BULGARIAN ACADEMY OF SCIENCES

INSTITUTE OF MATHEMATICS AND INFORMATICS

Subordination principle for generalized fractional evolution equations

Emilia Grigorova Bazhlekova

ABSTRACT OF DISSERTATION
for awarding of the scientific degree
"Doctor of Science"

Higher education area:
4. Natural Sciences, Mathematics and Informatics,
Professional field: 4.5. Mathematics,
Scientific specialty: "Mathematical analysis"

Sofia, 2022

I. INTRODUCTION

In recent decades, considerable scientific interest has been shown in the so-called *fractional calculus*, which allows integration and differentiation of arbitrary order, not necessarily integer. This is largely due to the applications of fractional calculus to the mathematical modeling of a number of processes with memory in different fields such as physics, biology, even sociology.

The first ideas about fractional calculus date back to the end of the 17th century and are associated with the name of Gottfried Leibniz. Almost three centuries later, fractional calculus began to be used intensively to describe the evolution of various real systems using *fractional evolution equations*, i.e. evolution equations in which integer time or space derivatives are replaced by fractional order operators. They are used to model anomalous diffusion, heat transfer in materials with memory, waves in viscoelastic media, etc.

For modeling of some complex systems, it turns out to be more appropriate to use generalizations of classical fractional derivatives, for example fractional derivatives of distributed order or more general integro-differential operators of convolutional type. This leads to a large variety of generalized fractional evolution equations, which gives rise to the need to organize the multitude of such equations and to find ways of investigating, solving and classifying them.

Very useful in this regard is the so-called *subordination principle*. In general, this principle consists of the following: given two Cauchy problems (P) and (P_*), the problem (P) is called subordinate to the problem (P_*) if it is solvable whenever (P_*) is solvable and the solution $u(x,t)$ of (P) is represented by the solution $u_*(x,t)$ of (P_*) by the integral relation

$$u(x,t) = \int_0^\infty \varphi(t,\tau) u_*(x,\tau)\, d\tau,$$

where the kernel $\varphi(t,\tau)$ is a probability density with respect to $\tau \geq 0$ when $t > 0$ is considered as a parameter, i.e.

$$\varphi(t,\tau) \geq 0, \quad \int_0^\infty \varphi(t,\tau)\, d\tau = 1. \tag{0.1}$$

By solvability of the problem we mean that it is well-posed, i.e. there exists a unique solution that depends continuously on the initial conditions.

The subordination principle makes it possible to represent the solutions of complex equations through the solutions of simpler classical equations and is a useful tool for proving the solvability of the problem, finding estimates for the solution, establishing its asymptotic behavior and other properties. In addition, the subordination principle establishes a "hierarchy" among the variety of generalized fractional evolution equations, which is important for the correct classification and evaluation of the physical meaning of the corresponding mathematical models.

The present dissertation is devoted to the study of the subordination principle for generalized fractional evolution equations. A methodology has been developed that allows establishing a subordination dependence between two equations and thus helps to classify these equations into two main groups: equations describing subdiffusion and diffusion-wave equations. A number of specific equations found in the scientific literature are studied.

The main mathematical tools that are used in the conducted research are the theory of operators and special functions of fractional calculus, the Laplace transform technique and the theory of Bernstein functions and special classes of functions related to them.

II. STRUCTURE AND BRIEF OVERVIEW OF THE DISSERTATION

The dissertation contains 200 pages. It consists of an Introduction, eight chapters (divided into 31 sections), Concluding remarks, Bibliography and Index. The Bibliography contains 110 titles. The theorems are numbered with two numbers, the first being the number of the chapter, and the second being the serial number in the chapter itself. The numbering of formulas, definitions, statements, remarks, examples and figures follows the same principle. The text is in English.

The dissertation is the result of the author's research conducted over the past seven years (2015-2021). It is based on 11 papers published during this period: [B1]-[B11] given in Section III of this abstract.

Next we give a short overview of the dissertation work, as for each chapter we will note which of the publications are used.

The Introduction contains the motivation for the research conducted, giving examples of different types of subordination principles. **Chapter 1** contains notations, definitions, and basic properties of the fractional integration and differentiation operators, the Laplace transform, Mittag-Leffler functions, and some Wright-type functions. In **Chapter 2**, after an introduction to the theory of Bernstein functions and Volterra integral equations, we prove two general subordination theorems. **Chapter 3** ([B5] and [B9]) is devoted to a detailed study of the subordination principle for evolution equations with fractional derivatives in time and in space. As an application, integral representations for the fundamental solution are obtained, as well as some explicit representations by means of special functions. The rest of the dissertation deals with generalized evolution equations with fractional operators in time. To demonstrate the important role of the subordination principle in the study of these equations, in **Chapter 4** ([B10]) Jeffrey's fractional heat equation is considered. In **Chapter 5** ([B1], [B2] and [B3]) results are obtained for subdiffusion equations of distributed order and for more general equations with memory kernels. Useful estimates in the scalar case are derived. In **Chapter 6** ([B6]) the multinomial Mittag-Leffler function is studied, which is related to the solution of relaxation equations with several time derivatives of different (fractional) orders. The last two chapters deal with equations describing phenomena that are intermediate between diffusion and wave propagation. In **Chapter 7** ([B4] and [B7]) an open problem concerning the interpretation of the fundamental solution of diffusion-wave equations of distributed order as a probability density is discussed and partially solved. This property of the fundamental solution is important both for the physical meaning of the model and for establishing subordination relation with respect to the wave equation. In **Chapter 8** ([B4], [B8] and [B11]) equations describing wave propagation in viscoelastic media with completely monotone relaxation moduli are considered. Generalized fractional Maxwell and Zener models are considered, as well as a new model with a relaxation modulus that is represented by a fcompletely monotone binomial Mittag-Leffler function. The particular case of a fractional Jeffrey model is studied in detail and the physical meaning of the subordination formula is discussed. The dissertation ends with a summary of the main scientific contributions.

III. PUBLICATIONS ON WHICH THE DISSERTATION IS BASED

[B1] E. Bazhlekova (2015), Completely monotone functions and some classes of fractional evolution equations. *Integral Transforms and Special Functions*, 26 (9) 737-752; **IF: 0.528** – **Q3** (Web of Science) – **30** points.
Cited **16** times in Scopus

[B2] E. Bazhlekova (2015), Subordination principle for a class of fractional order differential equations. *Mathematics* (MDPI), 3 (2) 412-427; индексирана в Web of Science и Scopus – **12** points.
Cited **13** times in Scopus

[B3] E. Bazhlekova (2018), Estimates for a general fractional relaxation equation and application to an inverse source problem. *Mathematical Methods in the Applied Sciences*, 41 (18) 9018-9026; **IF: 1.533** – **Q2** (Web of Science) – **40** points.
Cited **4** times in Scopus

[B4] E. Bazhlekova (2018), Subordination in a class of generalized time-fractional diffusion-wave equations. *Fractional Calculus and Applied Analysis*, 21 (4) 869-900; **IF: 3.514** – **Q1**(Web of Science) – **50** points.
Cited **19** times in Scopus

[B5] E. Bazhlekova (2019), Subordination principle for space-time fractional evolution equations and some applications. *Integral Transforms and Special Functions*, 30 (6) 431-452;**IF: 0.705** – **Q3** (Web of Science) – **30** points.
Cited **8** times in Scopus

[B6] E. Bazhlekova (2021), Completely monotone multinomial Mittag-Leffler type functions and diffusion equations with multiple time-derivatives. *Fractional Calculus and Applied Analysis*, 24 (1) 88-111; **IF: 3.17** – **Q1**(Web of Science) – **50** points.
Cited **3** times in Scopus

[B7] E. Bazhlekova, I. Bazhlekov (2018), Subordination approach to multi-term time-fractional diffusion-wave equations. *Journal of Computational and Applied Mathematics* (Elsevier), 339, 179-192; **IF: 1.883** – **Q1**(Web of Science) – **50** points.

Cited **14** times in Scopus

[B8] E. Bazhlekova, I. Bazhlekov (2018), Complete monotonicity of the relaxation moduli of distributed-order fractional Zener model. *AIP Conference Proceedings*, 2048, art.no. 050008; **SJR: 0.182** – **20** points.
Cited **3** times in Scopus

[B9] E. Bazhlekova, I. Bazhlekov (2019), Subordination approach to space-time fractional diffusion. *Mathematics* (MDPI), 7(5) art.no. 415; **IF: 1.747** – **Q1** (Web of Science) – **50** points.
Cited **7** times in Scopus

[B10] E. Bazhlekova, I. Bazhlekov (2020), Transition from diffusion to wave propagation in fractional Jeffreys-type heat conduction equation. *Fractal and Fractional* (MDPI), 4(3), art.no. 32; **IF: 3.313** – **Q1**(Web of Science) – **50** points.
Cited **3** times in Scopus

[B11] E. Bazhlekova, S. Pshenichnov (2021), Wave propagation in viscoelastic half-space with memory functions of Mittag-Leffler type. *International Journal of Applied Mathematics*, 34(3) 423-440; **SJR: 0.268** – Q3 (Scopus) – **20** points.

Each impact factor, impact rank or quartile refers to the year of publication of the respective article. The points are calculated according to the rules for professional area 4.5 Mathematics in the Regulations on the Conditions and Order for Acquiring Scientific Degrees and Occupying Academic Positions at the Bulgarian Academy of Sciences. For each article we give the corresponding number of citations found in the Scopus database, excluding self-citations.

Of the presented **11** publications, **8** are in journals with Impact Factor (total IF: 16.4). From them **5** are in Q1 (250 points), **1** in Q2 (40 points), **2** in Q3 (60 points) - a total of **350** points. Publications that are in journals with SJR: **2** – **40** points. One publication is in a publication indexed on Web of Science and Scopus, but without IF/SJR for the year of publication - **12** points. Total of all dissertation publications: **402** points.

The citations of the above publications found in the Scopus database are **90**. They carry a total of **540** points (6 points per citation according to the BAS Regulations).

Of the publications, 6 are written by the author alone and 5 are co-authored with one author. The analytical results in all publications are obtained by the dissertation author. The contribution of the co-authors in publications [B7]-[B11] consists in the definition of the mathematical model, numerical computations and visualization of the results. In articles [B9] and [B10] the contributions of individual authors are explicitly stated. In the dissertation, only results obtained by the author are described. The only exception is the presented figures (Figures 4.1, 7.1-7.3 and 8.1). They are given in order to visualize the behavior of the analytically derived solutions and are not considered as a contribution of the dissertation.

None of the above publications have been used in other dissertations or procedures of the authors. All 11 articles were published after the completion of the last procedure of the dissertation author (which was for the occupation of the academic position of associate professor in 2014).

