotone if the parameters satisfy the conditions [43]

$$0<\mu\le 1,\ 0<\mu\delta\le\beta\le 1.$$

A detailed proof can be found in [31]. This result is extended next to the multinomial case. To this end, we prove first an auxiliary statement.

Proposition 6.1. *Let $\alpha\in(0,1]$ and $0\le\alpha_j<\alpha\le 1$, $q_j>0$, $j=1,\dots,m$. Then*

$$\left(s^{\alpha}+\sum_{j=1}^{m}q_j s^{\alpha_j}\right)^{1/\alpha}\in\mathcal{CBF}\quad\textit{and}\quad\left(s^{-\alpha}+\sum_{j=1}^{m}q_j s^{-\alpha_j}\right)^{-1/\alpha}\in\mathcal{CBF}.$$

Proof. Property **(P14)** in Proposition 2.1 implies by induction that for any $f,f_j\in\mathcal{CBF}$, $j=1,\dots,m$, and $\alpha\in[-1,1]\backslash\{0\}$ there holds

$$\left(f^{\alpha}(s)+\sum_{j=1}^{m}f_j^{\alpha}(s)\right)^{1/\alpha}\in\mathcal{CBF}. \tag{6.13}$$

It remains to plug in (6.13) the complete Bernstein functions $f(s)=s$, $f_j(s)=q_j^{1/\alpha}s^{\alpha_j/\alpha}$, $j=1,\dots,m$, and use property **(P11)** in Proposition 2.1 □

Theorem 6.4. *Let $1\ge\mu_1>\mu_2>\ldots>\mu_m>0$, $0<\mu_1\delta\le\beta\le 1$, and $a_j>0$, $j=1,\dots,m$. Then*

$$\mathcal{E}^{\delta}_{(\mu_1,\dots,\mu_m),\beta}(t;a_1,\dots,a_m)\in\mathcal{CMF},\quad t>0. \tag{6.14}$$

Proof. We prove complete monotonicity of $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})$ by applying criterion **(P6)** in Proposition 2.1. To establish $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a})\in\mathcal{SF}$ we note that, according to **(P8)** in Proposition 2.1, it is equivalent to $\left(\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a})\right)^{-1}\in\mathcal{CBF}$, or, taking into account (6.6), to

$$s^{\beta-\mu_1\delta}\left(s^{\mu_1}+\sum_{j=1}^{m}a_j s^{\mu_1-\mu_j}\right)^{\delta}\in\mathcal{CBF}. \tag{6.15}$$

Let first $\beta \neq \mu_1\delta$. To prove (6.15) we apply **(P7)** to the function $\varphi^{\alpha_1}(s).\psi^{\alpha_2}(s)$ with $\alpha_1 = \beta - \mu_1\delta > 0,\ \alpha_2 = \mu_1\delta > 0$, and

$$\varphi(s) = s, \quad \psi(s) = \left(s^{\mu_1} + \sum_{j=1}^{m} a_j s^{\mu_1-\mu_j}\right)^{1/\mu_1},$$

where $\varphi \in \mathcal{CBF}$ and $\psi \in \mathcal{CBF}$ (according to Proposition 6.1).

If $\beta = \mu_1\delta$ then the function in (6.15) is $\psi^{\mu_1\delta}(s)$ and it is a complete Bernstein function as a composition of two complete Bernstein functions: $\psi(s)$ and $s^{\mu_1\delta}$, where $\mu_1\delta \le 1$, see **(P11)**.

In this way (6.15) is verified and, thus, we proved that $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a}) \in \mathcal{SF}$. Moreover, since $\beta > 0$, (6.6) implies $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a}) \to 0$ as $s \to \infty$. Therefore, the conditions in **(P6)** are established and the proof of the theorem is completed. □

Let us note that the condition $\beta \le 1$ is also necessary for complete monotonicity property (6.14). Indeed, $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) \in \mathcal{CMF}$ implies that the asymptotic expansions of this function for $t \to 0$ as well as for $t \to +\infty$ should be completely monotone functions. We see from (6.5) that at $t \to 0$ the function $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})$ behaves as $t^{\beta-1}/\Gamma(\beta)$, which is completely monotone only when $\beta \le 1$.

Next we derive the asymptotic expansion for $t \to \infty$. To this end we need the expansion of $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a})$ for $s \to 0$. Let $\mu_1 > \mu_2 > \ldots > \mu_m > 0$. Then for $s \to 0$

$$\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a}) = \frac{s^{\mu_1\delta-\beta}}{(s^{\mu_1} + a_m s^{\mu_1-\mu_m} + \ldots + a_2 s^{\mu_1-\mu_2} + a_1)^{\delta}} \sim \frac{s^{\mu_1\delta-\beta}}{(a_2 s^{\mu_1-\mu_2} + a_1)^{\delta}}$$

and, therefore

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) \sim a_2^{-\delta} t^{\beta-\mu_2\delta-1} E^{\delta}_{\mu_1-\mu_2,\beta-\mu_2\delta}\left(-a_1 a_2^{-1} t^{\mu_1-\mu_2}\right), \quad t \to +\infty.$$

From the asymptotic behavior of the Prabhakar function (see e.g. [43], eq. (3.13)) the leading term as $t \to +\infty$ is obtained as follows

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) \sim \begin{cases} a_1^{-\delta} \dfrac{t^{\beta-\mu_1\delta-1}}{\Gamma(\beta-\mu_1\delta)}, & \mu_1\delta \neq \beta, \\ -\delta a_1^{-\delta-1} a_2 \dfrac{t^{-\mu_1+\mu_2-1}}{\Gamma(-\mu_1+\mu_2)}, & \mu_1\delta = \beta. \end{cases}, \quad t \to +\infty. \tag{6.16}$$

We observe that the leading terms in (6.16) are completely monotone functions under the assumptions of Theorem 6.4.

Let us point out that (6.16) can be guaranteed only when $a_j > 0$ for each $j = 1, \ldots, m$. In the classical case $m = 1$ this is known [43]. A relevant counterexample concerning the two-term case is provided in [67], Remark 4.1.

We also note that, according to **(P6)** and **(P15)** in Proposition 2.1, the complete monotonicity property (6.14) implies that $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a})$ can be analytically extended to the whole complex plane cut along the negative real axis. Therefore, the function $s^{\mu_1} + a_m s^{\mu_1-\mu_m} + \ldots + a_2 s^{\mu_1-\mu_2} + a_1$ should not have any zeros there. This is guaranteed by the assumptions $\mu_j < \mu_1 \leq 1$ and $a_j > 0$. The question whether these conditions are also necessary for complete monotonicity property (6.14) in the multinomial case needs further investigation.

Further, let us note that identity (6.6) implies

$$\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a})\widehat{\mathcal{E}}^{-\delta}_{\vec{\mu},1-\beta}(s;\vec{a}) = 1/s, \quad s > 0. \tag{6.17}$$

Therefore, according to property **(P10)** in Proposition 2.1 $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a}) \in \mathcal{SF}$ if and only if $\widehat{\mathcal{E}}^{-\delta}_{\vec{\mu},1-\beta}(s;\vec{a}) \in \mathcal{SF}$. If $\beta \in (0,1)$ then both Laplace transforms vanish as $s \to \infty$ and according to **(P3)** $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) \in \mathcal{CMF}$ if and only if $\mathcal{E}^{-\delta}_{\vec{\mu},1-\beta}(t;\vec{a}) \in \mathcal{CMF}$. In other words, identity (6.17) implies that $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})$ and $\mathcal{E}^{-\delta}_{\vec{\mu},1-\beta}(t;\vec{a})$ are Sonine kernels, that is

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) * \mathcal{E}^{-\delta}_{\vec{\mu},1-\beta}(t;\vec{a}) = 1, \quad t > 0,$$

and the complete monotonicity of the one implies the complete monotonicity of the other. In this way we obtained the following

Corollary 6.1. *Under the assumptions of Theorem 6.4 and $\beta \neq 1$ there holds*

$$\mathcal{E}^{-\delta}_{(\mu_1,\ldots,\mu_m),1-\beta}(t;a_1,\ldots,a_m) \in \mathcal{CMF}, \quad t > 0. \tag{6.18}$$

6.3 Equations with multiple time-derivatives

Let ${}^{C}D^{\alpha}_{t}$ and D^{α}_{t} be the fractional time-derivatives in the Caputo and Riemann-Liouville sense, respectively, and let A be a generator of a bounded C_0- semigroup in a Banach space X. In this section we continue the study of the two types of multi-term generalizations of the fractional evolution equation

$${}^{C}D^{\alpha}_{t}u(t) = Au(t) + f(t), \quad t > 0, \quad 0 < \alpha \leq 1. \tag{6.19}$$

Let $1 \geq \alpha > \alpha_1 > ... > \alpha_m > 0$, $b_j > 0$, $j = 1, ..., m$. We consider the multi-term time-fractional differential equation in the Caputo form

$$ {}^C D_t^\alpha u(t) + \sum_{j=1}^m b_j \, {}^C D_t^{\alpha_j} u(t) = Au(t) + f(t), \quad t > 0, \tag{6.20} $$

and in the Riemann-Liouville form

$$ u'(t) = D_t^{1-\alpha} Au(t) + \sum_{j=1}^m b_j \, D_t^{1-\alpha_j} Au(t) + f(t), \quad t > 0. \tag{6.21} $$

For notational convenience equation (6.21) is written here in a slightly different form compared to (5.6) in the previous chapter. We point out that in our considerations of equations (6.20) and (6.21) the case $\alpha = 1$ is included in order to cover important models, such as the two time-scale mobile-immobile model for the subdiffusive transport of solutes in heterogeneous porous media [103], and the Rayleigh-Stokes problem for a generalized second grade fluid [24]. Therefore, it is not possible to use for the study of equations (6.20) and (6.21) the framework of general fractional derivative proposed in [62]. Indeed, if for example, the multi-term derivative operator in (6.20) with $\alpha = 1$ is represented as a general fractional derivative, the corresponding kernel of this derivative would contain a Dirac delta function, see also [51] for a related discussion.

For a unified approach to the two types of multi-term time-fractional differential equations, (6.20) and (6.21), we rewrite them for $f \equiv 0$ as a Volterra integral equation

$$ u(t) = u(0) + \int_0^t k(t-\tau) Au(\tau) \, d\tau, \quad t > 0, \tag{6.22} $$

where the kernel $k(t) = k_1(t)$ in the case of equation (6.20) and $k(t) = k_2(t)$ in the case of equation (6.21). The Laplace transforms of the kernels obey $\widehat{k}_i(s) = 1/g_i(s)$, $i = 1, 2$, where

$$ g_1(s) = s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}, \qquad g_2(s) = \left(s^{-\alpha} + \sum_{j=1}^m b_j s^{-\alpha_j} \right)^{-1}. \tag{6.23} $$

Therefore, taking into account (6.6), we deduce

$$ k_1(t) = \mathcal{E}_{(\alpha-\alpha_1,\dots,\alpha-\alpha_m),\alpha}(t; b_1, \dots, b_m), \tag{6.24} $$

$$ k_2(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \sum_{j=1}^m b_j \frac{t^{\alpha_j - 1}}{\Gamma(\alpha_j)}. \tag{6.25} $$

The kernels $k_i(t) \in C(\mathbb{R}_+) \cap L^1_{loc}(\mathbb{R}_+)$ are completely monotone functions, see Theorem 6.4.

6.3.1 Subordination

In order to apply the general subordination theorem (Theorem 2.4) we have to establish (2.24) for some α. We know from the previous chapter that $g_i(s) \in \mathcal{CBF},\ i = 1, 2$. According to Proposition 6.1 a stronger property is satisfied:

$$g_i(s)^{1/\alpha} \in \mathcal{CBF}, \quad i = 1, 2. \tag{6.26}$$

This together with property **(P11)** in Proposition 2.1 also implies

$$g_i(s)^{1/\beta} = \left(g_i(s)^{1/\alpha}\right)^{\alpha/\beta} \in \mathcal{CBF}, \quad 0 < \alpha \le \beta \le 1, \quad i = 1, 2,$$

as a composition of two complete Bernstein functions.

Proposition 6.2. *The functions $g_1(s)$ and $g_2(s)$ defined in (6.23) satisfy*

$$g_i(s)^{1/\beta} \in \mathcal{CBF}, \quad 0 < \alpha \le \beta \le 1, \quad i = 1, 2. \tag{6.27}$$

Theorem 2.4 and property (6.27) imply the following subordination result.

Theorem 6.5. *Let $0 < \alpha \le \beta \le 1$ and assume the single-term problem (6.19) of order β admits a bounded solution operator $S_\beta(t)$. Then the solution operator $S(t)$ of problem (6.20), resp. (6.21), satisfies the subordination identity*

$$S(t) = \int_0^\infty \varphi(t, \tau) S_\beta(\tau)\, d\tau, \quad t > 0,$$

with function $\varphi(t, \tau)$ defined by

$$\varphi(t, \tau) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \exp\left(st - \tau g^{1/\beta}(s)\right) \frac{g^{1/\beta}(s)}{s}\, ds, \quad \gamma, t, \tau > 0,$$

where $g(s) = g_1(s)$ in case of problem (6.20) and $g(s) = g_2(s)$ in case of problem (6.21). The function $\varphi(t, \tau)$ is a probability density function, i.e. it satisfies the properties (2.26).

Moreover, if $\alpha < \beta$, then there exists $\theta_0 \in (0, \pi/2)$ such that $\varphi(t, \tau)$ admits analytic extension to the sector $|\arg t| < \theta_0$ and is bounded on each subsector $|\arg t| \le \theta$, where $0 < \theta < \theta_0$.

Proof. We have to prove only the last part - the analyticity property of the subordination kernel $\varphi(t,\tau)$, since the rest follows from Theorem 2.4. Next, $g(s)$ stands for either $g_1(s)$ or $g_2(s)$. Applying property **(P15)** in Proposition 2.1 to the complete Bernstein functions $g(s)^{1/\alpha}$ (see (6.26)) it follows

$$|\arg g(s)| \leq \alpha |\arg s|, \;\; s \in \mathbb{C}\backslash(-\infty, 0].$$

This estimate can be also derived from the definitions $g_1(s)$ and $g_2(s)$ in (6.23) by applying the rules $|\arg(s^\beta)| = \beta|\arg s|$, $|\arg(s_1+s_2)| \leq \max\{|\arg s_1|, |\arg s_2|\}$ and $\arg(s^{-1}) = \arg s$. Therefore

$$|\arg g(s)^{1/\beta}| \leq \gamma |\arg s|, \;\; s \in \mathbb{C}\backslash(-\infty, 0],$$

where $\gamma = \alpha/\beta \in (0,1)$.

Define

$$\theta_0 = \min\{(1/\gamma - 1)\pi/2, \pi/2\} - \varepsilon, \tag{6.28}$$

where $\varepsilon > 0$ is small enough, such that $\theta_0 > 0$. According to Theorem 1.3 it suffices to prove that the function $\widehat{\varphi}(s,\tau)$ admits analytic extension to the sector $|\arg s| < \pi/2+\theta_0$ and the function $s\widehat{\varphi}(s,\tau)$ is bounded on each subsector $|\arg s| \leq \pi/2 + \theta, \;\theta < \theta_0$. The complete Bernstein function $g(s)^{1/\beta}$ can be extended analytically to $\mathbb{C}\backslash(-\infty, 0]$, see **(P15)** in Proposition 2.1, thus this holds also for the function

$$\widehat{\varphi}(s,\tau) = \frac{g^{1/\beta}(s)}{s} \exp\left(-\tau g^{1/\beta}(s)\right).$$

Take s such that $|\arg s| \leq \pi/2 + \theta, \;\theta < \theta_0$, where θ_0 is defined in (6.28). Then

$$|\arg g(s)^{1/\beta}| \leq \gamma|\arg s| < \pi/2 - \gamma\varepsilon.$$

Therefore, $g(s)^{1/\beta} = \rho e^{\mathrm{i}\phi}$, for some $\rho > 0$, $|\phi| < \pi/2 - \gamma\varepsilon$, and thus

$$|s\widehat{\varphi}(s,\tau)| = |g(s)e^{-\tau g(s)}| \leq \rho e^{-\tau\rho\cos\phi} \leq \rho e^{-a\rho} \leq (ea)^{-1},$$

where $a = \tau \sin\gamma\varepsilon > 0$. Here we have used that the function $f(\rho) = \rho e^{-a\rho}, \rho > 0$, admits its maximum for $\rho = 1/a$. Therefore, we can apply Theorem 1.3 to obtain the desired result. □

6.3.2 Relaxation functions

Setting $A = -\lambda$, $\lambda > 0$, in equations (6.20) and (6.21) leads to two forms of multi-term relaxation equations. In this section we study the properties of the relaxation functions, obtained as solutions of these equations.

By the use of Laplace transform we deduce that the solution of the relaxation equation in the Caputo form

$$ {}^{C}D_t^{\alpha} u(t) + \sum_{j=1}^{m} b_j \, {}^{C}D_t^{\alpha_j} u(t) + \lambda u(t) = f(t), \qquad t > 0; \quad u(0) = 1, \tag{6.29} $$

is given by

$$ u(t) = u_1(t;\lambda) + \int_0^t v_1(t-\tau;\lambda) f(\tau)\, d\tau, \tag{6.30} $$

and the solution of the relaxation equation in the Riemann-Liouville form

$$ u'(t) + \lambda D_t^{1-\alpha} u(t) + \lambda \sum_{j=1}^{m} b_j \, D_t^{1-\alpha_j} u(t) = f(t), \qquad t > 0; \quad u(0) = 1, \tag{6.31} $$

is represented as

$$ u(t) = u_2(t;\lambda) + \int_0^t u_2(t-\tau;\lambda) f(\tau)\, d\tau, \tag{6.32} $$

where the functions $u_1(t;\lambda)$, $v_1(t;\lambda)$, and $u_2(t;\lambda)$ satisfy the following Laplace transform identities

$$ \widehat{u}_i(s;\lambda) = \frac{g_i(s)}{s(g_i(s)+\lambda)}, \quad i = 1,2; \qquad \widehat{v}_1(s;\lambda) = \frac{1}{g_1(s)+\lambda}, \tag{6.33} $$

with $g_1(s)$ and $g_2(s)$ defined in (6.23).

The functions $u_1(t;\lambda)$ and $v_1(t;\lambda)$ are the relaxation functions related to problem (6.29) and $u_2(t;\lambda)$ is the relaxation function related to problem (6.31). Laplace transform inversion in (6.33) by the use of (6.6) yields the following explicit representations in terms of multinomial Mittag-Leffler functions

$$ u_1(t;\lambda) = 1 - \lambda \mathcal{E}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha+1}\left(t;\lambda,b_1,\ldots,b_m\right), \tag{6.34} $$

$$ u_2(t;\lambda) = \mathcal{E}_{(\alpha,\alpha_1,\ldots,\alpha_m),1}\left(t;\lambda,\lambda b_1,\ldots,\lambda b_m\right), \tag{6.35} $$

$$ v_1(t;\lambda) = \mathcal{E}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha}\left(t;\lambda,b_1,\ldots,b_m\right). \tag{6.36} $$

In the single term case the relaxation functions reduce to the classical Mittag-Leffler functions

$$u_i(t;\lambda) = E_\alpha(-\lambda t^\alpha),\ i = 1,2, \quad v_1(t;\lambda) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha).$$

Subordination identities for the relaxation functions $u_i(t;\lambda)$ can be derived from the scalar version of Theorem 6.5, where $S(t) = u_i(t;\lambda)$, $S_\beta(t) = E_\beta(-\lambda t^\beta)$. In particular, for $\beta = 1$ it follows

$$u_i(t;\lambda) = \int_0^\infty \varphi_i(t,\tau)e^{-\lambda\tau}\,d\tau, \quad t > 0,\ i = 1,2, \tag{6.37}$$

where the functions $\varphi_i(t,\tau)$ are nonnegative and normalized. A subordination result for the third relaxation function $v_1(t;\lambda)$ is given next.

Theorem 6.6. *The relaxation function $v_1(t;\lambda)$ obeys the identity*

$$v_1(t;\lambda) = \int_0^\infty \psi(t,\tau)e^{-\lambda\tau}\,d\tau, \quad t > 0, \tag{6.38}$$

where the kernel $\psi(t,\tau)$ is nonnegative and admits the representation

$$\psi(t,\tau) = h_\alpha(t,\tau) * h_{\alpha_1}(t,b_1\tau) * \ldots * h_{\alpha_m}(t,b_m\tau). \tag{6.39}$$

Here $*$ denotes the Laplace convolution and

$$h_\alpha(t,\sigma) = \sigma^{-1/\alpha}L_\alpha\left(t\sigma^{-1/\alpha}\right), \tag{6.40}$$

where $L_\alpha(\cdot)$ is the Lévy extremal stable density, defined in (1.41).

