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RELATIVE COMPACTNESS FOR HYPERSPACES

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Abstract. The present paper contains results characterizing relatively compact subsets of the space of the closed subsets of a metrizable space, equipped with various hypertopologies.

We investigate the hyperspace topologies that admit a representation as weak topologies generated by families of gap functionals defined on closed sets, as well as hit-and-miss topologies and proximal-hit and-miss topologies.

1. Preliminaries and introduction. Let (X, d) be a metric space; we denote by $CL(X)$ the family of all closed subsets of X and by $CL_0(X)$ the family of all nonempty closed subsets of X.

For $x \in X$ and a nonempty subset $A \subset X$, the *distance* between x and A is defined as

$$d(x, A) = \inf_{a \in A} d(a, x).$$

We adopt the convention that the distance from a point to the empty set is ∞.

The *gap* between nonempty subsets A and B of X is given by

$$D(A, B) = \inf_{a \in A} d(a, B) = \inf_{b \in B} d(b, A).$$

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Notice that gap reduces to ordinary distance when A is a singleton subset.

For a nonempty subset A of X, the (closed) ϵ-*neighborhood* of A is the set

$$V_\epsilon[A] = \{x \in X : d(x, A) \leqslant \epsilon\}.$$

We call *hypertopology* a topology τ defined on $CL(X)$ such that, if we restrict τ to the singleton subsets, the induced subspace agrees with the initial topology on X. The resulting topological space $(CL(X), \tau)$ is called a *hyperspace*.

A basic class of hyperspace topologies is the family of the hit-and-miss topologies. Let us introduce some notation. Let E be a subset of X; corresponding to E are these families of closed subsets of X:

$$\begin{aligned} E^- &= \{H \in CL(X) : H \cap E \neq \emptyset\} \\ E^+ &= \{H \in CL(X) : H \subset E\} \\ E^{++} &= \{H \in CL(X) : \exists \epsilon > 0 : V_\epsilon[H] \subset E\}. \end{aligned}$$

Suppose Ω is a subfamily of $CL_0(X)$. By the *hit-and-miss* topology determined by Ω, we mean the topology having as a subbase all sets of the form G^-, where G is an open subset of X, and all sets of the form $(F^c)^+$, where $F \in \Omega$. By the *proximal-hit-and-miss* topology determined by Ω, we mean the topology having as a subbase all sets of the form G^-, where G is an open subset of X, and all sets of the form $(F^c)^{++}$, where $F \in \Omega$.

The hit-and-miss topology determined by $CL_0(X)$ is the well-known *Vietoris topology*; choosing as Ω the family of the nonempty compact subsets of X gives rise to the *Fell topology*; using the family of nonempty closed and bounded sets gives the *bounded Vietoris topology*; using the closed balls gives the *ball topology*. The *proximal topology* is the proximal-hit-and-miss topology determined by $CL_0(X)$; the *ball proximal topology* is determined by the family of the closed balls, and the *bounded proximal topology* is determined by nonempty closed and bounded subsets of X.

In the last few years many basic hyperspace topologies have been presented as weak topologies, i.e., as the weakest topologies on $CL(X)$ for which each member of a given family of functionals is continuous. Moreover, most of hyperspace topologies admit a characterization as weak topologies generated by gap functionals.

The *Wijsman topology* is defined as the weak topology on $CL(X)$ determined by the family of distance functionals

$$\{d(x, \cdot) : x \in X\} = \{D(\{x\}, \cdot) : x \in X\}.$$

The Vietoris topology is generated by

$$\{d(x,\cdot) : x \in X, d \in \mathcal{D}\},$$

where $\mathcal{D}$ denotes the set of compatible metrics for X.

The proximal topology is the weak topology on $CL(X)$ determined by the family of gap functionals

$$\{D(\cdot,F) : F \in CL_0(X)\}.$$

The bounded proximal topology is generated by

$$\{D(\cdot,C) : C \in CLB_0(X)\},$$

where $CLB_0(X)$ denotes the class of the nonempty closed and bounded sets of X.

These results are special cases of a general phenomen.