IV. CONTENTS AND MAIN RESULTS

We will present the content and main results of the dissertation by chapter.

1 Fractional calculus operators and special functions

Chapter 1, consisting of 5 sections, contains preliminary information. Fractional integration and differentiation operators are defined, as well as some special functions closely related to fractional calculus. Their main properties are given.

The sets of positive integers, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{C}_+ = \{z \in \mathbb{C},\ \Re z > 0\}$.

Let X be a Banach space. For $-\infty \leq a < b \leq +\infty$ we denote by $C([a, b]; X)$ the space of continuous functions $f : [a, b] \to X$. The space of functions $f : \mathbb{R}_+ \to X$ that are Bochner integrable on any interval $[0, \tau]$, $\tau > 0$ is denoted by $L^1_{loc}(\mathbb{R}_+; X)$. For brevity, $L^1_{loc}(\mathbb{R}_+) := L^1_{loc}(\mathbb{R}_+; \mathbb{R})$.

The Laplace transform of a function $f \in L^1_{loc}(\mathbb{R}_+; X)$ is defined as follows

$$\mathcal{L}\{f(t)\}(s) = \widehat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad Res > 0.$$

Let $\alpha > 0$ and $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$. We define

$$\omega_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0,\ \alpha > 0, \tag{1.1}$$

where $\Gamma(\cdot)$ denotes the Gamma function. Furthermore, we put $\omega_0(t) = \delta(t)$ where δ is the Dirac delta function.

For $u : \mathbb{R}_+ \to X$ we define *fractional Riemann-Liouville derivative* of order α by the identity [15, 25]

$$(D_t^\alpha u)(t) = \frac{d^m}{dt^m} \int_0^t \omega_{m-\alpha}(t - \tau) u(\tau)\, d\ tau,\ t > 0.$$

The *Caputo fractional derivative* is defined by the identity [15, 25]

$$({}^C D_t^\alpha u)(t) = \int_0^t \omega_{m-\alpha}(t - \tau) u^{(m)}(\tau)\, d\tau,\ t > 0.$$

For $\alpha = m \in \mathbb{N}$, we have $D_t^m = {}^C D_t^m = \frac{d^m}{dt^m}$.

The Mittag-Leffler function is an entire function defined by the following series expansion [12, 15, 20]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\ \beta \in \mathbb{R}, \quad z \in \mathbb{C}. \tag{1.2}$$

In particular, we denote $E_\alpha(z) := E_{\alpha,1}(z)$.

The Prabhakar function (or Mittag-Leffler function with three parameters) is an entire function defined as follows [27, 10]

$$E^\delta_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad z \in \mathbb{C},\ \alpha \in \mathbb{R}_+,\ \beta, \delta \in \mathbb{R}, \tag{1.3}$$

where $(\delta)_k$ denotes the Pochhammer symbol

$$(\delta)_k = \frac{\Gamma(\delta + k)}{\Gamma(\delta)} = \delta(\delta+1)\dots(\delta+k-1), \quad k \in \mathbb{N}, \quad (\delta)_0 = 1. \tag{1.4}$$

In the special case $\delta = 1$ we obtain $E_{\alpha,\beta}(z) = E^1_{\alpha,\beta}(z)$.

The Mainardi function (also called M-Wright function) is an entire Wright-type function defined by the series [20, 12]

$$M_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\gamma n + 1 - \gamma)}, \quad 0 < \gamma < 1,\ z \in \mathbb{C}. \tag{1.5}$$

It is related to the Mitag-Leffler function $E_\gamma(\cdot)$ by the identity

$$\mathcal{L}\{M_\gamma(\tau)\}(s) = \int_0^\infty e^{-s\tau} M_\gamma(\tau)\, d\tau = E_\gamma(-s), \quad 0 < \gamma < 1. \tag{1.6}$$

The function $L_\gamma(\cdot)$ defined by its Laplace transform as follows

$$\mathcal{L}\{L_\gamma(\tau)\}(s) = \int_0^\infty e^{-s\tau} L_\gamma(\tau)\, d\tau = \exp(-s^\gamma), \quad 0 < \gamma < 1, \tag{1.7}$$

is called Lévy extremal stable density [9, 21, 22]. It is related to the Mainardi function by the equality [20, 26]

$$L_\gamma(z) = \gamma z^{-\gamma-1} M_\gamma\left(z^{-\gamma}\right), \quad 0 < \gamma < 1, \quad z \in \mathbb{C}\backslash(-\infty, 0]. \tag{1.8}$$

A function defined on $\mathbb{R}_+$ is said to be one-sided probability density function when it satisfies the conditions

$$\varphi(\tau) \geq 0, \quad \tau \geq 0; \qquad \int_0^\infty \varphi(\tau)\, d\tau = 1. \tag{1.9}$$

The functions $M_\gamma(\tau)$ and $L_\gamma(\tau)$ are one-sided probability densities.

Detailed information on fractional calculus and Mittag-Leffler functions is contained in the monographs [12, 15, 25]. An overview of special functions related to fractional calculus can be found in [16, 17].

2 Introduction to subordination principle

In Chapter 2 (4 sections), we first give the definitions and basic properties of Bernstein functions and related special classes of functions that play an important role in the thesis. In order to have a unified approach to the variety of evolution equations containing fractional derivatives, we use the theory of abstract Volterra equations, a brief introduction to which is given next. Finally, we prove two general subordination theorems that will be used later in the dissertation.

2.1 Bernstein functions

Four special classes of functions play an essential role in this thesis: the classes of completely monotone functions ($\mathcal{CMF}$), Bernstein functions ($\mathcal{BF}$), Stiltes functions ($\mathcal{SF}$), and complete Bernstein functions ($\mathcal{CBF}$). The last class is also found in the literature under other names, for example Nevanlinna functions. In the dissertation we use the terminology in the monograph [30].

In the following definitions of the function ϕ we assume $\phi : \mathbb{R}_+ \to \mathbb{R}$. A function ϕ is called a completely monotone function ($\mathcal{CMF}$) if it is infinitely differentiable and

$$(-1)^n \phi^{(n)}(t) \geq 0, \qquad t > 0, \quad n \in \mathbb{N}_0. \tag{2.1}$$

According to Bernstein's Theorem, a function is completely monotone if and only if it can be represented as the Laplace transform of a non-negative (generalized) function.

The class $\mathcal{BF}$ of Bernstein functions consists of all functions $\phi \geq 0$, such that $\phi'(t) \in \mathcal{CMF}$.

The class of Stiltjes functions ($\mathcal{SF}$) consists of all functions [18]

$$\phi(s) = \frac{a}{s} + b + \int_0^\infty e^{-s\tau} \psi(\tau)\, d\tau, \quad s > 0, \tag{2.2}$$

where $a, b \geq 0$, $\psi \in \mathcal{CMF}$ and the Laplace transform of ψ exists for every $s > 0$.

A function ϕ is called complete Bernstein function ($\phi \in \mathcal{CBF}$) if and only if $\phi(s)/s \in \mathcal{SF}$, $s > 0$.

The inclusions $\mathcal{SF} \subset \mathcal{CMF}$ and $\mathcal{CBF} \subset \mathcal{BF}$ hold.

Elementary examples of Stiltjes functions and complete Bernstein functions are as follows:

$$s^\alpha \in \mathcal{SF}, \quad s^\alpha \in \mathcal{CBF}, \quad \alpha \in [0, 1].$$

Functions defined in this way have a number of interesting properties [30]. A selection of properties that are essentially used in the thesis are given in Section 2.1.

2.2 Abstract Volterra integral equations

Let X be a Banach space with norm $\|.\|$ and let $A : D(A) \to X$ be a closed linear densely defined operator.

Let $\alpha > 0$ and $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$. We consider the Cauchy problem for the fractional evolution equation:

$$\begin{aligned} &{}^C D_t^\alpha u(t) = Au(t), \quad t > 0, \\ &u(0) = v \in X, \ u^{(k)}(0) = 0, \ k = 1, 2, \ldots, m - 1. \end{aligned} \tag{2.3}$$

In the scalar case when $X = \mathbb{R}$ and the operator A is simply multiplication by a constant, $A = -\lambda$, $\lambda > 0$, the solution of (2.3) is given by the Mittag-Leffler function: $u(t) = u(0)E_\alpha(-\lambda t^\alpha)$. It describes fractional (slow) relaxation for $\alpha \in (0,1)$ and damped oscillations for $\alpha \in (1,2)$.

The classical Cauchy problem for a first-order equation is a special case of (2.3) with $\alpha = 1$:

$$u'(t) = Au(t), \quad t > 0; \quad u(0) = v \in X, \tag{2.4}$$

and for $\alpha = 2$, the Cauchy problem for a second-order equation is obtained

$$u''(t) = Au(t), \quad t > 0; \quad u(0) = v \in X, \ u'(0) = 0. \tag{2.5}$$

Consider the abstract Volterra integral equation

$$u(t) = \int_0^t k(t-\tau)Au(\tau)\,d\tau + f(t), \quad t > 0, \tag{2.6}$$

with kernel $k(t) \in L^1_{loc}(\mathbb{R}_+)$.

Let us note that (2.3) can be rewritten as a Volterra integral equation with kernel $k(t) = \omega_\alpha(t)$, where the function $\omega_\alpha(t)$ is defined in (1.1). The more general equations considered in the dissertation also have an equivalent representation in the form of an integral equation (2.6). Therefore, for their study we use the theory of abstract Volterra equations developed in the monograph [28]. We will first give some basic definitions.

Definition 2.1. *A function $u \in C(\mathbb{R}_+; X)$ is called a strong solution of the equation (2.6) if $u \in C(\mathbb{R}_+; D(A))$ and (2.6) is fulfilled on $\mathbb{R}_+$.*

Definition 2.2. *The problem (2.6) is called well-posed if for each $v \in D(A)$ there exists a unique strong solution $u(t;v)$ of*

$$u(t) = v + \int_0^t k(t-\tau)Au(\tau)\,d\tau, \qquad t > 0, \quad v \in D(A), \tag{2.7}$$

and from $\{v_n\} \subset D(A)$, $v_n \to 0$ follows $u(t; v_n) \to 0$ in X, uniformly on every compact interval.

Let the problem (2.6) be well posed. Then the solution operator $S(t)$ for (2.6) is defined as usual:

$$S(t)v = u(t;v), \quad v \in X, \ t \geq 0.$$

The solution operator $S(t)$ is called bounded if there exists a constant $C \geq 1$ such that

$$\|S(t)\| \leq C \quad \text{for all } t \geq 0.$$

Definition 2.3. *The solution operator $S(t)$ is called a bounded analytic solution operator with angle $\theta_0 \in (0, \pi/2]$ if the function $S(\cdot)$ has an analytic continuation $S(z)$ to the sector $|\arg z| < \theta_0$ which is bounded on every subsector $|\arg z| \leq \theta$ where $\theta < \theta_0$.*

In the case when the classical problem (2.4) is well-posed, the solution operator $S_1(t)$ is a strongly continuous (C_0-) semigroup. The operator A is said to generate C_0 - semigroup. In the case of problem (2.5), the solution operator $S_2(t)$ is a strongly continuous cosine operator function [2].

