# Serdica
## Mathematical Journal
# Сердика
## Математическо списание

Serdica Math. J. **25** (1999), 83-90

**Serdica**
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

# SOME PARA-HERMITIAN RELATED COMPLEX STRUCTURES AND NON-EXISTENCE OF SEMI-RIEMANNIAN METRIC ON SOME SPHERES

Sadettin Erdem

*Communicated by V. Kanev*

Abstract. It is shown that the spheres $S^{2n}$ (resp: $S^k$ with $k \equiv 1 \bmod 4$) can be given neither an indefinite metric of any signature (resp: of signature $(r, k-r)$ with $2 \le r \le k-2$) nor an almost paracomplex structure. Further for every given Riemannian metric on an almost para-Hermitian manifold with the associated 2-form $\phi$ one can construct an almost Hermitian structure (under certain conditions, two different almost Hermitian structures) whose associated 2-form(s) is $\phi$.

**1. Introduction.** Semi-Riemannian metric is quite important in differential geometry as well as in physics in which it plays a central role in the theory of relativity, aspecially as a Lorentz metric. Almost paracomplex structure is one of the basic ingredient in the geometry resting on the ring of paracomplex numbers. It also provides a link between paraholomorphicity and harmonicty of maps (as solutions of hyperbolic systems) among certain almost para-Hermitian manifolds. These are the few things that one may site among many others. As

1991 *Mathematics Subject Classification:* 53C15, 53C50, 53C55
*Key words:* Hermitian, para-Hermitian and indefinite metric.

for this reason, we state here some non-existence results and introduce some para-Hermitian related complex structures.

**Acknowledgement.** I would like to thank to the referee for bringing out a work of Matsushita, [2] to my attention and making valuable suggestions.

**2. Definition and results.** Let $(V, h) \to M$ denote throughout a vector bundle of rank $k$ endowed with a semi-Riemannian metric $h$ of signature $(r, s)$ over a paracompact manifold $M$. *A semi-Riemannian metric $h$ of signature $(r, s)$ on $V$ with $r+s = k$* is a globally defined continous symmetric section of the bundle $(V \otimes V)^*$ such that for every $p \in M$ if $h_p(X, Y) = 0, \quad \forall Y \in V_p$ then $X = 0$. (This is the nondegeneracy condition for $h_p$ on $V_p$). It is the standard fact that if $e_1, \cdots e_k$ form an $h_p$-orthogonal basis for $V_p$ then there are exactly $r$ many of them with $h_p(e_i, e_i) < 0$ and the rest $s$ many are with $h_p(e_i, e_i) > 0$. When $r = 0$ or $r = k$ then $h$ is called a *Riemannian metric.* When $r = s$ then $h$ is called a *neutral metric.*

Let $F$ be a globally defined continous section of the bundle $V^* \otimes V$ over $M$. Then

$a^\circ$) $F$ is called a *product structure on $V$* if $F^2 = I$ with $F \neq \pm I$, where $I$ is the identity.

$b^\circ$) $F$ is called a *complex structure on $V$* if $F^2 = -I$.

A product structure $F$ gives rise two complementary subbundles $F^+$ and $F^-$ over $M$ which are eigensubbundles of $F$ corresponding to eigenvalues 1 and $-1$ respectively. That is

$$F^+ = \{v \in V : F(v) = v\} \quad \text{and} \quad F^- = \{v \in V : F(v) = -v\}$$

Clearly $F^+ \oplus F^- = V$ and for $rank(F^+) = \ell, \quad rank(F^-) = t$ we have $\ell + t = k = rank(V)$ , and that $F$ *is said to have signature* $(\ell, t)$. If $F$ is of signature $(\ell, \ell)$ then it is called a *paracomplex structure on $V$*. We designate the letters $P$ and $J$ for paracomplex and complex structures respectively.

Note that if $V$ can be endowed with a structure $P$ or $J$ then the $rank(V)$ is necessarily even. In the cases where $V = TM$ (the tangent bundle of $M$) the structures $F, P$ and $J$ are called *almost product, almost paracomplex and almost complex structure* of $M$ respectively. Also

$c^\circ$) A pair $(P, h)$ (resp: $(J, g)$) is called *para-Hermitian (resp: Hermitian) structure on $V$* if the semi-Riemannian metric $h$ (resp: Riemannian metric $g$) satisfies

$$h(P(X), P(Y)) = -h(X, Y), \quad \text{(resp:} \quad g(J(X), J(Y)) = g(X, Y))$$

for every sections $X, Y$ of $V$.

