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ON THE SENSE PRESERVING MAPPINGS IN THE HELM TOPOLOGY IN THE PLANE

Pavel Pyrih*

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ABSTRACT. We introduce the *helm topology* in the plane. We show that (assuming the helm local injectivity and the Euclidean continuity) each mapping which is *oriented at* all points of a helm domain U is *oriented* at U.

1. Introduction. We can easily observe that any continuous complex function $f : \mathbb{C} \to \mathbb{C}$ has the following property (see [3], Proposition 3.1). If a point $z \in \mathbb{C}$ is the point of *holomorphy* (i.e. there exists nonzero complex derivative, i.e. $\overline{\partial} f(z) = 0$ and $\partial f(z) \neq 0$), then there exists a neighborhood U of z such that for each positively oriented circle $T \subset U$ centered at z the curve $f(T)$ is homotopic in $C \setminus \{f(z)\}$ to a positively oriented circle L centered at $f(z)$.

Similarly in some neighborhood of a point of *antiholomorphy* (i.e. $\overline{\partial} f(z) \neq 0$ and $\partial f(z) = 0$) all positively oriented circles are mapped onto curves homotopic to the negatively oriented circles.

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The question now arises whether each continuous function f which has only points of either holomorphy or antiholomorphy is itself either holomorphic ($\overline{\partial} f = 0$) or antiholomorphic ($\partial f = 0$).

This relates to the following problem (still unsolved).

Problem 1.1. *Is it true for every finely harmonic morphism f, defined in a fine domain U in $\mathbb{C}$ (this already implies that $\overline{\partial} f \cdot \partial f = 0$ Lebesgue almost everywhere in U), that either f or $\overline{f}$ is a finely holomorphic function?*

(See [1], [2], [3]).

The formulation of the Problem 1.1 uses the fine topology from potential theory. We study a similar question for the *helm topology* in the plane. First we introduce the *sun topology* and the *circle topology* in the plane.

Definition 1.2. *The sun topology in the complex plane $\mathbb{C}$ is defined by declaring a set A to be open at x if for (Lebesgue) almost every $\alpha \in [0, 2\pi]$ there exists $t_\alpha > 0$ such that $\{z \in \mathbb{C} : z = x + t(\cos\alpha + i\sin\alpha), |t| < t_\alpha\} \subset A$.*

In other words the set A contains with any point x a segment on almost every line through x.

Definition 1.3. *The circle topology in the complex plane $\mathbb{C}$ is defined by declaring a set A to be open if for each $x \in A$ there exists a set $D_x \subset \mathbb{R}_+$ such that $\{z \in \mathbb{C} : |x - z| \in D_x\} \subset A$ and the $0 \in \mathbb{R}$ is the point of (one dimensional) right-sided density for the set D_x.*

In other words the set A contains with any point x circles (not discs) with radii $r \in D_x$ and the origin is the point of density for the set $D_x \cup (-D_x)$.

For any topology (e.g. blue) we use the terms blue open, blue closure ... with respect to this topology. Now we can define the *helm topology* in the plane.

Definition 1.4. *We say that $A \subset \mathbb{C}$ is helm open if and only if A is circle open and simultaneously A is sun open.*

We see that if a set A is helm open then it contains with any point a 'helm wheel' (the ship's steering wheel) with infinitely many circles and infinitely many radii (not of equal length).

2. Oriented mappings. Now we use the idea of holomorphy and antiholomorphy mentioned in Introduction.

We say that a helm continuous function $f : U \to \mathbb{C}$ is *oriented at a point* $z \in U$ if there is a helm neighborhood $V \subset U$ of z such that for each positively oriented circle $T \subset V$ centered at z the curve $f(T)$ is homotopic in $\mathbb{C} \setminus \{f(z)\}$ to an oriented circle L centered at $f(z)$. If the circle L is a positively oriented circle we say that f is *positively oriented* at z. If the circle L is a negatively oriented circle we say that f is *negatively oriented* at z.

We say that a helm continuous function $f : U \to \mathbb{C}$ is *oriented at* U if either it is positively oriented at all points of U or it is negatively oriented at all points of U.

We see that each point of holomorphy is the point of the positive orientation and each point of antiholomorphy is the point of the negative orientation for any continuous complex function.

Now we show that (assuming the helm local injectivity and the Euclidean continuity) a helm continuous function which is oriented at all points of a helm domain U is oriented at U.

Proposition 2.1. *Let U be a helm domain in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be a helm continuous function oriented at all points of U. If f is helm locally both injective and Euclidean continuous in U, then f is oriented in U.*

Proof. Fix an $a \in U$. Let f be positively oriented at a. There exists a helm neighborhood V of a such that $V \subset U$ and f is both Euclidean continuous and injective on V.

Let b be a point in V such that the linear or circle segment $K = ab$ is contained at V and f is negatively oriented at b. We will deduce from this a contradiction.

Put $W_a := D(f(a), |f(b) - f(a)|/10)$, $W_b := D(f(b), |f(b) - f(a)|/10)$ (here $D(z, r)$ denotes the open disc with center z and radius r). The set V is helm open, f is both helm and Euclidean continuous in V. There exists a helm neighborhood $V_a \subset V$ of a such that $f(V_a) \subset W_a$. We find in V_a a circle with center a and radius r_a, less than $|a - b|/10$, such that $\Phi(t) := f(a + r_a e^{it})$ is a positively oriented Jordan curve defined on $[0, 2\pi]$ enclosing $f(a)$. Similarly, there exists a helm neighborhood $V_b \subset V$ of b such that $f(V_b) \subset W_b$. We find in V_b a circle with center b and radius r_b, less than $|a-b|/10$, such that $\Psi(t) := f(b+r_b e^{it})$ is a negatively oriented Jordan curve defined on $[0, 2\pi]$ enclosing $f(b)$.