2.3 General subordination theorems

The next two general subordination theoremsв are proven in Section 2.4.

Theorem 2.1. *Let the Cauchy problem (2.3) be well-posed for some* α, $0 < \alpha \leq 2$, *and have a bounded solution operator* $S_\alpha(t)$. *For the kernel* $k(t)$ *of the Volterra integral equation (2.6) we assume that* $k(t) \in L^1_{loc}(\mathbb{R}_+)$, *the Laplace transform* $\widehat{k}(s)$ *exists for* $s > 0$, $\widehat{k}(s) \neq 0$ *and the function* $g(s) = (\widehat{k}(s))^{-1}$ *satisfies the condition*

$$g(s)^{1/\alpha} \in \mathcal{CBF}, \quad s > 0. \tag{2.8}$$

Then problem (2.6) is also well-posed and has a bounded solution operator $S(t)$ *which is related to* $S_\alpha(t)$ *by the identity*

$$S(t) = \int_0^\infty \varphi(t,\tau) S_\alpha(\tau)\, d\tau, \quad t > 0, \tag{2.9}$$

where

$$\varphi(t,\tau) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{g(s)^{1/\alpha}}{s} \exp\left(st - \tau g(s)^{1/\alpha}\right) ds, \quad c > 0.$$

The subordination kernel $\varphi(t,\tau)$ *is a one-sided probability density with respect to* $\tau \geq 0$ *(for* $t > 0$ *considered as a parameter), i.e.*

$$\varphi(t,\tau) \geq 0, \quad \int_0^\infty \varphi(t,\tau)\, d\tau = 1. \tag{2.10}$$

Let us note that Theorem 2.1 is based on the following representation for the Laplace transform of the kernel $\varphi(t,\tau)$

$$\widehat{\varphi}(s,\tau) = \frac{g(s)^{1/\alpha}}{s} \exp\left(-\tau g(s)^{1/\alpha}\right), \quad s, \tau > 0. \tag{2.11}$$

From the condition (2.8) it follows $\widehat{\varphi}(s,\tau) \in \mathcal{CMF}$ with respect to $s > 0$ (if τ is considered as a parameter), which by Bernstein's theorem is equivalent to $\varphi(t,\tau) \geq 0$. This observation is essential for the studies in the dissertation. The goal is to find the (smallest) α for which (2.8) is satisfied (let us keep in mind that if (2.8) is satisfied for some $\alpha = al_*$, it is also fulfilled for every $\alpha > \alpha_*$). More generally, Theorem 2.1 reduces the question of subordination to a problem of Bernstein functions.

In the second theorem, we consider the case when the subordinated operator of the solution is bounded analytic and define the sector of analyticity.

Theorem 2.2. *Let the conditions of Theorem 2.1 be satisfied and*

$$|\arg\{g(s)^{1/\beta}\}| \leq |\arg s|, \quad 0 < \beta < \alpha, \ s \in \mathbb{C}\backslash(-\infty, 0]. \tag{2.12}$$

Then $S(t)$ is a bounded analytic operator of angle

$$\theta_* = \min\left\{\left(\frac{\alpha}{\beta} - 1\right)\frac{\pi}{2}, \frac{\pi}{2}\right\}. \tag{2.13}$$

If, moreover, $S_\alpha(t)$ is a bounded analytic operator of angle $\phi_0 \in (0, \pi/2]$ then the angle of $S(t)$ is

$$\theta_0 = \min\left\{\frac{\alpha}{\beta}\phi_0 + \left(\frac{\alpha}{\beta} - 1\right)\frac{\pi}{2},\ frac\pi 2\right\}. \tag{2.14}$$

3 Space-time fractional evolution equations

Chapter 3 (consisting of 5 sections) is devoted to the subordination principle for the fractional differential equation with the Caputo time-derivative of order $\beta \in (0, 1)$ and operator $-(-A)^\alpha$, $\alpha \in (0, 1)$, where A generates a C_0 - semigroup in a Banach space. Some properties of the subordination kernel are established and representations by the Mainardi function M_β and the Lévy extremal stable densities L_α are derived. The sector of analyticity of the solution operator is found, taking into account the asymptotic behavior of the subordination kernel. Subordination formulae are applied to the multidimensional space-time fractional diffusion equation to obtain integral representations for the fundamental solutions as well as closed-form solutions in some special cases.

Consider the problem

$${}^C D_t^\beta u(t) = -(-A)^\alpha u(t),\ t > 0;\quad u(0) = v \in X;\qquad 0 < \alpha, \beta \leq 1, \tag{3.1}$$

where the operator A generates C_0 - a semigroup in Banach space X and the operator $-(-A)^\alpha$ is defined by Balakrishnan's formula [6, 31]

$$(-A)^\alpha v = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda - A)^{-1}(-Av)\, d\lambda,\quad v \in D(A). \tag{3.2}$$

In this chapter, we use a two-index notation $S_{\alpha,\beta}(t)$ for the solution operator of (3.1). According to the assumptions made about the operator A, the classical problem (2.4) is well-posed with corresponding solution operator $S_{1,1}(t)$.

3.1 Subordination formula

By successively applying two known results for subordination in space and in time ([31] and [7]), the following theorem is derived.

Theorem 3.1. *If the operator A generates a bounded C_0-semigroup $S_{1,1}(t)$ then the problem (3.1) is correctly posed and has a bounded solution operator $S_{\alpha,\beta}(t)$ which has the following integral representation*

$$S_{\alpha,\beta}(t) = \int_0^\infty \psi_{\alpha,\beta}(t, \tau) S_{1,1}(\tau)\, d\tau,\quad t > 0. \tag{3.3}$$

The subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ is a one-sided probability density with respect to $\tau \geq 0$ (i.e., it satisfies (2.10)). The following relations are satisfied

$$\int_0^\infty \psi_{\alpha,\beta}(t,\tau)e^{-\lambda\tau}\,d\tau = E_\beta(-\lambda^\alpha t^\beta), \tag{3.4}$$

and

$$\int_0^\infty \psi_{\alpha,\beta}(t,\tau)e^{-st}\,dt = s^{\beta-1}\tau^{\alpha-1}E_{\alpha,\alpha}(-s^\beta\tau^\alpha). \tag{3.5}$$

Efforts are directed next to obtain representations of the subordination kernel $\psi_{\alpha,\beta}(t,\tau)$ and to study its properties as well as those of the subordination operator $S_{\alpha,\beta}(t)$. The following results were obtained:

Theorem 3.2. *The subordination kernel is given by the formula*

$$\psi_{\alpha,\beta}(t,\tau) = t^{-\beta/\alpha}K_{\alpha,\beta}(\tau t^{-\beta/\alpha}),$$

where the function $K_{\alpha,\beta}$ has the following representations

$$K_{\alpha,\beta}(r) = \int_0^\infty \sigma^{-1/\alpha}L_\alpha(r\sigma^{-1/\alpha})M_\beta(\sigma)\,d\sigma, \tag{3.6}$$

$$K_{\alpha,\beta}(r) = \int_0^\infty \sigma^{\beta/\alpha}L_\alpha(r\sigma^{\beta/\alpha})L_\beta(\sigma)\,d\sigma, \tag{3.7}$$

$$K_{\alpha,\beta}(r) = \alpha r^{\alpha-1}\int_0^\infty \sigma M_\alpha(\sigma)M_\beta(\sigma r^\alpha)\,d\sigma. \tag{3.8}$$

Here, L_α is the Lévy extremal stable density (1.7) and M_β is the Mainardi function (1.5). Moreover, in the special case $\alpha = \beta$ it holds

$$K_{\alpha,\alpha}(r) = \frac{1}{\pi}\frac{r^{\alpha-1}\sin\alpha\pi}{r^{2\alpha}+2r^\alpha\cos\alpha\pi+1}. \tag{3.9}$$

Theorem 3.3. *Let $0<\alpha\leq 1$, $0<\beta\leq 1$, and $\alpha\beta\neq 1$. Then the kernel $\psi_{\alpha,\beta}(t,\tau)$ has the following integral representation*

$$\psi_{\alpha,\beta}(t,\tau) = \frac{\tau^{\alpha-1}}{\pi}\int_0^\infty r^{\beta-1}\left(C_\beta(r,t)I_{\alpha,\beta}(r,\tau) + S_\beta(r,t)R_{\alpha,\beta}(r,\tau)\right)\,dr, \tag{3.10}$$

where $C_\beta(r,t) = \cos(rt+\beta\pi/2)$, $S_\beta(r,t) = \sin(rt+\beta\pi/2)$ and

$$I_{\alpha,\beta}(r,\tau) = \Im\{\tau^{\alpha-1}E_{\alpha,\alpha}(-\tau^\alpha r^\beta e^{i\beta\pi/2})\} = \sum_{k=0}^\infty \frac{(-1)^k\tau^{\alpha k+\alpha-1}r^{\beta k}\sin k\beta\pi/2}{\Gamma(\alpha k+\alpha)},$$

$$R_{\alpha,\beta}(r,\tau) = \Re\{\tau^{\alpha-1}E_{\alpha,\alpha}(-\tau^\alpha r^\beta e^{i\beta\pi/2})\} = \sum_{k=0}^\infty \frac{(-1)^k\tau^{\alpha k+\alpha-1}r^{\beta k}\cos k\beta\pi/2}{\Gamma(\alpha k+\alpha)}.$$

Theorem 3.4. *Let $0<\alpha,\beta\leq 1$, $\alpha\beta\neq 1$ and*

$$\theta_0 = \min\left\{\frac{(2-\alpha-\beta)\pi}{2\beta}, \frac{\pi}{2}\right\}. \tag{3.11}$$

Then for each $\tau > 0$ the function $\psi_{\alpha,\beta}(t,\tau)$ as a function of t has an analytic extension in the sector $|\arg t| < \theta_0$ which is bounded on each subsector $|\arg t| \le \theta$, $0 < \theta < \theta_0$.

The last kernel analyticity result leads to the analyticity of the subordinate solution over a larger sector in the complex plane, compared to the solution of the original problem (2.4).