$d^\circ$) A skew-symmetric, nondegenerate continuous section $\phi$ of $(V \otimes V)^*$ is called a *symplectic structure on* $V$. The bundle $V$ with $\phi$ is then called a *symplectic vector bundle.*

When $V = TM$, $\phi$ is then called *almost symplectic structure* on $M$. Note that the structures $(P, h)$ and $(J, g)$ define an *associated symplectic structures on* $V$ via

$$\phi(X, Y) = h(X, P(Y)) \quad \text{and} \quad \Omega(X, Y) = g(X, J(Y))$$

respectively.

Consider now a vector bundle $V$ of rank $k$ endowed with a semi-Riemannian metric $h$ of signature (r,s). Let $G$ be a Riemannian metric on $V$, (which always exists). Define a global section $L$ of the bundle $V^* \otimes V \to M$ by $G_p(L_p(X), Y) = h_p(X, Y)$ for every $p \in M$ and $X, Y \in V_p$.

Now we set $V_p^+$ (resp: $V_p^-$) to be the sum of the eigenspaces corresponding to the positive (resp: negative) eigenvalues of the symmetric endomorphism $L$ of $V$. Then it is easy to prove the following:

**Lemma 2.1.** *A pair* $(h, G)$ *of a semi-Riemannian and a Riemannian metrics gives rise to subbundles* $V^+$ *and* $V^-$ *satiyfying*

*($a^\circ$)* $V = V^+ \oplus V^-$,

*($b^\circ$) rank* $(V^+) = s$ *and rank* $(V^-) = r$,

*($c^\circ$)* $h$ *is positive definite on* $V^+$ *and negative definite on* $V^-$,

*($d^\circ$)* $V^-$ *is* $h$*-othogonal (and also* $G$*-othogonal) complement of* $V^+$.

We call $V^+$ and $V^-$ the $(h, G)$*-induced subbundles.*

Our first result is the follwing:

**Theorem 2.2.** *The* $k$*-sphere* $S^k$ *does not admit any semi-Riemannian metric of signature* $(r, s)$ *if* $k$ *is even and* $1 \leq r \leq k-1$ *or* $k \equiv 1 \pmod 4$ *and* $2 \leq r \leq k-2$.

P r o o f. Suppose $S^k$ admits a semi-Riemannian metric of signature $(r, s)$. Then by Lemma 2.1, the tangent bundle $TS^k$ splits into two subbundles of ranks $r$ and $s$. But it is well known that (see e.g [3, Theorem 27.18]) $TS^k$ does not admit a subbundle of rank $r$ if $k$ satisfies the hypothesis of the theorem. So the result follows immediately. □

**Corollary 2.3.** *A sphere of even dimension does not admit a semi-Riemannian metric of any signature.*

It is now straightforward to see that

**Theorem 2.4.** *The* $k$*-sphere* $S^k$ *does not admit any almost product structure of signature* $(\ell, t)$ *if* $k$ *is even and* $1 \leq \ell \leq k-1$ *or* $k \equiv 1 \pmod 4$ *and*

$2 \leq \ell \leq k-2$.

**Corollary 2.5.** *A sphere of even dimension does not admit an almost product structure of any signature. Therefore it does not admit an almost para-complex structure.*

**Remark.** Recall also that $S^2$ and $S^6$ are the only spheres that admit almost complex structures.

**Theorem 2.6.** *Let $V$ be a symplectic vector bundle with a symplectic structure $\phi$ and let $G$ be a Riemannian metric on $V$. Then the pair $(\phi, G)$ induces a complex structure $J$ which is also compatible with $\phi$, that is, $\phi(J(X), J(Y)) = \phi(X, Y)$. If further $\phi$ is the associated one with a para-Hermitian structure $(P, h)$ on $V$ then the pair $(\phi, G)$ gives rise to two more complex structures $J_+$ and $J_-$ which are not, in general, $\phi$-compatible. However, the following are equivalent:*
*($a^\circ$) $J_+$ is $\phi$-compatiple,*
*($b^\circ$) $J_-$ is $\phi$-compatible,*
*($c^\circ$) $J_- = -J_+$,*
*($d^\circ$) $V^+ = P(V^-)$,*
*($e^\circ$) $V^- = P(V^+)$*
*where $G$ is an arbitrary Riemannian metric and $V^+$, $V^-$ are $(G, h)$-induced subbundles.*