Definition 1. *A family Ω of nonempty closed subsets of a metric space (X,d) is called* stable under enlargements *if*

$$\forall F \in \Omega \text{ and } \forall \epsilon > 0, \text{ we have } V_\epsilon[F] \in \Omega.$$

Theorem (Beer, Lucchetti [2]). *Let Ω be a family of nonempty closed subsets of X stable under enlargements and containing the singleton subsets of X. Then the proximal- hit-and-miss topology on $CL(X)$ determined by Ω is the weak topology generated by $\{D(\cdot,F) : F \in \Omega\}$.*

The purpose of this paper is a characterization of relative compactness for the hyperspace of a metric space X equipped both with the weak topologies generated by families of gap functionals, and with the hit-and-miss topologies.

The main reference for this work is the article of O'Brien, Watson [3]: here their ideas are applied to the setting of hypertopologies.

That paper contain a result characterizing relative compactness for capacities, and an Ascoli theorem for the hyperspace equipped with the Vietoris topology is deduced as corollary.

Let us introduce capacities as in [3].

Let X be a topological space, and denote by $\mathcal{P}(X)$ the collection of all subsets of X.

Definition 2. *A* capacity *on X is a function $c : \mathcal{P}(X) \to [0,\infty]$ such that*

(*i*) $c(\emptyset) = 0$

(ii) $c(A) = \sup\{c(K) : K$ *compact,* $K \subset A\}$ $\forall A \in \mathcal{P}(X)$

(iii) $c(K) = \inf\{c(G) : G$ *open,* $G \supset K\}$ $\forall K$ *compact subset of* X.

Definition 3. *We say that a capacity* c *is a* sup measure *if* $c(K_1 \cup K_2) = \max\{c(K_1), c(K_2)\}$ $\forall K_1, K_2$ *compact subsets of* X.

The class of sup measures is denoted by $\mathcal{SM}$.
We define $\mathcal{SM}_1$ to be

$$\{c \in \mathcal{SM} : c(K) \in \{0,1\}\} \ \forall K \text{ compact subset of } X.$$

There is a natural bijection between sup measures and upper semicontinuous functions.

If $c \in \mathcal{SM}$, then the function $f_c : X \to [0,\infty]$, such that $f_c(x) = c(\{x\})$ $\forall x \in X$, is upper semicontinuous. Conversely, if $f : X \to [0,\infty]$ is upper semicontinuous, we define an element of $\mathcal{SM}$ by setting $c_f(A) = \sup\{f(x) : x \in A\}$, $\forall A \in \mathcal{P}$.

Also there is a bijection between $\mathcal{SM}_1$ and then set consisting of all the closed subsets of X: if $c \in \mathcal{SM}_1$, the set $H_c = \{x \in X : c(x) = 1\}$ is closed, and if $H \subset X$ is closed we can define a sup measure $c \in \mathcal{SM}_1$ by setting $c(A) = \sup\ \{f(x) : x \in A\}$, where f is the characteristic function of H.

In this paper we prove some Ascoli theorems for subsets of $\mathcal{SM}_1$. With suitable topologies, $\mathcal{SM}_1$ and $CL(X)$ are homeomorphic; so we can translate the results for sup measures into results for closed sets.

2. Relative compactness for sup measures.

Definition 4. *Let* X *be a topological space and* $Y \subset X$. *We say* Y *is* relatively compact *in* X *if every net in* Y *has a cluster point in* X.

Notice that $Y \subset X$ being relatively compact does not imply that the closure of Y in X is compact (for a conterexample see [3]). However, if X is a regular space then $Y \subset X$ is relatively compact if and only if the closure of Y in X is compact.

Let Ω be a family of nonempty closed subsets of X; we consider on $\mathcal{SM}_1$ the topology σ_1 having as a subbase all sets of the following forms:

$$\{c \in \mathcal{SM}_1 : c(G) = 1\},$$

$$\{c \in \mathcal{SM}_1 : \exists \epsilon > 0 : c(V_{\alpha+\epsilon}[F]) = 0\},$$

where G is open, $F \in \Omega$, and $\alpha \in [0,\infty)$.

The next theorem is a characterization of relative compactness for subsets of $\mathcal{SM}_1$ with the topology just defined.