Proof. Consider a subordination kernel $\psi(t,\tau)$, which Laplace transform with respect to t satisfies

$$\widehat{\psi}(s,\tau) = \int_0^\infty e^{-st}\psi(t,\tau)\,dt = e^{-\tau g_1(s)}. \tag{6.41}$$

Then, the functions $v_1(t;\lambda)$ defined by identity (6.38) obeys

$$\int_0^\infty e^{-st}v_1(t;\lambda)\,dt = \int_0^\infty e^{-\tau g_1(s)}e^{-\lambda\tau}\,d\tau = \frac{1}{g_1(s)+\lambda}.$$

Comparing this result to (6.23), it follows by the uniqueness of the Laplace transform that $v_1(t;\lambda)$ defined by (6.38) is indeed the relaxation function (6.36). In this way (6.38) is established.

Since $g_1(s) \in \mathcal{CBF}$ then by applying **(P2)** it follows $e^{-\tau g_1(s)} \in \mathcal{CMF}$. The nonnegativity of $\psi(t,\tau)$ then follows by the Bernstein's theorem. From (6.41) and (6.23)

$$\widehat{\psi}(s,\tau) = e^{-\tau g_1(s)} = e^{-\tau\left(s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}\right)} = e^{-\tau s^\alpha} \prod_{j=1}^m e^{-\tau b_j s^{\alpha_j}},$$

which, after Laplace transform inversion, yields representation (6.39). □

By fractional integration of (6.38) and taking into account (6.36) and identity (6.10) we deduce the following representation for completely monotone multinomial Mittag-Leffler functions, which is of independent interest.

Corollary 6.2. *Let $0 < \alpha \le \beta \le 1$, $0 < \alpha_j < \alpha$, $\lambda > 0$, $b_j > 0$, $j = 1, \ldots, m$. Then*

$$\mathcal{E}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\beta}(t; \lambda, b_1, \ldots, b_m) = \int_0^\infty \phi(t,\tau) e^{-\lambda\tau}\, d\tau, \quad t > 0, \tag{6.42}$$

where the kernel $\phi(t,\tau)$ is nonnegative and admits the representation

$$\phi(t,\tau) = \frac{t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} * h_\alpha(t,\tau) * h_{\alpha_1}(t, b_1\tau) * \ldots * h_{\alpha_m}(t, b_m\tau)$$

if $\alpha < \beta$ and $\phi(t,\tau) = \psi(t,\tau)$, defined in (6.39), when $\alpha = \beta$. The functions $h_\alpha(t,\cdot)$ are defined in (6.40).

Some properties of the relaxation functions, including useful estimates, are collected in the next theorem. The proof is analogous to that of Theorem 5.8 and is omitted.

Theorem 6.7. *For any $\lambda > 0$ the relaxation functions $u_1(t;\lambda)$, $u_2(t;\lambda)$, and $v_1(t;\lambda)$, defined in (6.34)-(6.36), are positive, strictly decreasing, completely monotone for $t > 0$, and admit analytic extensions to the half-plane $\mathbb{C}_+$. The relation holds true*

$$\frac{\partial}{\partial t} u_1(t;\lambda) = -\lambda v_1(t;\lambda).$$

The following uniform bounds are satisfied

$$0 < u_i(t;\lambda) < 1, \quad t > 0, \; u_i(0;\lambda) = 1, \quad i = 1, 2,$$
$$u_i(t;\lambda) \le \frac{1}{1 + \lambda l_i(t)}, \quad i = 1, 2,$$

where

$$l_1(t) = (1 * k_1)(t) = \mathcal{E}_{(\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha+1}(t; b_1, \ldots, b_m), \quad (6.43)$$

$$l_2(t) = (1 * k_2)(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^m b_j \frac{t^{\alpha_j}}{\Gamma(\alpha_j+1)}. \quad (6.44)$$

For any $\lambda \geq \lambda_0 > 0$ *and* $t > 0$

$$u_i(t; \lambda) \leq u_i(t; \lambda_0), \ i = 1, 2, \qquad v_1(t; \lambda) \leq v_1(t; \lambda_0),$$

and there holds the estimate

$$C \leq \lambda \int_0^T v_1(t; \lambda)\, dt < 1, \quad T > 0,$$

where the constant $C = 1 - u_1(T; \lambda_0) > 0$ *is independent of* λ.

6.3.3 Moments of the fundamental solution

As an application of the multinomial Prabhakar type functions (6.4), in this section we derive expressions for the moments of the Green functions of the multiterm time-fractional differential equations in terms of such functions. Consider the Cauchy problem for the multi-term equations (6.20) and (6.21), where $A = \left(\frac{\partial}{\partial x}\right)^2$, $x \in \mathbb{R}$ (for the precise definition see (2.20)). The fundamental solution $\mathcal{G}(x,t)$ is defined by assuming the initial and boundary conditions

$$\mathcal{G}(x, 0) = \delta(x); \ x \in \mathbb{R}, \qquad \lim_{|x|\to\infty} \mathcal{G}(x,t) = 0, \quad t > 0,$$

where $\delta(\cdot)$ is the Dirac delta function. Applying as usual Laplace transform with respect to the temporal variable and Fourier transform with respect to the spatial variable, we derive for the Green function $\mathcal{G}(x,t)$ in Fourier-Laplace domain

$$\widehat{\widehat{\mathcal{G}}}(\xi, s) = \frac{g(s)/s}{g(s) + \xi^2}, \quad \xi \in \mathbb{R}, \quad s > 0. \quad (6.45)$$

Here $g(s) = g_1(s)$ in the case of equation (6.20) and $g(s) = g_2(s)$ in the case of equation (6.21), and the definitions of these functions are given in (6.23). By Fourier inversion in (6.45) the Laplace transform of the solution is obtained as follows

$$\widehat{\mathcal{G}}(x, s) = \frac{\sqrt{g(s)}}{2s} \exp\left(-|x|\sqrt{g(s)}\right), \quad x \in \mathbb{R}. \quad (6.46)$$

Let $\gamma > 0$. Next we derive the moments of the fundamental solution

$$\langle |x|^\gamma(t)\rangle = \int_{\mathbb{R}} x^\gamma \mathcal{G}(x,t)\,dx.$$

Representation (6.46) implies for the Laplace transforms of the moments $\langle |x|^\gamma(t)\rangle$ of order γ

$$\int_{\mathbb{R}} x^\gamma \widehat{\mathcal{G}}(x,s)\,dx = \frac{\sqrt{g(s)}}{s}\int_0^\infty x^\gamma \exp\left(-x\sqrt{g(s)}\right)\,dx = \frac{\Gamma(\gamma+1)}{s(g(s))^{\gamma/2}},$$

where the formula $\int_0^\infty x^{b-1}e^{-ax}\,dx = \Gamma(b)a^{-b}$ is used. Taking inverse Laplace transform, we obtain by the use of (6.6)

$$\langle |x|_1^\gamma(t)\rangle = C_1\,\mathcal{E}^{\gamma/2}_{(\alpha-\alpha_m,\alpha-\alpha_{m-1},\ldots,\alpha-\alpha_1),\frac{\alpha\gamma}{2}+1}(t;b_m,b_{m-1},\ldots,b_1)$$

for the equation (6.20), where $C_1 = \Gamma(\gamma+1)$, and

$$\langle |x|_2^\gamma(t)\rangle = C_2\,\mathcal{E}^{-\gamma/2}_{(\alpha-\alpha_m,\alpha_1-\alpha_m,\ldots,\alpha_{m-1}-\alpha_m),\frac{\alpha_m\gamma}{2}+1}\left(t;\frac{1}{b_m},\frac{b_1}{b_m},\ldots,\frac{b_{m-1}}{b_m}\right),$$

for the equation (6.21), where $C_2 = \Gamma(\gamma+1)b_m^{\gamma/2}$.

Let us note that the indices in the brackets of the above multinomial Prabhakar type functions are specially arranged, so that the first index, $\alpha-\alpha_m$, is the largest. The obtained representations for the moments, together with the properties (6.10), (6.14), and (6.18), imply that the moments of the Green functions of both equations are Bernstein functions (integrals of completely monotone functions) provided $\alpha\gamma \le 2$.

The corresponding mean squared displacements $\left\langle |x|_i^2(t)\right\rangle$ are derived by setting $\gamma = 2$. This yields

$$\left\langle |x|_i^2(t)\right\rangle = 2l_i(t), \quad i = 1,2,$$

where the functions $l_1(t)$ and $l_2(t)$ are defined in (6.43) and (6.44). As we see, $l_2(t)$ is a finite sum, and this is the case for all moments of even order for the equation in the Riemann-Liouville form (6.21).

The asymptotic behavior of the derived moments can be deduced from the asymptotic expansions (6.5) and (6.16) for the multinomial Prabhakar type functions. In this way we obtain $\langle |x|_1^\gamma(t)\rangle \sim ct^{\alpha\gamma/2}$ as $t \to 0$ and $\langle |x|_1^\gamma(t)\rangle \sim ct^{\alpha_m\gamma/2}$ as $t \to \infty$ for the equation (6.20), while for the equation (6.21) the opposite behavior is observed: $\langle |x|_2^\gamma(t)\rangle \sim ct^{\alpha_m\gamma/2}$ as $t \to 0$ and $\langle |x|_2^\gamma(t)\rangle \sim ct^{\alpha\gamma/2}$ as $t \to \infty$. Here c denotes different positive constants.

Chapter 7

Distributed-order diffusion-wave equations

In the last two chapters we study the subordination principle for generalized time-fractional diffusion-wave equations. Various linear generalizations of the fractional diffusion-wave equation have been proposed in the literature. The most studied examples are the distributed-order time-fractional diffusion-wave equation and various equations governing wave propagation in viscoelastic media. The present chapter is devoted to the distributed-order time-fractional diffusion-wave equation with discrete or continuous distribution of fractional Caputo time derivatives over the interval $(0, 2]$. We first discuss an open problem concerning the interpretation of the fundamental solution to the corresponding one-dimensional Cauchy problem as a spatial probability density. Then the subordination principle for multi-term time-fractional diffusion-wave equation is studied in detail.

7.1 Positivity of the fundamental solution

The time-fractional diffusion-wave equation

$$ {}^C D_t^\alpha u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad \alpha \in (1,2), \quad x \in \mathbb{R}, \ t > 0, \tag{7.1} $$

where ${}^C D_t^\alpha$ denotes the Caputo fractional time derivative of order $\alpha \in (1, 2)$ has been extensively studied as a model of evolution processes intermediate between diffusion and wave propagation, see e.g. [6, 72, 75, 78, 80].

Consider the equation derived by replacing the single fractional time derivative in (7.1) by a distribution of fractional Caputo time derivatives over the interval $(0, 2]$:

$$\int_0^2 \mu(\beta)^C D_t^\beta u(x,t)\, d\beta = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R}, t > 0, \tag{7.2}$$

where $\mu(\beta)$ is a nonnegative (generalized) weight function, such that

$$\operatorname{supp} \mu \cap (1, 2] \neq \emptyset.$$

The Cauchy problem for equation (7.2) with initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = 0$ is studied in [46] with the main focus on the interpretation of the fundamental solution $\mathcal{G}(x, t)$ (corresponding to $f(x) = \delta(x)$) as a spatial probability density function:

$$\mathcal{G}(x,t) \geq 0 \text{ for } x \in \mathbb{R}, t > 0; \qquad \int_{-\infty}^{\infty} \mathcal{G}(x,t)\, dx = 1 \text{ for } t \geq 0. \tag{7.3}$$

The importance of properties (7.3) for the stochastic interpretation of the distributed order wave equation and for its physical meaning is explained by Gorenflo in [44]. In addition, it appears to be essential for subordination of equation (7.2) to second order Cauchy problem, as we will see next.

The Laplace transform of the fundamental solution with respect to the time variable is given by the formula [46]

$$\widehat{\mathcal{G}}(x,s) = \frac{\sqrt{g(s)}}{2s} \exp\left(-|x|\sqrt{g(s)}\right), \quad x \in \mathbb{R}, \ s > 0, \tag{7.4}$$

where

$$g(s) = \int_0^2 \mu(\beta) s^\beta\, d\beta. \tag{7.5}$$

Therefore, the integral identity in (7.3) is easily established, see [46]. The more difficult part is the nonnegativity of $\mathcal{G}(x, t)$, that, according to Bernstein's theorem, is equivalent to complete monotonicity of the Laplace transform (7.4). Based on Bernstein functions technique, it is proven in [46] that the fundamental solution $\mathcal{G}(x, t)$ of (7.2) is non-negative if the weight $\mu(\beta)$ vanishes identically on the interval $(0, 1)$. The question whether this assumption can be relaxed is stated in [46] as an open problem. Here we discuss this problem together with its relation to subordination principle.

Consider the distributed-order diffusion-wave equation in abstract form

$$\int_0^2 \mu(\beta)\,{}^C\!D_t^\beta u(t)\,d\beta = Au(t),\ t>0;\quad u(0)=a\in X,\ u'(0)=0, \tag{7.6}$$

where the operator A is a generator of a strongly continuous cosine family in a Banach space X, that is, the second-order Cauchy problem (2.7) is well posed. By applying Laplace transform, the Cauchy problem (7.6) is reformulated as Volterra integral equation (2.10) with kernel $k(t)$ with characteristic function $g(s) = 1/\widehat{k}(s)$ given by (7.5). According to the general subordination theorem (Theorem 2.4) problem (7.6) is subordinated to the second-order Cauchy problem (2.7) provided

$$g(s)^{1/2} \in \mathcal{CBF},\quad s>0. \tag{7.7}$$

This condition ensures for any fixed $\tau > 0$

$$\widehat{\varphi}(s,\tau) = \frac{\sqrt{g(s)}}{s}\exp\left(-\tau\sqrt{g(s)}\right) \in \mathcal{CMF},\quad s>0, \tag{7.8}$$

which is equivalent to $\varphi(t,\tau) \geq 0$ ($\varphi(t,\tau)$ is the corresponding subordination kernel). A comparison to (7.4) shows that the problem of nonnegativity of the fundamental solution $\mathcal{G}(x,t)$ is equivalent to the above problem of subordination. Both problems reduce to proving complete monotonicity of the expression in (7.8). In this case the well-posedness of the second order Cauchy problem implies well-posedness of the distributed-order Cauchy problem (7.6).

In this section we discuss conditions on the weight functions μ, implying (7.7). In the next theorem we relax the condition $\operatorname{supp}\mu \subset [1,2]$, which was considered in [46]. We prove that the support of the function $\mu(\beta)$ can be any interval with length at most 1, not necessarily the interval $[1,2]$.

Proposition 7.1. *Assume*

$$\operatorname{supp}\mu \subseteq [\alpha-1,\alpha],\quad 1<\alpha\leq 2. \tag{7.9}$$

Then the function $g(s)$, defined in (7.5), satisfies (7.7).

Proof. Under the assumption (7.9)

$$g(s) = \int_{\alpha-1}^{\alpha} \mu(\beta)s^\beta\,d\beta = s^{\alpha-1}\int_{\alpha-1}^{\alpha}\mu(\beta)s^{\beta-\alpha+1}\,d\beta.$$

Since $s^{\beta-\alpha+1} \in \mathcal{CBF}$ for $\beta \in [\alpha-1,\alpha], \alpha \in (1,2]$, then also the integral of $s^{\beta-\alpha+1}$ with positive weight is a complete Bernstein function as a point-wise limit of positive linear combinations of complete Bernstein functions, see **(P2)** in Proposition 2.1. Moreover, $s^{\alpha-1} \in \mathcal{CBF}$ for $\alpha \in (1,2]$. Therefore, $g(s)$ is a product of two complete Bernstein functions and property (2.2) implies (7.7). □

The next example of power-law weight function on an interval with arbitrary length (≤ 2) shows that the assumption on the weight function in Proposition 7.1 is not necessary for (7.7).

Proposition 7.2. *Let $a > 0$ and $0 < \alpha_1 < \alpha_2 \leq 2$. Assume $\mu(\beta) = a^\beta$ for $\beta \in [\alpha_1, \alpha_2]$, and $\mu(\beta) = 0$ for $\beta \in (0,\alpha_1) \cup (\alpha_2, 2]$. Then the function $g(s)$, defined in (7.5), satisfies (7.7).*

Proof. Taking into account $s^\beta = e^{\beta \log s}$, we deduce

$$\begin{aligned} g(s) &= \int_{\alpha_1}^{\alpha_2} (as)^\beta \, d\beta = \frac{(as)^{\alpha_2} - (as)^{\alpha_1}}{\log(as)} \\ &= \left((as)^{\alpha_2/2} + (as)^{\alpha_1/2}\right) \frac{(as)^{\alpha_2/2} - (as)^{\alpha_1/2}}{\log(as)} \\ &= \left((as)^{\alpha_2/2} + (as)^{\alpha_1/2}\right) \int_{\alpha_1/2}^{\alpha_2/2} (as)^\beta \, d\beta. \end{aligned} \tag{7.10}$$

Since $\alpha_i/2 \in (0,1]$, then $(as)^{\alpha_i/2} \in \mathcal{CBF}$, $i = 1,2$, and their sum is again in $\mathcal{CBF}$. The integral in (7.10) is a complete Bernstein function as a point-wise limit of positive linear combinations of complete Bernstein functions $(as)^\beta$, $\beta \in (0,1]$. Therefore, $g(s)$ is a product of two complete Bernstein functions and (2.2) implies (7.7). □

The situation is different in the case of discrete distribution, as we will see in the next example..

Example 7.1. Let $\mu(\beta) = \delta(\beta-\alpha_1)+\delta(\beta-\alpha_2)$, where $0 < \alpha_1 < \alpha_2 \leq 2$. Then $g(s) = s^{\alpha_2} + s^{\alpha_1}$. If $\alpha_2 - \alpha_1 > 1$, then representation $g(s) = s^{\alpha_1}(s^{\alpha_2-\alpha_1} + 1)$ implies that $g(s)$ has a zero in $\mathbb{C}\backslash(-\infty, 0]$. Therefore, the multivalued complex function $g(s)^{1/2}$ has a branch point in $\mathbb{C}\backslash(-\infty, 0]$ and, according to property **(P15)** in Proposition 2.1, $g(s)^{1/2}$ cannot be a complete Bernstein function. This shows that, in order to have (7.7) for multi-term equations, the distance between the largest and the smallest fractional orders should not exceed 1.

In fact, also the weaker property $\sqrt{g(s)} \in \mathcal{BF}$ does not hold without a restriction on the distance $\alpha - \alpha_1$. Considering the above two-term example with different values of the parameters α and α_1 such that $\alpha - \alpha_1 > 1$ (e.g. $\alpha = 1.9, \alpha_1 \in (0, 0.5]$; $\alpha = 1.8, \alpha_1 \in (0, 0.3]$) we obtain by direct computation that the second derivative $d^2/ds^2(\sqrt{g(s)})$ admits positive values for some $s > 0$. Therefore, the function $\sqrt{g(s)}$ is not concave for all $s > 0$, which implies that $\sqrt{g(s)} \notin \mathcal{BF}$.

7.2 Multi-term diffusion-wave equation

This section is devoted to a detailed study of the class of multi-term time-fractional diffusion-wave equations

$$c\,{}^C\!D_t^{\alpha} u(t) + \sum_{j=1}^{m} c_j\,{}^C\!D_t^{\alpha_j} u(t) = Au(t), \quad u(0) = a \in X,\ u'(0) = 0, \qquad (7.11)$$

where A is a closed linear unbounded operator densely defined in a Banach space X, which generates a strongly continuous cosine family. We suppose that the parameters α, α_j, c, c_j, satisfy the following restrictions

$$\begin{aligned} &\alpha \in (1, 2], \quad \alpha > \alpha_1 > \cdots > \alpha_m > 0, \quad \alpha - \alpha_m \le 1, \\ &c > 0, \quad c_j > 0, \quad j = 1, \cdots, m. \end{aligned} \qquad (7.12)$$

7.2.1 Propagation function

Consider first the following problem for the spatially one-dimensional version of the multi-term equation in (7.11)

$$c\,{}^C\!D_t^{\alpha} w(x,t) + \sum_{j=1}^{m} c_j\,{}^C\!D_t^{\alpha_j} w(x,t) = \frac{\partial^2}{\partial x^2} w(x,t), \quad x, t > 0, \qquad (7.13)$$

$$w(x, 0) = w_t(x, 0) = 0, \quad x > 0, \qquad (7.14)$$

$$w(0, t) = \Theta(t), \quad w \to 0 \text{ as } x \to \infty, \quad t > 0, \qquad (7.15)$$

where the parameters $\alpha, \alpha_j, c, c_j, j = 1, ..., m$, satisfy conditions (7.12), and $\Theta(t)$ is the Heaviside unit step function.

Problem (7.13)-(7.14)-(7.15) models the propagation in time of a disturbance at $x = 0$. That is why the solution $w(x, t)$ is referred to as propagation function.