Proof. (For a detailed proof see [4, Theorem 2.3.3.]). Define a tensor field $W$, a section of $V^* \otimes V$, by $G(W(X), Y) = \phi(X, Y)$ for every sections $X, Y$. Choose a local frame field $B = \{v_1, \cdots, v_{2n}\}$ such that the matrix representation $[W]_B$ of $W$ in $B$ is

$$\begin{bmatrix} 0 & \vdots & N \\ \cdots & \cdots & \cdots \\ -N^t & \vdots & 0 \end{bmatrix} \qquad \text{where} \qquad N = \begin{bmatrix} 0 & \cdots & 0 & \lambda_1 \\ 0 & \cdots & \lambda_2 & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \lambda_n & 0 & \cdots & 0 \end{bmatrix}$$

with $\lambda_i \neq 0 \; \forall i = 1, \cdots, n$. (One can show the existence of such a basis). For a section $K$ of $V^* \otimes V$ with $[K]_B = \text{diag} \quad (\lambda_1, \cdots, \lambda_n, \lambda_n, \cdots, \lambda_1)$, set

$$J = WK^{-1} \quad \text{and} \quad g(X, Y) = \phi(X, J(Y)).$$

Then $(J, g)$ is the required Hermitian structure on $V$.

Further, since $V^\pm \cap P(V^\pm) = 0$, the vector bundle $V$ has two splittings

$$V = V^+ \oplus P(V^+) \quad \text{and} \quad V = V^- \oplus P(V^-),$$

where $V^+$ and $V^-$ are the $(h, G)$-induced subbundles. Thus we have two complex structures $J_+$ and $J_-$ on $V$ given by

$$J_\pm(X + P(Y)) = -Y + P(X) \quad \text{for} \quad X, Y \in C(V^\pm).$$

Moreover, for $u, v, w, z \in C(V^+)$ set $X = u + P(v)$ and $Y = w + P(z)$ and observe that

$$\phi(J_+X, J_+Y) = \phi(v, z) - \phi(u, w) + \phi(u, P(z)) + \phi(P(v), w),$$

$$\phi(X, Y) = \phi(u, w) - \phi(v, z) + \phi(u, P(z)) + \phi(P(v), w).$$

Thus we get $\phi(J_+X, J_+Y) = \phi(X, Y)$ if and only if $\phi \equiv 0$ on $V^+$, that is, $\phi(u, w) = h(u, P(w)) = 0\ \forall u, w \in C(V^+)$. Hence the equivalence of $(a^\circ)$, $(b^\circ), (c^\circ), (d^\circ) and (e^\circ)$ follows easily. □

We call a complex structure on a vector bundle induced by the pair $(\phi, G)$ with $(P, h)-$associated $\phi$ *the* $(P, h, G)$*-related complex structure.*

**Remarks.**

1°) The statement of the above theorem is no longer true for the case of paracomplex structures, i.e. that symplectic vector bundle may not admit any paracomplex structure at all, e.g. $S^2$ and $S^6$ admit almost symplectic structures and yet do not admit any almost paracomplex structure.

2°) The $\phi$-compatible complex structure $J$ in the above theorem gives rise to a Hermitian structure $(J, g)$ via $g(X, Y) = \phi(X, J(Y))$ whose associated symplectic structure coincides with $\phi$.

3°) If the $(P, h, G)-$related complex structures $J_+$ and $J_-$ on $V$ satisfy that $J_+ = -J_-$ then, by the above theorem, they are $\phi$-compatible. (For the case where $J_+ \neq -J_-$, see Example 3.1 at the end). Thus $J_\pm$ together with the metric $g_\pm(X, Y) = \phi(X, J_\pm(Y))$ defines a Hermitian structure $(J_\pm, g_\pm)$ whose associated symplectic structure coincides with $\phi$. Also the Hermitian metric has the following properties:

$a^\circ$) $g_+ = h$ on $V^+$, $g_+ = -h$ on $V^-$ and $V^+, V^-$ are $g_+$-orthogonal.

