Global regularity of nonlinear dispersive equations and Strichartz estimates.

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Chapter 1

Introduction

Strichartz estimates are a type of a-priori estimates for the solutions of a large class of linear partial differential equations whose common property is that their solutions tend to disperse over time. Originally, such estimates were proved by R. Strichartz [11] in the late 1970's for the wave equation but later researchers extended them to other dispersive equations. The original method of proof relied on the recently discovered by Stein and Tomas fundamental results on the restriction properties of the multidimensional Fourier transform. However, the techniques were based on heavy harmonic analysis and the estimates were limited to special cases. In his article [10], Pecher showed that the time and space exponents need not be equal and thus provided most of the Strichartz estimates for the homogeneous equation in the special context of the Klein-Gordon equation. The next major advancement in the method came out in Ginibre and Velo [5] who invented a simpler and more flexible proof that relied only on the duality principle in Functional Analysis. In the late 1980's, Yajima extended the method to equations with inhomogeneous terms to cover different time and space exponents. These ideas were finalized in the mid 1990's in the papers by Lindblad and Sogge [9] and Ginibre and Velo [6]. Today, the core of these techniques is known as the TT^*-method.

By the mid 1990's Strichartz estimates became a standard tool in the analysis of the Schrödinger and the wave equations and gradually became familiar to researches working outsides these two equations. For example, in 1996 came out Castella and Perthame's short article [2], where they prove some homogeneous Strichartz estimates for the kinetic transport equation.

The next breakthrough came in 1997 when Keel and Tao [8] brought a much awaited unification in the theory. The authors elucidated the fundamental property of scaling in the estimates, presented the method in the abstract level, and gave some new tools based on bilinear-form interpolation and scaling invariant decompositions which are today the core of studying the end-point estimates and the inhomogeneous estimates. In a paper of 2005, Foschi [4] gave a further refinement of the method by introducing a dyadic Whitney decomposition which is more effective than the original one of [8] in the inhomogeneous setting.

We denote by $U(t)$ the continuous linear evolution group of a linear homogeneous differential equation. The two most important properties of $U(t)$ are

- the dispersive estimate:

$$\|U(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^\sigma} \|f\|_{L_x^1}, \quad t \in \mathbb{R}, \forall f \in L^1(X; d\mu) \tag{1.1}$$

- the energy estimate

$$\|U(t)f\|_{L_x^2} \lesssim \|f\|_{L_x^2}, \quad t \in \mathbb{R}, \forall f \in L^2(X; d\mu) \tag{1.2}$$

where $\sigma > 0$ is the rate of decay, f is the initial profile of the wave, and by $L^p = L^p(X; d\mu)$ we denote the Lebesgue space L^p over some measure space $(X, d\mu)$. The two inequalities above reflect the physical phenomenon that the amplitude of the wave decays over time (equation (1.1)), while its total energy remains constant (in the case of equality in equation (1.2)).

The homogeneous Strichartz estimates have the form

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}, \qquad \forall f \in L_x^2.$$

To the inhomogeneous equation we associate the following operator

$$W(t)F = \int_{-\infty}^t U(t-s)F(s)ds. \tag{1.3}$$

Under the assumption that supp $F \subseteq [0, \infty) \times \mathbb{R}^n$, (1.3) gives the Duhamel's formula of the fundamental solution to the inhomogeneous PDE. The inhomogeneous Strichartz estimates have the form

$$\|W(t)F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{1.4}$$

where by $L_t^q L_x^r$ we denote the Lebesgue space $L^q(\mathbb{R}; L^r(X; d\mu))$. We show in the sequel that the homogeneous Strichartz estimates can be identified as a special subclass of the inhomogeneous ones, see Theorem 1.0.2. From this point of view, the study of the inhomogeneous Strichartz estimates shall be our prime goal. The Lebesgue norms in the dispersive and energy inequalities shall be suitably generalized to vector-valued Lebesgue norms and abstract Banach space norms in the subsequent chapters. We shall study the explicit form of the Strichartz estimates for concrete equations and shall prove new Strichartz estimates that shall help us prove existence of solutions to nonlinear PDE's.