By applying Laplace transform with respect to the temporal variable in (7.13) and (7.15) and taking into account initial conditions (7.14) we obtain using (1.13) the following problem

$$g(s)\widehat{w}(x,s) = \widehat{w}_{xx}(x,s),\ \widehat{w}(0,s) = 1/s,\ \widehat{w}(x,s) \to 0 \text{ as } x \to \infty, \tag{7.16}$$

where

$$g(s) = cs^{\alpha} + \sum_{j=1}^{m} c_j s^{\alpha_j}, \quad s > 0. \tag{7.17}$$

Solving (7.16) as ODE of second order (with s considered as a parameter) we deduce

$$\widehat{w}(x,s) = \frac{1}{s}\exp\left(-x\sqrt{g(s)}\right). \tag{7.18}$$

Properties

Problem (7.13)-(7.14)-(7.15) is physically meaningful when the propagation function $w(x,t)$ is nonnegative. By Bernstein's theorem this is equivalent to $\widehat{w}(x,s) \in \mathcal{CMF}$, which is guaranteed if $\sqrt{g(s)}$ is a Bernstein function. In fact, conditions (7.12) imply a stronger property: $\sqrt{g(s)} \in \mathcal{CBF} \subset \mathcal{BF}$.

Proposition 7.3. *Assume $g(s)$ is defined by (7.17) with parameters α, α_j, c, c_j, $j = 1, \cdots, m$, satisfying conditions (7.12). Then $\sqrt{g(s)} \in \mathcal{CBF}$.*

Proof. Consider first the case $\alpha_m \geq 1$. Set $f(s) = g(s)/s$. Since $2 > \alpha, \alpha_j \geq 1$ the function $f(s) = cs^{\alpha-1} + \sum_{j=1}^{m} c_j s^{\alpha_j - 1} \in \mathcal{CBF}$ as a sum of complete Bernstein functions. Also, $s \in \mathcal{CBF}$. Then, applying property (2.2) it follows that $\sqrt{g(s)} = \sqrt{s}\sqrt{f(s)} \in \mathcal{CBF}$.

In the case $\alpha_m < 1$ we set $f(s) = g(s)/s^{\alpha_m}$. The assumption $0 < \alpha - \alpha_m \leq 1$ implies again $f(s) \in \mathcal{CBF}$. Since also $s^{\alpha_m} \in \mathcal{CBF}$, we obtain in the same way as above $\sqrt{g(s)} = \sqrt{s^{\alpha_m}}\sqrt{f(s)} \in \mathcal{CBF}$. □

Proposition 7.3 implies important properties of the propagation function.

Theorem 7.1. *Under conditions (7.12) the propagation function $w(x,t)$ satisfies the properties*

$$w(x,t) \geq 0, \quad \frac{\partial}{\partial t} w(x,t) \geq 0, \quad -\frac{\partial}{\partial x} w(x,t) \geq 0, \qquad x, t > 0. \tag{7.19}$$

Proof. According to Bernstein's theorem it is sufficient to prove that the Laplace transforms of the three functions in (7.19) are completely monotone. We have from Proposition 7.3 that $\sqrt{g(s)} \in \mathcal{CBF} \subset \mathcal{BF}$. Then, by property **(P5)** in Proposition 2.1, the function $\exp\left(-x\sqrt{g(s)}\right) \in \mathcal{CMF}$ as a composition of the completely monotone exponential function and the Bernstein function $\sqrt{g(s)}$.

Since $1/s \in \mathcal{CMF}$, applying **(P1)** in Proposition 2.1 and taking into account (7.18) it follows $\widehat{w}(x,s) \in \mathcal{CMF}$ as a product of two completely monotone functions. Further, since $\lim_{t\to 0} w(x,t) = \lim_{s\to\infty} s\widehat{w}(x,s) = 0$, (7.18) and (7.17) imply

$$\mathcal{L}\{w_t\}(x,s) = s\widehat{w}(x,s) - w(x,0) = \exp\left(-x\sqrt{g(s)}\right) \in \mathcal{CMF}.$$

For the third function we obtain

$$\mathcal{L}\{-w_x\}(x,s) = -\frac{\partial}{\partial x}\widehat{w}(x,s) = \frac{\sqrt{g(s)}}{s}\exp\left(-x\sqrt{g(s)}\right) \in \mathcal{CMF} \qquad (7.20)$$

by applying (2.3). □

Theorem 7.1 implies that $w(x,t)$ is a nonincreasing function in x and nondecreasing function in t with limiting value found by applying Finite value theorem for Laplace transform

$$\lim_{t\to+\infty} w(x,t) = \lim_{s\to 0} s\widehat{w}(x,s) = 1. \qquad (7.21)$$

The fundamental solution $\mathcal{G}(x,t)$ of the Cauchy problem for equation (7.11) with $A = \partial^2/\partial x^2$ can be expressed in terms of the propagation function $w(x,t)$ as follows:

$$\mathcal{G}(x,t) = -\frac{1}{2}w_x(|x|,t), \quad x \in \mathbb{R}, \qquad (7.22)$$

which is deduced by comparison of (7.4) and (7.20). Therefore Theorem 7.1 implies that $\mathcal{G}(x,t)$ is a nonnegative function, as it was expected.

Next we distinguish two cases $\alpha < 2$ and $\alpha = 2$. For $\alpha < 2$ the propagation function $w(x,t)$ admits an analytic extension to a sector in the complex plane $t \in \mathbb{C}\backslash 0$, $|\arg t| < \theta_0$ (the proof is essentially the same as that of Theorem 7.6). Therefore, for any $x > 0$ the set of zeros of $w(x,t)$ on $t > 0$ can be only discrete. This together with (7.19) and (7.21) implies that $w(x,t) > 0$ for all $x, t > 0$, which means that a disturbance spreads infinitely fast.

Theorem 7.2. *If* $1 < \alpha < 2$ *then* $w(x,t) > 0$ *for all* $x, t > 0$.

On the other hand, in the case $\alpha = 2$, a disturbance spreads with finite speed as in the case of classical wave equation ($m = 0, \alpha = 2$) and classical telegraph equation ($m = 1, \alpha = 2, \alpha_1 = 1$). However, in contrast to the classical equations, in the case when there is at least one fractional time-derivative in equation (7.13) a phenomenon of coexistence of finite propagation speed and absence of wave front is established. This is a memory effect, not observed in linear integer-order differential equations.

We will prove that for $\alpha = 2$ a disturbance spreads with a finite propagation speed $1/\sqrt{c}$. Let us define the function

$$h(s) = \sqrt{g(s)} - \sqrt{c}s.$$

Then (7.18) implies

$$\begin{aligned} w(x,t) &= \mathcal{L}^{-1}\left\{\frac{1}{s}\exp\left(-xh(s)\right)\exp\left(-x\sqrt{c}s\right)\right\} \qquad (7.23)\\ &= w_0(x, t-\sqrt{c}x)\Theta(t-\sqrt{c}x), \end{aligned}$$

where

$$w_0(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\exp\left(-xh(s)\right)\right\}$$

and $\Theta(t)$ is the Heaviside unit-step function. Here we have used the property $\mathcal{L}\{f(t-a)\Theta(t-a)\}(s) = \exp(-as)\mathcal{L}\{f\}(s)$. Since $h(s), h'(s) \geq 0$ for $s > 0$ and $\sqrt{g(s)} \in \mathcal{BF}$, it follows that $h(s) \in \mathcal{BF}$. Therefore, $w_0(x,t) \geq 0$ by the same argument as in the proof of Theorem 7.1. Formula (7.23) implies that the propagation function $w(x,t)$ vanishes for $x > t/\sqrt{c}$, i.e. the propagation speed is $1/\sqrt{c}$.

Theorem 7.3. *If $\alpha = 2$ then $w(x,t) \equiv 0$ for $x > t/\sqrt{c}$.*

Except in the two classical cases (wave equation and classical telegraph equation) there holds $\lim_{s\to\infty} h(s) = \infty$, which implies that there is no wave front (jump discontinuity) at $x = t/\sqrt{c}$ (cf. [93], Chapter 5).

The behaviour of the propagation function $w(x,t)$ is illustrated in Figures 7.1 and 7.2. Three different cases for the equation (7.13) with two time-derivatives are considered: the classical telegraph equation (Fig. 7.1) which exhibits finite propagation speed and wave front, an equation with $\alpha = 2$ and $\alpha_1 \in (1,2)$ (Fig. 7.2(a)) exhibiting finite propagation speed and no wave front, and an equation with $\alpha < 2$ (Fig. 7.2(b)) exhibiting infinite propagation speed. The figures are from [18]. The plots are obtained by numerical computation based on the explicit integral representation for $w(x,t)$ derived next.

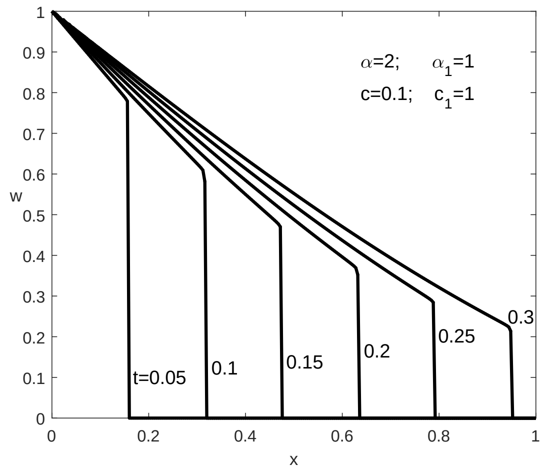


Figure 7.1: Propagation function $w(x,t)$ for the classical telegraph equation as a function of x for different values of t: finite propagation speed and wave front.

Explicit representation

To derive an explicit representation for the propagation function we apply the complex Laplace inversion formula to (7.18), which yields

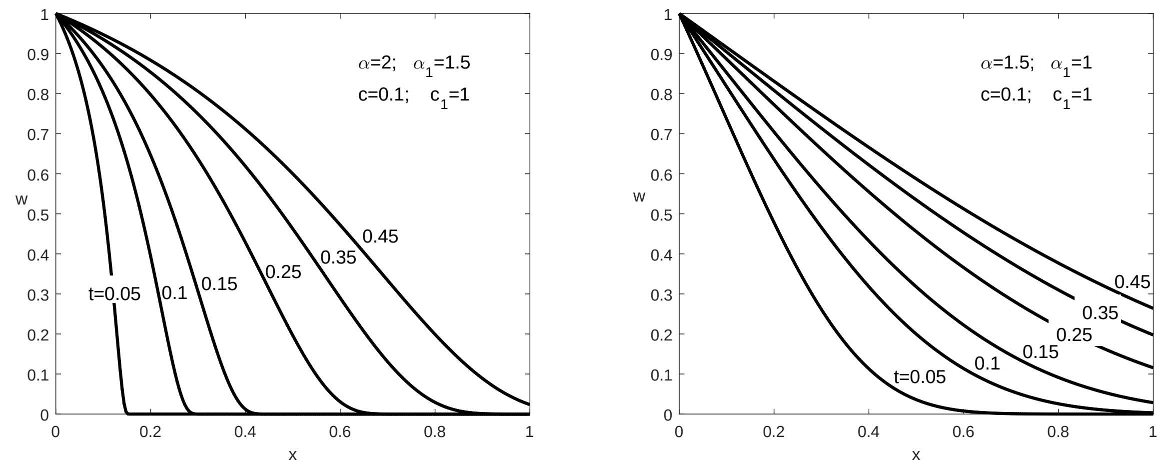
$$\begin{aligned} w(x,t) &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{st}\widehat{w}(x,s)\, ds \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \exp\left(st - x\sqrt{g(s)}\right) \frac{ds}{s}, \quad \gamma > 0. \end{aligned} \tag{7.24}$$

Since $\sqrt{g(s)} \in \mathcal{CBF}$ it can be analytically extended to $\mathbb{C}\backslash(-\infty,0]$. Therefore, this holds also for the function under the integral sign in (7.24). By the Cauchy's theorem, the integration on the contour $\{s = \gamma + \mathrm{i}r,\ r \in (-\infty,+\infty)\}$ can be replaced by integration on the contour $D_R^- \cup D \cup D_0 \cup D_R^+$, where (with appropriate orientation)

$$D = \{s = \mathrm{i}r,\ r \in (-\infty,-\varepsilon) \cup (\varepsilon,\infty)\},\ D_\varepsilon = \{s = \varepsilon e^{\mathrm{i}\theta},\ \theta \in [-\pi/2,\pi/2]\},$$

$$D_R^+ = \{|s| = R,\ \Re s \in [0,\gamma],\ \Im s > 0\},\ \ D_R^- = \{|s| = R,\ \Re s \in [0,\gamma],\ \Im s < 0\}.$$

To prove that the integrals on the arcs D_R^- and D_R^+ vanish for $R \to \infty$ it is sufficient to show that for any $x > 0$ the function $\widehat{w}(x,s)$ is uniformly bounded on $D_{R_n}^+$ and $D_{R_n}^-$, where $R_n \to \infty$, and that $\widehat{w}(x,s) \to 0$ for $s \in D_R^\pm$ and $R \to \infty$, see e.g. [35], Chapter 2, Lemma 2. This follows from the fact that



(**a**) (**b**)

Figure 7.2: Propagation function $w(x,t)$ for a two-term equation as a function of x for different values of t; (**a**) $\alpha = 2$, $\alpha_1 = 1.5$: finite propagation speed, no wave front; (**b**) $\alpha = 1.5$, $\alpha_1 = 1$: infinite propagation speed.

$\Re\sqrt{g(s)} \geq 0$ for $\Re s \geq 0$ and therefore

$$\left|\frac{1}{s}\exp\left(-x\sqrt{g(s)}\right)\right| \leq \frac{1}{R}\exp\left(-x\Re\sqrt{g(s)}\right) \leq \frac{1}{R}, \quad s \in D_R^{\pm}. \tag{7.25}$$

The integral on the semi-circular contour D_ε equals $1/2$ when $\varepsilon \to 0$. This can be obtained by direct check using that

$$\lim_{s\to 0} s\left(\frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\right) = 1.$$

Integration on the contour D yields after letting $\varepsilon \to 0$ and $R \to \infty$:

$$\frac{1}{2\pi \mathrm{i}}\int_D \frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\,ds = \frac{1}{\pi}\int_0^\infty \frac{1}{r}\Im\exp\left(\mathrm{i}rt - x\sqrt{g(\mathrm{i}r)}\right)\,dr.$$

Here we have used the fact that $\sqrt{g(s^*)} = \left(\sqrt{g(s)}\right)^*$, where $*$ denotes the complex conjugate, see property **(P15)** in Proposition 2.1. Applying the formula (4.17) for real and imaginary parts of the square root of a complex number we obtain the following result.

Theorem 7.4. *The propagation function $w(x,t)$ admits the integral representation:*

$$w(x,t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \exp(-xK^+(r))\sin(rt - xK^-(r))\,\frac{dr}{r}, \quad x,t > 0, \tag{7.26}$$

where

$$K^{\pm}(r) = \frac{1}{\sqrt{2}}\left(\left(A^2(r)+B^2(r)\right)^{1/2} \pm A(r)\right)^{1/2} \tag{7.27}$$

with

$$A(r) = \Re g(ir) = cr^{\alpha}\cos(\alpha\pi/2) + \sum_{j=1}^{m} c_j r^{\alpha_j}\cos(\alpha_j\pi/2),$$
$$B(r) = \Im g(ir) = cr^{\alpha}\sin(\alpha\pi/2) + \sum_{j=1}^{m} c_j r^{\alpha_j}\sin(\alpha_j\pi/2).$$

To check that the obtained integral in (7.26) is convergent we note that $K^{\pm}(r) > 0$, $K^{\pm}(r) \sim r^{\alpha_m/2}$ as $r \to 0$ and $K^{\pm}(r) \sim r^{\alpha/2}$ as $r \to \infty$. Therefore, the function under the integral sign in (7.26) has an integrable singularity at $r = 0$, while at $r \to \infty$ the term $\exp(-xK^{+}(r))$ ensures integrability not only of this function, but also of all derivatives with respect to t. Therefore, $w(x,t)$ is well defined function, which is infinitely differentiable in t.

Corollary 7.1. *In the single-term case $m = 0$ and $c = 1$ the propagation function $w(x,t)$ admits the integral representation for $x, t > 0$:*

$$w(x,t) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \exp\left(-xr^{\alpha/2}\cos(\alpha\pi/4)\right)\sin\left(rt - xr^{\alpha/2}\sin(\alpha\pi/4)\right)\frac{dr}{r}.$$

7.2.2 Subordination results

We suppose first that A generates a cosine family. This means that the second-order Cauchy problem (2.7) is well posed with solution operator $S_2(t)$.

We establish subordination results for the Cauchy problem (7.11) for the multi-term diffusion-wave equation with parameters satisfying (7.12). Applying Laplace transform we rewrite problem (7.11) as the abstract Volterra integral equation

$$u(t) = a + \int_0^t k(t-\tau)Au(\tau)\,d\tau. \tag{7.28}$$

The scalar kernel $k(t)$ is defined by its Laplace transform

$$\widehat{k}(s) = 1/g(s), \tag{7.29}$$

where the function $g(s)$ is defined in (7.17).

Problem (7.11) is well posed iff the corresponding Volterra integral equation (7.28) is well posed. In this case the solution operator of problem (7.28) coincides with the solution operator of problem (7.11). Denote by $S(t)$ this solution operator.

Since $\sqrt{g(s)} \in \mathcal{CBF}$ (see Proposition 7.3), applying Theorem 2.4 we obtain that problem (7.11) is subordinated to the second order Cauchy problem.

Theorem 7.5. *If A is a generator of a bounded cosine family $S_2(t)$ in X then problem (7.11) admits a bounded solution operator $S(t)$. It is related to $S_2(t)$ via the subordination identity*

$$S(t) = \int_0^\infty \varphi(t,\tau) S_2(\tau)\, d\tau, \quad t > 0. \tag{7.30}$$

The function $\varphi(t,\tau)$ is a PDF in τ (i.e. satisfies conditions (2.26)) and admits the following integral representation

$$\begin{aligned}\varphi(t,\tau) &= \frac{1}{\pi}\int_0^\infty \exp\left(-\tau K^+(r)\right)\left(K^+(r)\sin\left(rt - \tau K^-(r)\right)\right. \\ &+ \left. K^-(r)\cos\left(rt - \tau K^-(r)\right)\right)\frac{dr}{r}, \quad t,\tau > 0, \end{aligned} \tag{7.31}$$

where $K^{\pm}(r)$ are the functions defined in (7.27).

Proof. We have to prove only integral representation (7.31). Indeed, the subordination kernel $\varphi(t,\tau)$ is related to the propagation function $w(x,t)$ via the identity

$$\varphi(t,\tau) = -w_x(x,t)|_{x=\tau}, \quad t,\tau > 0, \tag{7.32}$$

which can be deduced by comparing their Laplace transforms. Then the integral representation (7.31) follows after easily justified differentiation under the integral sign in (7.26). □

Plots of the subordination kernel $\varphi(t,\tau)$ in (7.30) in the case of some two-term equations are shown in Figure 7.3. The numerical computations are based on the integral representation (7.31). The figure is from [18].

In the case $\alpha = 2$ identity (7.32) and Theorem 7.3 imply that $\varphi(t,\tau) \equiv 0$ for $\tau > t/\sqrt{c}$. Therefore in this case the integral in (7.30) is finite.

Corollary 7.2. *Let $\alpha = 2$. Under the hypotheses of Theorem 7.5 the subordination relation (7.30) has the form*

$$S(t) = \int_0^{t/\sqrt{c}} \varphi(t,\tau) S_2(\tau)\, d\tau, \quad t > 0. \tag{7.33}$$

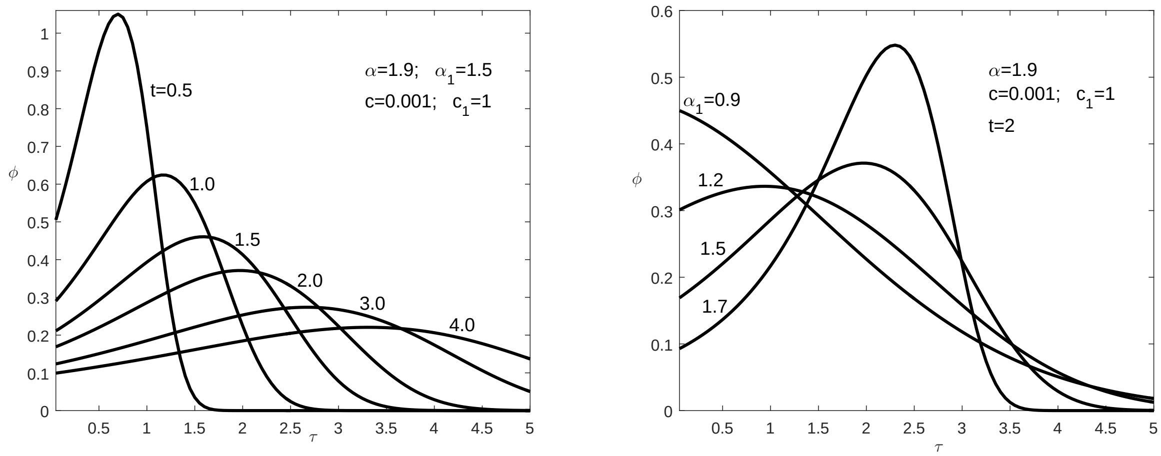


Figure 7.3: Subordination function $\varphi(t,\tau)$ for a two-term equation as a function of τ; (**a**) $\alpha = 1.9$, $\alpha_1 = 1.5$, and different values of t; (**b**) $\alpha = 1.9$, $t = 2$ and different values of α_1.