Theorem 3.5. *If $0 < \alpha, \beta \le 1$, $\alpha\beta \ne 1$ and the operator A generates a bounded analytic semigroup $S_{1,1}(t)$ of angle $\phi_0 \in (0, \pi/2]$, then $S_{\alpha,\beta}(t)$ is a bounded analytic solution operator of angle θ_0, where*

$$\theta_0 = \min\left\{\frac{\alpha\phi_0}{\beta} + \frac{(2-\alpha-\beta)\pi}{2\beta}, \frac{\pi}{2}\right\}. \tag{3.12}$$

In the limiting case $\phi_0 = 0$ ($S_{1,1}(t)$ is only of class C_0 and not an analytic semigroup) the subordinated solution operator $S_{\alpha,\beta}(t)$ is again analytic. From (3.12) we obtain the analyticity sector of $S_{\alpha,\beta}(t)$, which at $\phi_0 = 0$ coincides with the analyticity sector of the kernel (3.11). Therefore, the subordinate solution is always analytic, regardless of whether the solution of (2.4) has this property.

3.2 Multi-dimensional fundamental solution

Let us apply the subordination formula (3.3) to find the solution of the following basic special case of the abstract Cauchy problem (3.1)

$${}^{C}D_t^{\beta} u(\mathbf{x},t) = -(-\Delta)^{\alpha} u(\mathbf{x},t), \quad t>0, \;\; \mathbf{x} \in \mathbb{R}^n; \quad u(\mathbf{x},0) = v(\mathbf{x}); \tag{3.13}$$

where $0 < \alpha, \beta \le 1$, ${}^{C}D_t^{\beta}$ is the Caputo derivative, and Δ is the Laplace operator in $\mathbb{R}^n$. The solution operator $S_{\alpha,\beta}(t)$ of the Cauchy problem (3.13) is defined by

$$(S_{\alpha,\beta}(t)v)(\mathbf{x}) = \int_{\mathbb{R}^n} \mathcal{G}_{\alpha,\beta,n}(\mathbf{y},t) v(\mathbf{x}-\mathbf{y})\, d\mathbf{y}, \quad v \in X, \quad t>0, \; \mathbf{x} \in \mathbb{R}^n,$$

where $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$ is the corresponding Green function (fundamental solution). Therefore, the subordination formula (3.3) can be written as a relation between Green functions as follows

$$\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t) = \int_0^{\infty} \psi_{\alpha,\beta}(t,\tau) \mathcal{G}_{1,1,n}(\mathbf{x},\tau)\, d\tau, \quad \mathbf{x} \in \mathbb{R}^n. \tag{3.14}$$

It is known that [2]

$$\mathcal{G}_{1,1,n}(\mathbf{x},t) = \frac{1}{(4\pi t)^{n/2}} e^{-|\mathbf{x}|^2/4t}, \quad \mathbf{x} \in \mathbb{R}^n, \; t>0. \tag{3.15}$$

The formulas (3.14) and (3.15) have been used to obtain representations for the Green's function $\mathcal{G}_{\alpha,\beta,n}(\mathbf{x},t)$. Thus, in the special case $\alpha = \beta = 1/2$ we obtain the explicit representation

$$\mathcal{G}_{1/2,1/2,n}(\mathbf{x},t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2^n\; pi^{n/2+1} t^{n/2}} U\left(\frac{n+1}{2}, \frac{n+1}{2}, \frac{|\mathbf{x}|^2}{4t}\right), \quad \mathbf{x} \in \mathbb{R}^n, \tag{3.16}$$

where U is the confluent Tricomi hypergeometric function ([1], Eq. 13.2.5)

$$U(a,c,z)=\frac{1}{\Gamma(a)}\int_0^\infty \xi^{a-1}(1+\xi)^{c-a-1}e^{-z\xi}\,d\xi,\ a>0, z>0. \tag{3.17}$$

From the formulas (3.14) and (3.15), the following integral representations of $\mathcal{G}_{\alpha,\beta,n}$ for $n=1,2,3$ are derived:

Theorem 3.6. *Let $0<\alpha,\beta\le 1$ and $\alpha\beta\neq 1$. Then*

$$\mathcal{G}_{\alpha,\beta,1}(x,t)=\frac{2\alpha}{\beta}\frac{t^\beta}{\pi|x|}\int_0^\infty \sin(|x|\sigma)\sigma^{2\alpha-1}E_{\beta,\beta}(-\sigma^{2\alpha}t^\beta)\,d\sigma,$$
$$\mathcal{G}_{\alpha,\beta,2}(\mathbf{x},t)=-\frac{1}{2\pi^2|\mathbf{x}|^2}\int_0^\pi \frac{1}{\cos^2\theta}\left(1+\int_0^\infty \cos(|\mathbf{x}|\sigma\cos\theta)H_{\alpha,\beta}(\sigma,t)\,d\sigma\right)d\theta,$$
$$\mathcal{G}_{\alpha,\beta,3}(\mathbf{x},t)=\frac{1}{2\pi^2|\mathbf{x}|^3}\int_0^\infty \sin(|\mathbf{x}|\sigma)H_{\alpha,\beta}(\sigma,t)\,d\sigma.$$

The function $H_{\alpha,\beta}$ is defined as follows in terms of Mittag-Leffler functions

$$H_{\alpha,\beta}(\sigma,t)=\mu\sigma^{2\alpha-1}t^\beta\left((1+\mu)\,E_{\beta,\ be}(-\sigma^{2\alpha}t^\beta)+\mu E_{\beta,\beta-1}(-\sigma^{2\alpha}t^\beta)\right), \tag{3.18}$$

where $\mu=2\alpha/\beta$.

4 Transition from diffusion to wave propagation

In Chapter 4 (consisting of 4 sections)the Jeffrey-type fractional heat conduction equation is studied. It is an evolution equation containing fractional Riemann-Liouville derivatives in time, which, depending on the model parameters, satisfies two different subordination principles and, accordingly, two fundamentally different types of behavior: the diffusion regime and wave propagation regime. Integral representations for the Green function of the one-dimensional Cauchy problem are derived. It is shown that the Green function represents a probability density in the spatial variable that evolves with time. It is unimodal in the diffusion regimee and bimodal in the wave propagation regime. The considered example illustrates how the principle of subordination is closely related to the physical properties of a given model.

4.1 Subordination theorems

Consider the equation

$$(1+aD_t^\alpha)\,u'(t)=(1+bD_t^\alpha)\,Au(t),\quad t>0, \tag{4.1}$$

with initial conditions $u(0)=v$ and $u'(0)=0$, where D_t^α is the Riemann-Liouville fractional derivative of order $\alpha\in(0,1]$, $a,b\ge 0$, and A is a linear closed operator in a Banach space. When A is the second derivative in space, then (4.1) is the Jeffrey-type fractional heat conduction equation [5], Chapter 7.

The problem (4.1) is written as a Volterra integral equation (2.7) with a characteristic function

$$g(s) = (\widehat{k}(s))^{-1} = \frac{s(1+as^\alpha)}{1+bs^\alpha}, \quad s > 0,$$

and for $a = b$ the classical problem (2.4) is obtained. The following properties play a decisive role in the study:

$$g(s)/s \in \mathcal{SF} \ \text{ for } \ a < b; \qquad g(s)/s \in \mathcal{CBF} \ \text{ for } \ a > b.$$

From them follows:

Proposition 4.1. *Let $0 < \alpha \le 1$ and $a, b \ge 0$. Then $\sqrt{g(s)} \in \mathcal{CBF}$. If we further assume $0 \le a < b$, then $g(s) \in \mathcal{CBF}$.*

This statement together with Theorem 2.1 leads to the following two subordination theorems.

Theorem 4.1. *Let $a, b \ge 0$ and $0 < \alpha \le 1$. Let the operator A generate a bounded strongly continuous cosine operator function $S_2(t)$. Then the problem (4.1) is well-posed and the solution operator $S(t)$ satisfies the identity*

$$S(t) = \int_0^\infty \varphi_1(t,\tau) S_2(\tau)\, d\tau, \quad t > 0, \tag{4.2}$$

where the kernel $\varphi_1(t,\tau)$ is a one-sided probability density defined by means of the Laplace transform

$$\widehat{\varphi}_1(s,\tau) = \frac{\sqrt{g(s)}}{s} \exp\left(-\tau\sqrt{g(s)}\right), \quad s, \tau > 0. \tag{4.3}$$

Theorem 4.2. *Let $0 \le a < b$ and $0 < \alpha \le 1$. Suppose that A generates a bounded C_0-semigroup $S_1(t)$. Then the problem (4.1) is well-posed and the solution operator $S(t)$ satisfies the identity*

$$S(t) = \int_0^\infty \varphi_2(t,\tau) S_1(\tau)\, d\tau, \quad t > 0.$$

The kernel $\varphi_2(t,\tau)$ is a one-sided probability density defined by means of the Laplace transform

$$\widehat{\varphi}_2(s,\tau) = \frac{g(s)}{s} \exp\left(-\tau g(s)\right), \quad s, \tau > 0.$$

These two theorems show that for $a < b$ the equation (4.1) obeys the first-order equation (2.4), while for $a > b$ it obeys the second-order equation (2.5). This corresponds to substantially different properties of the solution, as we will see with the example of the one-dimensional Cauchy problem below.

Since at $a > b \ge 0$ the property is satisfied

$$g(s)^{1/(\alpha+1)} \in \mathcal{CBF}, \quad s > 0, \tag{4.4}$$

then the following more precise result holds in this case:

Theorem 4.3. *Let $a > b \geq 0$ and $0 < \alpha \leq 1$. We assume that the Cauchy problem for the fractional evolution equation (2.3) of order $\alpha + 1$ is well-posed and has a bounded solution operator $S_{\alpha+1}(t)$. Then the problem (4.1) is correctly posed with a solution operator $S(t)$ satisfying the equality*

$$S(t) = \int_0^\infty \varphi(t,\tau) S_{\alpha+1}(\tau)\, d\tau, \quad t > 0.$$

The kernel $\varphi(t,\tau)$ is a one-sided probability density defined by means of the Laplace transform

$$\widehat{\varphi}(s,\tau) = \frac{g(s)^{1/(\alpha+1)}}{s} \exp\left(-\tau g(s)^{1/(\alpha+1)}\right), \quad s, \tau > 0.$$

In the case of a fractional model, $0 < \alpha < 1$, this result is stronger than the one formulated in Theorem 4.1.

4.2 One-dimensional fundamental solution

As a particular example of (4.1) we consider the task

$$(1 + aD_t^\alpha) \frac{\partial}{\partial t} u(x,t) = (1 + bD_t^\alpha) \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{4.5}$$

$$u(x,0) = u_0(x); \quad \lim_{t\to 0^+} \frac{\partial}{\partial t} u(x,t) = 0, \quad x \in \mathbb{R}, \tag{4.6}$$

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad t > 0. \tag{4.7}$$

Applying Laplace transform in t and Fourier transform in x gives the following result:

$$\widehat{\mathcal{G}}(x,s) = \frac{\sqrt{g(s)}}{2s} \exp\left(-|x|\sqrt{g(s)}\right), \quad x \in \mathbb{R}. \tag{4.8}$$

Let us point out the relation with the subordination kernel, whose Laplace transform is given in (4.3).