$b^\circ$) For a $G$ and $h$ orthogonal local frame field $\{u_1, \cdots, u_{2n}\}$ of $V$ we have $g_+(u_i, u_j) = \lambda_i G(u_i, u_j)$ for some nonzero $\lambda_i(p) \in \mathbb{R}$; $\forall\ i, j = 1, 2, \cdots, (2n)$; $p \in M$.

$c^\circ$) $g_+ = -g_-$, where $g_-(X, Y) = \phi(X, J_-(Y))$

**Theorem 2.7.** *For a vector bundle $V$ the following are equivalent:*
*i) $V$ admits a product structure $F$ of signature $(\ell, t)$*
*ii) $V$ admits a semi-Riemannian metric $h$ of signature $(\ell, t)$.*

Proof. Assume $(i)$, then we have a splitting $V = F^+ \oplus F^-$. For a

Riemannian metric $g$ on $V$ set $h(X,Y) = g(X,Y)$ if $X, Y \in F^+$; $h(X,Y) = -g(X,Y)$ if $X, Y \in F^-$ and $h(X,Y) = 0$ otherwise. Then $h$ is the required semi-Riemannian metric on $V$.

Conversely assume (ii), then $V$ admits a splitting $V = V^+ \oplus V^-$ with $\dim V^+ = \ell$ and $\dim V^- = t$ by Lemma (1.1). So set $F(X) = X$ if $X \in V^+$ and $F(X) = -X$ if $X \in V^-$. Then $F$ is the required product structure. □

**Remark.** In the case where $\ell = t$ the above theorem states that:

*$N$ admits an almost paracomplex structure $Q$ (of signature $(\ell,\ell)$) if and only if $N$ admits a neutral metric $H$ (of signature $(\ell,\ell)$).*

**3. A special case and an example.** Let $N$ be an oriented 4-manifold and set $V = TN$. Then the following two conditions are equivalent, [2]:

*i) $N$ admits two distinct mutually commuting almost complex structures*

*ii) $N$ admits non-degenarate field of oriented 2-planes.*

On the other hand, from the last Remark, we have the following two statements that are equivalent to each other

*iii) $N$ admits an almost paracomplex structure $Q$ (of signature $(2,2)$)*

*iv) $N$ admits a neutral metric $H$ (of signature $(2,2)$)*

Also note that $(Q,H)$ does not form an almost para-Hermitian structure on $N$. It is easy to see that $(i)$ (and therefore $(ii)$) implies $(iii)$ (and therefore $(iv)$). But the converse is not true as the field of 2-planes induced by either $Q$ or $H$ is not necessarily orientable. However, the fact that "*$(i)$ implies $(iii)$*" enables us to consider some complex surfaces such as *some certain classes of minimal rational surfaces, Hopf surfaces and Inoue surfaces, ruled surfaces of genus $\mu \geq 1$, Enriques surfaces, hyperelliptic surfaces, Kodaira surfaces, $K3$ surfaces, Tori and minimal property elliptic surfaces* as examples of paracomplex (and also semi-Riemannian) compact 4-manifolds which are not product of real surfaces, (for detail see [2]). Note that the tangent bundle of a product manifold $M$ has a canonical splitting and therefore $M$ has a product structure and a semi-Riemannian metric.

**Example 3.1.** For $X = (x_i),\ Y = (y_i) \in \mathbb{R}^7$ set

$$H(X,Y) = -\sum_{i=1}^{3} x_i y_i + \sum_{i=4}^{7} x_i y_i \quad \textit{and} \quad S^6_3 = \{X \in \mathbb{R}^7 : H(X,X) = 1\}$$

and denote by $h_\circ$ the neutral metric (which is naturally of signature (3,3)) on $S^6_3$ induced by $H$.

Also for $X = (x_i) \in S^6_3 \subseteq \mathbb{R}^7$ and $Y = (y_i) \in T_X S^6_3 \subseteq \mathbb{R}^7$ Let $Q \in C(T^* S^6_3 \otimes T S^6_3)$

be a tensor field such that at $X \in S^6_3$, $\quad Q_X : T_X S^6_3 \to T_X S^6_3$ is given by