Taking into account the asymptotic expansions of the functions $K^{\pm}(r)$, it is clear that the function under the integral sign in (7.31) can be infinitely differentiated in t. Therefore, this should hold also for the function $\varphi(t,\tau)$. In the next theorem we prove a stronger regularity property in the case $\alpha < 2$.

Theorem 7.6. *Assume* $1 < \alpha < 2$ *and let*

$$\theta_0 = \frac{(2-\alpha)\pi}{2\alpha} - \varepsilon, \tag{7.34}$$

where $\varepsilon > 0$ *is arbitrarily small. For any* $\tau > 0$ *the function* $\varphi(t,\tau)$ *as a function of* t *admits analytic extension to the sector* $\Sigma(\theta_0)$ *and is bounded on each sector* $\overline{\Sigma(\theta)}$, $0 < \theta < \theta_0$.

Proof. First note that $\alpha > 1$ implies $\theta_0 < \pi/2$. It suffices to prove that for any $\tau > 0$ the Laplace transform $\widehat{\varphi}(s,\tau)$ of the function $\varphi(t,\tau)$ admits analytic extension for $s \in \Sigma(\pi/2 + \theta_0)$, such that $s\widehat{\varphi}(s,\tau)$ is bounded for $s \in \overline{\Sigma(\pi/2+\theta)}$, $0 < \theta < \theta_0$, see Theorem 1.3.

Indeed, since $\sqrt{g(s)} \in \mathcal{CBF}$, it can be extended analytically to $\mathbb{C}\backslash(-\infty, 0]$. Therefore this holds also for the function

$$\widehat{\varphi}(s,\tau) = \frac{\sqrt{g(s)}}{s}\exp(-\tau\sqrt{g(s)}).$$

For $s \in \overline{\Sigma(\pi/2+\theta)}$, $\theta < \theta_0$, the definition (7.17) of $g(s)$ together with the property $|\arg(s_1+s_2)| \le \max\{|\arg s_1|, |\arg s_2|\}$ and (7.34) implies

$$|\arg\sqrt{g(s)}| \le \frac{\alpha}{2}|\arg s| < \pi/2 - \varepsilon\alpha/2. \tag{7.35}$$

Therefore,

$$\begin{aligned} |s\widehat{\varphi}(s,\tau)| &= \left|\sqrt{g(s)}\exp\left(-\tau\sqrt{g(s)}\right)\right| \\ &\le \rho\exp\left(-\tau\rho\cos\left(\arg\sqrt{g(s)}\right)\right) \\ &\le \rho e^{-a\rho} \le (ea)^{-1}, \end{aligned} \tag{7.36}$$

where $\rho = \left|\sqrt{g(s)}\right|$ and $a = \tau\sin(\varepsilon\alpha/2) > 0$. □

Theorem 7.7. *Let $1 < \alpha < 2$. Under the hypotheses of Theorem 7.5 the solution operator $S(t)$ of problem (7.11) is a bounded analytic solution operator of angle θ_0, defined in (7.34).*

Proof. Roughly speaking, since $S_2(t)$ is bounded, according to Theorem 7.6 the function under the integral sign in (7.30) is analytic in $t \in \Sigma(\theta_0)$ and the integral is absolutely and uniformly convergent on compact subsets of $\Sigma(\theta_0)$. Therefore, $S(t)$ given by (7.30) is analytic in $\Sigma(\theta_0)$ and bounded in the subsectors.

Strictly, this result follows from Theorem 2.5 taking into account (7.35). □

Theorem 7.7 is in agreement with Theorem 3.2. in [9], where the same property is established for the solution operators $S_\alpha(t)$.

In fact, problem (7.11) is not only subordinate to the second order Cauchy problem, but also to the fractional Cauchy problem (2.8) of order α, which is a stronger result for $\alpha < 2$. To deduce this fact, according to Theorem 2.4 we only need to prove the following property of $g(s)$.

Proposition 7.4. *If $g(s)$ is defined as in (7.17) with parameters $\alpha, \alpha_j, c, c_j, j = 1, ..., m$, satisfying (7.12), then $g(s)^{1/\alpha} \in \mathcal{CBF}$.*

Proof. According to property **(P7)** in Proposition 2.1 it is sufficient to prove that

$$f(s) = \frac{g(s)^{1/\alpha}}{s} \in \mathcal{SF}. \tag{7.37}$$

Since $0 < \alpha - \alpha_m \leq 1$ the function

$$f^\alpha(s) = \frac{g(s)}{s^\alpha} = c + \sum_{j=1}^{m} c_j s^{\alpha_j - \alpha}$$

is a Stieltjes function. Moreover, $s^{1/\alpha} \in \mathcal{CBF}$ for $\alpha > 1$. This together with property **(P12)** in Proposition 2.1 gives (7.37). □

As discussed earlier, the property $g(s)^{1/\alpha} \in \mathcal{CBF}$ is stronger than the property $g(s)^{1/2} \in \mathcal{CBF}$ proven in Proposition 7.3. This follows from the representation $g(s)^{1/2} = (g(s)^{1/\alpha})^{\alpha/2}$ as a composition of two complete Bernstein functions, which by property **(P11)** in Proposition 2.1 is again a complete Bernstein function.

Theorem 7.8. *Assume problem (2.8) has a bounded solution operator $S_\alpha(t)$. Then problem (7.11) admits a bounded solution operator $S(t)$, which is related to $S_\alpha(t)$ by the subordination identity*

$$S(t) = \int_0^\infty \psi(t,\tau) S_\alpha(\tau)\, d\tau, \quad t > 0, \tag{7.38}$$

where the function $\psi(t,\tau)$ is a unilateral PDF in τ, defined as the inverse Laplace transform

$$\psi(t,\tau) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \frac{g(s)^{1/\alpha}}{s} \exp\left(st - \tau g(s)^{1/\alpha}\right) ds, \quad \gamma, t, \tau > 0.$$

Theorem 7.8 implies that the solution operator $S(t)$ has (at least) the same regularity as $S_\alpha(t)$. This result is in agreement with Theorem 3.4 in [65].

Let us note that the subordination results of this section can be generalized to the case of continuous distribution, depending on the support of the weight function $\mu(\beta)$ in (7.5). For instance, if condition (7.9) is satisfied then the distributed-order equation (7.6) is subordinated to the single-term equation (2.8) of order α. To prove this, it suffices to check that $g(s)^{1/\alpha} \in \mathcal{CBF}$ for $g(s) = \int_{\alpha-1}^{\alpha} \mu(\beta) s^\beta d\beta$. If we define $f(s) = g(s)^{1/\alpha}/s$ then

$$(f(s))^\alpha = \frac{g(s)}{s^\alpha} = \int_{\alpha-1}^{\alpha} \mu(\beta) s^{\beta-\alpha} d\beta \in \mathcal{SF}.$$

Since $s^{1/\alpha} \in \mathcal{CBF}$ for $\alpha > 1$ the composition rule **(P12)** in Proposition 2.1 yields $f(s) \in \mathcal{SF}$, which implies $g(s)^{1/\alpha} \in \mathcal{CBF}$.

7.2.3 Applications

Simple examples of application of the subordination theorems are given in this section.

Example 7.2. Let $X = L^p(\mathbb{R}), 1 \leq p < \infty$. Define the operator A by means of $(Au)(x) = u''(x)$, with domain $D(A) = \{u \in X : u'' \in X\}$. Then A generates a bounded cosine family given by the d'Alembert formula

$$(S_2(t)v)(x) = \frac{1}{2}\left(v(x+t) + v(x-t)\right). \tag{7.39}$$

Inserting (7.39) in the subordination formula (7.30) we obtain for the solution of problem (7.11)

$$\begin{aligned} u(x,t) = (S(t)v)(x) &= \int_0^\infty \varphi(t,\tau)(S_2(\tau)v)(x)\, d\tau \\ &= \frac{1}{2}\int_{-\infty}^\infty \varphi(t,|\xi|)v(x-\xi)\, d\xi. \end{aligned} \tag{7.40}$$

In this way the relation between the fundamental solution of the spatially one-dimensional Cauchy problem and the subordination kernel $\mathcal{G}(x,t) = \frac{1}{2}\varphi(t,|x|)$ is reestablished. It is remarkable that, due to the specific form of the d'Alembert formula (7.39), the convolution in time in subordination relation is transformed to a convolution relation for the space variable in (7.40).

Example 7.3. Assume $\Omega \subset \mathbb{R}^n$ is an open set and let $X = L^2(\Omega)$. Let A be the Laplace operator with Dirichlet boundary conditions: $A = \Delta$, $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ (see also Section 5.4.3). It is known that the operator A generates a bounded cosine family, see e.g [2], Section 7.2.

If $\{-\lambda_n, \varphi_n\}_{n=1}^\infty$ is the eigensystem of the operator A, then $0 < \lambda_1 \leq \lambda_2 \leq ...$, $\lambda_n \to \infty$ as $n \to \infty$, and $\{\varphi_n\}_{n=1}^\infty$ form an orthonormal basis of $L^2(\Omega)$. The cosine family $S_2(t)$ admits the following eigenfunction decomposition

$$S_2(t)v = \sum_{n=1}^\infty v_n \cos(\sqrt{\lambda_n}t)\varphi_n, \tag{7.41}$$

with $v_n = (v, \varphi_n)$, where $(.,.)$ is the inner product in $L^2(\Omega)$.

Therefore, applying Theorem 7.5 we obtain the solution of problem (7.11) in the form:

$$S(t)v = \sum_{n=1}^\infty v_n u_n(t)\varphi_n, \tag{7.42}$$

where the eigenmodes $u_n(t)$ admit the integral representation

$$u_n(t) = \int_0^\infty \varphi(t,\tau)\cos(\sqrt{\lambda_n}\tau)\,d\tau. \tag{7.43}$$

The eigenmodes $u_n(t)$ can be numerically computed by the use of (7.43) and (7.31).

In particular, in the one-dimensional case, $\Omega = (0,1)$, the eigensystem is $\lambda_n = n^2\pi^2$, $\varphi_n = \sqrt{2}\sin(n\pi x)$, $n = 1, 2, ...$

The following example illustrates the application of the stronger subordination result in Theorem 7.8.

Example 7.4. Consider the neutral-fractional telegraph equation [30]

$${}^C\!D_t^\alpha v(x,t) + b\,{}^C\!D_t^{\alpha/2} v(x,t) = R_x^\alpha v(x,t), \quad v(x,0) = \delta(x),\ v_t(x,0) = 0, \tag{7.44}$$

where $x \in \mathbb{R}, t > 0$, $\alpha \in (1,2)$, $b > 0$, and R_x^α denotes the spatial Riesz fractional pseudo-differential operator (see Section 3.4). It is known that the solution to the Cauchy problem for the neutral fractional wave equation (obtained from (7.44) for $b = 0$)

$${}^C\!D_t^\alpha u(x,t) = R_x^\alpha u(x,t), \quad u(x,0) = \delta(x),\ u_t(x,0) = 0, \quad x \in \mathbb{R},\ t > 0, \tag{7.45}$$

is a spatial probability density function [44, 76]. Since $(s^\alpha + bs^{\alpha/2})^{1/\alpha} \in \mathcal{CBF}$ (see Proposition 7.4), Theorem 7.8 implies that the solutions of equations (7.44) and (7.45) are related by the identity

$$v(x,t) = \int_0^\infty \psi(t,\tau) u(x,t)\,\mathrm{d}\tau, \tag{7.46}$$

where $\psi(t,\tau)$ is a unilateral probability density in τ, defined in Laplace domain by

$$\widehat{\psi}(s,\tau) = \frac{(s^\alpha + bs^{\alpha/2})^{1/\alpha}}{s}\exp\left(-\tau(s^\alpha + bs^{\alpha/2})^{1/\alpha}\right).$$

Representation (7.46) of the solution $v(x,t)$ to the neutral-fractional telegraph equation (7.44) in particular implies that it is as well a spatial PDF.

Chapter 8

Wave propagation in linear viscoelastic media

In this chapter subordination principle is established for equations modelling the propagation of waves in linear viscoelastic media. Various constitutive models are considered, which are fractional-order generalizations of the classical ones. All of them have completely monotone relaxation moduli. In particular, equations modelling unidirectional flows of fractional Jeffreys' fluids are studied in detail. Applications of the subordination relation, as well as its physical interpretation are discussed. The chapter is closed with a short comment on the definition of the class of generalized fractional diffusion-wave equations.

8.1 Evolution equation and subordination

Phenomena intermediate between diffusion and wave propagation, and therefore intrinsically related to some kind of attenuated waves, are known to occur in viscoelastic media that combine the characteristics of elastic solids exhibiting wave propagation and viscous fluids that support diffusion processes. Rheological constitutive equations involving fractional derivatives in time play an important role in linear viscoelasticity and have a long history [55, 75, 95]. It appears that using fractional derivatives in time, the damping behaviour of viscoelastic media can be modelled with much less parameters, compared to the integer-order models [75, 95]. For a review of the main aspects of wave propagation in linear homogeneous viscoelastic media and the simplest and most used fractional constitutive models we refer to [75].

In linear viscoelasticity the rheological properties of a viscoelastic medium are described through a linear constitutive relation between stress σ and strain ε. Following [75], we restrict our considerations to the uniaxial case, in which $\sigma = \sigma(x,t)$ and $\varepsilon = \varepsilon(x,t)$, and consider systems quiescent for all times prior to some starting time, $t = 0$. The constitutive equation in this case admits the form [75]

$$\sigma(x,t) = \int_0^t G(t-\tau)\dot{\varepsilon}(x,\tau)\, d\tau, \quad t > 0, \tag{8.1}$$

where $G(t)$ is the so-called relaxation modulus and the over-dot denotes the first derivative in time.

In a physically meaningful model the relaxation modulus $G(t)$ should be a non-negative and non-increasing function for $t > 0$. This is related to the physical phenomenon of stress relaxation, an inherent property of real materials. If $G(+\infty) = 0$ (full relaxation), the model governs fluid-like behaviour, otherwise the behaviour is solid-like [75].

Based on the stress-strain relation (8.1), the equation of motion, and the kinematic equation, the uniaxial wave equations in different viscoelastic media can be written in a unified manner in terms of the relaxation modulus $G(t)$ [92, 93]

$$u(t) = \int_0^t k(t-\tau)Au(\tau)\, d\tau + f(t), \qquad t > 0, \tag{8.2}$$

where the kernel $k(t)$ is defined by the identity

$$k(t) = \int_0^t G(\tau)\, d\tau. \tag{8.3}$$

Then the characteristic function $g(s) = (\widehat{k}(s))^{-1}$ is expressed in terms of the Laplace transform of the relaxation modulus $\widehat{G}(s)$ as follows

$$g(s) = \frac{s}{\widehat{G}(s)}, \qquad s > 0. \tag{8.4}$$

Let us note that the function u in (8.2) is either particle displacement or velocity, depending whether the model exhibits solid-like or fluid-like behaviour.

All constitutive equations considered in this chapter have completely monotone relaxation moduli, more exactly they satisfy $G(t) \in \mathcal{CMF}_0$, where the class $\mathcal{CMF}_0$ is defined in (2.4). This property follows from the thermodynamic restrictions on their parameters. For the simplest models this is related to the complete monotonicity of the Mittag-Leffler function. Let us also note that $G(t) \in \mathcal{CMF}_0$ implies that the kernel $k(t)$ of the corresponding Volterra equation, defined in (8.3), is a Bernstein function. Next we establish a property of the characteristic function $g(s)$, necessary to apply the subordination Theorem 2.4.

Proposition 8.1. *If $G(t) \in \mathcal{CMF}_0$ then $g(s)^{1/2} \in \mathcal{CBF}$.*

Proof. Under the assumption of the proposition $\widehat{G}(s) \in \mathcal{SF}$ which, by property **(P8)** in Proposition 2.1, is equivalent to $1/\widehat{G}(s) \in \mathcal{CBF}$. Then the function $g(s) = s/\widehat{G}(s)$ is a product of two complete Bernstein functions (s and $1/\widehat{G}(s)$) and property (2.2) implies $g(s)^{1/2} \in \mathcal{CBF}$. □

Applying Theorem 2.4 we formulate the following result.

Theorem 8.1. *Let A be a generator of a strongly continuous bounded cosine family $S_2(t)$ in a Banach space X. Assume $G(t) \in \mathcal{CMF}_0$. Then the abstract Volterra equation (8.2) with kernel $k(t)$ defined in (8.3) is well posed with bounded solution operator $S(t)$, which satisfies the subordination relation*

$$S(t) = \int_0^\infty \varphi(t,\tau) S_2(\tau)\, d\tau, \quad t > 0. \tag{8.5}$$

The subordination kernel $\varphi(t,\tau)$ is a unilateral probability density function, i.e. it obeys (2.26), and its Laplace transform is given by

$$\widehat{\varphi}(s,\tau) = \frac{g(s)^{1/2}}{s} \exp\left(-\tau g(s)^{1/2}\right), \tag{8.6}$$

where $g(s)$ is defined in (8.4).

As in the previous chapter, the integral in the subordination relation (8.5) is finite in the case of finite propagation speed of a disturbance. From general theory, see e.g. [75, Chapter 4], [93, Chapter 5], the velocity of propagation of a disturbance c is

$$c = \lim_{s\to\infty} \frac{s}{g(s)^{1/2}}. \tag{8.7}$$

In the case of finite propagation speed, a jump discontinuity at the planar surface $x = ct$ exists if and only if

$$\eta = \lim_{s\to\infty} \left(g(s)^{1/2} - s/c\right) < \infty. \tag{8.8}$$

The relations (8.4), (8.7), (8.8), and the initial and final value theorems for Laplace transform yield

$$c = G(0)^{1/2}, \quad \eta = -\frac{G'(0)}{2G(0)^{3/2}}, \tag{8.9}$$

where $G(t)$ is the relaxation modulus.

In the case of finite propagation speed ($c < \infty$) and absence of wave front ($\eta = \infty$) the subordination to cosine families (8.5) is given by a finite integral relation:

$$S(t) = \int_0^{ct} \varphi(t,\tau) S_2(\tau)\, d\tau, \quad t > 0, \tag{8.10}$$

since in this case the subordination kernel $\varphi(t,\tau)$ vanishes for $\tau > ct$ (the proof is the same as that of Corollary 7.2.

8.2 Analysis of fractional viscoelastic models

Application of Laplace transform with respect to time variable in (8.1) leads to a stress-strain relation in Laplace domain [75]

$$\widehat{\sigma}(\cdot, s) = s\widehat{G}(s)\widehat{\varepsilon}(\cdot, s). \tag{8.11}$$

The representations of the relaxation moduli for the specific constitutive relations below are derived by applying Laplace transform to the constitutive equation and comparing the result to (8.11). Some properties of the relaxation moduli can be directly derived on the basis of their representation in Laplace domain.

To characterize a viscoelastic medium whose mechanical properties are intermediate between those of pure elastic solid (Hooke model: $\sigma = b\varepsilon$) and of pure viscous fluid (Newton model: $\sigma = b\dot{\varepsilon}$), the fractional Scott-Blair stress-strain law was introduced [75]

$$\sigma(x,t) = bD_t^\alpha \varepsilon(x,t), \quad 0 < \alpha < 1. \tag{8.12}$$

Here b is a positive constant and D_t^α denotes fractional time derivative in the Riemann-Liouville sense. This fractional-order model has led to various fractional-order generalizations of the classical integer-order constitutive models.

In this section we give a short analysis of some basic constitutive models with time fractional derivatives, derive the corresponding relaxation moduli, and study their complete monotonicity.

8.2.1 Fractional Kelvin-Voigt model

As a first example we consider the fractional Kelvin-Voigt model [75]

$$\sigma(x,t) = (1 + bD_t^\alpha)\varepsilon(x,t), \quad 0 < \alpha < 1, \quad b > 0. \tag{8.13}$$

Applying Laplace transform and using (1.19) we obtain $\widehat{\sigma} = (1 + bs^\alpha)\widehat{\varepsilon}$, which, compared to (8.11) yields $\widehat{G}(s) = s^{-1} + bs^{\alpha-1}$. By the use of (1.4) we obtain the corresponding relaxation modulus

$$G(t) = 1 + b\omega_{1-\alpha}(t).$$

It is completely monotone under the assumptions on the parameters $0 < \alpha < 1$, $b > 0$. Moreover, $G(+\infty) = 1$ and $G(0) = +\infty$ indicate that the fractional Kelvin-Voigt model governs solid-like behaviour with infinite propagation speed of a disturbance.