Applying the Laplace transform inversion formula to (4.8) yields an integral representation for the fundamental solution in terms of elementary functions.

Theorem 4.4. *The fundamental solution $\mathcal{G}(x,t)$ of the Cauchy problem (4.5)-(4.6)-(4.7) has the integral representation for $x \in \mathbb{R}\backslash\{0\}$ and $t > 0$:*

$$\begin{aligned} \mathcal{G}(x,t) &= \frac{1}{2\pi} \int_0^\infty \exp\left(-|x|K^-(r)\right) \left(K^-(r) \sin\left(rt - |x|K^+(r)\right)\right. \\ &+ \left. K^+(r) \cos\left(rt - |x|K^+(r)\right)\right) \frac{dr}{r}. \end{aligned} \tag{4.9}$$

The functions $K^\pm(r)$ are defined by the equalities

$$K^\pm(r) = \left(\frac{r}{2}\right)^{1/2} \left(\left(A^2(r) + B^2(r)\right)^{1/2} \pm A(r)\right)^{1/2} \tag{4.10}$$

where

$$A(r) = \frac{(a-b)r^{\alpha}\sin(\alpha\pi/2)}{1+2br^{\alpha}\cos(\alpha\pi/2)+b^2r^{2\alpha}},$$

$$B(r) = \frac{1+(a+b)r^{\alpha}\cos(\alpha\pi/2)+abr^{2\alpha}}{1+2br^{\alpha}\cos(\alpha\pi/2)+b^2r^{2\alpha}}.$$

The integral representation (4.9) is suitable for numerical computation and visualization of the solution. At $a < b$ a diffusion regime is observed, while at $a > b$ the behavior has a character of diffusive waves. The observed behavior is analogous to the behavior of the solution of the fractional diffusion-wave equation (2.3) with $A = \partial^2/\partial x^2$, where the diffusion mode is at $0 < \alpha \leq 1$, and the wave mode is at $1 < \alpha \leq 2$.

For brevity, in the rest of the dissertation we call generalized subdiffusion equations those subordinated to the first-order equation (2.4), and equations that are subordinated to the second-order equation (2.5), but are not generalized subdiffusion equations, we call generalized diffusion-wave equations.

5 Generalized subdiffusion equations

Chapter 5 (consisting of 4 sections) considers first the abstract Cauchy problem for the time-fractional evolution equations of distributed order with continuous or discrete distribution over the interval $[0, 1]$. The special case of problem (4.1) with $a = 0$ and $b > 0$ is studied in detail. Then, the problem with general convolutional derivative is investigated and two kinds of subordination theorems are established. The subordination principle in the scalar case is applied to derive useful estimates for relaxation functions that generalize some results for Mittag–Leffler functions. As an illustration of the application of these estimates, uniqueness and stability are proved for an inverse problem.

5.1 Equations with generalized convolutional derivative

The Caputo-type generalized convolutional derivative is introduced in [18] in the form

$$({}^{C}\mathbb{D}_t^{(\kappa)} f)(t) = \frac{d}{dt}\int_0^t \kappa(t-\tau)f(\tau)\,d\tau - \kappa(t)f(0), \quad t > 0, \tag{5.1}$$

where $\kappa(t) \in L^1_{loc}(\mathbb{R}_+)$ is a nonnegative function. For the kernel $\kappa(t)$, we assume that its Laplace transform $\widehat{\kappa}(s)$ exists for every $s > 0$ and

$$\widehat{\kappa}(s) \in \mathcal{SF} \quad \text{and} \quad \lim_{s\to+\infty} s\widehat{\kappa}(s) = +\infty, \tag{5.2}$$

where $\mathcal{SF}$ is the class of Stiltjes functions.

We consider the Cauchy problem

$${}^{C}\mathbb{D}_t^{(\kappa)} u(t) = Au(t), \qquad t > 0; \quad u(0) = a \in X, \tag{5.3}$$

where ${}^C\mathbb{D}_t^{(\kappa)}$ is the generalized convolutional derivative (5.1), and A an operator that generates a bounded C_0-semigroup.

Let us note that in the original definition of the generalized fractional derivative in [18] additional restrictions are imposed on the boundary behavior of the function $\widehat{\kappa}(s)$. To include some practically important equations (such as (4.1) with $0 \le a < b$ or the discussed in the next chapter (6.1) and (6.2) with $\alpha = 1$), we assume only the requirements (5.2). They are sufficient for the general subordination theorem Theorem 2.1 to hold in the special case $\alpha = 1$.

The following additional result follows from the Post-Wither formula for the inversion of the Laplace transform:

Theorem 5.1. *If the operator A generates a bounded C_0-semigroup and the kernel $\kappa(t)$ satisfies (5.2) then the problem (5.3) is well-posed and its solution $u(t)$ has the representation*

$$u(t) = \lim_{n\to\infty} \frac{1}{n!} (n/t)^{n+1} \sum_{k=0}^{n} \sum_{p=1}^{k} b_{n,k,p} (n/t) \left(g (n/t) - A\right)^{-(p+1)} u(0). \tag{5.4}$$

Here $g(s) = s\widehat{\kappa}(s)$ and the functions $b_{n,k,p}(s) \ge 0$ are defined as follows

$$b_{n,k,p}(s) = (-1)^{n+p} \binom{n}{k} \left(\frac{g(s)}{s}\right)^{(n-k)} a_{k,p}(s) p!, \quad s > 0, \tag{5.5}$$

where $a_{k,p}(s)$ are defined by the recurrence relations

$$\begin{aligned} &a_{k+1,p}(s) = a_{k,p-1}(s) g'(s) + a'_{k,p}(s), \quad 1 \le p \le k+1, \ k \ge 1, \\ &a_{k,0} = a_{k,k+1} \equiv 0, \quad a_{1,1}(s) = g'(s). \end{aligned} \tag{5.6}$$

Formula (5.4) summarizes the exponential representation

$$u(t) = \lim_{n\to\infty} \left(I - \frac{t}{n} A\right)^{-n} u(0)$$

for the solution of the classical Cauchy problem (2.4).

5.2 Relaxation functions

Let us now consider the general relaxation equation with convolutional derivative (5.1)

$${}^C\mathbb{D}_t^{(\kappa)} u(t) + \lambda u(t) = f(t), \quad \lambda > 0, \ \ t > 0; \qquad u(0) = a \in \mathbb{R}. \tag{5.7}$$

We denote by $u(t;\lambda)$ and $v(t;\lambda)$ the solutions corresponding to $a = 1$, $f \equiv 0$, and $a = 0$, $f(t) = \delta(t)$, where $\delta(t)$ is the Dirac delta function. The functions $u(t;\lambda)$ and $v(t;\lambda)$ are called relaxation functions.

The solution of (5.7) is presented as follows

$$u(t) = a u(t;\lambda) + \int_0^t v(\tau;\lambda) f(t-\tau)\, d\tau. \tag{5.8}$$

In the special case where ${}^C\mathbb{D}_t^{(\kappa)}$ is the Caputo fractional derivative ${}^CD_t^\alpha$, $0 < \alpha < 1$, the relaxation functions are represented by Mittag-Leffler functions: $u(t;\lambda) = E_\alpha(-\lambda t^\alpha)$ and $v(t;\lambda) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$. The next two theorems summarize some properties of these functions.

Theorem 5.2. *For the relaxation functions $u(t;\lambda)$ and $v(t;\lambda)$ the integral representations hold*

$$u(t;\lambda) = \int_0^\infty \varphi(t,\tau)e^{-\lambda\tau}\,d\tau, \quad t > 0, \tag{5.9}$$

$$v(t;\lambda) = \int_0^\infty \psi(t,\tau)e^{-\lambda\tau}\,d\tau, \quad t > 0, \tag{5.10}$$

where the functions $\varphi(t,\tau)$ and $\psi(t,\tau)$ satisfy the properties

$$\varphi(t,\tau) \geq 0,\ \ \psi(t,\tau) \geq 0; \quad \int_0^\infty \varphi(t,\tau)\,d\tau = 1,\ \int_0^\infty \psi(t,\tau)\,d\tau = k(t). \tag{5.11}$$

Here, $k(t)$ is the resolvent kernel of $\kappa(t)$, i.e. $(k * \kappa)(t) = 1$.

Theorem 5.3. *For each $\lambda > 0$, the functions $u(t;\lambda)$ and $v(t;\lambda)$ as functions of t have an analytic continuation in $\mathbb{C}_+$. For $t > 0$ they have the properties*

$$u(t;\lambda), v(t;\lambda) \in \mathcal{CMF}; \tag{5.12}$$

$$u(0;\lambda) = 1; \quad 0 < u(t;\lambda) < 1, \quad v(t;\lambda) > 0; \tag{5.13}$$

$$\frac{d}{dt}u(t;\lambda) = -\lambda v(t;\lambda). \tag{5.14}$$

Furthermore

$$u(t;\lambda) \leq \frac{1}{1 + \lambda(1 * k)(t)}, \tag{5.15}$$

where $k(t)$ is the resolving kernel of $\kappa(t)$, i.e. $(k * \kappa)(t) = 1$.

For each $\lambda \geq \lambda_0 > 0$ and $t > 0$

$$u(t;\lambda) \leq u(t;\lambda_0), \quad v(t;\lambda) \leq v(t;\lambda_0), \tag{5.16}$$

and

$$C \leq \lambda \int_0^T v(t;\lambda)\,dt < 1, \quad T > 0, \tag{5.17}$$

where the constant $C = 1 - u(T;\lambda_0) > 0$ does not depend on λ.

The estimates in Theorem 5.3 are very useful in studying boundary value problems by applying eigenfunction expansion.

6 Multinomial functions of Mittag-Leffler type

In Chapter 6 (consisting of 3 sections) we continue the study of evolution equations with several time derivatives of different orders in the interval $(0,1]$. The main emphasis is now on the Mittag-Leffler multinomial function that appears in the representation of their solutions. The basic properties of this function and its Prabhakar-type generalization are investigated. Conditions are found for the parameters under which the function is completely monotone. Some subordination equalities are established. As specific examples, relaxation functions for equations with several time derivatives are studied in detail. The obtained results generalize known properties of the classical Mittag-Leffler function.