$$Q_X(Y) = (a_1(X,Y), \cdots, a_7(X,Y))$$

where

$$\begin{aligned}
a_1 &= x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4 - x_6y_7 + x_7y_6 \\
a_2 &= -x_1y_3 + x_3y_1 + x_4y_7 - x_7y_4 - x_5y_6 + x_6y_5 \\
a_3 &= x_1y_2 - x_2y_1 + x_4y_6 - x_6y_4 + x_5y_7 - x_7y_5 \\
a_4 &= x_1y_5 - x_5y_1 + x_2y_7 - x_7y_2 + x_3y_6 - x_6y_3 \\
a_5 &= -x_1y_4 + x_4y_1 - x_2y_6 + x_6y_2 + x_3y_7 - x_7y_3 \\
a_6 &= -x_1y_7 + x_7y_1 + x_2y_5 - x_5y_2 - x_3y_4 + x_4y_3 \\
a_7 &= x_1y_6 - x_6y_1 - x_2y_4 + x_4y_2 - x_3y_5 + x_5y_3
\end{aligned}$$

This tensor field $Q$, together with the metric $h_\circ$ defines an almost para-Hermitian structure $(Q, h_\circ)$ on $S^6_3$ which is first described by Libermann [1] by using the Cayley's split octaves. Denote the $(Q_\circ, h_\circ)-$ associated almost symplectic structure on $S^6_3$ by $\phi_\circ$

Now let $\alpha = (0,0,1;1,1,0,0) \in S^6_3$, then the set $U = \{E_1, \cdots, E_6\}$ forms a basis for $T_\alpha = T_\alpha S^6_3$ where

$$E_1 = (1,0,\cdots,0), \quad E_2 = (0,1,0,\cdots,0), \quad E_3 = (0,0,1;1,0,0,0)$$

$$E_4 = (0,0,1;0,1,0,0), \quad E_5 = (0,\cdots,0,1,0), \quad E_6 = (0,\cdots,0,1)$$

The matrix representation $A = [W_\alpha]_U$ in $U$ of a linear transformation $W_\alpha : T_\alpha \to T_\alpha$ defined by the equation

$$G_\circ(W(Z), Y) = h_\circ(Z, Y)$$

takes the form

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & -2/3 & 0 & 0 \\ 0 & 0 & -2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $G_\circ$ is the Riemannian metric on $S^6_3$ obtained by restricting the standard inner product on $\mathbb{R}^7$ to $T_X S^6_3$. Hence the $(h_\circ, G_\circ)$-induced subbundles $V^+, V^-$ of $TS^6_3$ have the fibres at $\alpha$ given by:

$$V^+_\alpha \;=\; \{Z \in T_\alpha S^6_3 : W(Z) = Z\} = \text{span } \{E_3 - E_4, E_5, E_6\}$$

$$V_\alpha^- = \{Z \in T_\alpha S_3^6 : W(Z) = -Z\} = \text{span } \{E_1, E_2, E_3 + E_4\}$$

We see that $Q_\alpha(E_1) = (0,1,0;1,1,0,0) \notin V^+$ while $E_1 \in V^-$. So $Q(V^-) \neq V^+$, thus by Theorem 2.6, the pair $(\phi_\circ, G_\circ)$ induces three different almost complex structures $J_\circ, J_\circ^+$ and $J_\circ^-$ on $S_3^6$ and that $(J_\circ, g_\circ)$ defines an almost Hermitian structure whose associated almost symplectic structure coincides with $\phi_\circ$, where $g_\circ(X,Y) = h_\circ(X, QJ_\circ(Y)) = \phi(X, J_\circ(Y))$ which is positive definite. On the other hand $(J_\circ^\pm, g_\circ^\pm)-$ associated symplectic structure $\phi_\circ^\pm$ does not coincide with $\phi_\circ$.

## REFERENCES

[1] P. Libermann. Sur le probleme d'equivalence de certaines structures infinitesimales. *Ann. Mat. Pura Appl. (4)* **36** (1954), 27-120.

[2] Y. Matsushita. Fields of 2-planes and two kinds of almost complex structures on compact 4-dimensional manifolds. *Math. Z.* **207** (1991), 281-291.

[3] N. Steenrod. The Topology of Fibre Bundles. Princeton University Press, 1951, Princeton, New Jersey.

[4] I. Vaisman. Symplectic geometry and secondary characteristic classes. Progress in Mathematics vol. **72**, Birkhauser Boston, Basel, 1987.

*Department of Mathematics*
*Middle East Technical University*
*06531 Ankara*
*Turkey*
*e-mail:* `serdem@arf.math.metu.edu.tr`

*Received September 7, 1998*
*Revised January 28, 1999*