A distributed-order generalization of constitutive laws (8.12) and (8.13) is proposed in [32] in the form

$$\sigma(x,t) = \int_0^1 p_\varepsilon(\beta) D_t^\beta \varepsilon(x,t)\, d\beta, \tag{8.14}$$

where $p_\varepsilon(\cdot)$ is a weight function. The corresponding relaxation modulus

$$G(t) = \int_0^1 p_\varepsilon(\beta)\omega_{1-\alpha}(t)\, d\beta \tag{8.15}$$

is again a completely monotone function, which follows by the use of property **(P1)** in Proposition 2.1. The limiting behaviour of the functions of the form (8.15) is studied in Theorem 5.1. In the case of continuous weight function $G(+\infty) = 0$, i.e. constitutive equation (8.14) models fluid-like behaviour. Since $G(0) = +\infty$, the propagation speed of a disturbance is infinite.

8.2.2 Fractional Maxwell model

Consider the fractional Maxwell constitutive equation [55, Chapter 7]

$$(1 + aD_t^\alpha)\sigma(x,t) = bD_t^\beta \varepsilon(x,t), \quad 0 < \alpha \le \beta \le 1, \quad a, b > 0. \tag{8.16}$$

Applying Laplace transform to (8.16) we obtain by the use of (1.19) and (8.11)

$$\widehat{G}(s) = \frac{bs^{\beta-1}}{1 + as^\alpha}.$$

Therefore, applying (1.27) we obtain

$$G(t) = \frac{b}{a} t^{\alpha-\beta} E_{\alpha,\alpha-\beta+1}\left(-t^{\alpha}/a\right), \tag{8.17}$$

In the next proposition we prove that under the constraints on the parameters in (8.16) the relaxation modulus $G(t)$ is completely monotone. In fact, it appears that the assumption on the fractional parameters $\alpha \leq \beta$ is also necessary for thermodynamic compatibility of the fractional Maxwell model.

Proposition 8.2. *Assume* $0 < \alpha, \beta \leq 1$, $a, b > 0$, $t > 0$. *The following assertions are equivalent:*
(a) $0 < \alpha \leq \beta \leq 1$;
(b) $G(t)$ *is monotonically non-increasing;*
(c) $G(t)$ *is a completely monotone function.*

Proof. Using representation (8.17) for the function $G(t)$, we prove that condition (a) is equivalent to any of the conditions (b) and (c). First we show that if $1 \geq \alpha > \beta > 0$ then (b) and (c) are not satisfied. The definition of Mittag-Leffler function (1.21) implies $G(t) \sim Ct^{\alpha-\beta}$ for $t \to 0$. Therefore, if $\alpha > \beta$, $G(t)$ is increasing function for t near 0, i.e., (b) and (c) are violated. Therefore, any of the conditions (b) and (c) implies (a). It remains to prove that (a) implies (b) and (c). Indeed, if $0 < \alpha \leq \beta \leq 1$ then $t^{\alpha-\beta} \in \mathcal{CMF}$ and $E_{\alpha,\alpha-\beta+1}\left(-t^{\alpha}/a\right) \in \mathcal{CMF}$ as a composition of the completely monotone Mittag-Leffler function of negative argument (see (1.26)) and the Bernstein function t^{α}. Therefore, $G(t) \in \mathcal{CMF}$ as a product of two completely monotone functions. The proof is completed. □

Representation (8.17) together with the asymptotic expansion (1.22) yields $G(+\infty) = 0$. Therefore, the fractional Maxwell constitutive equation models fluid-like behavior. More precisely, (1.33) implies for $t \to +\infty$ that $G(t) \sim Ct^{-\alpha}$ if $\beta < 1$ and $G(t) \sim Ct^{-\alpha-1}$ if $\beta = 1$. This means that for $\beta = 1$ the relaxation function $G(t)$ is integrable at infinity and the integral over $(0, \infty)$ is finite.

The asymptotic expansion for $t \to 0^{+}$ is $G(t) \sim Ct^{\alpha-\beta}$. Therefore, if $\alpha < \beta$ then $G(0^{+}) = +\infty$, while if $\alpha = \beta$ different behavior is observed: $G(0^{+}) = b/a < \infty$. This means that the fractional Maxwell viscoelastic model supports finite propagation speed if and only if $\alpha = \beta$.

8.2.3 Distributed order fractional Zener model

The fractional Zener constitutive equation [75]

$$(1 + aD_t^{\alpha})\sigma(x,t) = (1 + bD_t^{\alpha})\varepsilon(x,t), \quad 0 < \alpha \leq 1, \quad 0 < a < b, \tag{8.18}$$

is extensively studied as a model of solid-like viscoelastic behaviour [6, 7, 55]. The constraints on the parameters imply the complete monotonicity of the corresponding relaxation modulus, which admits the representation in terms of Mittag-Leffler function [75],

$$G(t) = 1 + (b/a - 1)\, E_{\alpha}\left(-t^{\alpha}/a\right), \ 0 < \alpha \leq 1, \quad 0 < a < b. \tag{8.19}$$

Consider the multi-term stress-strain relation [5]

$$\sum_{n=0}^{N} a_n D_t^{\alpha_n} \sigma(x,t) = \sum_{n=0}^{N} b_n D_t^{\alpha_n} \varepsilon(x,t), \tag{8.20}$$

where $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_N < 1,\ a_n, b_n > 0, \quad n = 0, 1, ..., N$. This multi-term viscoelastic model a generalization of (8.18). It is consistent with the second law of thermodynamics if the following restrictions on the parameters are satisfied [5]

$$\frac{a_0}{b_0} \geq \frac{a_1}{b_1} \geq \cdots \geq \frac{a_N}{b_N}. \tag{8.21}$$

Note that if $a_n = a^{\alpha_n},\ b_n = b^{\alpha_n}$, and $0 < a < b$, then conditions (8.21) are automatically satisfied.

A more general distributed-order stress-strain relation is proposed in [4]

$$\int_0^1 p_{\sigma}(\alpha) D_t^{\alpha} \sigma(x,t)\, d\alpha = \int_0^1 p_{\varepsilon}(\alpha) D_t^{\alpha} \varepsilon(x,t)\, d\alpha, \tag{8.22}$$

where $p_{\sigma}(\alpha)$ and $p_{\varepsilon}(\alpha)$ are non-negative (generalized) weight functions. The distributed-order constitutive equation (8.22) and the related mechanical models are studied in [6], Chapter 3.

The power type distributed-order model (8.22) is obtained when the weight functions are power functions

$$p_{\sigma}(\alpha) = a^{\alpha}, \quad p_{\varepsilon}(\alpha) = b^{\alpha}, \qquad a, b > 0. \tag{8.23}$$

It is found in [5] that rheological model (8.22) with weight functions (8.23) is thermodynamically compatible under the constraint

$$a < b. \tag{8.24}$$

Next we study the distributed-order fractional Zener model in the cases of discrete distribution (8.20) and continuous distribution with power-law weight functions (8.22)-(8.23). We prove the complete monotonicity of the relaxation moduli provided the thermodynamic restrictions on the parameters (8.21) and (8.24), respectively, are satisfied. We discuss the asymptotic behaviour of the relaxation moduli. In addition, we show that for model (8.20) with $N \leq 2$ and for the power type model (8.22)-(8.23) conditions (8.21), resp. (8.24), are also necessary for physical acceptability. In the course of the proof of complete monotonicity we derive integral representations for the relaxation moduli.

Relaxation modulus in the case of discrete distribution

Applying Laplace transform to constitutive law (8.20) we obtain by the use of (1.19):

$$\sum_{n=0}^{N} a_n s^{\alpha_n} \widehat{\sigma}(x,s) = \sum_{n=0}^{N} b_n s^{\alpha_n} \widehat{\varepsilon}(x,s).$$

Comparing this result to (8.11), the following representation in Laplace domain for the relaxation modulus is deduced

$$\widehat{G}(s) = \frac{\sum_{n=0}^{N} b_n s^{\alpha_n}}{s\left(\sum_{n=0}^{N} a_n s^{\alpha_n}\right)}. \tag{8.25}$$

Let us first discuss the small- and large-time behaviour of this function. Initial and final value theorems for Laplace transform yield

$$\begin{aligned} \lim_{t\to 0} G(t) &= \lim_{s\to+\infty} s\widehat{G}(s) = \lim_{s\to+\infty} \frac{\sum_{n=0}^{N} b_n s^{\alpha_n}}{\sum_{n=0}^{N} a_n s^{\alpha_n}} = \frac{b_N}{a_N}, \\ \lim_{t\to+\infty} G(t) &= \lim_{s\to+0} s\widehat{G}(s) = \frac{b_0}{a_0}. \end{aligned} \tag{8.26}$$

As expected, the initial value is greater than the final value, due to (8.21). Since $G(t)$ has a finite value at $t = 0$, waves in a viscoelastic medium with constitutive model (8.20) propagate with finite speed $c = \sqrt{G(0)} = \sqrt{b_N/a_N}$. The second limit in (8.26) shows that there is no full relaxation ($G(+\infty) > 0$). This means that the constitutive equation indeed models solid-like behaviour.

More precise asymptotic expressions for the relaxation modulus follow by taking into account the dominant terms in (8.25). For small s, neglecting all

s^α with $\alpha > \alpha_1$ in (8.25), we deduce using the expansion $(1+z)^{-1} \sim 1 - z$ for $z \to 0$,

$$\widehat{G}(s) \sim \frac{b_0 s^{\alpha_0} + b_1 s^{\alpha_1}}{s(a_0 s^{\alpha_0} + a_1 s^{\alpha_1})} \sim \frac{b_0}{a_0 s} + \frac{b_0}{a_0}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right) s^{\alpha_1 - \alpha_0 - 1}, \quad s \to 0.$$

This implies by the use of (1.4) the following large-time asymptotic expression for the relaxation modulus

$$G(t) \sim \frac{b_0}{a_0} + \frac{b_0}{a_0}\left(\frac{b_1}{b_0} - \frac{a_1}{a_0}\right) \frac{t^{\alpha_0 - \alpha_1}}{\Gamma(1 + \alpha_0 - \alpha_1)}, \quad t \to +\infty. \tag{8.27}$$

Note that the second term in the expansion (8.27) is positive due to constraints (8.21). On the other hand, for large s, neglecting s^α for all $\alpha < \alpha_{N-1}$ in (8.25), we obtain in a similar way

$$\begin{aligned} \widehat{G}(s) &\sim \frac{b_{N-1} s^{\alpha_{N-1}} + b_N s^{\alpha_N}}{s(a_{N-1} s^{\alpha_{N-1}} + a_N s^{\alpha_N})} \\ &\sim \frac{b_N}{a_N s} + \frac{b_N}{a_N}\left(\frac{b_{N-1}}{b_N} - \frac{a_{N-1}}{a_N}\right) s^{\alpha_{N-1} - \alpha_N - 1}, \quad s \to +\infty, \end{aligned}$$

which implies by the use of (1.4) the following small-time asymptotic expansion for the relaxation modulus

$$G(t) \sim \frac{b_N}{a_N} + \frac{b_N}{a_N}\left(\frac{b_{N-1}}{b_N} - \frac{a_{N-1}}{a_N}\right) \frac{t^{\alpha_N - \alpha_{N-1}}}{\Gamma(1 + \alpha_N - \alpha_{N-1})}, \quad t \to 0. \tag{8.28}$$

Let us note that the second term in the expansion (8.28) has a negative sign due to the thermodynamic restrictions (8.21).

The behaviour of $G'(t)$ for small times indicates whether there is a jump discontinuity at the wave front: such a discontinuity appears when $G'(0)$ is finite, see (8.9). In our case

$$\begin{aligned} \lim_{t \to +0} G'(t) &= \lim_{s \to +\infty} s\mathcal{L}\{G'(t)\}(s) = \lim_{s \to +\infty} s\left(s\widehat{G}(s) - G(0)\right) \\ &= \lim_{s \to +\infty} s\left(\frac{\sum_{n=0}^N b_n s^{\alpha_n}}{\sum_{n=0}^N a_n s^{\alpha_n}} - \frac{b_N}{a_N}\right) = -\infty, \end{aligned}$$

because the dominant term in the expression (up to positive multiplicative constant) is $(a_N b_{N-1} - b_N a_{N-1}) s^{1 + \alpha_{N-1} - \alpha_N}$ and taking into account (8.21). Therefore, waves in this medium propagate with finite wave speed, and the wave front is smooth.

Consider the particular case of (8.20) with $N = 2$ and $\alpha_0 = 0$, $a_0 = b_0 = 1$:

$$(1 + a_1 D_t^{\alpha_1} + a_2 D_t^{\alpha_2})\, \sigma(x,t) = (1 + b_1 D_t^{\alpha_1} + b_2 D_t^{\alpha_2})\, \varepsilon(x,t). \tag{8.29}$$

We prove next that in this simpler case of constitutive equation thermodynamic constraints (8.21) are also necessary for physical admissibility of the model. Assume that (8.29) is thermodynamically compatible, that is,

$$G(t) \geq 0 \;\text{ and } G'(t) \leq 0 \quad \text{for all } t > 0. \tag{8.30}$$

We will prove that

$$1 \geq a_1/b_1 \geq a_2/b_2. \tag{8.31}$$

First, condition $G'(t) \leq 0$ for all $t > 0$, implies $\mathcal{L}\{G'(t)\}(s) \leq 0$ for all $s > 0$. Since

$$\begin{aligned} \mathcal{L}\{G'(t)\}(s) &= s\widehat{G}(s) - G(0) = \frac{1 + b_1 s^{\alpha_1} + b_2 s^{\alpha_2}}{1 + a_1 s^{\alpha_1} + a_2 s^{\alpha_2}} - \frac{b_2}{a_2} \\ &= \frac{(a_2 - b_2) + (a_2 b_1 - b_2 a_1) s^{\alpha_1}}{a_2(1 + a_1 s^{\alpha_1} + a_2 s^{\alpha_2})} \end{aligned}$$

and it should be non-positive for small as well for large s, then $a_2 - b_2 \leq 0$ and $a_2 b_1 - b_2 a_1 \leq 0$, i.e. $a_1/b_1 \geq a_2/b_2$. To prove that $b_1 \geq a_1$ we use the asymptotic expansion (8.27) for large t, which in this particular case implies

$$G(t) \sim 1 + (b_1 - a_1) \frac{t^{-\alpha_1}}{\Gamma(1 - \alpha_1)}, \quad t \to +\infty$$

and take into account that $G(t) \geq 1$ for any $t > 0$ (since $G(+\infty) = 1$ and $G(t)$ is non-increasing). In this way we deduced the thermodynamic constraints (8.31) from the conditions (8.30).

It remains to prove the main result in this subsection: the complete monotonicity of the relaxation modulus.

Theorem 8.2. *Assume that constraints (8.21) are satisfied. Then the relaxation modulus $G(t)$ of the constitutive equation (8.20) is completely monotone.*

Proof. We will prove that the relaxation modulus $G(t)$ is a completely monotone function by establishing the following integral representation:

$$G(t) = \frac{b_0}{a_0} + \int_0^\infty e^{-rt} K(r)\, dr, \tag{8.32}$$

where the function $K(r) \geq 0$.

To establish (8.32) we apply the inverse Laplace integral in (8.25) and obtain

$$G(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{st} \frac{\sum_{n=0}^{N} b_n s^{\alpha_n}}{s \left(\sum_{n=0}^{N} a_n s^{\alpha_n} \right)} \, ds, \tag{8.33}$$

where $\gamma > 0$. For the multivalued complex function s^{α} we take the principal branch. The function under the integral sign in (8.33) has no poles in the complex plane cut along the negative real axis, since $\Im \left\{ \sum_{n=0}^{N} a_n s^{\alpha_n} \right\} \neq 0$ for $s \in \mathbb{C} \backslash (-\infty, 0]$. This is due to the fact that for $a_n > 0$ and $\alpha_n \in (0,1)$ the imaginary part of any term in this sum ($a_n \sin(\alpha_n \arg s)$) has the same sign. Let us bend the contour in (8.33) into the Hankel path $Ha(\rho)$, which starts from $-\infty$ along the lower side of the negative real axis, encircles the disc $|s| = \rho$ counterclockwise and ends at $-\infty$ along the upper side of the negative real axis. The integral on the circular contour $|s| = \rho$ equals b_0/a_0 when $\rho \to 0$. This can be obtained by direct check, taking into account that

$$\lim_{s \to 0} s \frac{\sum_{n=0}^{N} b_n s^{\alpha_n}}{s \left(\sum_{n=0}^{N} a_n s^{\alpha_n} \right)} = \frac{b_0}{a_0}.$$

The sum of the integrals along the lower and the upper sides of the negative real axis yields the integral in (8.32) where

$$K(r) = -\frac{1}{\pi} \Im \left\{ \left. \frac{\sum_{n=0}^{N} b_n s^{\alpha_n}}{s \left(\sum_{n=0}^{N} a_n s^{\alpha_n} \right)} \right|_{s = r e^{\mathrm{i}\pi}} \right\},$$

which implies

$$K(r) = \frac{1}{\pi r} \frac{\sum_{0 \leq i < j \leq N} (a_i b_j - a_j b_i) r^{\alpha_i + \alpha_j} \sin(\alpha_j - \alpha_i)\pi}{\left(\sum_{n=0}^{N} a_n r^{\alpha_n} \cos \alpha_n \pi \right)^2 + \left(\sum_{n=0}^{N} a_n r^{\alpha_n} \sin \alpha_n \pi \right)^2}. \tag{8.34}$$

Thermodynamic constraints (8.21) imply that all terms in the sum in the numerator in (8.34) are non-negative. Therefore, $K(r) \geq 0$, and representation (8.32) implies that the function $G(t)$ is completely monotone (by applying the Bernstein theorem or by direct check). □

An alternative way to prove the complete monotonicity property of the relaxation modulus is directly from the properties of $\widehat{G}(s)$ by applying property **(P6)** in Proposition 2.1. For this we will need to prove that $\widehat{G}(s) \in \mathcal{SF}$ for $s > 0$. An advantage of Theorem 8.2 is that it also provides an integral representation for $G(t)$.

On the other hand, Theorem 8.2 implies by the use of property **(P6)** in Proposition 2.1 that under the conditions (8.21) the function $\widehat{G}(s)$ defined in (8.25) is a Stieltjes function.

Relaxation modulus of the power type distributed order model

Consider the distributed order constitutive equation (8.22) with weight functions defined in (8.23). Applying Laplace transform to the constitutive equation and comparing the result to (8.11) results in the following representation in Laplace domain

$$\widehat{G}(s) = \frac{\int_0^1 (bs)^\alpha \, d\alpha}{s \int_0^1 (as)^\alpha \, d\alpha} = \frac{(bs-1)\ln as}{s(as-1)\ln bs}, \tag{8.35}$$

where the integration is performed taking into account that $z^\alpha = e^{\alpha \ln z}$. The initial and final value theorems for Laplace transform pairs imply

$$\begin{aligned} \lim_{t\to+0} G(t) &= \lim_{s\to+\infty} s\widehat{G}(s) = \lim_{s\to+\infty} \frac{(bs-1)\ln as}{(as-1)\ln bs} = \frac{b}{a}, \\ \lim_{t\to+\infty} G(t) &= \lim_{s\to+0} s\widehat{G}(s) = 1. \end{aligned}$$

First, from the two limits we see that constraint (8.24), i.e. $b > a$, is not only sufficient, but also a necessary condition for physical acceptability of this model, taking into account that in a physically meaningful model $G(t)$ is monotonically decreasing function, i.e. $G(0) > G(+\infty)$. Moreover, the two limits show again that this is a model for solid-like behaviour and the propagation speed of a disturbance is finite $c = \sqrt{b/a}$. In addition,

$$\begin{aligned} \lim_{t\to+0} G'(t) &= \lim_{s\to+\infty} s\mathcal{L}\{G'(t)\}(s) = \lim_{s\to+\infty} s\left(s\widehat{G}(s) - G(0)\right) \\ &= \lim_{s\to+\infty} \frac{bs^2(\ln a - \ln b)}{(as-1)\ln bs} = -\infty, \end{aligned}$$

since $\ln a < \ln b$. This implies again that also in this medium there is no discontinuity at the wave front, see (8.9).