6.1 Multinomial Mitag-Leffler function

Various types of multi-index generalizations of the classical Mittag-Leffler function (1.2) are discussed in the literature (e.g. [16, 17, 24] and the monographs [12, 23]). One such generalization is the multinomial Mittag-Leffler function

$$E_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m)=\sum_{k=0}^{\infty}\sum_{\substack{k_1+\ldots+k_m=k\\ k_1\geq 0,\ldots,k_m\geq 0}}\frac{k!}{k_1!\ldots k_m!}\frac{\prod_{j=1}^m z_j^{k_j}}{\Gamma\left(\beta+\sum_{j=1}^m \mu_j k_j\right)},$$

where $z_j\in\mathbb{C}$, $\mu_j>0$, $\beta\in\mathbb{R}$, $j=1,\ldots,m$. This function was introduced in [14, 19], where it is used to solve multi-term fractional differential equations with constant coefficients. More precisely, the following function of one variable t is involved in expressing the solutions:

$$\mathcal{E}_{(\mu_1,\ldots,\mu_m),\beta}(t;a_1,\ldots,a_m)=t^{\beta-1}E_{(\mu_1,\ldots,\mu_m),\beta}(-a_1t^{\mu_1},\ldots,-a_mt^{\mu_m}).$$

Consider the following two multi-term equations: with Caputo derivatives

$$^{C}D_t^{\alpha}u(t)+\sum_{j=1}^m b_j\, {}^{C}D_t^{\alpha_j}u(t)=Au(t)+f(t),\quad t>0, \tag{6.1}$$

and with Riemann-Liouville derivatives

$$u'(t)=D_t^{1-\alpha}Au(t)+\sum_{j=1}^m b_j\, D_t^{1-\alpha_j}Au(t)+f(t),\quad t>0, \tag{6.2}$$

where $1\geq\alpha>\alpha_1>\ldots>\alpha_m>0$, $b_j>0$, $j=1,\ldots,m$, and A is an operator generating a C_0 semigroup. From Theorem 2.1 we derive the following subordination result:

Theorem 6.1. *If the Cauchy problem (2.3) has a bounded operator of the solution $S_\alpha(t)$, then the problems (6.1), respectively (6.2), are well-posed and their solution operators are represented as follows*

$$S(t)=\int_0^\infty \varphi(t,\tau)S_\alpha(\tau)\,d\tau,\quad t>0.$$

The kernel $\varphi(t,\tau)$ is a one-sided probability density in $\tau\geq 0$ defined by means of the Laplace transform (2.11), where

$$g(s)=s^\alpha+\sum_{j=1}^m b_js^{\alpha_j}$$

in the case of equation (6.1) and

$$g(s) = \left(s^{-\alpha} + \sum_{j=1}^{m} b_j s^{-\alpha_j} \right)^{-1}$$

in the case of equation (6.2).

In the scalar case ($A = -\lambda$, where $\lambda > 0$ is a constant), the solutions of (6.1) and (6.2) are presented in the form

$$u(t) = u_n(t;\lambda) + \int_0^t v_n(t-\tau;\lambda) f(\tau)\, d\tau, \quad n = 1, 2,$$

where $n = 1$ for equation (6.1), $n = 2$ for equation (6.2) and

$$u_1(t;\lambda) = 1 - \lambda \mathcal{E}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha+1}\left(t;\lambda,b_1,\ldots,b_m\right), \tag{6.3}$$

$$v_1(t;\lambda) = \mathcal{E}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha}\left(t;\lambda,b_1,\ldots,b_m\right), \tag{6.4}$$

$$u_2(t;\lambda) = v_2(t;\lambda) = \mathcal{E}_{(\alpha,\alpha_1,\ldots,\alpha_m),1}\left(t;\lambda,\lambda b_1,\ldots,\lambda b_m\right). \tag{6.5}$$

The following representation is obtained as an additional result of the considerations in this chapter.

Theorem 6.2. *Let $0 < \alpha \le \beta \le 1$, $0 < \alpha_j < \alpha$, $\lambda > 0$, $b_j > 0$, $j = 1, \ldots, m$. Then*

$$\mathcal{E}_{(\alpha,\alpha_1,\ldots,\alpha_m),\beta}\left(t;\lambda,b_1,\ldots,b_m\right) = \int_0^\infty \phi(t,\tau) e^{-\lambda\tau}\, d\tau, \quad t > 0, \tag{6.6}$$

where the kernel $\phi(t,\tau) \ge 0$ has the representation

$$\phi(t,\tau) = \omega_{\beta-\alpha}(t) * h_\alpha(t,\tau) * h_{\alpha-\alpha_1}(t, b_1\tau) \; ast \ldots * h_{\alpha-\alpha_m}(t, b_m\tau).$$

Here $*$ is the Laplace convolution with respect to the variable t, the function $\omega_\alpha(t)$ is defined in (1.1) and

$$h_\alpha(t,\sigma) = \sigma^{-1/\alpha} L_\alpha\left(t\sigma^{-1/\alpha}\right),$$

where $L_\alpha(\cdot)$ is the Lévy stable extremal density (1.7).

6.2 Multinomial Prabhakar function

For brevity we use the vector notation $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$.

Multinomial Prabhakar function is defined as follows [8]

$$E^{\delta}_{\vec{\mu},\beta}(z_1,\ldots,z_m) = \sum_{k=0}^{\infty} \sum_{\substack{k_1+\ldots+k_m=k \\ k_1\ge 0,\ldots,k_m\ge 0}} \frac{(\delta)_k}{k_1!\cdots k_m!} \frac{\prod_{j=1}^m z_j^{k_j}}{\Gamma\left(\beta + \sum_{j=1}^m \mu_j k_j\right)}, \tag{6.7}$$

where $z_j \in \mathbb{C}$, $\mu_j, \beta, \delta \in \mathbb{R}$, $\mu_j > 0$, $j = 1, \ldots, m$. Here $(\delta)_k$ denotes the Pochhamer symbol (1.4). We set

$$\mathcal{E}^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(t;a_1,\ldots,a_m) = t^{\beta-1} E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(-a_1 t^{\mu_1},\ldots,-a_m t^{\mu_m}). \tag{6.8}$$

Some important results for the function (6.8) are given next.

Theorem 6.3. *The Laplace transform* $\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a})$ *of the function* $\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})$ *is defined by the equality*

$$\widehat{\mathcal{E}}^{\delta}_{\vec{\mu},\beta}(s;\vec{a}) := \mathcal{L}\left\{\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})\right\}(s) = \frac{s^{-\beta}}{\left(1+\sum_{j=1}^{m} a_j s^{-\mu_j}\right)^{\delta}}, \quad s \in \mathbb{C}_+. \tag{6.9}$$

Based on Theorem 6.3 and the properties of Bernstein functions, the following complete monotonicity result is proved:

Theorem 6.4. *Let* $0 < \mu_j \leq 1$, $a_j > 0$, $j = 1, \ldots, m$, *and* $0 < \mu_* \delta \leq \beta \leq 1$, *where* $\mu_* = \max_{j=1,\ldots,m}\{\mu_j\}$. *Then*

$$\mathcal{E}^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(t; a_1, \ldots, a_m) \in \mathcal{CMF}, \quad t > 0. \tag{6.10}$$

This is one of the main results in this chapter.

As an example of the application of Prabhakar's functions (6.8) with $\delta \neq 1$, the moments of the fundamental solutions of the Cauchy problems for equations (6.1) and (6.2) in the case $A = \left(\frac{\partial}{\partial x}\right)^2$, $x \in \mathbb{R}$.

In the last two chapters, we study the subordination principle for generalized diffusion-wave equations with fractional time derivatives. Various linear generalizations of the fractional diffusion-wave equation (2.3) with $1 < \alpha < 2$ have been considered in the literature, the most studied examples being the distributed-order fractional diffusion-wave equations and various equations modeling the propagation of waves in viscoelastic media.

7 Distributed-order diffusion-wave equations

Chapter 7 (consisting of 2 sections) is devoted to diffusion-wave equations with Caputo fractional derivatives whose orders are discretely or continuously distributed in the interval $(0, 2]$. We first discuss an open problem concerning the interpretation of the fundamental solution of the corresponding one-dimensional Cauchy problem as a spatial probability density. Then, the subordination principle for the multi-term diffusion-wave equation is studied in detail.

7.1 When the fundamental solution is a probability density?

We consider the distributed order equation

$$\int_0^2 \mu(\beta)\, {}^C D_t^{\beta} u(x,t)\, d\beta = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R}, t > 0, \tag{7.1}$$

where $\mu(\beta)$ is a nonnegative function such that

$$\operatorname{supp} \mu \,\cap\, (1, 2] \neq \emptyset.$$

The Cauchy problem for (7.1) with initial conditions $u(x, 0) = v(x)$ and $u_t(x, 0) = 0$ is studied in [13] with the main focus on the interpretation of the fundamental solution $\mathcal{G}(x,t)$ as a probability density in $x \in \mathbb{R}$ (for $t > 0$ considered as a parameter), i.e. satisfying:

$$\mathcal{G}(x,t) \geq 0, \qquad \int_{-\infty}^{\infty} \mathcal{G}(x,t)\, dx = 1. \tag{7.2}$$

The importance of the properties (7.2) for the stochastic interpretation of the equation (7.1) and for its physical meaning is explained in [11]. At the same time, the fulfillment of these conditions ensures the existence of subordination for equations of the type (7.1) with respect to the second-order Cauchy problem. This is so because of the following relationship between the kernel $\varphi(t,\tau)$ in the subordination identity (7.6) below and the fundamental solution $\mathcal{G}(x,t)$:

$$\varphi(t,\tau) = 2\mathcal{G}(x,t), \quad \text{at} \quad \tau = x \geq 0.$$

Let us note that $\mathcal{G}(-x,t) = \mathcal{G}(x,t)$, $x \in \mathbb{R}$.

A sufficient condition for the fundamental solution of (7.1) to satisfy properties (7.2) is $g(s)^{1/2} \in \mathcal{CBF}$, where

$$g(s) = \int_0^2 \mu(\beta) s^\beta \, d\beta, \quad s > 0. \tag{7.3}$$

In [13] it is proved that if $\operatorname{supp}\mu \subseteq [1,2]$ then $g(s)^{1/2} \in \mathcal{CBF}$ and therefore the properties (refGGppddff) are met. At the same time, the question arises whether this condition for the weight function μ can be relaxed. We prove that the support of the weight function can be an arbitrary subinterval of $[0,2]$ with length no more than 1.

Proposition 7.1. *Let* $\operatorname{supp}\mu \subseteq [\alpha-1,\alpha]$, $1 < \alpha \leq 2$. *Then* $g(s)^{1/2} \in \mathcal{CBF}$.

Proposition 7.1 applies to both continuous and discrete weight function μ. The following statement gives a special example of a weight function for which the condition on its support can be further relaxed. In the considered case, the support of the weight function can be any subinterval of the interval $[0,2]$.