In this section we present some instances of equivalence between two given Strichartz estimates. To do so, let us first introduce the setting. Consider two abstract Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$. Suppose that the duality pairing $\langle \cdot, \cdot \rangle$ for these two spaces is the same and that $\mathcal{B}_1$ and ${\mathcal{B}_2}^*$ have a common dense subset $\mathcal{S}$. We define the adjoint $U^*(t) : \mathcal{S} \to {\mathcal{B}_1}^*$ to $U(t) : \mathcal{S} \to \mathcal{B}_2$ by

$$\langle U(t)f, g \rangle = \langle f, U^*(t)g \rangle \quad \forall f, g \in \mathcal{S}.$$

A typical example is $\mathcal{B}_1 = L^p$, $\mathcal{B}_2 = L^q$, which have the same duality pairing $\langle f, g \rangle = \int fg dx$, and $\mathcal{S}$ being taken as the Schwartz class on $\mathbb{R}^n$.

Lemma 1.0.1 (The Duality lemma)**.** *The following two estimates for $W(t)$ are equivalent*

$$\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L^p_t(\mathbb{R};\mathcal{B}_1)},$$
$$\|W(t)F\|_{L^{p'}(\mathbb{R};{\mathcal{B}_1}^*)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)},$$

for $1 \leq p, q \leq \infty$, whenever they are both invariant to the transformation $U(t) \leftrightarrow U(-t)$.

Theorem 1.0.2 (The Equivalence theorem)**. A.** *The following three estimates are equivalent*

$$\|U(t)f\|_{L^q_t(\mathbb{R};\mathcal{B}_2)} \lesssim \|f\|_{\mathcal{B}_1}, \qquad \forall f \in \mathcal{B}_1,$$
$$\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L^1_t(\mathbb{R};\mathcal{B}_1)}, \qquad \forall F \in L^1(\mathbb{R};\mathcal{B}_1),$$
$$\|W(t)F\|_{L^\infty(\mathbb{R};{\mathcal{B}_1}^*)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)}, \qquad \forall F \in L^{q'}(\mathbb{R};{\mathcal{B}_2}^*).$$

B. *If $\mathcal{B}_1$ is a Hilbert space, the homogeneous estimate above is equivalent to*

$$\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)}, \qquad \forall F \in L^{q'}(\mathbb{R};{\mathcal{B}_2}^*).$$

whenever $q > 2$. In the case when $q = 2$ we can only claim that the homogeneous estimate is implied from the latter inhomogeneous estimate.

Chapter 2

Strichartz Estimates for the Kinetic Transport Equation

The main goal of this chapter is to study the range of validity of the Strichartz estimates for the kinetic transport (KT) equation

$$\partial_t u(t,x,v) + v \cdot \nabla_x u(t,x,v) = F(t,x,v), \quad (t,x,v) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, \quad (2.1)$$
$$u(0,x,v) = f(x,v). \quad (2.2)$$

All estimates that we prove in the sequel involve the following two basic operators

$$U(t)f = f(x - tv, v), \quad W(t)F = \int_0^t U(t-s)F(s)ds, \tag{2.3}$$

that decompose the solution u to the Cauchy problem for the linear KT equation (2.1), (2.2) into a homogeneous and inhomogeneous part

$$u(t) = U(t)f + W(t)F.$$

The homogeneous Strichartz estimates have the form

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L^a_{x,v}}, \tag{2.4}$$

where by $L_t^q L_x^r L_v^p$ we mean $L^q([0,\infty); L^r(\mathbb{R}^n; L^p(\mathbb{R}^n)))$. The inhomogeneous estimates have the form

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}. \tag{2.5}$$

Let us now describe the range of validity of the homogeneous estimates. Following Keel and Tao [8], we shall call the Lebesgue exponents for which estimate (2.4) holds for every $f \in L^a_{x,v}$ *admissible.*

Definition 2.0.1. We say that the exponent triplet (q, r, p) is *KT-admissible* if

$$\frac{1}{q} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \quad a \overset{def}{=} \mathrm{HM}(p, r), \tag{2.6}$$

$$1 \le p, q, r \le \infty, \quad p^*(a) \le p \le a, \quad a \le r \le r^*(a), \tag{2.7}$$

except in the case $n = 1$, $(q, r, p) = (a, \infty, a/2)$.