Theorem 1. *A closed (arbitrary) subset $\mathcal{A}$ of $\mathcal{SM}_1$ is σ_1 (relatively) compact if and only if the following condition is true:*

(1) *for all $F \in \Omega$, all $p < q$ in $(0, \infty)$ and all open covers $\mathcal{G}_0$ of $V_q[F]$, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $c \in \mathcal{A}$ and $c(V_p[F]) = 1$, there exists $G \in \mathcal{G}_1$ such that $c(G) = 1$.*

Proof. ($\Rightarrow$:) Suppose there are $F \in \Omega$, $p < q$ and an open cover $\mathcal{G}_0$ of $V_q[F]$ such that, for each finite collection $\{G_1, G_2, \ldots, G_m\} \subset \mathcal{G}_0$, there is a $c \in \mathcal{A}$ with $c(V_p[F]) = 1$ and $c(G_i) = 0$ for $i = 1 \ldots m$. We order the finite subcollections of $\mathcal{G}_0$ by saying that $\mathcal{G}_1 \preceq \mathcal{G}_2$ if every element of $\mathcal{G}_1$ belongs to $\mathcal{G}_2$; in this way we obtain a directed set J. To each $j \in J$, i.e. a finite collection $\{G_1, G_2, \ldots, G_m\} \subset \mathcal{G}_0$, it is associated a sup measure $c_j \in \mathcal{A}$ with $c_j(V_p[F]) = 1$ and $c(G_i) = 0$ for $i = 1 \ldots m$; these sup measures form a net in $\mathcal{A}$: $\{c_j : j \in J\}$. Being $\mathcal{A}$ relatively compact in $\mathcal{SM}_1$, the net has a cluster point $d \in \mathcal{SM}_1$. For all $j \in J$ $c_j \in \{c \in \mathcal{SM}_1 : \forall \epsilon > 0 \ c(V_{p+\epsilon}[F]) = 1\}$, and this set is closed, so $d(V_q[F]) = 1$ and then there will be an open set $G \in \mathcal{G}_0$ such that $d(G) = 1$. The open set $\{c \in \mathcal{SM}_1 : c(G) = 1\}$ contains d and does not contains c_j for all $j \succeq G$. This is a contradiction.

($\Leftarrow$:) To prove that $\mathcal{A}$ is relatively compact in $\mathcal{SM}_1$ we show that every subbasic open cover of $\mathcal{SM}_1$ has a finite subfamily which covers $\mathcal{A}$ (see Lemma 1 in [3]). So let $\mathcal{U}$ be a subbasic open cover of $\mathcal{SM}_1$. Define a function λ on X in the following way: $\lambda(x) = 0$ if there is an open neighborhood V_x of x such that $\{c \in \mathcal{SM}_1 : c(V_x) = 1\} \in \mathcal{U}$, otherwise $\lambda(x) = 1$. It is easy to see that λ is upper semicontinuous, so if we put $\lambda(A) = \sup\{\lambda(x) : x \in A\}$ for all $A \subset X$, we obtain an element of $\mathcal{SM}_1$. There is a subbasic $W \in \mathcal{U}$ such that $\lambda \in W$. If W had the form $\{c \in \mathcal{SM}_1 : c(G) = 1\}$ for some open G of X, then $\lambda(G) = 1$, so $W \notin \mathcal{U}$. Thus W must have the form $\{c \in \mathcal{SM}_1 : \exists \epsilon > 0 : c(V_{\alpha+\epsilon}[F]) = 0\}$ for some $F \in \Omega$ and $\alpha > 0$. There is $q > \alpha$ such that $\lambda(V_q[F]) = 0$, then for all $x \in V_q[F]$ there exists an open set $V_x \ni x$ such that $\{c \in \mathcal{SM}_1 : c(V_x) = 1\} \in \mathcal{U}$. Choosing $p \in (\alpha, q)$, there is a finite subcollection $\{V_{x_1}, \ldots, V_{x_m}\}$ of $\{V_x : x \in V_q[F]\}$ such that, for all $c \in \mathcal{A}$ with $c(V_p[F]) = 1$, $c(V_{x_i}) = 1$ for some $i = 1 \ldots m$. Since $\{c \in \mathcal{A} : c(V_p[F]) = 0\} \subset W$, the finite subcollection of $\mathcal{U}$ consisting of $\{c \in \mathcal{SM}_1 : c(V_{x_i}) = 1\}, i = 1 \ldots m$, and W, covers $\mathcal{A}$. $\square$

Again let Ω be an arbitrary family of nonempty closed subsets of X. We define another topology σ_2 on $\mathcal{SM}_1$ as the smallest topology such that, for each open set $G \subset X$ and each closed set $F \in \Omega$, the following sets are open

$$\{c \in \mathcal{SM}_1 : c(G) = 1\},$$

$$\{c \in \mathcal{SM}_1 : \exists \epsilon > 0 : c(V_\epsilon[F]) = 0\}.$$

With this topology, the characterization of relative compactness is expressed by the following theorem, whose proof is omitted being similar to the former one.