Proposition 7.2. *Let* $a > 0$ *and* $0 < \alpha_1 < \alpha_2 \leq 2$. *If* $\mu(\beta) = a^\beta$ *for* $\beta \in [\alpha_1, \alpha_2]$ *and* $\mu(\beta) = 0$ *for* $\beta \in (0,\alpha_1) \cup (\alpha_2, 2]$, *then* $g(s)^{1/2} \in \mathcal{CBF}$.

These two statements partially answer the question posed in [13]. For more precise results, the cases of continuous and discrete distribution should be considered separately.

In the rest of this chapter, we study in detail equations with a discrete distribution of the derivative orders in an interval $[\alpha-1,\alpha]$, where $\alpha \in (1,2]$.

7.2 Multi-term diffusion-wave equation

Consider the problem

$$c\,{}^C\!D_t^\alpha u(t) + \sum_{j=1}^m c_j\,{}^C\!D_t^{\alpha_j} u(t) = Au(t), \quad u(0) = a \in X, \ u'(0) = 0, \tag{7.4}$$

where the operator A generates a strongly continuous cosine function $S_2(t)$. For the parameters α, α_j, c, c_j, we assume that they satisfy the conditions

$$\begin{aligned} &\alpha \in (1,2], \quad \alpha > \alpha_1 > \cdots > \alpha_m > 0, \quad \alpha - \alpha_m \leq 1, \\ &c > 0, \quad c_j > 0, \quad j = 1, \cdots, m. \end{aligned} \tag{7.5}$$

Applying the general Theorem 2.1 we obtain the following result:

Theorem 7.1. *The problem (7.4) is well-posed and has a bounded solution operator $S(t)$ which is related to $S_2(t)$ by the identity*

$$S(t) = \int_0^\infty \varphi(t,\tau) S_2(\tau)\, d\tau, \quad t > 0. \tag{7.6}$$

The kernel $\varphi(t,\tau)$ is a probability density in τ and admits the representation

$$\begin{aligned} \varphi(t,\tau) &= \frac{1}{\pi} \int_0^\infty \exp\left(-\tau K^+(r)\right) \left(K^+(r) \sin\left(rt - \tau K^-(r)\right)\right. \\ &+ \left. K^-(r) \cos\left(rt - \tau K^-(r)\right)\right) \frac{dr}{r}, \quad t, \tau > 0, \end{aligned} \tag{7.7}$$

where $K^\pm(r)$ are defined as follows

$$\begin{aligned} K^\pm(r) &= \frac{1}{\sqrt{2}} \left(\left(A^2(r) + B^2(r)\right)^{1/2} \pm A(r) \right)^{1/2}; \\ A(r) &= c r^\alpha \cos(\alpha\pi/2) + \sum_{j=1}^m c_j r^{\alpha_j} \cos(\alpha_j \pi/2), \\ B(r) &= c r^\alpha \sin(\alpha\pi/2) + \sum_{j=1}^m c_j r^{\alpha_j} \sin(\alpha_j \pi/2). \end{aligned}$$

Again, the stronger statement holds true that the solution operator $S(t)$ is subordinated to $S_\alpha(t)$, where α is the largest order of time derivative in the equation (7.4). In addition, we have the following different propertes in the cases $\alpha = 2$ and $1 < \alpha < 2$:

Theorem 7.2. *If $1 < \alpha < 2$, then the solution operator $S(t)$ of problem (7.4) is bounded analytic with angle*

$$\theta_0 = \frac{(2-\alpha)\pi}{2\alpha}.$$

If $\alpha = 2$ then the kernel $\varphi(t,\tau)$ in (7.6) satisfies $\varphi(t,\tau) = 0$ for $\tau > t/\sqrt{c}$.

8 Wave propagation in linear viscoelastic media

Chapter 8 (consisting of 4 sections) discusses the subordination principle for equations modeling wave propagation in linear viscoelastic media. Various constitutive laws are considered, which are fractional generalizations of some classical models. For all models, the relaxation modulus is proved to be a completely monotone function. Problems of wave propagation in a viscoelastic fluid with the fractional Jeffrey model are studied in more detail and some applications of the subordination principle as well as its physical interpretation are given. The chapter concludes with a brief comment on the definition of the class of generalized fractional diffusion-wave equations.

The properties of a linear viscoelastic model are defined by a linear relationship between the stress σ and the strain ε. We consider the one-dimensional case where $\sigma = \sigma(x,t)$ and $\varepsilon = \varepsilon(x,t)$. The relaxation modulus $G(t)$ is defined by the identity

$$\sigma(x,t) = \int_0^t G(t-\tau)\dot{\varepsilon}(x,\tau)\,d\tau, \quad t>0, \tag{8.1}$$

where the dot traditionally means the first derivative in time.

For a model to make physical sense, the relaxation modulus $G(t)$ must satisfy the conditions $G(t) \geq 0$ and $G'(t) \leq 0$. In many cases, it turns out that these two conditions imply the stronger property $G(t) \in \mathcal{CMF}$. In the dissertation, this is proved for the fractional Maxwell, Jeffreys' and Zener generalized fractional laws.

8.1 Distributed-order fractional Zener model

Let us take a closer look at the generalized fractional distributed-order Zener model, which is defined by the constitutive equation [4]

$$\int_0^1 p_\sigma(\alpha) D_t^\alpha \sigma(x,t)\,d\alpha = \int_0^1 p_\varepsilon(\alpha) D_t^\alpha \varepsilon(x,t)\,d\alpha, \tag{8.2}$$

where $p_\sigma(\alpha)$ and $p_\varepsilon(\alpha)$ are nonnegative weight functions. This model is studied in [5], Chapter 3, without discussing the complete monotonicity of the corresponding relaxation modulus. Two cases are considered: the multi-term fractional Zener model

$$\sum_{n=0}^N a_n D_t^{\alpha_n} \sigma(x,t) = \sum_{n=0}^N b_n D_t^{\alpha_n} \varepsilon(x,t), \tag{8.3}$$

where $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_N < 1,\ a_n, b_n > 0, \quad n = 0,1,...,N$, and

$$\frac{a_0}{b_0} \geq \frac{a_1}{b_1} \geq \cdots \geq \frac{a_N}{b_N}, \tag{8.4}$$

and the case of power weight functions

$$p_\sigma(\alpha) = a^\alpha, \quad p_\varepsilon(\alpha) = b^\alpha, \qquad 0 < a < b. \tag{8.5}$$

We prove the following two theorems, which establish that $G(t) \in \mathcal{CMF}$ by means of its representation as the Laplace transform of a nonnegative function.

Theorem 8.1. *If the conditions (8.4) are satisfied, then the relaxation modulus $G(t)$ of the model (8.3) is a completely monotone function, which admits the representation*

$$G(t) = \frac{b_0}{a_0} + \int_0^\infty e^{-rt} K(r)\, dr, \tag{8.6}$$

where

$$K(r) = \frac{1}{\pi r} \frac{\sum_{0\le i<j\le N} (a_i b_j - a_j b_i) r^{\alpha_i+\alpha_j} \sin(\alpha_j - \alpha_i)\pi}{\left(\sum_{n=0}^N a_n r^{\alpha_n} \cos \alpha_n \pi\right)^2 + (\ sum_{n=0}^N a_n r^{\alpha_n} \sin \alpha_n \pi)^2} \ge 0. \tag{8.7}$$

Theorem 8.2. *The relaxation modulus $G(t)$ of model (8.2)-(8.5) is a completely monotone function, which admits the representation*

$$G(t) = 1 + \int_0^\infty e^{-rt} K(r)\, dr, \tag{8.8}$$

where

$$K(r) = \frac{(br+1)(\ln b - \ln a)}{r(ar+1)\left(\ln^2(br) + \pi^2\right)}\ ge0. \tag{8.9}$$

8.2 Subordination

Thanks to the property $G(t) \in \mathcal{CMF}$, it is possible to formulate a subordination dependence between the corresponding diffusion-wave equation and the classical wave equation. This is due to the fact that the propagation of waves in a viscoelastic medium with relaxation function $G(t)$ is defined by an integral equation of the form

$$u(x,t) = \int_0^t k(t-\tau) u_{xx}(x,\tau)\, d\tau + f(x,t), \tag{8.10}$$

where $k(t) = \int_0^t G(\tau)\, d\tau$. Then for the function $g(s)$ in Theorem 2.1 we have

$$g(s) = (\widehat{k}(s))^{-1} = \frac{s}{\widehat{G}(s)}. \tag{8.11}$$

According to the properties of Bernstein functions, $G(t) \in \mathcal{CMF}$ implies $\widehat{G}(s) \in \mathcal{SF}$. Therefore, $g(s)$ is the product of two functions of class $\mathcal{CBF}$ (s and $1/\widehat{G}(s)$), which gives $g(s)^{1/2} \in \mathcal{CBF}$. This means that the conditions of the general Theorem 2.1 are fulfilled for $\alpha = 2$ and the equation (8.10) is subordinated to the classical wave equation $u_{tt} = u_{xx}$ with the same initial and boundary conditions.

Therefore, the physical meaning of the subordination formula in this case is that it splits the solution of the diffusion-wave equation (8.10) into two parts, one (the probability density) depending only on the parameters of the viscoelastic medium, and the second (the solution of the classical wave equation) depending only on the imposed initial and boundary conditions.

8.3 Generalized diffusion-wave equations

By analogy with the generalized fractional subdiffusion equations discussed in Chapter 5, it is proposed in the work [29] to use the term generalized diffusion-wave equations for equations of the form

$$\int_0^t \eta(t-\tau)\frac{\partial^2}{\partial\tau^2}u(x,\tau)\,d\tau = \frac{\partial^2}{\partial x^2}u(x,t),$$

where $\widehat{\eta}(s) \in \mathcal{SF}$. This suggestion is based on the fact that the convolutional time derivative on the left is a generalization of the Caputo derivative of order $\alpha \in (1,2)$ which corresponds to $\widehat{\eta}(s) = s^{\alpha-2}$.

It turns out that this definition corresponds exactly to an equation of the form (8.10) with $G(t) \in \mathcal{CMF}$. However, this is only a specific class of diffusion-wave equations, which does not include many equations describing diffusion-wave processes. At the end of the chapter, examples of equations are given that are not of this type, but are nevertheless subordinated to the classical wave equation according to the principle of subordination and therefore describe intermediate processes between diffusion and wave propagation. One such example is an equation with two fractional time derivatives of orders α and α_1 such that $\alpha \in (1,2), \alpha_1 \in (0,1), \alpha - \ al_1 \leq 1$.

These observations suggest an extension of the definition of a generalized diffusion-wave equation proposed in [29]. One possible way can be based on the principle of subordination as follows: generalized diffusion-wave equations are all equations that are subordinated by the subordination principle to the classical wave equation, but are not subdiffusion equations.