In the above definition we use the abbreviation $\mathrm{HM}(p, r)$ to denote the harmonic mean of p and r, i.e. $a = \mathrm{HM}(p, r)$ whenever

$$\frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right).$$

For convenience we have also computed the exact lower boundary p^* to p and the exact upper boundary r^* to r which are given in

Definition 2.0.2. Set

$$\begin{cases} p^*(a) = \frac{na}{n+1}, & r^*(a) = \frac{na}{n-1}, \quad \text{if} \quad \frac{n+1}{n} \le a \le \infty, \\ p^*(a) = 1, & r^*(a) = \frac{a}{2-a}, \quad \text{if} \quad 1 \le a \le \frac{n+1}{n}. \end{cases} \tag{2.8}$$

We have used the convention that $1/0 = \infty$, i.e. for $n = 1$, $r^*(a) = \infty$. Furthermore, throughout this text we shall always use the convention $1/\infty = 0$ and $1/0 = \infty$ in the context of Lebesgue exponents.

To describe the range of the inhomogeneous estimates we shall need the next two definitions. Following Foschi [4], we shall call the exponent triplet (q, r, p) *KT-acceptable* if it satisfies a certain condition that is necessary for the validity of the inhomogeneous estimates of the form (2.5) for any right hand side $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

Definition 2.0.3. We say that the exponent triplet (q, r, p) is *KT-acceptable* if

$$\frac{1}{q} < n\left(\frac{1}{p} - \frac{1}{r}\right), \quad 1 \leq q \leq \infty, \quad 1 \leq p < r \leq \infty, \tag{2.9}$$

or if $q = \infty$, $1 \leq p = r \leq \infty$.

Note that a KT-acceptable triplet is always KT-admissible. We shall later see that this condition is necessary both for the validity of the generalized homogeneous estimates and the inhomogeneous estimates. To further describe the range of validity of the inhomogeneous estimates we give the following

Definition 2.0.4. We say that the two KT-acceptable exponent triplets (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are *jointly KT-acceptable* if

$$\frac{1}{q} + \frac{1}{\tilde{q}} = n\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \frac{1}{q} + \frac{1}{\tilde{q}} \leq 1, \tag{2.10}$$

$$\mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}'), \tag{2.11}$$

and if the exponents satisfy further the additional restrictions

(i)
$$\frac{n-1}{p'} < \frac{n}{\tilde{r}}, \quad \frac{n-1}{\tilde{p}'} < \frac{n}{r}, \tag{2.12}$$

for $r, \tilde{r} \neq \infty$.

(ii) if $r = \infty$ then the point $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_1 \cup B$,

$$\Sigma_1 = \{(\mu, 0, \kappa, \nu, 1-\kappa, 1) \; : \; 0 < \mu, \nu < 1, \; 0 < \mu + \nu < 1, \; \kappa = (\mu+\nu)/n\},$$
$$B = (0, 0, 0, 0, 1, 1). \tag{2.13}$$

(iii) if $\tilde{r} = \infty$ then the point $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_2 \cup C$,

$$\Sigma_2 = \{(\mu, 1-\kappa, 1, \nu, 0, \kappa) \; : \; 0 < \mu, \nu < 1, \; 0 < \mu + \nu < 1, \; \kappa = (\mu+\nu)/n\},$$
$$C = (0, 1, 1, 0, 0, 0). \tag{2.14}$$

Theorem 2.0.5. *Let $u(t)$ be the solution to the Cauchy problem for (2.1), (2.2). Then the estimate*

$$\|u(t)\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_{x,v}} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \tag{2.15}$$

holds for all $f \in L^a(\mathbb{R}^{2n})$ and all $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v$ if and only if (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two KT-admissible exponent triplets and $a = \mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}')$, apart from the case when $n > 1$ and (q, r, p) is being an endpoint triplet for which the corresponding estimates in higher dimensions remain unresolved.