Theorem 2. *A closed (arbitrary) subset $\mathcal{A}$ of $\mathcal{SM}_1$ is σ_2 (relatively) compact if and only if the following condition is true:*

(2) *for all $F \in \Omega$, all $q > 0$ and all open covers $\mathcal{G}_0$ of $V_q[F]$, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $c \in \mathcal{A}$ and $c(V_p[F]) = 1 \ \forall\ p > 0$, there exists $G \in \mathcal{G}_1$ such that $c(G) = 1$.*

Proposition 1. *If $\Omega \subset CL_0(X)$ is stable under enlargements then both conditions* (1) *and* (2) *in the previous theorems are equivalent to the following:*

(3) *for all $F \in \Omega$, all $q > 0$ and all open covers $\mathcal{G}_0$ of $V_q[F]$, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $c \in \mathcal{A}$ and $c(F) = 1$, there exists $G \in \mathcal{G}_1$ such that $c(G) = 1$.*

P r o o f. Obviously condition (2) implies condition (3). Conversely, suppose condition (3) holds and let $F \in \Omega$, $q > 0$ and $\mathcal{G}_0$ be an open cover of $V_q[F]$. It is easy to see that $V_{\frac{q}{3}}(V_{\frac{q}{3}}[F]) \subset V_q[F]$, so $\mathcal{G}_0$ covers $V_{\frac{q}{3}}[H]$ where $H = V_{\frac{q}{3}}[F] \in \Omega$. Thus there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, for all $c \in \mathcal{A}$ with $c(H) = 1$, $c(G) = 1$ for some $G \in \mathcal{G}_1$. The same finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ covers $\{c \in \mathcal{A} : \forall p > 0\ c(V_p[F]) = 1\}$.

We prove now that conditions (1) and (3) are equivalent.

Suppose condition (3) holds and let $F \in \Omega$, $p < q$ and $\mathcal{G}_0$ be an open cover of $V_q[F]$. If $q = p + \epsilon$, then $\mathcal{G}_0$ covers $V_{\frac{\epsilon}{3}}[H]$ where $H = V_{p+\frac{\epsilon}{3}}[F] \in \Omega$. Thus there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $c \in \mathcal{A}$ and $c(H) = 1$ then $c(G) = 1$ for some $G \in \mathcal{G}_1$. As $H \supset V_p[F]$, $c(V_p[F]) = 1$ implies $c(H) = 1$. $\square$

The last topology on $\mathcal{SM}_1$ we want to consider is the one generated by the collection of the following sets:

$$\{c \in \mathcal{SM}_1 : c(G) = 1\},$$

$$\{c \in \mathcal{SM}_1 : c(F) = 0\},$$

where G is an open set of X and F belongs to a subfamily Ω of $CL_0(X)$.

We denote this topology by σ_3.

Theorem 3. *A closed (arbitrary) subset $\mathcal{A}$ of $\mathcal{SM}_1$ is σ_3 (relatively) compact if and only if for all $F \in \Omega$ and all open covers $\mathcal{G}_0$ of F, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $c \in \mathcal{A}$ and $c(F) = 1$, there exists $G \in \mathcal{G}_1$ such that $c(G) = 1$.*

The proof of this theorem is very similar to the proof of Theorem 1, so we omit it.

3. Relative compactness for hyperspaces. Given a family Ω of nonempty closed subsets of X containing the singleton subsets of X, we consider on $CL(X)$ the weak topology generated by the family of functionals $\{D(\cdot,F) : F \in \Omega\}$, that is, the weakest topology on $CL(X)$ such that for every $F \in \Omega$, $H \mapsto D(H,F)$ is a continuous function from $CL(X)$ to $\mathbb{R}$.