V. SCIENTIFIC CONTRIBUTIONS

In the opinion of the author the main contributions in the dissertation and the related publications are the following:

- A unified methodology is developed for establishing a subordination relation between a linear evolution equation of general type and a linear evolution equation of fractional or integer order. The problem of subordination between the two equations is reduced to proving that a characteristic function belongs to the class of complete Bernstein functions.
- Subordination relations are established for a number of equations with fractional time derivatives that are found in the scientific literature. These representations split the solution into two parts: subordination kernel (a probability density function, containing all information about the operators acting on time); the solution of a simpler equation of integer or fractional order (containing information about the geometry of the problem: the imposed initial and boundary conditions).
- The subordination principle for space-time fractional equations is studied (Chapter 3). Different representations of the subordination kernel are obtained and its properties are studied. The sector of the complex plane is found, in which the subordinate solution is bounded analytical. Using the subordination identity, integral representations are derived for the n-dimensional fundamental solution, n=1,2,3, as well as explicit formulas in some special cases.
- Evolution equations with Jeffrey's fractional constitutive law are studied in detail (Chapter 4 and Section 8.3). Based on this model, the relationship between the principle of subordination and the physical character of an evolution equation is shown.
- Subordination dependences are established for the solutions of equations describing anomalous diffusion (Chapter 5). An explicit approximation formula is derived that generalizes the exponential formula for C0-semigroups. As an application of the subordination formula, a useful two-sided estimate for the solution of the generalized relaxation equation is derived, which is applied in the study of an inverse problem.
- A multinomial function of Prabhakar type is introduced and studied (Chapter 6). Conditions are found under which the function is completely monotone. This property allows the multinomial Prabhakar-type function to be used to propose a model that generalizes known relaxation laws (Section 8.2.4).
- The question of the conditions under which the one-dimensional fundamental solution of the distributed-order diffusion-wave equation is a probability density is partially resolved (Section 7.1). The class of admissible weight functions is extended from functions with support contained in the interval [1,2] to functions

with support contained in the interval [a,a+1], 0<a? 1. It is proven that in special cases this condition can be further relaxed.

- The subordination principle for diffusion-wave equations with several time derivatives of different (fractional) orders is studied in detail (Section 7.2). An integral representation of the subordination kernel is derived. The cases of finite and infinite propagation speed are considered.

- The relaxation modulus for some generalized fractional viscoelastic models defined in the literature is investigated. The fractional Maxwell, Jeffrey, and distributed-order Zener models are shown to make physical sense if and only if the corresponding relaxation moduli are completely monotone functions (Section 8.2). This property plays an important role in establishing a subordination principle for the corresponding wave equations.

- Based on the subordination principle, two main classes of generalized fractional evolution equations are defined: equations describing subdiffusion (subordinated to the classical diffusion equation) and diffusion-wave equations (subordinated to the classical wave equation, which are not subdiffusion equations). This way of classification is physically correct and extends the definitions of these two classes proposed in the literature, allowing some important physically meaningful models to be covered.

VI. APPROVAL OF THE RESULTS

The results in this dissertation has been presented at more than 10 international scientific conferences, including:

- International Conference on "Fractional Differentiation and its Applications", Novi Sad, Serbia (2016);

- 8th International conference "Transform Methods & Special Functions", Sofia (2017);

- "Mathematics Days in Sofia" (2017);

- 15th International Conference of Numerical Analysis and Applied Mathematics, Thessaloniki, Greece (2017);

as well as at the following international forums held in Novi Sad, Serbia:

"Pannonian Mathematical Modelling" (2015), "Applications of Generalized Functions in Harmonic Analysis, Mechanics, Stochastics and PDE" (2017) and "Topics in Fractional Calculus and Time-Frequency Analysis" (2020).

In addition, the obtained results were reported at the joint seminar "Analysis, Geometry and Topology" at IMI-BAS (2015), at the annual scientific sessions of the "Analysis, Geometry and Topology" section at IMI-BAS, as well as at the Mathematical Modeling seminar of FMI-Sofia University in 2015, 2017 and 2019.

Corectness of the results

Different methods were used in the dissertation to confirm the reliability of some of the derived analytical formulas. First, it is checked whether there is a qualitative agreement with the expected behavior of the considered quantity, so that the obtained formula makes physical sense (for example, the fundamental solution must always represent a probability distribution, the relaxation modulus must be a positive and non-increasing function). Second, some of the derived analytical results have been verified by being used for numerical computations and the obtained numerical results have been compared to those found by other authors (for example, results from Chapter 4 have been compared in publication [B10] with results from the monograph [5]) or with results obtained by another numerical method (see e.g. the comparisons given in Fig. 8.1). In addition, numerical comparisons with already known analytical formulas are performed in some particular cases.

Participation in scientific projects

The results included in the dissertation were established within the framework of the following scientific projects:

- project of the Scientific Research Fund (FNI): "Theoretical and numerical study of nonlinear mathematical models" (2014-2017);

- project in the frames of the national scientific program "Information and communication technologies for a single digital market in science, education and security" (2018-2021);

- international project under operational program "Science and education for intelligent growth": "Center for excellence in informatics and information and communication technologies" (2018-2023);

- project of FNI with Russia "Investigation of the dynamic behavior of deformable bodies taking into account the effects of heredity of the material" (2020-2023);

- three projects under a bilateral scientific agreement between the Bulgarian Academy of Sciences and the Serbian Academy of Sciences and Arts: "Mathematical modeling by integral transform methods, partial differential equations, special and generalized functions" (2012-2016), "Analytical and numerical methods for differential and integral equations and mathematical models of arbitrary (fractional or integer) order" (2017-2019) and "Operators, differential equations and special functions of fractional calculus - numerical methods and applications" (2020-2022).

Acknowledgements

First of all, I would like to express my gratitude to Corr. Member I. Dimovski and Prof. V. Kiryakova, who directed my scientific interests to the field of fractional calculus more than a quarter of a century ago and have continuously supported and encouraged me since then.

I am especially grateful to Prof. V. Kiryakova for the creation of an international center of fractional calculus at IMI (the scientific journal "Fractional Calculus and Applied Analysis", the traditional conference "Transform Methods & Special Functions", bilateral collaborations), which had a very positive impact on my career.

Many thanks to Prof. J. Paneva-Konovska for the useful recommendations, responsiveness and valuable help in preparing this dissertation.

I am also grateful to all other colleagues from the section "Analysis, Geometry and Topology" for the created favorable working environment and for the support during the preliminary presentation of the dissertation.

Emilia Bazhlekova, Associate Professor, PhD
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

References

[1] M. Abramowitz, I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* New York, Dover (1964).

[2] W. Arendt, C. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems.* Birkhäuser, Basel (2011).

[3] G.B. Arfken, H.J. Weber, *Mathematical Methods for Physicists.* Elsevier, Amsterdam (2005).

[4] T. Atanacković, On a distributed derivative model of a viscoelastic body. *C. R. Mécanique*, **331** (2003), 687-692.

[5] T. Atanacković, S. Pilipović, B. Stanković, D. Zorica, *Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes.* John Wiley & Sons, London (2014).

[6] A.V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them. *Pacific J. Math.* **10** (1961), 419-437.

[7] E. Bazhlekova, Subordination principle for fractional evolution equations. *Fract. Calc. Appl. Anal.* **3**, No 3 (2000), 213-230.

[8] E. Bazhlekova, I. Bazhlekov, Identification of a space-dependent source term in a nonlocal problem for the general time-fractional diffusion equation. *J. Comput. Appl. Math.* **386** (2021), Art. # 113213.

[9] W. Feller, *An Introduction to Probability Theory and its Applications.* vol. 2, Wiley, New York (1971).

[10] A. Giusti, I. Colombaro, R. Garra, R. Garrappa, F. Polito, M. Popolizio, F. Mainardi, A practical guide to Prabhakar fractional calculus. *Fract. Calc. Appl. Anal.* **23** (2020) 9-54.

[11] R. Gorenflo, Stochastic processes related to time-fractional diffusion-wave equation. *Commun. Appl. Ind. Math.* **6**, No 2 (2015), e-531.

[12] R. Gorenflo, A. Kilbas, F. Mainardi, S. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications.* 2nd Ed., Springer, Berlin- Heidelberg (2020).

[13] R. Gorenflo, Y. Luchko, M. Stojanović, Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density. *Fract. Calc. Appl. Anal.* **16** (2013), 297-316.

[14] S.B. Hadid, Y. Luchko, An operational method for solving fractional differential equations of an arbitrary real order. *Panam. Math. J.* **6**, No 1 (1996), 57-73.

[15] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics studies, Elsevier, Amsterdam (2006).

[16] V. Kiryakova, Unified approach to fractional calculus images of special functions - a survey. *Mathematics,* **8**, No 12 (2020), Art. # 2260.

[17] V. Kiryakova, A guide to special functions in fractional calculus. *Mathematics,* **9**, No 1 (2021), Art. # 106.

[18] A. Kochubei, General fractional calculus, evolution equations, and renewal processes. *Integr. Equ. Oper. Theory*, **71** (2011), 583-600.

[19] Yu. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Math. Vietnamica,* **24** (1999), 207-233.

[20] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity.* Imperial College Press, London (2010).

[21] F. Mainardi, Yu. Luchko, G. Pagnini, The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **4** (2001), 153-192.

[22] J. Mikusiński, On the function whose Laplace-transform is $\exp(-s^\alpha)$. *Stud. Math.* **18** (1959), 191-198.

[23] J. Paneva-Konovska, *From Bessel to Multi-Index Mittag-Leffler Functions: Enumerable Families, Series in them and Convergence.* World Sci. Publ., London (2016).

[24] J. Paneva-Konovska, V. Kiryakova, On the multi-index Mittag-Leffler functions and their Mellin transforms. *Int. J. Appl. Math.* **33** No 4 (2020), 549-571.

[25] I. Podlubny, *Fractional Differential Equations.* San Diego: Academic Press (1999).

[26] H. Pollard, The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. *Bull Amer Math Soc.* **54** No 12 (1948), 1115-1116.

[27] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Math. J.* **19**, No 1 (1971), 7-15.

[28] J. Prüss, *Evolutionary Integral Equations and Applications.* Birkhäuser, Basel, Boston, Berlin (1993).

[29] T. Sandev, Z. Tomovski, J.L.A. Dubbeldam, A. Chechkin, Generalized diffusion-wave equation with memory kernel. *J. Phys. A Math. Theor.* **52** No 1 (2019), Art. # 015201.

[30] R. Schilling, R. Song, Z. Vondraček, *Bernstein Functions: Theory and Applications.* De Gruyter, Berlin (2010).

[31] K. Yosida, *Functional Analysis.* Berlin: Springer-Verlag (1965).