Note that Theorem 2.0.5 allows the second triplet $(\tilde{q}, \tilde{r}, \tilde{p})$ to be endpoint and excludes only the estimates where the first triplet (q, r, p) is endpoint. Despite the fact that Theorem 2.0.5 is essentially optimal it does not give the full range of validity of the estimates for the operator $W(t)$.

Theorem 2.0.6. *Suppose that (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two jointly KT-acceptable exponent triplets that further satisfy the following conditions $1 < q, \tilde{q} < \infty$, $q > \tilde{q}'$, then the estimate*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \tag{2.16}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$. Conversely, if estimate (2.16) holds for all $F \in L_t^{q'} L_x^{r'} L_v^{p'}$, then (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ must be two jointly KT-acceptable exponent triplets, apart from conditions (2.12), (2.13), (2.14), *whose necessity is not fully verified.*

Chapter 3

Application to Kinetic Chemotaxis

Chemotaxis is a process in which bacteria, or, more generally, cells, change their state of movement, reacting to the presence of chemical substance called chemoattractant, approaching chemically favorably environments and avoiding unfavorable ones. Generally, the movement of bacteria is composed of two different phases, a "run" phase and a "tumble" phase. The run phase consists of a directed movement in a straight line, while the "tumble" phase is the reorientation to a new direction.

We denote the chemoattractant $S(t,x)$ at time $t \in [0,\infty)$ and position $x \in \mathbb{R}^n$. The cell density in phase space is denoted by $u(t,x,v)$ and its integral over all possible velocities, which is assumed to be the bounded set $V \subset \mathbb{R}^n$, is the cell density

$$\rho(t,x) = \int_{v\in V} u(t,x,v)dv$$

in physical space.

The kinetic model of chemotaxis proposed by Othmer-Dunbar-Alt, see e.g. [3], reads

$$\partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) = \int_{v'\in V} T[S](t,x,v,v')u(t,x,v')dv', \tag{3.1}$$

$$- \int_{v'\in V} T[S](t,x,v',v)u(t,x,v)dv', \quad t>0,\ x\in\mathbb{R}^n, v\in V$$

$$-\Delta_x S(t,x) + S(t,x) = \rho(t,x) \overset{def}{=} \int_{v\in V} u(t,x,v)dv, \tag{3.2}$$

$$u(0,x,v) = f(x,v) \geq 0. \tag{3.3}$$

Here, the free transport operator $\partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v)$ describes the free runs of the bacteria which have velocity $v \in V$. The right hand side of (3.1) denotes a scattering operator whose first term describes turning into direction

v, and the second term turning away from v. More specifically, in this model the tumble (the reorientation) is a Poisson process with rate

$$\lambda[S] = \int_V T[S](t,x,v',v)dv',$$

and $T[S](t,x,v',v)/\lambda[S]$ is the probability density for a change in velocity from v to v', given that a reorientation occurs for a cell at position x, velocity v, and time t.

The Cauchy problem (3.1)-(3.3) was first studied in [3] (2004) where global existence was proved in dimension $n = 3$ for nonnegative initial data $f \in L^1 \cap L^\infty$ under the assumption that the turning kernel satisfies the structural condition

$$0 \le T[S](t,x,v,v') \le C(1 + S(t,x+v) + S(t,x-v')).$$

The meaning of the term $S(t,x-v')$ is that the cells measure the concentration of the chemical S at position $x - v'$ before changing their direction at position x, because of an internal memory effect. The meaning of the term $S(t,x+v)$ is that the cells are able to measure the concentration at location $x + v$ thanks to censorial protrusions.