More precise asymptotic expansion of the relaxation modulus for $t \to +\infty$ can be obtained by the use of the Karamata-Feller Tauberian theorem, see Theorem 1.2. Taking $f(s) = (\ln a + \ln s)/(s(\ln b + \ln s))$ and the slowly varying function $L(x) = (\ln a - \ln x)/(\ln b - \ln x)$ we deduce

$$G(t) \sim 1 - \frac{\ln b - \ln a}{\ln b - \ln t} - \frac{\ln b - \ln a}{(\ln b - \ln t)^2} \sim 1 + \frac{\ln(b/a)}{\ln t} \quad \text{as} \quad t \to \infty, \tag{8.36}$$

i.e. the relaxation modulus exhibits a very slow logarithmic decay to its final value.

Theorem 8.3. *If $a < b$ then the relaxation modulus $G(t)$ defined in (8.35) is completely monotone.*

Proof. To prove that the relaxation modulus $G(t)$ is a completely monotone function we apply the same technique as in the proof of Theorem 8.2. We will find representation of the form

$$G(t) = 1 + \int_0^\infty e^{-rt} K(r)\, dr \tag{8.37}$$

with appropriate function $K(r)$, such that $K(r) \geq 0$. Taking the inverse Laplace integral in (8.35) we obtain

$$G(t) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{st} \frac{(bs-1)\ln(as)}{s(as-1)\ln(bs)}\, ds, \tag{8.38}$$

where $\gamma > 0$. For the multivalued complex logarithmic function we take the principal branch. The function under the integral sign in (8.33) has no poles in the complex plane cut along the negative real axis. This is implied by the fact that the imaginary part of the denominator, $\Im\left\{\int_0^1 (as)^\alpha\, d\alpha\right\} \neq 0$ for $s \in \mathbb{C}\backslash(-\infty, 0]$ (since $a > 0, \alpha \in (0,1)$). Let us also note that the integrand in (8.35) has finite limits when $s \to 1/a$ and $s \to 1/b$. Bending the contour Br into the Hankel path $Ha(\rho)$, we obtain for the integral on the circle $|s| = \rho$ when taking $\rho \to 0$

$$\lim_{s \to 0} s \frac{(bs-1)\ln(as)}{s(as-1)\ln(bs)} = 1.$$

The sum of the two complex integrals along the lower and the upper sides of the negative real axis gives the real integral in (8.37) with

$$K(r) = -\frac{1}{\pi} \Im\left\{ \left. \frac{(bs-1)\ln(as)}{s(as-1)\ln(bs)} \right|_{s = re^{\mathrm{i}\pi}} \right\} = \frac{(br+1)(\ln b - \ln a)}{r(ar+1)\left(\ln^2(br) + \pi^2\right)}. \tag{8.39}$$

Representation (8.39) implies $K(r) \geq 0$ when the thermodynamic constraint $a < b$ is satisfied. Therefore, under this condition, $G(t)$ is a completely monotone function. □

Compared to other methods of proof of complete monotonicity of the relaxation modulus, an advantage of Theorem 8.3 is that at the same time it provides an integral representation for $G(t)$.

Theorem 8.3 and property **(P6)** in Proposition 2.1 also imply that if $a < b$ then the function $\widehat{G}(s)$ defined in (8.35) is a Stieltjes function.

8.2.4 Binomial Mittag-Leffler type relaxation

Next we propose a constitutive model with relaxation modulus in the form of a binomial Mittag-Leffler type function, which appears to generalize known relaxation laws.

Let $0 < \alpha_0 < \alpha \leq 1$, $0 < \delta \leq 1$, and $\lambda, \lambda_0 > 0$, and $0 < C < \lambda^\delta$. Consider the relaxation modulus

$$G(t) = 1 - Ct^{\alpha\delta} E^{\delta}_{(\alpha,\alpha_0),\alpha\delta+1} \left(-\lambda t^{\alpha}, -\lambda_0 t^{\alpha_0}\right), \tag{8.40}$$

where $E^{\delta}_{(\alpha,\alpha_0),\alpha\delta+1}$ is the binomial Prabhakar function, see (6.1). The relaxation function (8.40) is completely monotone, which follows from Theorem 6.4.

The asymptotic expansions of $G(t)$ for $t \to 0$ and $t \to +\infty$ are obtained from (8.40) by applying (6.5) and (6.16), respectively:

$$\begin{aligned} G(t) &\sim 1 - C\frac{t^{\alpha\delta}}{\Gamma(\alpha\delta+1)} + C\delta\lambda_0 \frac{t^{\alpha\delta+\alpha_0}}{\Gamma(\alpha\delta+\alpha_0+1)}, \quad t \to 0, \\ G(t) &\sim 1 - \frac{C}{\lambda^\delta} + C\delta\lambda^{-\delta-1}\lambda_0 \frac{t^{-\alpha+\alpha_0-1}}{\Gamma(-\alpha+\alpha_0)}, \quad t \to +\infty. \end{aligned}$$

In particular, $G(0) = 1$, $G(+\infty) = 1 - C/\lambda^\delta$, i.e. the function $G(t)$ is monotonically decreasing in $(0, +\infty)$ from $G(0) = 1$ to $G(+\infty) \in (0, 1)$. Therefore, this is a model for solid-like behaviour.

Applying (6.6) we obtain the Laplace transform of the relaxation modulus

$$\widehat{G}(s) = \frac{1}{s}\left(1 - \frac{d}{(as^{\alpha} + a_1 s^{\alpha-\alpha_0} + 1)^{\delta}}\right), \tag{8.41}$$

where $a = \lambda^{-1}$, $a_1 = \lambda_0\lambda^{-1}$, and $d = C/\lambda^\delta < 1$. From (8.41) we recognize some known viscoelastic models as particular cases. For instance, if $\delta = 1$ then (8.41) is a particular case of the two-term fractional Zener model (8.29). For $\lambda_0 = 0$ ($a_1 = 0$) we recognize from (8.41) the Havriliak-Negami relaxation model (see e.g. [53]). For $\delta = 1$ and $\lambda_0 = 0$ the fractional Zener relaxation modulus (8.19) (up to a multiplicative constant) is recovered from (8.40):

$$G(t) = 1 - Ct^\alpha E_{\alpha,\alpha+1}(-\lambda t^\alpha) = \left(1 - \frac{C}{\lambda}\right) + \frac{C}{\lambda} E_\alpha(-\lambda t^\alpha).$$

For the viscoelastic model with relaxation modulus (8.40) the propagation speed of a disturbance is finite, $c = 1$. The model with $\delta = 1$, $\alpha = 1$ and $\alpha_0 < 1$ provides a (nonclassical) example, for which there is a jump discontinuity at the wave front, since in this case $\eta = C/2 < \infty$, where η is defined in (8.8). Except this special case, in all other cases the wave front is smooth.

8.2.5 Fractional Jeffreys' model

The fractional Jeffreys' constitutive equation

$$(1 + aD_t^\alpha)\,\sigma(x,t) = (1 + bD_t^\beta)\dot{\varepsilon}(x,t), \tag{8.42}$$

where $a, b > 0$ and $0 < \alpha, \beta \leq 1$, is introduced in the experimental work [102] as a model for viscoelastic fluid-like behaviour.

In the next theorem we formulate conditions, which are necessary and sufficient for thermodynamic compatibility of model (8.42). In particular, we derive the following thermodynamic restrictions on the parameters

$$\alpha = \beta \text{ and } a \geq b. \tag{8.43}$$

from the monotonicity properties of the relaxation function.

Theorem 8.4. *Assume $\alpha, \beta \in (0,1)$, $a, b > 0$, $t > 0$. The following assertions are equivalent:*
(a) *$\alpha = \beta$ and $a \geq b$;*
(b) *$G(t)$ is non-negative;*
(c) *$G(t)$ is non-increasing;*
(d) *$G(t) \in \mathcal{CMF}_0$.*
If any of the above conditions is satisfied then $G(t)$ admits the representation

$$G(t) = \mu\delta(t) + (1-\mu)\frac{t^{\alpha-1}}{a} E_{\alpha,\alpha}\left(-t^\alpha/a\right), \ \mu = b/a, \tag{8.44}$$

where $\delta(t)$ is the Dirac delta function.

Proof. Applying Laplace transform to (8.42) we obtain by the use of (1.19) and (8.11) the following identity for the relaxation function of the fractional Jeffreys' model

$$\widehat{G}(s) = \frac{1 + bs^{\beta}}{1 + as^{\alpha}}. \tag{8.45}$$

Taking inverse Laplace transform in (8.45) the following explicit expression for $G(t)$ is derived by the use of (1.27):

$$G(t) = \frac{1}{a} t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{1}{a}t^{\alpha}\right) + \frac{b}{a} t^{\alpha-\beta-1} E_{\alpha,\alpha-\beta}\left(-\frac{1}{a}t^{\alpha}\right). \tag{8.46}$$

We will prove that (a) is equivalent to any of the conditions (b)-(d).

First we prove that any of conditions (b) and (c) implies $\alpha = \beta$. Indeed, if we assume that $\alpha < \beta$, then taking the first terms of the expansions of the Mittag-Leffler functions in (8.46) we obtain

$$G(t) \sim \frac{b}{a} \frac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}, \ t \to 0. \tag{8.47}$$

Since in this case $-1 < \alpha - \beta < 0$ and thus $\Gamma(\alpha - \beta) < 0$ it follows from (8.47) that any of conditions (b) and (c) is violated. On the other hand, if we suppose $\alpha > \beta$ then the asymptotic expansion of the Mittag-Leffler function (1.22) implies

$$G(t) \sim b \frac{t^{-\beta-1}}{\Gamma(-\beta)}, \ t \to +\infty,$$

which indicates violation of conditions (c) and (d) for large t. Therefore $\alpha = \beta$.

To prove that $a \geq b$ we deduce representation (8.44) first. To this end we take $\alpha = \beta$ in (8.45) and obtain

$$\widehat{G}(s) = \frac{1 + bs^{\alpha}}{1 + as^{\alpha}} = \frac{b}{a} + \frac{1}{a}\left(1 - \frac{b}{a}\right) \frac{1}{s^{\alpha} + 1/a}. \tag{8.48}$$

Applying inverse Laplace transform to (8.48) and using the identities (1.27) we deduce representation (8.44). Further, since $t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha}/a)$ is completely monotone for $t > 0$ (see (1.26)), representation (8.44) shows that any of (b) and (c) implies $a \geq b$. In this way we finished the nontrivial part of the proof: that any of the conditions (b) and (c) implies (a). In addition, representation (8.44) shows that (a) implies (d). To finish the proof, we note that (d) implies both (b) and (c) by definition. □

Representation (8.44) and the asymptotic expansion (1.33) give

$$G(t) = O(t^{-\alpha-1}), \quad t \to +\infty,$$

i.e. $\lim_{t\to+\infty} G(t) = 0$ and $G(t)$ is integrable on $(0, \infty)$. In fact

$$\int_0^\infty G(t)\, dt = 1,$$

which follows from (8.44) by applying the identity (1.25).

This behavior of the relaxation modulus confirms that the fractional Jeffreys' constitutive equation indeed models fluid-like behavior.

Let us note that, in general, thermodynamic compatibility of a constitutive equation does not necessarily imply complete monotonicity of the relaxation modulus.

8.3 Unidirectional flows of fractional Jeffreys' fluids

This section is devoted to a detailed study of evolution equations with the fractional Jeffreys' constitutive model.

8.3.1 Stokes' first problem

Consider a plane Couette flow of an incompressible viscoelastic fluid with the thermodynamically compatible fractional Jeffreys' constitutive equation

$$(1 + aD_t^\alpha)\,\sigma(x,t) = (1 + bD_t^\alpha)\dot{\varepsilon}(x,t), \quad 0 < \alpha < 1, \quad a \geq b > 0. \tag{8.49}$$

Assume the fluid fills a half-space $x > 0$ and is set into motion by a sudden movement of the bounding plane $x = 0$. Denote by $u(x,t)$ the induced velocity field. Noting that $\dot{\varepsilon} = \partial u/\partial x$ and eliminating σ between Eq. (8.49) and Cauchy's first law $\partial u/\partial t = \partial\sigma/\partial x$ we obtain the following problem

$$(1 + aD_t^\alpha)\,\frac{\partial}{\partial t}u(x,t) = (1 + bD_t^\alpha)\frac{\partial^2}{\partial x^2}u(x,t), \quad x, t > 0, \tag{8.50}$$

$$u(x,0) = 0, \ \lim_{t\to 0^+} \frac{\partial}{\partial t}u(x,t) = 0, \quad x > 0, \tag{8.51}$$

$$u(0,t) = \Theta(t), \quad u \to 0 \text{ as } x \to \infty, \quad t > 0, \tag{8.52}$$

where $\Theta(t)$ is the Heaviside unit step function.

Problem (8.50)-(8.51)-(8.52) is referred to as Stokes' first problem. Let us note that the solution to this problem is exactly the propagation function, considered in Section 7.2.1.

By applying Laplace transform with respect to the temporal variable in (8.50) and (8.52) and using (8.51) we obtain for the Laplace transform of $u(x,t)$ with respect to t

$$\widehat{u}(x,s) = \frac{1}{s}\exp\left(-x\sqrt{g(s)}\right), \tag{8.53}$$

where

$$g(s) = \frac{s(1+as^{\alpha})}{1+bs^{\alpha}}. \tag{8.54}$$

Let us note that equations with the same characteristic function (8.54) are studied in Chapter 4 in the context of fractional Jeffreys' heat conduction equation. Therefore, we can use here some results already obtained in Chapter 4.

To find explicit integral expression for the solution $u(x,t)$ we apply Bromwich integral inversion formula to (8.53):

$$u(x,t) = \frac{1}{2\pi \mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\,ds, \quad \gamma>0.$$

By the Cauchy's theorem, the integration on the Bromwich path can be replaced by integration on the contour $D\cup D_0$, where

$$D = \{s=\mathrm{i}r,\ r\in(-\infty,-\varepsilon)\cup(\varepsilon,\infty)\},\ D_0=\{s=\varepsilon e^{\mathrm{i}\theta},\theta\in[-\pi/2,\pi/2]\}.$$

This is possible since the integrals on the contours $\{s=\sigma\pm\mathrm{i}R,\sigma\in(0,\gamma)\}$ vanish for $R\to\infty$ due to the following asymptotic expression

$$\Re\sqrt{g(s)} \sim \sqrt{\frac{a}{b}|s|}\cos\frac{\arg s}{2} \sim \left(\frac{a}{b}(\sigma^2+R^2)^{1/2}\right)^{1/2}\cos(\pm\pi/4),\ R\to\infty.$$

Further, since

$$\lim_{s\to 0} s\left(\frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\right) = 1,$$

it follows that the integral on the semi-circular contour D_0 equals $1/2$. Integration on the contour D yields after letting $\varepsilon\to 0$ and $R\to\infty$:

$$\frac{1}{2\pi\mathrm{i}}\int_D \frac{1}{s}\exp\left(st - x\sqrt{g(s)}\right)\,ds = \frac{1}{\pi}\int_0^{\infty}\Im\exp\left(\mathrm{i}rt - x\sqrt{g(\mathrm{i}r)}\right)\frac{dr}{r}.$$

Applying the formula (4.17) for the real and imaginary parts of the square root of a complex number we obtain after some standard manipulations the following result.

Theorem 8.5. *The solution of the Stokes' first problem (8.50)-(8.51)-(8.52) admits the integral representation:*

$$u(x,t) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \exp(-xK^-(r))\sin(rt - xK^+(r))\,\frac{dr}{r}, \quad x,t>0, \quad (8.55)$$

where the functions $K^{\pm}(r)$ are defined in (4.19).

According to Proposition 4.1, the function $\sqrt{g(s)}$ is a complete Bernstein function, where $g(s)$ is defined by (8.54). This implies the following

Theorem 8.6. *The solution of the Stokes' first problem (8.50)-(8.51)-(8.52) satisfies*

$$u(x,t) \geq 0, \quad \frac{\partial}{\partial x}u(x,t) \leq 0, \quad \frac{\partial}{\partial t}u(x,t) \geq 0 \quad x,t>0. \qquad (8.56)$$

The proof is the same as that of Theorem 7.1.

In fact, for $0<\alpha<1$, all inequalities in Theorem 8.6 are strict. To prove this we will show that $u(x,t)$ considered as a function of t admits an analytic extension to some sector in the complex plane by applying Theorem 1.3. Set $\gamma = \frac{\alpha+1}{2} < 1$. Let $s\in\mathbb{C}$ is such that $|\arg s| < \pi/2+\theta$, $0<\theta<\theta_0$, where $\theta_0 = (1/\gamma - 1)\,\pi/2 - \varepsilon_0$. Then, by the use of (4.10)

$$|\arg\sqrt{g(s)}| \leq \frac{\alpha+1}{2}|\arg s| \leq \pi/2 - \gamma\varepsilon_0.$$

Taking into account (8.53) we obtain

$$|s\widehat{u}(x,s)| \leq \exp\left(-x|g(s)|^{1/2}\sin(\gamma\varepsilon_0)\right) \leq 1.$$

Therefore, the conditions of Theorem 1.3 are satisfied and $u(x,t)$ is analytic function in t in the sector $\Sigma(\theta_0)$. Analyticity and monotonicity of the function $u(x,t)$ imply that $u \neq 0$.

Theorem 8.6 implies that the solution to the Stokes' first problem has a physically acceptable behavior (positive, decreasing in x, and increasing in t), see Figure 8.1. The figure is from [16], where for the numerical computation formula (8.55) is used.

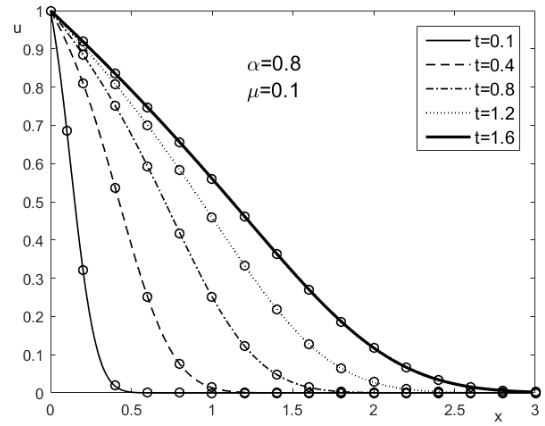


Figure 8.1: Solution (8.55) of Stokes' first problem as a function of x for different values of t, compared to the numerical solution by finite difference method (circles).

8.3.2 Subordination relation

Consider the abstract Cauchy problem

$$(1 + aD_t^\alpha)\, u'(t) = (1 + bD_t^\beta)Au(t) + (1 + aD_t^\alpha)\, f(t), \quad t > 0, \tag{8.57}$$
$$u(0) = u'(0) = 0, \tag{8.58}$$

where A is a linear operator densely defined in a suitably chosen Banach space X and f is a continuous X-valued function, $f \in C(\mathbb{R}_+; X)$. Problems for the velocity distribution u of a unidirectional flow of fractional Jeffreys' fluid usually can be written in abstract form as (8.57)-(8.58), where A is a one- or two-dimensional realization of the Laplace operator, or a more general elliptic operator, see e.g. [109].

We study the general problem (8.57)-(8.58) assuming that A is a generator of a bounded strongly continuous cosine family. We consider only thermodynamically compatible models, i.e. in equation (8.57) we assume

$$0 < \alpha = \beta \le 1, \quad a > b > 0,$$

which implies that the properties listed in Theorem 8.4 are satisfied.

Applying Laplace transform to (8.57) we deduce by the use of (1.19) and (8.58)

$$(1 + as^\alpha)s\widehat{u}(s) = (1 + bs^\alpha)A\widehat{u}(s) + (1 + as^\alpha)\widehat{f}(s)$$

and hence

$$\widehat{u}(s) = \frac{g(s)}{s}\left(g(s) - A\right)^{-1}\widehat{f}(s),$$

where $g(s)$ is defined in (8.54). Therefore, taking into account (2.13), the solution of (8.57)-(8.58) is given by

$$u(t) = \int_0^t S(t-\tau)f(\tau)\,d\tau, \tag{8.59}$$

where $S(t)$ is the solution operator of the abstract Cauchy problem for the homogeneous equation

$$(1 + aD_t^\alpha)\,u'(t) = (1 + bD_t^\alpha)Au(t), \quad t > 0, \tag{8.60}$$

$$u(0) = v \in X, \;\; u'(0) = 0. \tag{8.61}$$

It is equivalent to the abstract Volterra equation $u(t) = v + \int_0^t k(t-\tau)Au(\tau)\,d\tau$ with characteristic function $g(s) = (\widehat{k}(s))^{-1}$ defined by (8.54). The kernel $k(t)$ admits the explicit representation

$$k(t) = \int_0^t G(\tau)\,d\tau = 1 - (1 - b/a)\,E_\alpha(-t^\alpha/a),$$

where we have used (8.44) and (1.25). Since $G(t) \in \mathcal{CMF}_0$ (see Theorem 8.4), we can apply subordination Theorem 8.1. The subordination kernel is proportional to the fundamental solution of the corresponding one-dimensional Cauchy problem (see their Laplace transforms (8.6) and (4.16)). Therefore, explicit representation of the subordination kernel can be found in Theorem 4.5. In this way we establish the following subordination result.