This topology is denoted by τ_1.

One can verify that the topology just defined on $CL(X)$ admits as a subbase the collection of the following sets:

$$\{H \in CL(X) : H \cap G \neq \emptyset\},$$

$$\{H \in CL(X) : \exists \epsilon > 0 : H \cap V_{\alpha+\epsilon}[F] = \emptyset\},$$

where G is open, $F \in \Omega$ and $\alpha \geq 0$.

The bijection between $\mathcal{SM}_1$ and $CL(X)$ is a homeomorphism when $\mathcal{SM}_1$ is equipped with the σ_1 topology and $CL(X)$ with the topology defined above.

Theorem 1 yields the following

Corollary 1. *A closed (arbitrary) subset $\mathcal{A}$ of $CL(X)$ is τ_1 (relatively) compact if and only if for all $F \in \Omega$, all $r < s$ in $(0,\infty)$ and all open covers $\mathcal{G}_0$ of $V_s[F]$, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $H \in \mathcal{A}$ and $H \cap V_r[F] \neq \emptyset$, there exists $G \in \mathcal{G}_1$ such that $H \cap G \neq \emptyset$.*

Let Ω be an arbitrary subfamily of $CL_0(X)$; we consider now the topology τ_2 on $CL(X)$ having has a subbase all sets of the forms:

$$\{H \in CL(X) : H \cap G \neq \emptyset\},$$

$$\{H \in CL(X) : \exists \epsilon > 0 : H \cap V_{\epsilon}[F] = \emptyset\},$$

for each open set G and each closed set $F \in \Omega$.

Theorem 2 yields the following

Corollary 2. *A closed (arbitrary) subset $\mathcal{A}$ of $CL(X)$ is τ_2 (relatively) compact if and only if for all $F \in \Omega$, all $s > 0$ and all open covers $\mathcal{G}_0$ of $V_s[F]$, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $H \in \mathcal{A}$ and $H \cap V_r[F] \neq \emptyset$ $\forall r > 0$, there exists $G \in \mathcal{G}_1$ such that $H \cap G \neq \emptyset$.*

Suppose now Ω is stable under enlargements and contains the singleton subsets of X. It has been proved that the topology on $CL(X)$ generated by all sets of the form $\{H \in CL(X) : H \cap G \neq \emptyset\}$ where G is open, and all sets of the form $\{H \in CL(X) : \exists \epsilon > 0 : H \cap V_\epsilon[F] = \emptyset\}$ where $F \in \Omega$ (i.e. the proximal-hit-and-miss topology determined by Ω), is the weakest topology on $CL(X)$ such that for every $F \in \Omega$, $H \mapsto D(H, F)$ is continuous (see [2]). With this topology the characterization of relative compactness for closed sets is given by the following

Corollary 3. *A closed (arbitrary) subset $\mathcal{A}$ of $CL(X)$ is (relatively) compact if and only if for all $F \in \Omega$, all $s > 0$ and all open covers $\mathcal{G}_0$ of $V_s[F]$, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $H \in \mathcal{A}$ and $H \cap F \neq \emptyset$, there exists $G \in \mathcal{G}_1$ such that $H \cap G \neq \emptyset$.*

Finally we consider on $CL(X)$ the topology τ_3 generated by the following sets:

$$\{H \in CL(X) : H \cap G \neq \emptyset\}$$

$$\{H \in CL(X) : H \cap F = \emptyset\},$$

where G is open and $F \in \Omega \subset CL_0(X)$ (i.e. the hit-and-miss topology determined by Ω).

Corollary 4. *A closed (arbitrary) subset $\mathcal{A}$ of $CL(X)$ is τ_3 relatively compact if and only if for all $F \in \Omega$ and all open covers $\mathcal{G}_0$ of F, there is a finite subcollection $\mathcal{G}_1$ of $\mathcal{G}_0$ such that, if $H \in \mathcal{A}$ and $H \cap F \neq \emptyset$, there exists $G \in \mathcal{G}_1$ such that $H \cap G \neq \emptyset$.*

From Corollary 4 we immediatly notice that, choosing as Ω the class of the nonempty compact subsets of X, i.e. the Fell topology, then $CL(X)$ is a compact space. So we obtain with a different proof a well-known result (see [1]).

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