However, based on experimental data, it is believed that the reorientation of the bacteria depends on the changes in concentration of the chemoattractant. Thus, in a more realistic model the turning kernel should depend not only on S, but also on its gradient ∇S (the x variables). Let us consider the most general condition on T

$$\begin{aligned} 0 \le T[S](t,x,v,v') \le C_1 + C_2 S(t,x+v) + C_3 S(t,x-v') + \\ C_4 |\nabla S(t,x+v)| + C_5 |\nabla S(t,x-v')|. \end{aligned} \tag{3.4}$$

The method of [3] was adapted in [7] to include turning kernels satisfying

$$0 \le T[S](t,x,v,v') \le C(1 + S(t,x+v) + |\nabla S(t,x+v)|),$$

or

$$0 \le T[S](t,x,v,v') \le C(1 + S(t,x-v') + |\nabla S(t,x-v')|),$$

under the same assumptions for the initial data.

The first successful attempt to consider the most general kernel in 3d, i.e. (3.4), was made in [1]. The authors replace condition (3.4) (with $C_1 = 0$) with the more general

$$\|T[S](t,\cdot,\cdot,\cdot)\|_{L^r_x L^{p_1}_v L^{p_2}_{v'}} \lesssim_{|V|,p_1,p_2} \|S(t,\cdot)\|_{L^r_x} + \|\nabla S(t,\cdot)\|_{L^r_x}, \tag{3.5}$$

whenever $r \ge p_1$, p_2, see [1, Theorem 3]. They establish the existence of a global weak solution for small enough initial data $f \in L^1 \cap L^a$, where $a \in [3/2, 2]$. However, the authors do not prove uniqueness of the solution and work in data classes that are not preserved by the evolution of the system. The new feature of their approach is the use of Strichartz estimates for the kinetic transport equation derived in [2]. We shall adapt their method but based on the larger set of inhomogeneous Strichartz estimates that we derive in the present work. Our proof shall use more delicate spacetime estimates on the chemoattractant S, unlike the proof in [1] that uses estimates on S only for fixed time, and use a double bootstrap argument. Because our aim is to show global well-posedness of the solution we need to consider differences $T[S_1] - T[S_2]$, together with structural condition (3.5) we impose the natural condition

$$\|T[S_1](t,\cdot,\cdot,\cdot) - T[S_2](t,\cdot,\cdot,\cdot)\|_{L_x^r L_v^{p_1} L_{v'}^{p_2}} \lesssim_{|V|,p_1,p_2} \|S_1(t,\cdot) - S_2(t,\cdot)\|_{L_x^r} + \|\nabla S_1(t,\cdot) - \nabla S_2(t,\cdot)\|_{L_x^r}, \quad (3.6)$$

whenever $r \geq p_1,\, p_2$.

Our result is presented in

Theorem 3.0.1. *The Cauchy problem* (3.1)-(3.3), (3.5), (3.6)*, is globally well-posed for small data $f \geq 0$ in the class $f \in L^1(\mathbb{R}^n \times V) \cap L^a(\mathbb{R}^n \times V)$ for $3n/4 < a < 3$ and $n = 2, 3$. More specifically, for every $3n/4 < a < 3$ there exist a fixed positive constant M, depending only on the space dimension n, the constants in the structural conditions* (3.5), (3.6)*, and on the Lebesgue exponent a, such that whenever $\|f\|_{L^a_{x,v}} < M$ the considered problem admits a unique nonnegative solution*

$$u(t) \in C([0,\infty); L^1(\mathbb{R}^n \times V) \cap L^a(\mathbb{R}^n \times V))$$

for which

$$\|\rho\|_{L_t^3 L_x^{3na/(3n-a)}} < \infty, \quad \|S\|_{L_t^3 L_x^\infty} + \|\nabla S\|_{L_t^3 L_x^\infty} < \infty.$$

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