The curves Φ and Ψ are disjoint and mutually outside, since $\Phi^* \subset W_a$, $\Psi^* \subset W_b$ and $W_a \cap W_b = \emptyset$. (For a Jordan curve ρ, ρ^* denotes the range of ρ, i.e. the geometrical image $\rho([0, 2\pi])$, Int ρ denotes the bounded component of $\mathbb{C} \setminus \rho^*$ and Ext ρ denotes the unbounded component of $\mathbb{C} \setminus \rho^*$. We say that two Jordan curves ρ and τ are mutually outside if Int ρ is disjoint with Int τ. Similarly, ρ is inside τ means that Int $\rho \subset$ Int τ, ρ and τ are disjoint means that $\rho^* \cap \tau^* = \emptyset$.)

Put $K' := K \setminus D(a, r_a) \setminus D(b, r_b)$. For each $z \in K'$, we construct similarly a circle, with center z and radius $r_z < \min(r_a, r_b)/10$, contained in V. Obviously

$$K \subset \bigcup_{z \in K' \cup \{a,b\}} D(z, r_z).$$

Since K is a compact set, we can find a finite set $M = \{a, z_1, \ldots, z_n, b\}$ such that

$$K \subset \bigcup_{z \in M} D(z, r_z).$$

We denote by L the boundary of

$$\bigcup_{z \in M} D(z, r_z).$$

The construction shows that $L \cap K = \emptyset$ and $L \subset V$. L is a circle-polygon composed of finitely many arcs of circles (L contains arcs from the circles $T(a, r_a)$ and $T(b, r_b)$ with a central angle greater than π). Denote by λ a parametrical representation of L, defined on $[0, 2\pi]$, such that λ is a positively oriented Jordan curve. Put $\Lambda := f \circ \lambda$. Λ is a Jordan curve, since f is injective on V. The curves Λ and Φ form together a Θ-curve (see [4], Proposition V.2.4), the same is true for Λ and Ψ. Recall that Φ is a positively oriented Jordan curve, Ψ is a negatively oriented Jordan curve and they are disjoint and mutually outside.

There are 4 possible situations:

(1) : Λ is mutually outside to both Ψ and Φ. Then the orientation of Λ cannot agree with the orientations of both Φ and Ψ. So we obtain a contradiction.

(2) : both Φ and Ψ are inside Λ. Then the orientation of Λ cannot agree with the orientations of both Φ and Ψ. So we obtain a contradiction.

(3) : Φ is inside Λ, Ψ and Λ are mutually outside. We shall show that this case is impossible. Indeed, put $A := T(a, r_a) \cap K$. Then $A \notin L$ and $f(A) \in \Phi^* \setminus \Lambda^*$. So we have $f(A) \in \operatorname{Int} \Lambda$. Similarly, we denote $B := T(b, r_b) \cap K$, then $B \notin L$ and $f(B) \in \Psi^* \setminus \Lambda^*$, so we have $f(B) \in \operatorname{Ext} \Lambda$.

By the construction, $f(K)$ in connected and $f(K) \cap \Lambda^* = \emptyset$. But $f(A) \in f(K)$, $f(B) \in f(K)$, $f(A) \in \operatorname{Int} \Lambda$, $f(B) \in \operatorname{Ext} \Lambda$ – a contradiction.

(4) : Ψ is inside Λ, Φ and Λ are mutually outside. In a similar way as in (3), we show that this case is impossible.

Therefore f cannot be positively oriented at a and negatively oriented at b. Hence f is positively oriented at all points z with linear or circle segment $az \subset V$. The same argument gives the positive orientation in any point that can be joined with a by a (both circle and linear) polygonal path in V. Due to the construction of the helm topology we see that the set of points where f is positively oriented is helm open.

So we conclude that, f is oriented in the helm domain U. $\square$

The proof is a simple modification of the proof of Theorem 3.3 in [3].

We have a simple corollary.

Corollary 2.2. *Let f be a continuous (helm) locally injective complex function in a complex domain U. Let each point of U be either the point of holomorphy for f or the point of antiholomorphy for f. Then f is either holomorphic* ($\overline{\partial} f = 0$) *or antiholomorphic* ($\partial f = 0$) *in U.*

Finally we can formulate a problem.

Problem 2.3. *We do not know whether the condition of helm local injectivity in Proposition* 2.1 *can be avoided, i.e. whether the same holds without this assumption.*

REFERENCES

[1] B. Fuglede. Finely harmonic mappings and finely holomorphic functions. *Ann. Acad. Sci. Fenn. Math.* **2** (1976), 113-127.

[2] B. Fuglede. Finely holomorphic functions and finely harmonic morphisms. *Aequations Math.* **34** (1987), 167-173.

[3] P. Pyrih. Finely locally injective finely harmonic morphisms. *Potential Anal.* **1** (1992), 373-378.

[4] G. T. Whyburn. Topological Analysis. Princeton University Press, Princeton, 1958.

Department of Mathematical Analysis
Charles University
Sokolovská 83
CZ-18675 Prague
Czech Republic
`e-mail: pyrih@karlin.mff.cuni.cz`

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