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SOME PROPERTIES OF $\gamma-$ AND $P-$SPACES

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Abstract. A γ-space with a strictly positive measure is separable. An example of a non-separable $\gamma-$space with c.c.c. is given. A $P-$space with c.c.c. is countable and discrete.

In this paper by a space we mean a Tychonoff space. Our terminology is the standard one: any undefined term can be found in [1] or [2].

Is X is a space, then $C_p(X)$ is the space of all real-valued continuous functions on X, with the topology of pointwise convergence.

It is well known that $C_p(X)$ is $1^{st}-$countable iff X is countable [2]. On the other hand, there exist uncountable X's with $C_p(X)$ satisfying a weaker property, the Frechet-Urysohn (F.U.) property (i.e. if $A \subset C_p(X)$, $f \in \overline{A}$ implies $\lim f_n = f$, for a suitable sequence (f_n) with $f_n \in A$). Such spaces are the compact scattered, the Lindelöf $P-$spaces e.t.c. [3]. Spaces X for which $C_p(X)$ has the F.U. property are exactly the spaces with the so-called $\gamma-$property that is the expression of the F.U. property on $C_p(X)$ in terms of covering axioms of X [2].

It is easy to see that a compact scattered space X with the countable chain condition (c.c.c.) is separable. Indeed the set $A = \{x \in X \ : \ \{x\} \text{ is clopen}\}$ is countable and dense. Below we prove the same result for a Lindelöf $P-$space (or simply a $P-$space). The question arises whether this is also true on the general class of $\gamma-$spaces or not.

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Theorem 1. *There exists a non-separable $\gamma-$space with c.c.c..*

P r o o f. Consider an uncountable set Γ and the space

$$\Sigma_\omega = \{x \in \{0,1\}^\Gamma \ : \ |\{\gamma \in \Gamma \ : \ \pi_\gamma(x) \neq 0\}| < \omega\}$$

(π_γ is the usual $\gamma-$projection).

Then Σ_ω has c.c.c. since it is dense in $\{0,1\}^\Gamma$, but it is not separable since the family $\{\pi_\gamma^{-1}\{1\} \ : \ \gamma \in \Gamma\}$ does not contain any infinite subfamily with non-empty intersection. Now we shall prove that Σ_ω is a $\gamma-$space.

For $k = 1, 2, \ldots$ set

$$\Sigma_k = \{x \in \{0,1\}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| \leq k\}.$$

Every Σ_k is compact and $\Sigma_\omega = \cup \Sigma_k$. It follows that Σ_ω is $\sigma-$compact, so it preserves the Lindelöf property on finite powers and consequently $t(C_p(\Sigma_\omega)) = \omega$ (t = tightness). Besides the generated algebra of $\{\pi_\gamma : \gamma \in \Gamma\} \cup \{\mathbf{1}\}$ is dense in $C_p(\Sigma_\omega)$. From these facts it follows that every continuous function on Σ_ω depends on a countable subset of Γ. So if $f \in \overline{\{f_n : n = 1, 2, \ldots\}} \subset C_p(\Sigma_\omega)$ and A is the countable subset of Γ on which all f, f_n depend, we may suppose that these functions are defined on $\Sigma_\omega \cap (\{0,1\}^A \times (0)_{\Gamma \setminus A})$ which is countable, and the result is immediate.

Remarks. (i) It is well known that $\{0,1\}^\Gamma$ does not have a countable dense subset, in case that $|\Gamma| > 2^\omega$. However, $\{0,1\}^\Gamma$ (hence every dyadic space too) contains a dense $\gamma-$subset.

(ii) Every Σ_k, is a $\gamma-$space as a closed subspace of a $\gamma-$space. It follows that Σ_k, is compact scattered [2]. Certainly that can also be proved directly. In this case we have that every continuous function on Σ_ω has countable range and consequently Theorem 5 of [2] does also imply that Σ_ω is a $\gamma-$space.

(iii) In fact we proved that Σ_ω does not have a strictly positive measure (s.p.m) (i.e. a probability measure μ that is defined on the $\sigma-$field generated by a pseudobase B for its topology and such that $\mu(B) > 0$, for all non-empty $B \in \mathbf{B}$). The next theorem shows that here is a crucial point for the separability.

Theorem 2. *Let X be a $\gamma-$space.*

(a) If X has a s.p.m. then X is separable.

(b) A Borel finite measure μ on X, with the property $\mu\{x\} = \inf\{\mu(U) :$ U is clopen, $x \in U\}$ is of the form $\sum \alpha_k \delta_{x_k}$.

P r o o f. (a) Let μ be a s.p.m. on X, that is defined on the $\sigma-$field generated by the pseudobase $\mathbf{B}$. Notice that X has a base of clopen subsets [2].