Theorem 8.7. *Let $a \geq b > 0$ and $0 < \alpha \leq 1$. Assume A is a generator of a bounded cosine family $S_2(t)$ in X. Then problem (8.60)-(8.61) admits a bounded solution operator $S(t)$. It is related to $S_2(t)$ by the identity*

$$S(t) = \int_0^\infty \varphi(t,\tau)S_2(\tau)\,d\tau, \quad t > 0, \tag{8.62}$$

where the function $\varphi(t,\tau)$ is a PDF in τ (i.e. conditions (2.26) hold) and admits the integral representation

$$\begin{aligned}\varphi(t,\tau) \;&=\; \frac{1}{\pi}\int_0^\infty \exp\left(-\tau K^-(r)\right)\left(K^-(r)\sin\left(rt - \tau K^+(r)\right)\right.\\ &+\; \left.K^+(r)\cos\left(rt - \tau K^+(r)\right)\right)\frac{dr}{r}, \quad t,\tau > 0,\end{aligned} \tag{8.63}$$

where $K^{\pm}(r)$ are defined in (4.19).

We note that the convergence of the integral in (8.63) is guaranteed by the fact that $K^{\pm}(r) > 0$ and the asymptotic properties $K^{\pm}(r) \sim r^{(1-\alpha)/2}$ as $r \to +\infty$ and $K^{\pm}(r) \sim r^{(1+\alpha)/2}$ as $r \to 0$. These properties imply as well that infinite differentiation under the integral sign with respect to t is allowed by the use dominated convergence theorem.

Let us note that $\varphi(t,\tau)$ is also related to the solution $u(x,t)$ of the Stokes' first problem via the identity

$$\varphi(t,\tau) = -\left.\frac{\partial}{\partial x}u(x,t)\right|_{x=\tau}, \quad t,\tau > 0, \tag{8.64}$$

which together with the expression (8.55) gives an alternative way to obtain the explicit representation (8.63) for $\varphi(t,\tau)$. The differentiation under the integral sign is allowed by dominated convergence theorem.

Let $0 < \alpha < 1$, that is, $\gamma = \frac{\alpha+1}{2} < 1$. Then for any $\tau > 0$ the function $\varphi(t,\tau)$ as a function of t admits analytic extension to the sector $\Sigma(\theta_0)$ and is bounded on each sector $\overline{\Sigma(\theta)}$, $0 < \theta < \theta_0$, where $\theta_0 = (1/\gamma - 1)\,\pi/2 - \varepsilon_0$. This can be proven in the same way as Theorem 7.6.

Let us note that the subordination identity (8.62) splits the solution of problem (8.60)-(8.61) into two parts. The first part (the PDF) depends only on the constitutive model (8.49) and the second part (the cosine family, which gives the solution of a related wave equation) depends only on the flow geometry.

In the limiting case $a = b$ problem (8.60)-(8.61) reduces to the classical first-order Cauchy problem and

$$\varphi(t,\tau) = \frac{1}{\sqrt{\pi t}} \exp\left(-\tau^2/(4t)\right).$$

In this case the subordination relation (8.62) reduces to the abstract Weierstrass formula (0.4).

8.3.3 The scalar equation

Let us consider first the scalar variant of Cauchy problem (8.60)-(8.61), where $X = \mathbb{R}$ and operator A is defined as multiplication by a scalar $-\lambda$, $v = 1$. We denote by $u(t;\lambda)$ the solution of the scalar equation. To

For the solution we employ Laplace transform technique and obtain by the use of (1.19)

$$(1 + as^\alpha)(s\widehat{u}(s;\lambda) - 1) = -\lambda(1 + bs^\alpha)\widehat{v}(s;\lambda).$$

Therefore, the functions $u(t;\lambda)$ are defined by their Laplace transforms $\widehat{u}(s;\lambda)$ as follows

$$\widehat{u}(s;\lambda) = \frac{1 + as^\alpha}{s(1 + as^\alpha) + \lambda_n(1 + bs^\alpha)} = \frac{g(s)}{s}\left(g(s) + \lambda_n\right)^{-1}, \tag{8.65}$$

where $g(s)$ is given in (8.54). Representation (8.65) implies

$$\widehat{u}(s;\lambda) = \frac{1}{s}\left(1 + \frac{\lambda(1 + bs^\alpha)}{s(1 + as^\alpha)}\right)^{-1}. \tag{8.66}$$

Using the same rearrangement as in (8.48) we rewrite expression (8.66) for large $|s|$ in a series form

$$\widehat{u}(s;\lambda) = \sum_{k=0}^{\infty}\sum_{m=0}^{k}(-\lambda\mu)^k \binom{k}{m}\left(\frac{1-\mu}{a\mu}\right)^m \frac{s^{-k-1}}{(s^\alpha + 1/a)^m}, \tag{8.67}$$

where $\mu = b/a$. Taking the inverse Laplace transform in (8.67) and using the identity (1.34) the representation is deduced

$$\begin{aligned} u(t;\lambda) &= \exp\left(-\mu\lambda t\right) \\ &+ \sum_{k=1}^{\infty}\sum_{m=1}^{k}(-\lambda\mu)^k \binom{k}{m}\left(\frac{1-\mu}{a\mu}\right)^m t^{\alpha m+k}E^m_{\alpha,\alpha m+k+1}\left(-t^\alpha/a\right) \end{aligned} \tag{8.68}$$

where $\mu = b/a$ and $E^m_{\alpha,\beta}(\cdot)$ is the three-parameter Mittag-Leffler function (1.32).

Let us note that in the limiting case $a = b$ ($\mu = 1$) corresponding to a Newtonian fluid Eq. (8.68) reduces to $u(t;\lambda) = \exp(-\lambda t)$.

We deduce next the asymptotic behavior of the solution $u(t;\lambda)$. First, (8.68) implies

$$u(t;\lambda) \sim 1 - \mu\lambda t, \quad t \to 0.$$

Further, it follows from (8.66)

$$\widehat{u}(s;\lambda) \sim \frac{1 + as^\alpha}{\lambda(1 + bs^\alpha)} = \frac{1}{\lambda}\left(\frac{a}{b} + \frac{b-a}{b}\frac{1}{1 + bs^\alpha}\right), \quad |s| \to 0,$$

and taking the inverse Laplace transform yields

$$u(t;\lambda) \sim \frac{b-a}{\lambda b^2} t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{1}{b}t^{\alpha}\right), \quad t \to +\infty,$$

which by the use of (1.33) gives the asymptotic behavior of the eigenmodes for large t

$$u(t;\lambda) \sim \frac{a-b}{\lambda\Gamma(-\alpha)t^{\alpha+1}}, \quad t \to +\infty. \tag{8.69}$$

Therefore, the functions $u(t;\lambda)$ admit the following behavior: starting from 1 at $t = 0$, after some oscillations, they become permanently negative and vanish.

To find another representation of the solution $u(t;\lambda)$ we can apply the subordination formula (8.62). Since the cosine family in the scalar case is given by the function $S_2(t) = \cos(\sqrt{\lambda}t)$, formula (8.62) implies

$$u(t;\lambda) = \int_0^{\infty} \varphi(t,\tau) \cos(\sqrt{\lambda}\tau)\, d\tau, \tag{8.70}$$

where the function $\varphi(t,\tau)$ is given in (8.63). The integral representation (8.70) is appropriate for numerical computation.

8.3.4 Applications

Example 8.1. Consider a problem governing the velocity distribution of a plane Poiseuille flow between two parallel plates set in motion due to sudden application of a constant pressure gradient ($P = const$). The corresponding initial-boundary-value problem is

$$\begin{aligned} &(1 + aD_t^{\alpha})\, u_t = (1 + bD_t^{\alpha})u_{xx} + (1 + aD_t^{\alpha})\, P, \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = u_t(x,0) = 0. \end{aligned} \tag{8.71}$$

Let us set now $X = L^2([0,1])$ and define A by $(Au)(x) = u''(x),\ x \in [0,1]$ with domain $D(A) = \{u \in X : u', u'' \in X, u(0) = u(1) = 0\}$. The corresponding cosine family $S_2(t)$ is defined by the solution of the following problem for the wave equation

$$\begin{aligned} &u_{tt} = u_{xx}, \\ &u(0,t) = u(1,t) = 0, \\ &u(x,0) = v(x), \quad u_t(x,0) = 0. \end{aligned} \tag{8.72}$$

Therefore, if $v(x)$ has the eigenexpansion $v(x) = \sum_{n=1}^{\infty} v_n \sin(n\pi x)$ then

$$(S_2(t)v)(x) = \sum_{n=1}^{\infty} v_n \sin(n\pi x)\cos(n\pi t). \tag{8.73}$$

From the subordination identity (8.62) and the variation of parameters formula (8.59) we have

$$u(x,t) = \int_0^t S(\tau)P\mathrm{d}\tau = \int_0^t \int_0^\infty \varphi(\tau,\sigma)S_2(\sigma)P\,d\sigma d\tau. \tag{8.74}$$

Here $S_2(t)P$ is the solution of problem (8.72) with $v = P$. Applying (8.73) it follows

$$S_2(t)P = \frac{2P}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(n\pi x)\cos(n\pi t). \tag{8.75}$$

Inserting (8.75) in (8.74) and using (8.70) we derive the following explicit representation of the solution of problem (8.71)

$$\begin{aligned} u(x,t) &= \frac{2P}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(n\pi x)\int_0^t\int_0^\infty \varphi(\tau,\sigma)\cos(n\pi\sigma)\,d\sigma d\tau, \\ &= \frac{2P}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin(n\pi x)\int_0^t u(\tau;n^2\pi^2)\,d\tau, \end{aligned}$$

where the function $u(t;\lambda)$ is the solution of the scalar problem, see (8.68) and (8.70).

In the next example we consider a two-dimensional variant of problem (8.71), governing Poiseuille flow in a channel.

Example 8.2. Poiseuille flow of a fractional Jeffreys' fluid in a rectangular channel with cross-section $\Omega = (0,1)\times(0,1)$ is governed by the equation for the velocity field $u(x,y,t)$ [17]

$$(1+aD_t^\alpha)\frac{\partial u}{\partial t} = (1+bD_t^\alpha)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + f(t), \quad (x,y)\in\Omega,\ t>0, \tag{8.76}$$

subject to homogeneous Dirichlet boundary condition

$$u(x,y,t) = 0, \quad (x,y)\in\partial\Omega,\ t>0, \tag{8.77}$$

and initial conditions

$$u(x,y,0) = u_t(x,y,0) = 0, \quad (x,y) \in \overline{\Omega}. \tag{8.78}$$

Here $\alpha \in (0,1)$, $a > b \geq 0$, are given constant parameters and

$$f(t) = 1 + a\omega_{1-\alpha}(t),$$

where the notation (1.3) is used.

Applying eigenfunction decomposition, the solution of problem (8.76)-(8.77)-(8.78) admits the form

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\sin(m\pi x)\sin(n\pi y), \tag{8.79}$$

where the time-dependent components $u_{mn}(t)$ satisfy the following ordinary differential equations

$$\begin{aligned} &(1 + aD_t^{\alpha})u'_{mn}(t) = -\mu_{mn}^2(1 + bD_t^{\alpha})u_{mn}(t) + f_{mn}(t), \\ &u_{mn}(0) = u'_{mn}(0) = 0 \end{aligned} \tag{8.80}$$

with $\mu_{mn} = \pi\sqrt{m^2+n^2}$ and

$$f_{mn}(t) = B_{mn}f(t), \quad B_{mn} = \frac{4}{mn\pi^2}\left(1 - (-1)^m\right)\left(1 - (-1)^n\right). \tag{8.81}$$

By applying Laplace transform we solve problem (8.80) and obtain

$$u_{mn}(t) = B_{mn}G_{mn}(t),$$

where B_{mn} are given in (8.81) and the functions $G_{mn}(t)$ are defined through their Laplace transforms

$$\widehat{G}_{mn}(s) = \frac{1 + as^{\alpha}}{s[s(1+as^{\alpha}) + \mu_{mn}^2(1+bs^{\alpha})]}. \tag{8.82}$$

Therefore

$$G_{mn}(t) = \int_0^t u\left(\sigma; \mu_{mn}^2\right)\, d\sigma = \int_0^t \int_0^{\infty} \varphi(\sigma,\tau)\cos(\mu_{mn}\tau)\, d\tau d\sigma, \tag{8.83}$$

where the function $\varphi(t,\tau)$ admits the integral representation (8.63). Interchanging the order of integration in (8.83) we obtain

$$G_{mn}(t) = \int_0^\infty \psi(t,\tau)\cos(\mu_{mn}\tau)\,d\tau, \tag{8.84}$$

where

$$\begin{aligned}\psi(t,\tau) &= \frac{1}{\pi}\int_0^\infty \exp\left(-\tau K^-(r)\right)\left[K^+(r)\left(\sin\left(rt-\tau K^+(r)\right)+\sin\left(\tau K^+(r)\right)\right)\right.\\ &- \left.K^-(r)\left(\cos\left(\tau K^+(r)\right)+\cos\left(rt-\tau K^+(r)\right)\right)\right]\frac{dr}{r^2},\quad t,\tau>0.\end{aligned}$$

Here $K^{\pm}(r)$ are defined by (4.19).

In this way the following representation of the solution of problem (8.76)-(8.77)-(8.78) is derived

$$u(x,y,t) = \frac{4}{\pi^2}\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{(1-(-1)^m)\,(1-(-1)^n)}{mn}G_{mn}(t)\sin(m\pi x)\sin(n\pi y),$$

with functions $G_{mn}(t)$ given in (8.84).

8.4 Generalized diffusion-wave equation

After the study in the last two chapters of various generalizations of the fractional diffusion-wave equation, we conclude with a short discussion on the definition of generalized diffusion-wave equation.

A generalization of the Caputo fractional derivative (1.7) of order $\alpha\in(0,1)$ in the form

$$({}^C\mathbb{D}_t^{(\kappa)}f)(t) = \int_0^t \kappa(t-\tau)f'(\tau)\,d\tau,\quad t>0, \tag{8.85}$$

is discussed in Section 4.2. The kernel $\kappa\in L^1_{loc}(\mathbb{R}_+)$ is a nonnegative function, such that $\widehat{\kappa}(s)\in\mathcal{SF}$ for $s>0$. The generalized relaxation/subdiffusion equation with convolutional derivative (8.85) corresponds to a Volterra integral equation with kernel $k(t)$ such that

$$\widehat{k}(s)\widehat{\kappa}(s) = 1/s. \tag{8.86}$$

Pairs of kernels satisfying relation (8.86) (Sonine kernels) have the property that $\widehat{\kappa}(s)\in\mathcal{SF}$ if and only if $\widehat{k}(s)\in\mathcal{SF}$. Moreover, assumption $\widehat{\kappa}(s)\in\mathcal{SF}$

is equivalent to $g(s) = 1/\widehat{k}(s) = s\widehat{\kappa}(s) \in \mathcal{CBF}$ (see properties **(P7)** and **(P10)** in Proposition 2.1). Therefore, according to Theorem 2.4, the class of generalized subdiffusion equations consists of equations which are subordinated to the classical diffusion equation.

By analogy with the above considerations, the notion of generalized diffusion-wave equation is introduced in [98] in the following one-dimensional form

$$\int_0^t \eta(t-\tau)\frac{\partial^2}{\partial\tau^2}u(x,\tau)\,d\tau = \frac{\partial^2}{\partial x^2}u(x,t), \tag{8.87}$$

where the integro-differential operator in time is supposed to generalize the Caputo fractional derivative (1.7) of order $\alpha \in (1,2)$. In the case of Caputo derivative $\eta(t) = t^{1-\alpha}/\Gamma(2-\alpha)$, $\alpha \in (1,2)$. Therefore, it is natural to assume again for the kernel $\eta(t)$, see [98],

$$\widehat{\eta}(s) \in \mathcal{SF}. \tag{8.88}$$

Denote by $\xi(t)$ the corresponding Sonine kernel, i.e. $\widehat{\xi}(s)\widehat{\eta}(s) = 1/s$. Therefore, (8.88) is equivalent to $\widehat{\xi}(s) \in \mathcal{SF}$. By applying the operator $(1 * \xi)*$ to both sides of (8.87), we deduce that the generalized diffusion-wave equation (8.87) with initial conditions $u(x,0) = v(x)$ and $u_t(x,0) = 0$ is equivalent to the Volterra integral equation

$$u(x,t) = v(x) + \int_0^t k(t-\tau)\frac{\partial^2}{\partial x^2}u(x,\tau)\,d\tau$$

with kernel $k(t) = (1 * \xi)(t)$. Therefore, $\widehat{k}(s) = \widehat{\xi}(s)/s$ and

$$\widehat{k}(s)\widehat{\eta}(s) = 1/s^2. \tag{8.89}$$

We first note that (8.88) and property **(P10)** in Proposition 2.1 imply

$$s\widehat{\eta}(s) \in \mathcal{CBF}.$$

Relation (8.89) yields $g(s) = 1/\widehat{k}(s) = s^2\widehat{\eta}(s)$. Therefore, assumption (8.88) implies that $g(s)$ is a product of two complete Bernstein functions (s and $s\widehat{\eta}(s)$) and, thus, by (2.2), $g(s)^{1/2} \in \mathcal{CBF}$. Hence, according to Theorem 2.4, equation (8.87) is subordinated to the classical one-dimensional wave equation.

Wave equations in viscoelastic media with completely monotone relaxation moduli $G(t)$, considereed in this chapter, satisfy $g(s) = s/\widehat{G}(s)$, see (8.4).

Therefore, they are generalized diffusion-wave equations of the form (8.87), where $\eta(t)$ is the Sonine kernel of $G(t)$. Vice versa, any generalized diffusion-wave equation (8.87) can be interpreted as a wave equation in viscoelastic medium with completely monotone relaxation modulus $G(t) = \xi(t)$.

However, the class of equations, subordinated to the classical wave equation, is larger. Namely, there exists equations with characteristic function $g(s)$, satisfying $g(s)^{1/2} \in \mathcal{CBF}$, which are not of the form (8.87)-(8.88). Next we give two such examples.

Example 8.3. Consider a distributed-order diffusion-wave equation (7.2), such that $\operatorname{supp}\mu \not\subseteq [1,2]$. For example, let us consider the two-term time-fractional diffusion-wave equation

$$ {}^{C}D_t^{\alpha} u(x,t) + {}^{C}D_t^{\alpha_1} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t) $$

where $\alpha \in (1,2), \alpha_1 \in (0,1), \alpha - \alpha_1 \leq 1$. According to Theorem 7.5, this equation is subordinated to the classical wave equation. It is equivalent to Volterra integral equation with $g(s) = s^{\alpha} + s^{\alpha_1}$. If we rewrite it in the form (8.87), then $\widehat{\xi}(s) = s/g(s) = s^{1-\alpha_1}/(s^{\alpha-\alpha_1} + 1)$ and (1.27) yields

$$ \xi(t) = t^{\alpha-2} E_{\alpha-\alpha_1,\alpha-1}(-t^{\alpha-\alpha_1}). $$

The asymptotic formula for Mittag-Leffler function (1.24) implies

$$ \xi(t) \sim t^{\alpha_1-2}/\Gamma(\alpha_1 - 1) \quad \text{as } t \to \infty, $$

which is negative for $\alpha_1 \in (0,1)$. Therefore $\widehat{\xi}(s) \notin \mathcal{SF}$, which is equivalent to $\widehat{\eta}(s) \notin \mathcal{SF}$. Therefore, the considered two-term diffusion-wave equation is not of the form (8.87)-(8.88).

Example 8.4. The second example is Jeffreys' type equation (8.57) with $f(t) = 0$ and standard initial conditions $u(0) = v, u'(0) = 0$, where

$$ 0 < \alpha, \beta \leq 1, \ \alpha \neq \beta, \quad a > b > 0. $$

Even though it is not thermodynamically well behaved for $\alpha \neq \beta$ (see Theorem 8.4), the corresponding equation (8.57) is actually well posed and subordinated to wave equation. Indeed, for this equation condition $g(s)^{1/2} \in \mathcal{CBF}$ is satisfied for all $0 < \alpha, \beta < 1$. This follows from the representation

$$ g(s) = \frac{s(1 + as^{\alpha})}{1 + bs^{\beta}}, \tag{8.90} $$

which implies that $g(s)$ is a product of two complete Bernstein functions: $1+as^\alpha$ and $s/(1+bs^\beta)$ (see property **(P9)**). Then property (2.2) yields $g(s)^{1/2} \in \mathcal{CBF}$. Therefore, the conditions of Theorem 2.4 are satisfied with $\alpha = 2$ and equation (8.57) is subordinated to a wave equation. On the other hand, according to Theorem 8.4, in the considered case $\xi(t) = G(t) \notin \mathcal{CMF}$, that is, $\widehat{\eta}(s) \notin \mathcal{SF}$, and equation (8.57) is not of the form (8.87)-(8.88).