For $k = 1, 2, \ldots$, set

$$\mathcal{J}_k = \left\{ U \subset X \ : \ U \text{ is clopen and } \exists\, V \in \mathbf{B} \text{ such that } V \subset U \text{ and } \mu(V) \geq \frac{1}{k} \right\}.$$

Then every infinite family in $\mathcal{J}_k$ contains an infinite subfamily with non-empty intersection.

We claim that there exists a finite $F_k \subset X$ such that $F_k \cap U \neq \emptyset$, for every $U \in \mathcal{J}_k$. Suppose not. Then $0 \in \overline{\{\chi_U : U \in \mathcal{J}_k\}}$, so $\chi_{U_n} \to 0$, for a sequence U_n in $\mathcal{J}_k$, contradictory to the property mentioned before for $\mathcal{J}_k$.

(b) Consider the countable set $A = \{x \in X : \mu\{x\} > 0\}$ (because of the finiteness of μ). It is enough to prove that $\mu(X) = \mu(A)$.

Suppose that $\mu(A) < \mu(X)$ and let $0 < \delta < \mu(X) - \mu(A)$. For every finite $F \subset X$, choose a clopen U_F such that $F \subset U_F$ and $\mu(U_F) < \mu(A) + \dfrac{\delta}{2}$. Then $\mathbf{1} \in \overline{\{\chi_{U_F} : F \subset X, \text{ finite}\}}$, so $\chi_{U_{F_n}} \to \mathbf{1}$ for a sequence (U_{F_n}). It follows that $\mu(X) = \int \mathbf{1} d\mu \leq \sup\limits_n \mu(U_{F_n}) \leq \mu(A) + \dfrac{\delta}{2} < \mu(X)$, which is absurd.

Remark. Theorem 2(b) extends a previous result of Rudin for the class of compact scattered spaces [3].

If X has a Baire s.p.m μ, a metric can be defined in a natural way on $C(X)$ by the type, $\rho(f,g) = \displaystyle\int \frac{|f-g|}{1+|f-g|} d\mu$.

In case that X is a $\gamma-$space the identity map $id : C_p(X) \to C_\rho(X)$ is continuous. What about the continuity of id^{-1}? Certainly for uncountable $\gamma-$spaces the answer is negative. But if X is a $P-$space (where $G_\delta-$sets are open) then id^{-1} is continuous.

[Suppose that $\displaystyle\int \frac{|f_n - f|}{1+|f_n - f|} d\mu \to 0$. We claim that $f_n \to 0$. If not, then $\dfrac{|f_{n_k}(x_0) - f(x_0)|}{1 + |f_{n_k}(x_0) - f(x_0)|} > \delta$, for some $x_0 \in X$, $\delta > 0$ and a subsequence (f_{n_k}). For $k = 1, 2, \ldots$ set

$$U_k = \left\{ x \in X : \frac{|f_{n_k}(x) - f(x)|}{1 + |f_{n_k}(x) - f(x)|} > \delta \right\}.$$

Then $\cap U_k$ is a non-empty open subset of X, so $\mu(\cap U_k) > 0$. On the other hand, $\mu(\cap U_k) \leq \dfrac{1}{\delta} \displaystyle\int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|} d\mu \to 0$, contradiction.]

Consequently, a Lindelöf $P-$space with a Baire s.p.m. is countable and discrete. Certainly, this is also followed by Theorem 2(a). In fact, a more general result is valid:

Theorem 3. *A $P-$space with c.c.c. is countable and discrete.*

Proof. Let X be a $P-$space with c.c.c. and $x \in X$.

We claim that $\{x\}$ is clopen.

Suppose not. Then we construct pairwise disjoint clopen sets V_ξ, $\xi < \omega^+$ such that $x \notin V_\xi$ in the following way. If V_ξ, $\xi < \ell$ have been defined, then $\bigcap_{\xi<\ell} V_\xi^c$ is a clopen neighbourhood of x. Since $\bigcap_{\xi<\ell} V_\xi^c \neq \{x\}$, we find a clopen (since X has a base of clopen subsets) subset $V_\ell \subset \bigcap_{\xi<\ell} V_\xi^c$ with $x \notin V_\ell$. The result follows from the fact that X has c.c.c.

Remarks. (i) From the proof of Theorem 3 it follows that a $P_{k^+}-$space X with k^+.c.c. has cardinality $|X| \leq k$. We mention that this is not true if X is simply a $P-$space. For example, the space $X = \left(\{0,1\}^{(2^\omega)^+}\right)_{\omega^+}$ (= the space $\{0,1\}^{(2^\omega)^+}$ with the ω^+-box topology) is a $P-$space with $(2^\omega)^+$.c.c. [2] and $|X| \geq (2^\omega)^+$.

(ii) If X is a $P-$space and $A \subset X$ is countable then A is closed. We note that Shakhmatov constructed a non-separable, c.c.c. space, all countable subsets of which are closed [4].

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