The above observations suggest a revision of the definition of generalized fractional diffusion-wave equations, which should include equations subordinated to the classical wave equation.

Main scientific contributions

In this dissertation we developed a unified methodology for establishing a subordination relation between a linear evolution equation in a general form and a linear fractional or integer-order evolution equation. The problem of subordination is reduced to proving that a characteristic function belongs to the class of complete Bernstein functions.

We establish subordination relations for a number of equations with fractional derivatives, which have been recently proposed in the literature. The subordination relations for generalized time-fractional evolution equations split the solution into two parts: a probability density function, containing all information about the operators acting in time, and the solution of a simpler (integer-order or single-term fractional-order) problem.

Subordination principle for space-time fractional evolution equations is studied. Various representations for the subordination kernel are derived and its properties are studied. As an application, integral representations for the n-dimensional fundamental solution are established for $n = 1, 2, 3$ (Chapter 3).

Evolution equations with the fractional Jeffreys' constitutive law are studied in detail (it appears in heat conduction equations, as well as in equations modelling wave propagation in viscoelastic fluids, see Chapter 4, Section 8.3, and Section 5.1.2 for a particular case). Based on this model, the relationship between the subordination principle and the physical character of an evolution equation is illustrated.

Subordination relations are established for the generalized time-fractional subdiffusion equations (Chapter 5). An explicit approximation formula for the solution is obtained, which generalizes the exponential formula for C_0-semigroups. As an application of subordination principle, useful estimates for the solution of the generalized relaxation equation are derived. They are applied in the study of an inverse-source problem.

A multinomial function of Prabhakar type is introduced and studied (Chapter 6). It is related to differential equations with multiple time-derivatives.

Along with other properties, we formulate sufficient conditions for complete monotonicity of this function. As an application of this function, we propose a viscoelastic constitutive model, which generalizes some well-known relaxation laws (Section 8.2.4).

An open problem concerning the interpretation of the fundamental solution to distributed-order time-fractional diffusion-wave equation as a probability density is partly solved (Section 7.1). The class of allowed weight functions is extended from functions with support contained in the interval $[1, 2]$ to functions with support contained in the interval $[a, a+1]$, $0 < a \leq 1$. An example shows that this condition can be further relaxed.

Subordination principle for the multi-term time-fractional diffusion-wave equation is studied in detail (Section 7.2). Integral representation for the subordination kernel is derived. The cases of finite and infinite propagation speed are considered.

The relaxation modulus of a number of generalized fractional viscoelastic constitutive models is studied, such as fractional Maxwell, Jeffreys' and distributed-order Zener models. It is proven for these models that the thermodynamic constraints imply complete monotonicity of the relaxation modulus (Section 8.2). This property plays an important role in establishing a subordination principle for the corresponding wave equation.

Based on subordination principle, the considered in the dissertation generalized fractional evolution equations are divided into two main classes: subdiffusion equations (subordinated to the first order abstract Cauchy problem, see Chapters 5 and 6) and diffusion-wave equations (subordinated to the second order abstract Cauchy problem, which are not subdiffusion equations, see Chapters 7 and 8). Thus, we propose a new way to define those two classes of equations, which allows to cover important physically meaningful models.

Bibliography

[1] M. Abramowitz, I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* New York, Dover (1964).

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems.* Birkhäuser, Basel (2011).

[3] G.B. Arfken, H.J. Weber, *Mathematical Methods for Physicists.* Elsevier, Amsterdam (2005).

[4] T. Atanacković, On a distributed derivative model of a viscoelastic body. *C. R. Mécanique*, **331** (2003), 687-692.

[5] T. Atanacković, S. Konjik, L. Oparnica, D. Zorica, Thermodynamical restrictions and wave propagation for a class of fractional order viscoelastic rods. *Abstr. Appl. Anal.* **2011** (2011), Art. # 975694-1-32.

[6] T. Atanacković, S. Pilipović, B. Stanković, D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes.* John Wiley & Sons, London (2014).

[7] R.L. Bagley, P.J. Torvik, On the fractional calculus model of viscoelastic behavior. *J. Rheol.* **30** (1986), 137-148.

[8] A.V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them. *Pacific J. Math.* **10** (1961), 419-437.

[9] E. Bazhlekova, Subordination principle for fractional evolution equations. *Fract. Calc. Appl. Anal.* **3**, No 3 (2000), 213-230.

[10] E. Bazhlekova, Completely monotone functions and some classes of fractional evolution equations, *Integr. Transf. Spec. F.* **26** (2015), 737-752.

[11] E. Bazhlekova, Subordination principle for a class of fractional order differential equations. *Mathematics,* **3**, No 2 (2015), 412-427.

[12] E. Bazhlekova, Estimates for a general fractional relaxation equation and application to an inverse source problem. *Math. Met. Appl. Sci.* **4** (2018), 9018-9026.

[13] E. Bazhlekova, Subordination in a class of generalized time-fractional diffusion-wave equations. *Fract. Calc. Appl. Anal.* **21**, No 4 (2018), 869-900.

[14] E. Bazhlekova, Subordination principle for space-time fractional evolution equations and some applications. *Integr. Transf. Spec. Funct.* **30**, No 6 (2019), 431-452.

[15] E. Bazhlekova, Completely monotone multinomial Mittag-Leffler type functions and diffusion equations with multiple time-derivatives. *Fract. Calc. Appl. Anal.* **24**, No 1 (2021), 88-111.

[16] E. Bazhlekova, I. Bazhlekov, Unidirectional flows of fractional Jeffreys fluids: Thermodynamic constraints and subordination. *Comput. Math. Appl.* **73** (2017), 1363-1376.

[17] E. Bazhlekova, I. Bazhlekov, Poiseuille flow of fractional Jeffreys' fluid in a rectangular channel - Analytical results. *AIP Conf. Proc.* **1978** (2018), Art. # 140008.

[18] E. Bazhlekova, I. Bazhlekov, Subordination approach to multi-term time-fractional diffusion-wave equation. *J. Comput. Appl. Math.* **339** (2018), 179-192.

[19] E. Bazhlekova, I. Bazhlekov, Complete monotonicity of the relaxation moduli of distributed-order fractional Zener model. *AIP Conf. Proc.* **2048** (2018), Art. # 050008.

[20] E. Bazhlekova, I. Bazhlekov, Subordination approach to space-time fractional diffusion. *Mathematics* **7**, No 5 (2019), Art. # 415.

[21] E. Bazhlekova, I. Bazhlekov, Analyticity of solution operators to space-time fractional evolution equations. *AIP Conf. Proc.* **2172** (2019), Art. # 050004.

[22] E. Bazhlekova, I. Bazhlekov, Transition from diffusion to wave propagation in fractional Jeffreys-type heat conduction equation. *Fractal Fract.* **4** (2020), Art. # 32.

[23] E. Bazhlekova, I. Bazhlekov, Identification of a space-dependent source term in a nonlocal problem for the general time-fractional diffusion equation. *J. Comput. Appl. Math.* **386** (2021), Art. # 113213.

[24] E. Bazhlekova, B. Jin, R. Lazarov, Z. Zhou, An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid. *Numer. Math.* **131** (2015), 1-31.

[25] E. Bazhlekova, S. Pshenichnov, Wave propagation in viscoelastic half-space with memory functions of Mittag-Leffler type. *Int. J. Appl. Math.* **34**, No 3 (2021), 423-440.

[26] M.N. Berberan-Santos, Relation between the inverse Laplace transforms of $I(t^\beta)$ and $I(t)$: application to the Mittag-Leffler and asymptotic inverse power law relaxation functions. *J. Math. Chem.* **38** No 2 (2005), 265-270.

[27] N. Bleistein, R.A. Handelsman, *Asymptotic Expansions of Integrals.* New York, Dover (1986).

[28] S. Bochner, Diffusion equation and stochastic processes. *PNAS USA*, **35** (1949), 368-370.

[29] S. Bochner, *Harmonic Analysis and the Theory of Probability.* California Monographs in Mathematical Sciences, University of California Press, Berkeley, Calif, USA (1955).

[30] L. Boyadjiev, Yu. Luchko, The neutral-fractional telegraph equation. *Math. Model. Nat. Phenom.* **12** (2017), 51-67.

[31] E. Capelas de Oliveira, F. Mainardi, J. Vaz Jr, Models based on Mittag-Leffler functions for anomalous relaxation in dielectrics. *Eur. Phys. J. Spec. Top.* **193**, No 1 (2011), 161-171.

[32] M. Caputo, J.M. Carcione, Wave simulation in dissipative media described by distributed-order fractional time derivatives. *J. Vib. Control*, **17** (2010), 1121-1130.

[33] A. Chechkin, I.M. Sokolov, *Relation between generalized diffusion equations and subordination schemes. Phys. Rev. E* **103** (2021), Art. # 032133.

[34] C.-S. Deng, R.L. Schilling, Exact asymptotic formulas for the heat kernels of space and time-fractional equations. *Fract. Calc. Appl. Anal.* **22** No 4, (2019), 968-989.

[35] V.A. Ditkin, A.P. Prudnikov, *Integral Transforms and Operational Calculus.* Pergamon Press, Oxford, New York (1965).

[36] K.J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations.* Springer-Verlag: New York/Berlin/Heidelberg (1999).

[37] A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Tables of Integral Transforms.* Vol. 1, McGraw-Hill: New York, NY, USA, (1954).

[38] W. Feller, *An Introduction to Probability Theory and its Applications.* vol. 2, Wiley, New York (1971).

[39] A. Fernandez, C. Kürt, M. Özarslan, A naturally emerging bivariate Mittag-Leffler function and associated fractional calculus operators. *Comp. Appl. Math.* **39** (2020), Art. # 200.

[40] A. Fitouhi, I. Jebabli, E.L. Shishkina, S.M. Sitnik, Applications of integral transforms composition method to wave-type singular differential equations and index shift transmutations. *Electron. J. Differ. Eq.* **2018**, No 130 (2018), 1–27.

[41] Y. Fujita, Integrodifferential equations which interpolate the heat equation and the wave equation. *Osaka J. Math.* **27** (1990), 309-321.

[42] I.M. Gelfand, G.E. Shilov, *Generalized Functions.* vol.1, Academic Press, New York (1964).

[43] A. Giusti, I. Colombaro, R. Garra, R. Garrappa, F. Polito, M. Popolizio, F. Mainardi, A practical guide to Prabhakar fractional calculus. *Fract. Calc. Appl. Anal.* **23** (2020) 9-54.

[44] R. Gorenflo, Stochastic processes related to time-fractional diffusion-wave equation. *Commun. Appl. Ind. Math.* **6**, No 2 (2015), e-531.

[45] R. Gorenflo, A. Kilbas, F. Mainardi, S. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications.* 2nd Ed., Springer, Berlin- Heidelberg (2020).

[46] R. Gorenflo, Y. Luchko, M. Stojanović, Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density. *Fract. Calc. Appl. Anal.* **16** (2013), 297-316.

[47] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order. In: A. Carpinteri, F. Mainardi (Eds.) *Fractals and Fractional Calculus in Continuum Mechanics,* Springer-Verlag, Wien/New York (1997), 223-276.

[48] R. Gorenflo, F. Mainardi, Subordination pathways to fractional diffusion. *Eur. Phys. J. Special Topics,* **193** (2011), 119-132.

[49] G. Gripenberg, S.O. Londen, O.J. Staffans, *Volterra Integral and Functional Equations.* Cambridge University Press, Cambridge (1990).

[50] S.B. Hadid, Y. Luchko, An operational method for solving fractional differential equations of an arbitrary real order. *Panam. Math. J.* **6**, No 1 (1996), 57-73.

[51] A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations. *Proc. R. Soc. Lond. A*, **458** (2002), 933-957.

[52] A. Hanyga, Physically acceptable viscoelastic models. In: K. Hutter and Y. Wang (Eds.), *Trends in Applications of Mathematics to Mechanics*, Shaker Verlag GmbH, Aachen (2005), 125-136.

[53] A. Hanyga, Wave propagation in linear viscoelastic media with completely monotonic relaxation moduli. *Wave Motion,* **50** (2013), 909-928.

[54] A. Heibig, Existence of solutions for a fractional derivative system of equations. *Integral Equ. Oper. Theory,* **72** (2012), 483-508.

[55] R. Hilfer, *Applications of Fractional Calculus in Physics.* World Sci. Publ., Singapore (2000).

[56] B. Jin, R. Lazarov, Y. Liu, Z. Zhou, The Galerkin finite element method for a multi-term time-fractionl diffusion equation. *J. Comput. Phys.* **281** (2015), 825-843.

[57] L. Kexue, P. Jigen, G. Jinghuai, Existence results for semilinear fractional differential equations via Kuratowski measure of noncompactness. *Fract. Calc. Appl. Anal.* **15** (2012), 591-610.

[58] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics studies, Elsevier, Amsterdam (2006).

[59] V. Kiryakova, Unified approach to fractional calculus images of special functions - a survey. *Mathematics,* **8**, No 12 (2020), Art. # 2260.

[60] V. Kiryakova, A guide to special functions in fractional calculus. *Mathematics,* **9**, No 1 (2021), Art. # 106.

[61] A. Kochubei, Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.* **340** (2008), 252-281.

[62] A. Kochubei, General fractional calculus, evolution equations, and renewal processes. *Integr. Equ. Oper. Theory*, **71** (2011), 583-600.

[63] A. Kochubei, Asymptotic properties of solutions of the fractional diffusion-wave equation. *Fract. Calc. Appl. Anal.* **17** (2014), 881-896.

[64] M. Kwaśnicki, Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.* **20**, No 1 (2017), 7-51.

[65] C.G. Li, M. Kostić, M. Li, S. Piskarev, On a class of time-fractional differential equations. *Fract. Calc. Appl. Anal.* **15**, No 4 (2012), 639-668.

[66] M. Li, C. Chen, F.B. Li, On fractional powers of generators of fractional resolvent families. *J. Func. Anal.* **259**, No 10 (2010), 2702-2726.

[67] Z. Li, Y. Liu, M. Yamamoto, Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients. *Appl. Math. Comp.* **257** (2015), 381-397.

[68] Y. Liu, Strong maximum principle for multi-term time-fractional diffusion equations and its application to an inverse source problem. *Comput. Math. Appl.* **73**, No 1 (2017), 96-108.

[69] Yu. Luchko, Entropy production rate of a one-dimensional alpha-fractional diffusion process. *Axioms,* **15**, No 1 (2016), Art. # 6.

[70] Yu. Luchko, Subordination principles for the multi-dimensional space-time-fractional diffusion-wave equation. *Theory Probab. Math. Stat.* **98** (2018), 121-141.

[71] Yu. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Math. Vietnamica,* **24** (1999), 207-233.

[72] Yu. Luchko, F. Mainardi, Y. Povstenko, Propagation speed of the maximum of the fundamental solution to the fractional diffusion-wave equation. *Comput. Math. Appl.* **66** (2013), 774-784.

[73] J.A.T. Machado, V. Kiryakova, The chronicles of fractional calculus. *Fract. Calc. Appl. Anal.* **20**, No 2 (2017), 307-336.

[74] F. Mainardi, Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos Soliton. Fract.* **7** (1996), 1461-1477.

[75] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity.* Imperial College Press, London (2010).

[76] F. Mainardi, Yu. Luchko, G. Pagnini, The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **4** (2001), 153-192.

[77] F. Mainardi, G. Pagnini, R. Gorenflo, Mellin transform and subordination laws in fractional diffusion processes. *Fract. Calc. Appl. Anal.* **6**, No 4 (2003), 441-459.

[78] F. Mainardi, P. Paradisi, Fractional diffusive waves. *J. Comput. Acoust.* **9** (2001), 1417-1436.

[79] C. Martínez, M. Sanz, *The Theory of Fractional Powers of Operators.* North-Holland Math. Stud., vol. 187, Amsterdam (2001).

[80] M. Meerschaert, R. Schilling, A. Sikorskii, Stochastic solutions for fractional wave equations. *Nonlinear Dynam.* **80** (2015), 1685-1695.

[81] M. Meerschaert, B. Toaldo, Relaxation patterns and semi-Markov dynamics. *Stoch Process Their Appl.* **129** No 8 (2019), 2850-2879.

[82] J. Mikusiński, On the function whose Laplace-transform is $\exp(-s^{\alpha})$. *Stud. Math.* **18** (1959), 191-198.

[83] K.S. Miller, A note on the complete monotonicity of the generalized Mittag-Leffler function. *Real Anal. Exch.* **23** (1999), 753-756.

[84] L. Miller, M. Yamamoto, Coefficient inverse problem for a fractional diffusion equation. *Inverse Probl.* **29** (2013), Art. # 075013.

[85] J. Paneva-Konovska, *From Bessel to Multi-Index Mittag-Leffler Functions: Enumerable Families, Series in them and Convergence.* World Sci. Publ., London (2016).

[86] J. Paneva-Konovska, Differential and integral relations in the class of multi-index Mittag-Leffler functions. *Fract. Calc. Appl. Anal.* **21** No 1 (2018), 254-265.

[87] J. Paneva-Konovska, V. Kiryakova, On the multi-index Mittag-Leffler functions and their Mellin transforms. *Int. J. Appl. Math.* **33** No 4 (2020), 549-571.

[88] R.S. Phillips, On the generation of semigroups of linear operators. *Pacific J. Math.* **2** (1952), 343-369.

[89] I. Podlubny, *Fractional Differential Equations.* San Diego: Academic Press (1999).

[90] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. *Bull Amer Math Soc.* **54** No 12 (1948), 1115-1116.

[91] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* **19**, No 1 (1971), 7-15.

[92] L. Preziosi, D.D. Joseph, Stokes first problem for viscoelastic fluids. *J. Non Newtonian Fluid Mech.* **25** (1987), 239-259.

[93] J. Prüss, *Evolutionary Integral Equations and Applications.* Birkhäuser, Basel, Boston, Berlin (1993).

[94] A.V. Pskhu, The Stankovich integral transform and its applications, *Special Functions and Analysis of Differential Equations*, vol. 9, P. Agarwal, R. Agarwal, M. Ruzhansky (Eds.) Boca Raton (2020), 197-211.

[95] Y.A. Rossikhin, M.V. Shitikova, Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results. *Appl. Mech. Rev.* **63** (2010), Art. # 010801-1-25.

[96] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.* **382** (2012), 426-447.

[97] T. Sandev, A. Chechkin, N. Korabel, H. Kantz, I.M. Sokolov, R. Metzler, Distributed-order diffusion equations and multifractality: Models and solutions. *Phys. Rev. E*, **92** (2015), Art. # 042117.

[98] T. Sandev, I.M. Sokolov, R. Metzler, A. Chechkin, Beyond monofractional kinetics. *Chaos Soliton. Fact.* **102** (2017), 210-217.

[99] T. Sandev, Z. Tomovski, J.L.A. Dubbeldam, A. Chechkin, Generalized diffusion-wave equation with memory kernel. *J. Phys. A Math. Theor.* **52** No 1 (2019), Art. # 015201.

[100] M. Savov, B. Toaldo, Semi-Markov processes, integro-differential equations and anomalous diffusion-aggregation. *ANN I H POINCARE-PR* **56** No 4 (2020), 2640-2671.

[101] R. Schilling, R. Song, Z. Vondraček, *Bernstein Functions: Theory and Applications.* De Gruyter, Berlin (2010).

[102] D. Song, T. Jiang, Study on the constitutive equation with fractional derivative for the viscoelastic fluids - Modified Jeffreys model and its application. *Rheol. Acta*, **37** (1998), 512-517.

[103] R. Schumer, D. Benson, M. Meerschaert, B. Baeumer, Fractal mobile/immobile solute transport. *Water Resour. Res.* **39** No 10 (2003), 1296.

[104] B. Stanković, On a class of singular integral equations. *Zbornik Radova SAN,* **43** No 4 (1955), 81-130 (in Serbian).

[105] P. Todorov, New explicit formulas for the nth derivative of composite functions. *Pacific Journal of Mathematics*, **92** No 1 (1981), 217-236.

[106] E.M. Wright, On the coefficients of power series having exponential singularities. *Journal of the London Mathematical Society* **8** (1933), 71-79.

[107] E.M. Wright, The generalized Bessel function of order greater than one. *Quarterly Journal of Mathematics (Oxford ser.)* **11** (1940), 36-48.

[108] K. Yosida, *Functional Analysis.* Berlin: Springer-Verlag (1965).

[109] L. Zheng, X. Zhang, *Modeling and Analysis of Modern Fluid Problems.* Academic Press, Cambridge, MA, USA (2017).

[110] Y. Zhou, J.R. Wang, L. Zhang, *Basic Theory of Fractional Differential Equations*. World Sci. Publ., London (2016).

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